CHAPTER 1

Functions and Their Graphs

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On a sales rack of clothes at a department store, you see a shirt you like. The original price of the shirt was $100, but it has been discounted 30%. As a preferred shopper, you get an automatic additional 20% off the sale price at the register. How much will you pay for the shirt?

Naïve shoppers might be lured into thinking this shirt will cost $50 because they add the 20% and 30% to get 50% off, but they will end up paying more than that. Experienced shoppers know that they first take 30% off of $100, which results in a price of $70, and then they take an additional 20% off of the sale price, $70, which results in a final discounted price of $56. Experienced shoppers have already learned the concept of *composition of functions*.

A *composition of functions* can be thought of as a function of a function. One function takes an input (original price, $100) and maps it to an output (sale price, $70), and then another function takes that output as its input (sale price, $70) and maps that to an output (checkout price, $56).

FUNCTIONS AND THEIR GRAPHS

LEARNING OBJECTIVES

- Find the domain and range of a function.
- Sketch the graphs of common functions.
- Sketch graphs of general functions employing translations of common functions.
- Perform composition of functions.
- Find the inverse of a function.

In This Chapter

We will establish what a relation is, and then we will determine whether a relation is a function. We will discuss common functions, domain and range of functions, and graphs of functions. We will determine whether a function is increasing or decreasing on an interval and calculate the average rate of change of a function. We will perform operations on functions and composition of functions. Finally, we will discuss one-to-one functions and inverse functions.

1.1 Functions

SKILLS OBJECTIVES

- Determine whether a relation is a function.
- Determine whether an equation represents a function.
- Use function notation to evaluate functions for particular arguments.
- Determine the domain and range of a function.

CONCEPTUAL OBJECTIVES

- Understand that all functions are relations but not all relations are functions.
- Understand why the vertical line test determines if a relation is a function.
- Think of function notation as a placeholder or mapping.
- Understand the difference between implicit domain and explicit domain.

1.1.1 Relations and Functions

1.1.1 Skill Determine whether a relation is a function.

1.1.1 Conceptual Understand that all functions are relations but not all relations are functions.

What do the following sentences have in common?

- Every person has a blood type.
- Temperature is some specific value at a particular time of day.
- Every working household phone in the United States has a 10-digit phone number.
- First-class postage rates correspond to the weight of a letter.
- Certain times of the day are start times of sporting events at a university.

They all describe a particular correspondence between two groups. A **relation** is a correspondence between two sets. The first set is called the **domain**, and the corresponding second set is called the **range**. Members of these sets are called **elements**.

Relation

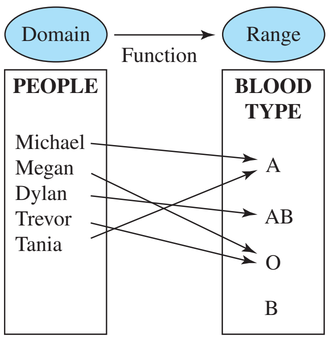
A **relation** is a correspondence between two sets where each element in the first set, called the **domain**, corresponds to *at least* one element in the second set, called the **range**.

A relation is a set of ordered pairs. The domain is the set of all the first components of the ordered pairs, and the range is the set of all the second components of the ordered pairs.

Person	Blood Type	Ordered Pair
Michael	A	(Michael, A)
Tania	A	(Tania, A)
Dylan	AB	(Dylan, AB)
Trevor	O	(Trevor, O)
Megan	O	(Megan, O)

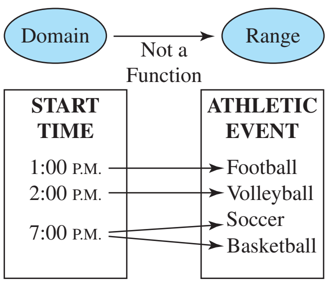
Words	**Math**
The domain is the set of all the first components.	{Michael, Tania, Dylan, Trevor, Megan}
The range is the set of all the second components.	{A, AB, O}

A *relation in* which *each element in the domain corresponds to exactly one element in the range* is a **function**.



Function

A **function** is a correspondence between two sets where each element in the first set, called the **domain**, corresponds to *exactly* one element in the second set, called the **range**.



Note that the definition of a function is more restrictive than the definition of a relation. For a relation, each input corresponds to *at least* one output, whereas, for a function, each input corresponds to *exactly* one output. The blood-type example given is both a relation and a function.

Also note that the range (set of values to which the elements of the domain correspond) is a subset of the set of all blood types. However, although all functions are relations, not all relations are functions.

For example, at a university, four primary sports typically overlap in the late fall: football, volleyball, soccer, and basketball. On a given Saturday, the accompanying table indicates the start times for the competitions.

Time of Day	Competition
1:00 P.M.	Football
2:00 P.M.	Volleyball
7:00 P.M.	Soccer
7:00 P.M.	Basketball

Words	Math
The 1:00 start time corresponds to exactly one event, football.	(1:00 P.M., football)
The 2:00 start time corresponds to exactly one event, volleyball.	(2:00 P.M., volleyball)
The 7:00 start time corresponds to two events, soccer and basketball.	(7:00 P.M., soccer) (7:00 P.M., basketball)

Because an element in the domain, 7:00 P.M., corresponds to more than one element in the range, soccer and basketball, this is not a function. It is, however, a relation.

Concept Check

If the domain consists of all physical (home) addresses in a particular county and the range is the persons living in that county, does this describe a relation? And if so, is that relation a function?

Answer: This is a relation but not a function.

STUDY TIP

All functions are relations but not all relations are functions.

EXAMPLE 1 | Determining Whether a Relation Is a Function

Determine whether the following relations are functions.

a. $\{(-3, 4), (2, 4), (3, 5), (6, 4)\}$

b. $\{(-3, 4), (2, 4), (3, 5), (2, 2)\}$

c. Domain = Set of all items for sale in a grocery store; Range = Price

Solution

a. No x-value is repeated. Therefore, each x-value corresponds to exactly one y-value.

This relation is a function.

b. The value $x = 2$ corresponds to *both* $y = 2$ and $y = 4$. This relation is not a function.

c. Each item in the grocery store corresponds to exactly one price. This relation is a function.

Your Turn Determine whether the following relations are functions.

a. $\{(1, 2), (3, 2), (5, 6), (7, 6)\}$

b. $\{(1, 2), (1, 3), (5, 6), (7, 8)\}$

c. {(11:00 A.M., 83°F), (2:00 P.M., 89°F), (6:00 P.M., 85°F)}

Answer

a. function

b. not a function

c. function

All of the examples we have discussed thus far are **discrete** sets in that they represent a countable set of distinct pairs of (x, y). A function can also be defined algebraically by an equation.

1.1.2 Functions Defined by Equations

1.1.2 Skill Determine whether an equation represents a function.

1.1.2 Conceptual Understand why the vertical line test determines if a relation is a function.

Let's start with the equation $y = x^2 - 3x$, where x can be any real number. This equation assigns to each x-value exactly one corresponding y-value.

x	$y = x^2 - 3x$	y
1	$y = (1)^2 - 3(1)$	-2
5	$y = (5)^2 - 3(5)$	10
$-\frac{2}{3}$	$y = \left(-\frac{2}{3}\right)^2 - 3\left(-\frac{2}{3}\right)$	$\frac{22}{9}$
1.2	$y = (1.2)^2 - 3(1.2)$	-2.16

Caution

Not all equations are functions.

Since the variable y *depends* on what value of x is selected, we denote y as the **dependent variable**. The variable x can be any number in the domain; therefore, we denote x as the **independent variable**.

Although functions are defined by equations, it is important to recognize that *not all equations are functions*. The requirement for an equation to define a function is that each element in the domain corresponds to exactly one element in the range. Throughout the ensuing discussion, we assume x to be the independent variable and y to be the dependent variable.

Equations that represent functions of x: $y = x^2$ $\quad y = |x|$ $\quad y = x^3$

Equations that *do not* represent functions of x: $x = y^2$ $\quad x^2 + y^2 = 1$ $\quad x = |y|$

STUDY TIP

We say that for $x = y^2$, y is not a function of x. However, if we reverse the independent and dependent variables, then for $x = y^2$, x is a function of y.

In the "equations that represent functions of x," every x-value corresponds to exactly one y-value. Some ordered pairs that correspond to these functions are

$y = x^2$: $(-1, 1)\ (0, 0)\ (1, 1)$

$y = |x|$: $(-1, 1)\ (0, 0)\ (1, 1)$

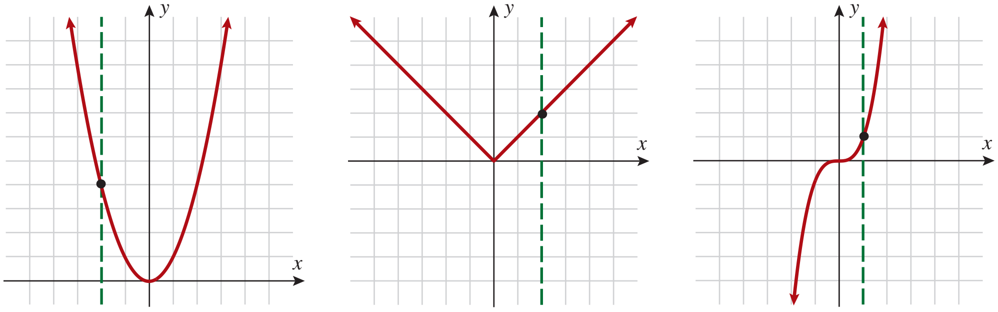
$y = x^3$: $(-1, -1)\ (0, 0)\ (1, 1)$

The fact that $x = -1$ and $x = 1$ both correspond to $y = 1$ in the first two examples does not violate the definition of a function.

In the "equations that do not represent functions of x," some x-values correspond to *more than one* y-value. Some ordered pairs that correspond to these equations are:

Relation	Solve Relation for y	Points that Lie on the Graph	
$x = y^2$	$y = \pm\sqrt{x}$	$(1, -1)\ (0, 0)\ (1, 1)$	$x = 1$ maps to **both** $y = -1$ and $y = 1$
$x^2 + y^2 = 1$	$y = \pm\sqrt{1 - x^2}$	$(0, -1)\ (0, 1)\ (-1, 0)\ (1, 0)$	$x = 0$ maps to **both** $y = -1$ and $y = 1$
$x = \|y\|$	$y = \pm x$	$(1, -1)\ (0, 0)\ (1, 1)$	$x = 1$ maps to **both** $y = -1$ and $y = 1$

Let's look at the graphs of the three **functions of x**:



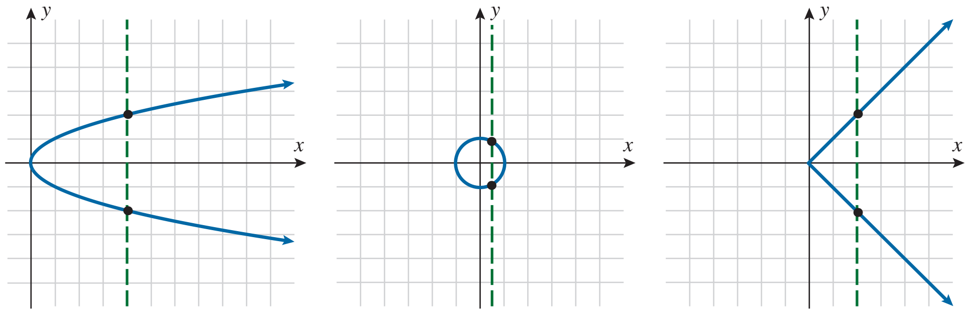
$y = x^2$ $\quad y = |x|$ $\quad y = x^3$

Let's take any value for x, say $x = a$. The graph of $x = a$ corresponds to a vertical line. A function of x maps each x-value to exactly one y-value; therefore, there should be at most one point of intersection with any vertical line. We see in these three graphs of functions that if a vertical line is drawn at *any* value of x on any of the three graphs, the vertical line only intersects the graph in one place. Look at the graphs of the three equations that do **not** represent **functions of** $\boldsymbol{x}$.

Concept Check

Draw two vertical lines ($x = 0$ and $x = 1$) on the graph of $x = y^2$. What three points do these lines intersect on the graph of the equation? Explain why the equation $x = y^2$ cannot be a function.

Answer: (0, 0) (1, −1), and (1, 1): the vertical line $x = 1$ intersects the graph at two places (fails the vertical line test).



$x = y^2$ $\qquad$ $x^2 + y^2 = 1$ $\qquad$ $x = |y|$

A vertical line can be drawn on any of the three graphs such that the vertical line will intersect each of these graphs at two points. Thus, there are two y-values that correspond to some x-value in the domain, which is why these equations do not define y as a function of x.

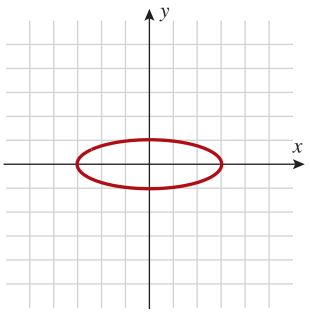
Vertical Line Test

Given the graph of an equation, if any vertical line that can be drawn intersects the graph at no more than one point, the equation defines y as a function of x. This test is called the **vertical line test**.

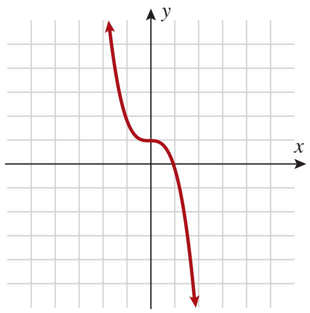
Video EXAMPLE 2 | Using the Vertical Line Test

Use the vertical line test to determine whether the graphs of equations define functions of x.

a.



b.



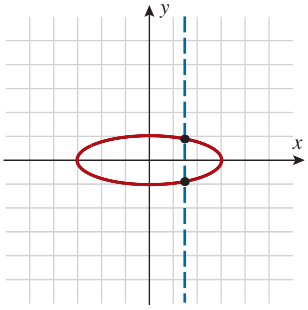
STUDY TIP

If any x-value corresponds to more than one y-value, then y is *not* a function of x.

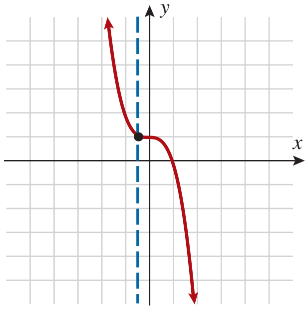
Solution

Apply the vertical line test.

a.



b.



a. Because the vertical line intersects the graph of the equation at two points, this equation does not represent a function.

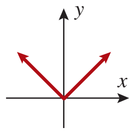
b. Because any vertical line will intersect the graph of this equation at no more than one point, this equation represents a function.

Your Turn Determine whether the equation $(x - 3)^2 + (y + 2)^2 = 16$ is a function of x.

Answer

The graph of the equation is a circle, which does not pass the vertical line test. Therefore, the equation does not define a function.

To recap, a function can be expressed one of four ways: verbally, numerically, algebraically, and graphically. This is sometimes called the Rule of 4.

Expressing a Function			
Verbally	**Numerically**	**Algebraically**	**Graphically**
Every real number has a corresponding absolute value.	$\{(-3, 3), (-1, 1), (0, 0), (1, 1), (5, 5)\}$	$y = \lvert x \rvert$	

1.1.3 Function Notation

1.1.3 Skill Use function notation to evaluate functions for particular arguments.

1.1.3 Conceptual Think of function notation as a placeholder or mapping.

We know that the equation $y = 2x + 5$ defines y as a function of x because its graph is a nonvertical line and thus passes the vertical line test. We can select x-values (input) and determine unique corresponding y-values (output). The output is found by taking 2 times the input and then adding 5. If we give the function a name, say, "f," then we can use **function notation**:

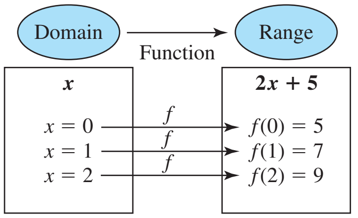
$$f(x) = 2x + 5$$

The symbol $f(x)$ is read "f evaluated at x" or "f of x" and represents the y-value that corresponds to a particular x-value. In other words, $y = f(x)$.

Input	Function	Output	Equation
x	f	$f(x)$	$f(x) = 2x + 5$
Independent variable	Mapping	Dependent variable	Mathematical rule

It is important to note that f is the function name, whereas $f(x)$ is the value of the function. In other words, the function f maps some value x in the domain to some value $f(x)$ in the range.

x	$f(x) = 2x + 5$	$f(x)$
0	$f(0) = 2(0) + 5$	$f(0) = 5$
1	$f(1) = 2(1) + 5$	$f(1) = 7$
2	$f(2) = 2(2) + 5$	$f(2) = 9$



The independent variable is also referred to as the **argument** of a function. To evaluate functions, it is often useful to think of the independent variable, or argument, as a placeholder. For example, $f(x) = x^2 - 3x$ can be thought of as

$$f(\square) = (\square)^2 - 3(\square)$$

In other words, "f of the argument is equal to the argument squared minus 3 times the argument." Any expression can be substituted for the argument:

$$f(1) = (1)^2 - 3(1)$$

$$f(x + 1) = (x + 1)^2 - 3(x + 1)$$

$$f(-x) = (-x)^2 - 3(-x)$$

It is important to note:

- $f(x)$ does *not* mean f times x.
- The most common function names are f and F since the word *function* begins with an "f." Other common function names are g and G, but any letter can be used.
- The letter most commonly used for the independent variable is x. The letter t is also common because in real-world applications it represents time, but any letter can be used.
- Although we can think of y and $f(x)$ as interchangeable, the function notation is useful when we want to consider two or more functions of the same independent variable.

STUDY TIP

It is important to note that $f(x)$ does not mean f times x.

Video EXAMPLE 3 | Evaluating Functions by Substitution

Given the function $f(x) = 2x^3 - 3x^2 + 6$, find $f(-1)$.

Solution

Consider the independent variable x to be a placeholder.	$f(\square) = 2(\square)^3 - 3(\square)^2 + 6$
To find $f(-1)$, substitute $x = -1$ into the function.	$f(-1) = 2(-1)^3 - 3(-1)^2 + 6$
Evaluate the right side.	$f(-1) = -2 - 3 + 6$
Simplify.	$\boxed{f(-1) = 1}$

Concept Check

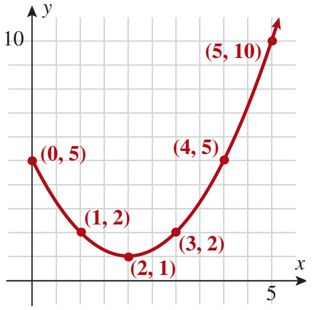
For the function in Example 3, find f(☺).

Answer: f(☺) = 2☺^{3} − 3☺^{2} + 6

EXAMPLE 4 | Finding Function Values from the Graph of a Function

The graph of f is given on the right.

a. Find $f(0)$.
b. Find $f(1)$.
c. Find $f(2)$.
d. Find $4f(3)$.
e. Find x such that $f(x) = 10$.
f. Find x such that $f(x) = 2$.



Solution

Solution (a): The value $x = 0$ corresponds to the value $y = 5$. $\boxed{f(0) = 5}$

Solution (b): The value $x = 1$ corresponds to the value $y = 2$. $\boxed{f(1) = 2}$

Solution (c): The value $x = 2$ corresponds to the value $y = 1$. $\boxed{f(2) = 1}$

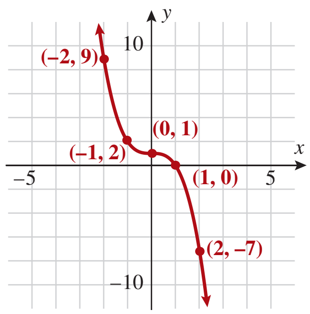
Solution (d): The value $x = 3$ corresponds to the value $y = 2$. $4f(3) = 4 \cdot 2 = \boxed{8}$

Solution (e): The value $y = 10$ corresponds to the value $\boxed{x = 5}$.

Solution (f): The value $y = 2$ corresponds to the values $\boxed{x = 1}$ and $\boxed{x = 3}$.

Your Turn For the following graph of a function, find:

a. $f(-1)$ **b.** $f(0)$ **c.** $3f(2)$
d. the value of x that corresponds to $f(x) = 0$



Answer

a. $f(-1) = 2$ **b.** $f(0) = 1$
c. $3f(2) = -21$ **d.** $x = 1$

Video

EXAMPLE 5 | Evaluating Functions with Variable Arguments (Inputs)

For the given function $f(x) = x^2 - 3x$, evaluate $f(x + 1)$ and simplify if possible.

Common Mistake

A common misunderstanding is to interpret the notation $f(x + 1)$ as a sum: $f(x + 1) \neq f(x) + f(1)$.

Correct

Write the original function.
$f(x) = x^2 - 3x$

Replace the argument x with a placeholder.
$f(\square) = (\square)^2 - 3(\square)$

Substitute $x + 1$ for the argument.
$f(x + 1) = (x + 1)^2 - 3(x + 1)$

Eliminate the parentheses.
$f(x + 1) = x^2 + 2x + 1 - 3x - 3$

Combine like terms.
$\boxed{f(x + 1) = x^2 - x - 2}$

Incorrect

The **ERROR** is in interpreting the notation as a sum.

$f(x + 1) \neq f(x) + f(1)$
$\neq x^2 - 3x - 2$

Caution
$f(x + 1) \neq f(x) + f(1)$

Your Turn For the given function $g(x) = x^2 - 2x + 3$, evaluate $g(x-1)$.

Answer

$g(x-1) = x^2 - 4x + 6$

EXAMPLE 6 | Evaluating Functions: Sums

For the given function $H(x) = x^2 + 2x$, evaluate:

a. $H(x+1)$ **b.** $H(x) + H(1)$

Solution (a)

Write the function H in placeholder notation. $H(\square) = (\square)^2 + 2(\square)$

Substitute $x + 1$ for the argument of H. $H(x+1) = (x+1)^2 + 2(x+1)$

Eliminate the parentheses on the right side. $H(x+1) = x^2 + 2x + 1 + 2x + 2$

Combine like terms on the right side. $\boxed{H(x+1) = x^2 + 4x + 3}$

Solution (b)

Write $H(x)$. $H(x) = x^2 + 2x$

Evaluate H at $x = 1$. $H(1) = (1)^2 + 2(1) = 3$

Evaluate the sum $H(x) + H(1)$. $H(x) + H(1) = x^2 + 2x + 3$

$\boxed{H(x) + H(1) = x^2 + 2x + 3}$

Note: Comparing the results of part (a) and part (b), we see that

$$H(x+1) \neq H(x) + H(1).$$

EXAMPLE 7 | Evaluating Functions: Negatives

For the given function $G(t) = t^2 - t$, evaluate:

a. $G(-t)$ **b.** $-G(t)$

Solution (a)

Write the function G in placeholder notation. $G(\square) = (\square)^2 - (\square)$

Substitute $-t$ for the argument of G. $G(-t) = (-t)^2 - (-t)$

Eliminate the parentheses on the right side. $\boxed{G(-t) = t^2 + t}$

Solution (b)

Write $G(t)$. $G(t) = t^2 - t$

Multiply by -1. $-G(t) = -(t^2 - t)$

Eliminate the parentheses on the right side. $\boxed{-G(t) = -t^2 + t}$

Note: Comparing the results of part (a) and part (b), we see that $G(-t) \neq -G(t)$.

EXAMPLE 8 | Evaluating Functions: Quotients

For the given function $F(x) = 3x + 5$, evaluate:

a. $F\left(\frac{1}{2}\right)$ **b.** $\frac{F(1)}{F(2)}$

Solution (a)

Write F in placeholder notation. $F(\square) = 3(\square) + 5$

Replace the argument with $\frac{1}{2}$. $F\left(\frac{1}{2}\right) = 3\left(\frac{1}{2}\right) + 5$

Simplify the right side. $\boxed{F\left(\frac{1}{2}\right) = \frac{13}{2}}$

Solution (b)

Evaluate $F(1)$. $F(1) = 3(1) + 5 = 8$

Evaluate $F(2)$. $F(2) = 3(2) + 5 = 11$

Divide $F(1)$ by $F(2)$. $\boxed{\frac{F(1)}{F(2)} = \frac{8}{11}}$

Note: Comparing the results of part (a) and part (b), we see that $F\left(\frac{1}{2}\right) \neq \frac{F(1)}{F(2)}$.

Your Turn Given the function $G(t) = 3t - 4$, evaluate:

a. $G(t - 2)$ **b.** $G(t) - G(2)$ **c.** $\frac{G(1)}{G(3)}$ **d.** $G\left(\frac{1}{3}\right)$

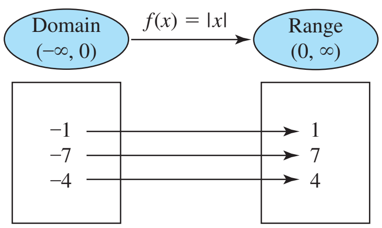
Answer

a. $G(t - 2) = 3t - 10$
b. $G(t) - G(2) = 3t - 6$
c. $\frac{G(1)}{G(3)} = -\frac{1}{5}$
d. $G\left(\frac{1}{3}\right) = -3$

1.1.4 Domain of a Function

1.1.4 Skill Determine the domain and range of a function.

1.1.4 Conceptual Understand the difference between implicit domain and explicit domain.



Sometimes the domain of a function is stated *explicitly*. For example,

$$f(x) = |x| \qquad \underbrace{x < 0}_{\text{domain}}$$

Here the **explicit domain** is the set of all negative real numbers $(-\infty, 0)$. Every negative real number in the domain is mapped to a positive real number in the range through the absolute value function.

If the expression that defines the function is given but the domain is not stated explicitly, then the domain is implied. The **implicit domain** is the largest set of real numbers for which the function is defined and the output value $f(x)$ is a real number. For example,

$$f(x) = \sqrt{x}$$

does not have the domain explicitly stated. There is, however, an implicit domain. Note that if the argument is negative, that is, if $x < 0$, then the result is an imaginary number. In order for the output of the function, $f(x)$, to be a real number, we must restrict the domain to nonnegative numbers, that is, if $x \geq 0$.

Function	Implicit Domain
$f(x) = \sqrt{x}$	$[0, \infty)$

In general, we ask the question, "What can x be?" The implicit domain of a function excludes values that cause a function to be undefined or have outputs that are not real numbers.

Expression That Defines the Function	Excluded x-Values	Example	Implicit Domain
Polynomial	None	$f(x) = x^3 - 4x^2$	All real numbers
Rational	x-values that make the denominator equal to 0	$g(x) = \dfrac{2}{x^2 - 9}$	$x \neq \pm 3$ or $(-\infty, -3) \cup (-3, 3) \cup (3, \infty)$
Radical	x-values that result in a square (even) root of a negative number	$h(x) = \sqrt{x - 5}$	$x \geq 5$ or $[5, \infty)$

Video EXAMPLE 9 | Determining the Domain of a Function

State the domain of the given functions.

a. $F(x) = \dfrac{3}{x^2 - 25}$ **b.** $H(x) = \sqrt[4]{9 - 2x}$ **c.** $G(x) = \sqrt[3]{x - 1}$

Solution (a)

Write the original equation. $F(x) = \dfrac{3}{x^2 - 25}$

Determine any restrictions on the values of x. $x^2 - 25 \neq 0$

Solve the restriction equation. $x^2 \neq 25$ or $x \neq \pm\sqrt{25} = \pm 5$

State the domain restrictions. $x \neq \pm 5$

Write the domain in interval notation. $\boxed{(-\infty, -5) \cup (-5, 5) \cup (5, \infty)}$

Solution (b)

Write the original equation. $H(x) = \sqrt[4]{9 - 2x}$

Determine any restrictions on the values of x. $9 - 2x \geq 0$

Solve the restriction inequality. $9 \geq 2x$

State the domain restrictions. $x \leq \dfrac{9}{2}$

Write the domain in interval notation. $\boxed{\left(-\infty, \dfrac{9}{2}\right]}$

Solution (c)

Write the original equation.	$G(x) = \sqrt[3]{x-1}$
Determine any restrictions on the values of x.	no restrictions
State the domain.	$\mathbb{R}$
Write the domain in interval notation.	$\boxed{(-\infty, \infty)}$

Your Turn State the domain of the given functions.

a. $f(x) = \sqrt{x-3}$ **b.** $g(x) = \dfrac{1}{x^2 - 4}$

Answer

a. $x \geq 3$ or $[3, \infty)$

b. $x \neq \pm 2$ or $(-\infty, -2) \cup (-2, 2) \cup (2, \infty)$

Concept Check

Find the implicit domain for $f(x) = \dfrac{1}{\sqrt{x-a}}$.

Answer: (a, ∞)

Applications Functions that are used in applications often have restrictions on the domains due to physical constraints. For example, the volume of a cube is given by the function $V(x) = x^3$, where x is the length of a side. The function $f(x) = x^3$ has no restrictions on x, and therefore the domain is the set of all real numbers. However, the volume of any cube has the restriction that the length of a side can never be negative or zero.

EXAMPLE 10 | The Dimensions of a Pool

Express the volume of a 30 ft × 10 ft rectangular swimming pool as a function of its depth.

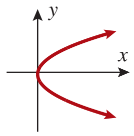
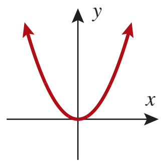
Solution

The volume of any rectangular box is $V = lwh$, where V is the volume, l is the length, w is the width, and h is the height. In this example, the length is 30 ft, the width is 10 ft, and the height represents the depth d of the pool.

Write the volume as a function of depth d.	$V(d) = (30)(10)d$
Simplify.	$\boxed{V(d) = 300d}$
Determine any restrictions on the domain.	$d > 0$

Section 1.1 Summary

Relations and Functions (*Let x represent the independent variable and y the dependent variable.*)

Type	Mapping/ Correspondence	Equation	Graph
Relation	Every x-value in the domain maps to **at least one** y-value in the range.	$x = y^2$	
Function	Every x-value in the domain maps to **exactly one** y-value in the range.	$y = x^2$	 Passes vertical line test

All functions are relations, but not all relations are functions. Functions can be represented by equations. In the following table, each column illustrates an alternative notation.

Input	Correspondence	Output	Equation
x	Function	y	$y = 2x + 5$
Independent Variable	Mapping	Dependent Variable	Mathematical Rule
Argument	f	$f(x)$	$f(x) = 2x + 5$

The **domain** is the set of all inputs (x-values), and the **range** is the set of all corresponding outputs (y-values). Placeholder notation is useful when evaluating functions.

$$f(x) = 3x^2 + 2x$$
$$f(\square) = 3(\square)^2 + 2(\square)$$

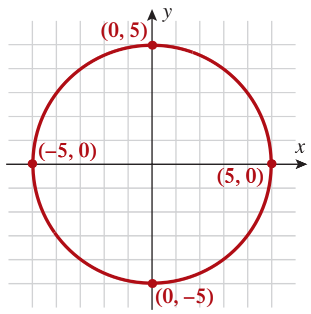
Explicit domain is stated, whereas **implicit domain** is found by *excluding* x-values that

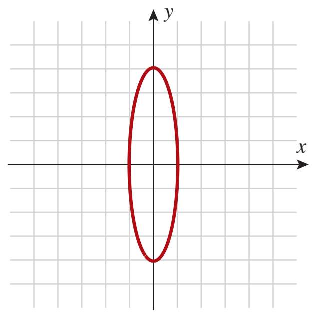
- make the function undefined (denominator = 0).
- result in a nonreal output (even roots of negative real numbers).

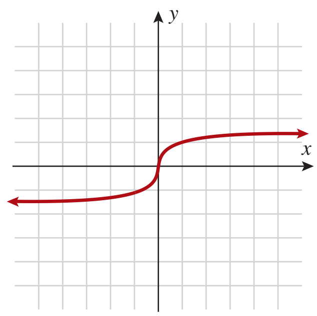
Section 1.1 Exercises

Skills

In Exercises 1–18, determine whether each relation is a function. Assume that the coordinate pair (x, y) represents the independent variable x and the dependent variable y.

1. $\{(0, -3), (0, 3), (-3, 0), (3, 0)\}$
2. $\{(2, -2), (2, 2), (5, -5), (5, 5)\}$
3. $\{(0, 0), (9, -3), (4, -2), (4, 2), (9, 3)\}$
4. $\{(0, 0), (-1, -1), (-2, -8), (1, 1), (2, 8)\}$
5. $\{(0, 1), (1, 0), (2, 1), (-2, 1), (5, 4), (-3, 4)\}$
6. $\{(0, 1), (1, 1), (2, 1), (3, 1)\}$
7. $x^2 + y^2 = 9$
8. $x = |y|$
9. $x = y^2$
10. $y = x^3$
11. $y = |x - 1|$
12. $y = 3$
13. 

14. 

15. 

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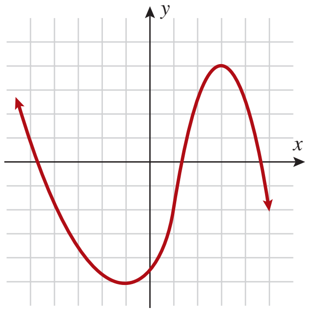

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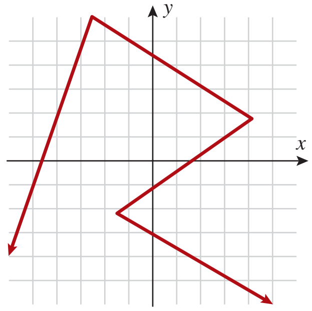
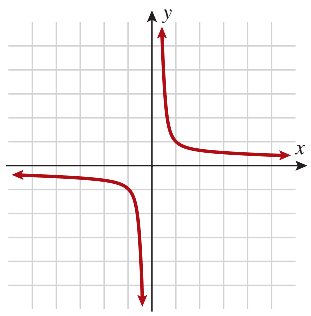
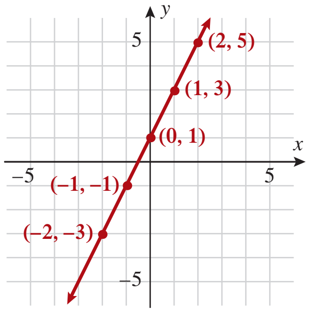


18.



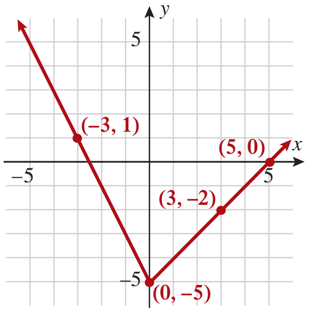
In Exercises 19–34, use the given graphs to evaluate the functions.

19. $y = f(x)$



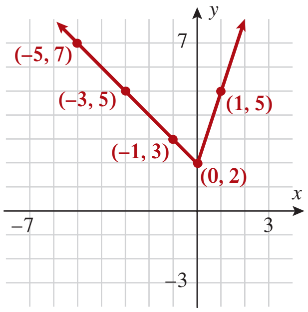
a. $f(2)$ **b.** $f(0)$ **c.** $f(-2)$

20. $y = g(x)$



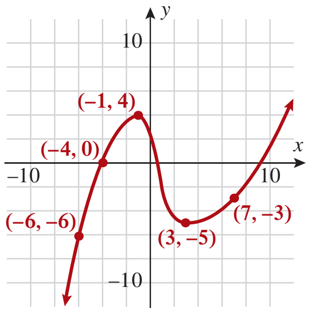
a. $g(-3)$ **b.** $g(0)$ **c.** $g(5)$

21. $y = p(x)$



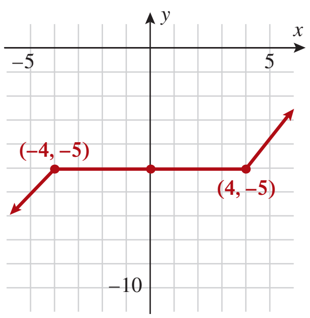
a. $p(-1)$ **b.** $p(0)$ **c.** $p(1)$

22. $y = r(x)$



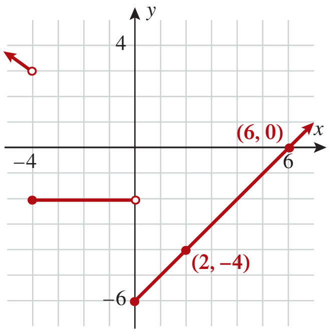
a. $r(-4)$ **b.** $r(-1)$ **c.** $r(3)$

23. $y = C(x)$



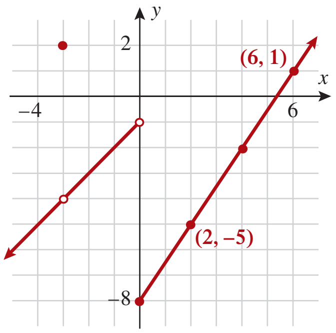
a. $C(2)$ **b.** $C(0)$ **c.** $C(-2)$

24. $y = q(x)$



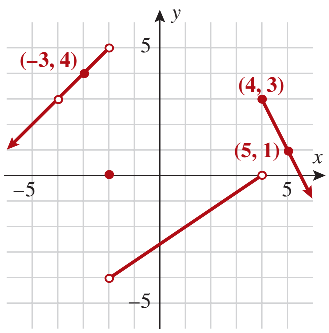
a. $q(-4)$ **b.** $q(0)$ **c.** $q(2)$

25. $y = S(x)$



a. $S(-3)$ **b.** $S(0)$ **c.** $S(2)$

26. $y = T(x)$



a. $T(-5)$ **b.** $T(-2)$ **c.** $T(4)$

27. Find x if $f(x) = 3$ in Exercise 19.

28. Find x if $g(x) = -2$ in Exercise 20.

29. Find x if $p(x) = 5$ in Exercise 21.

30. Find x if $r(x) = -7$ in Exercise 22.

31. Find x if $C(x) = -5$ in Exercise 23.

32. Find x if $q(x) = -2$ in Exercise 24.

33. Find x if $S(x) = 1$ in Exercise 25.

34. Find x if $T(x) = 4$ in Exercise 26.

In Exercises 35–54, evaluate the given quantities applying the following four functions.

$f(x) = 2x - 3$ $\quad F(t) = 4 - t^2$ $\quad g(t) = 5 + t$ $\quad G(x) = x^2 + 2x - 7$

35. $f(-2)$

36. $G(-3)$

37. $g(1)$

38. $F(-1)$

39. $f(-2) + g(1)$

40. $G(-3) - F(-1)$

41. $3f(-2) - 2g(1)$

42. $2F(-1) - 2G(-3)$

43. $\dfrac{f(-2)}{g(1)}$

44. $\dfrac{G(-3)}{F(-1)}$

45. $\dfrac{f(0) - f(-2)}{g(1)}$

46. $\dfrac{G(0) - G(-3)}{F(-1)}$

47. $f(-x)$

48. $-F(t)$

49. $-g(-t)$

50. $G(-x)$

51. $f(x+1) - f(x-1)$

52. $F(t+1) - F(t-1)$

53. $g(x+a) - f(x+a)$

54. $G(x+b) + F(b)$

In Exercises 55–90, find the domain of the given function. Express the domain in interval notation.

55. $f(x) = 2x - 5$

56. $f(x) = -2x - 5$

57. $g(t) = t^2 + 3t$

58. $h(x) = 3x^4 - 1$

59. $P(x) = \dfrac{x+5}{x-5}$

60. $Q(t) = \dfrac{2-t^2}{t+3}$

61. $T(x) = \dfrac{2}{x^2-4}$

62. $R(x) = \dfrac{1}{x^2-1}$

63. $F(x) = \dfrac{1}{x^2+1}$

64. $G(t) = \dfrac{2}{t^2+4}$

65. $q(x) = \sqrt{7-x}$

66. $k(t) = \sqrt{t-7}$

67. $f(x) = \sqrt{2x+5}$

68. $g(x) = \sqrt{5-2x}$

69. $G(t) = \sqrt{t^2-4}$

70. $F(x) = \sqrt{x^2-25}$

71. $F(x) = \dfrac{1}{\sqrt{x-3}}$

72. $G(x) = \dfrac{2}{\sqrt{5-x}}$

73. $f(x) = \sqrt[3]{1-2x}$

74. $g(x) = \sqrt[5]{7-5x}$

75. $P(x) = \dfrac{1}{\sqrt[5]{x+4}}$

76. $Q(x) = \dfrac{x}{\sqrt[3]{x^2-9}}$

77. $R(x) = \dfrac{x+1}{\sqrt[4]{3-2x}}$

78. $p(x) = \dfrac{x^2}{\sqrt{25-x^2}}$

79. $H(t) = \dfrac{t}{\sqrt{t^2-t-6}}$

80. $f(t) = \dfrac{t-3}{\sqrt[4]{t^2+9}}$

81. $f(x) = (x^2-16)^{1/2}$

82. $g(x) = (2x-5)^{1/3}$

83. $r(x) = x^2(3-2x)^{-1/2}$

84. $p(x) = (x-1)^2(x^2-9)^{-3/5}$

85. $f(x) = \dfrac{2}{5}x - \dfrac{2}{4}$

86. $g(x) = \dfrac{2}{3}x^2 - \dfrac{1}{6}x - \dfrac{3}{4}$

87. Let $g(x) = x^2 - 2x - 5$ and find the values of x that correspond to $g(x) = 3$.

88. Let $g(x) = \frac{5}{6}x - \frac{3}{4}$ and find the value of x that corresponds to $g(x) = \frac{2}{3}$.

89. Let $f(x) = 2x(x-5)^3 - 12(x-5)^2$ and find the values of x that correspond to $f(x) = 0$.

90. Let $f(x) = 3x(x+3)^2 - 6(x+3)^3$ and find the values of x that correspond to $f(x) = 0$.

Applications

91. Temperature. The average temperature in Tampa, Florida, in the springtime is given by the function $T(x) = -0.7x^2 + 16.8x - 10.8$, where T is the temperature in degrees Fahrenheit and x is the time of day in military time and is restricted to $6 \le x \le 18$ (sunrise to sunset). What is the temperature at 6 A.M.? What is the temperature at noon?

92. Temperature. The average temperature in Orlando, Florida in the summertime is given by the function $T(x) = -0.5x^2 + 14.2x - 2.8$, where T is the temperature in degrees Fahrenheit and x is the time of the day in military time and is restricted to $7 \le x \le 20$ (sunrise to sunset). What is the temperature at 9 A.M.? What is the temperature at 3 P.M.?

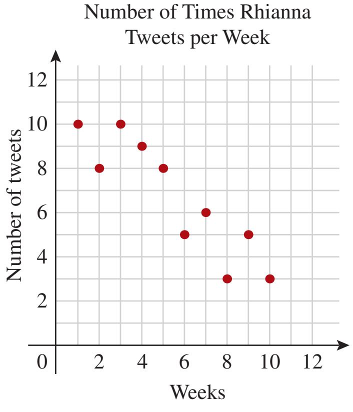
93. Falling Objects: Baseballs. A baseball is hit and its height is a function of time, $h(t) = -16t^2 + 45t + 1$, where h is the height in feet and t is the time in seconds, with $t = 0$ corresponding to the instant the ball is hit. What is the height after 2 seconds? What is the domain of this function?

94. Falling Objects: Firecrackers. A firecracker is launched straight up, and its height is a function of time, $h(t) = -16t^2 + 128t$, where h is the height in feet and t is the time in seconds with $t = 0$ corresponding to the instant it launches. What is the height 4 seconds after launch? What is the domain of this function?

95. Volume. An open box is constructed from a square 10-inch piece of cardboard by cutting squares of length x inches out of each corner and folding the sides up. Express the volume of the box as a function of x, and state the domain.

96. Volume. A cylindrical water basin will be built to harvest rainwater. The basin is limited in that the largest radius it can have is 10 feet. Write a function representing the volume of water V as a function of height h. How many additional gallons of water will be collected if you increase the height by 2 feet? *Hint:* 1 cubic foot = 7.48 gallons.

For Exercises 97 and 98, refer to the following:
Suppose that the number of times per week that Rihanna tweets over a 10-week period after releasing a new single is shown in the graph. Assume the number of tweets $T(t)$ is a function of time (week); let $T(1)$ be the number of tweets during the first week of release.



97. Music. Estimate the number of tweets Rihanna sent during weeks 4, 7, and 8.

98. Music. Find the increase or decrease in the number of tweets from (a) week 2 to 3 and (b) week 5 to 6.

For Exercises 99 and 100, refer to the following:

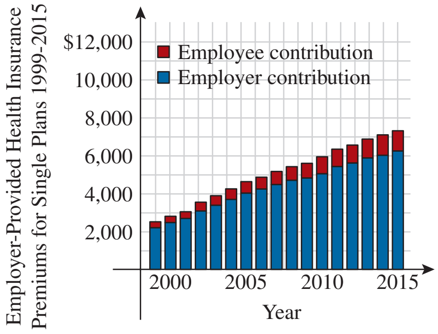
An epidemiological study of the spread of malaria in a rural area finds that the total number P of people who contracted malaria t days into an outbreak is modeled by the function

$$P(t) = -\frac{1}{4}t^2 + 7t + 180 \quad 1 \le t \le 14$$

99. Medicine/Health. How many people have contracted malaria 14 days into the outbreak?

100. Medicine/Health. How many people have contracted malaria 6 days into the outbreak?

For Exercises 101 and 102, use the following figure:



Source: Data from 2015 Employer Health Benefits Survey, Kaiser Family Foundation Health Research and Education Trust.

101. Health Care Costs. Fill in the following table. Round dollars to the nearest $1000.

Year	Total Health Care Cost for Family Plans
1999	
2003	
2007	
2011	
2015	

Write the five ordered pairs resulting from the table.

102. Health Care Costs. Using the table found in Exercise 101, let the years correspond to the domain and the total costs correspond to the range. Is this relation a function? Explain.

For Exercises 103 and 104, use the following information:

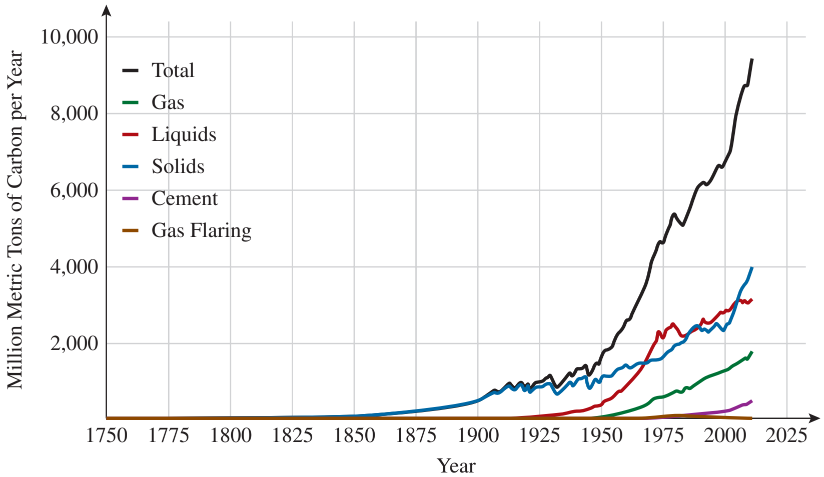
Let the functions f, F, g, G, h, and H represent the number of tons of carbon emitted per year as a function of year corresponding to gas flaring, cement, solids, liquids, gas, and the total amount respectively. Let t represent the year, with $t = 0$ corresponding to 1925.

103. Environment: Estimate (to the nearest thousand) the value of

a. $F(50)$ **b.** $g(50)$ **c.** $H(50)$

104. Environment: Explain what the sum $F(100) + g(100) + G(100)$ represents.

Total and Individual Contributors

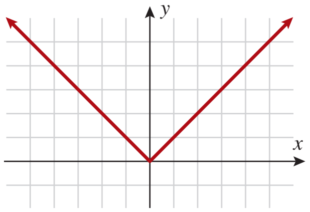


Source: Boden T; Marland G; Andres R J (1999): Global, Regional, and National Fossil-Fuel CO_2 Emissions (1751–2014) (V. 2017). Carbon Dioxide Information Analysis Center (CDIAC), Oak Ridge National Laboratory (ORNL), Oak Ridge, TN (United States). Licensed under CC-BY-4.0. https://serc.carleton.edu/details/images/46489.html

Catch the Mistake

In Exercises 105–108, explain the mistake that is made.

105. Determine whether the relationship is a function.

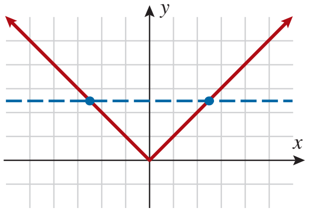


Solution

Apply the horizontal line test.

Because the horizontal line intersects the graph in two places, this is not a function.

This is incorrect. What mistake was made?



106. Given the function $H(x) = 3x - 2$, evaluate the quantity $H(3) - H(-1)$.

Solution

$H(3) - H(-1) = H(3) + H(1) = 7 + 1 = 8$

This is incorrect. What mistake was made?

107. Given the function $f(x) = x^2 - x$, evaluate the quantity $f(x+1)$.

Solution

$f(x+1) = f(x) + f(1) = x^2 - x + 0$

$f(x+1) = x^2 - x$

This is incorrect. What mistake was made?

108. Determine the domain of the function $g(t) = \sqrt{3-t}$ and express it in interval notation.

Solution

What can t be? Any nonnegative real number.

$3 - t > 0$

$3 > t \quad \text{or} \quad t < 3$

Domain: $(-\infty, 3)$

This is incorrect. What mistake was made?

Conceptual

In Exercises 109–112, determine whether each statement is true or false.

109. If a vertical line does not intersect the graph of an equation, then that equation does not represent a function.

110. If a horizontal line intersects a graph of an equation more than once, the equation does not represent a function.

111. For $x = y^2$, x is a function of y.

112. For $y = x^2$, y is a function of x.

113. If $f(x) = Ax^2 - 3x$ and $f(1) = -1$, find A.

114. If $g(x) = \dfrac{1}{b-x}$ and $g(3)$ is undefined, find b.

Challenge

115. If $F(x) = \dfrac{C-x}{D-x}$, $F(-2)$ is undefined, and $F(-1) = 4$, find C and D.

116. Construct a function that is undefined at $x = 5$ and whose graph passes through the point $(1, -1)$.

In Exercises 117 and 118, find the domain of each function, where *a* is any positive real number.

117. $f(x) = \dfrac{-100}{x^2 - a^2}$

118. $f(x) = -5\sqrt{x^2 - a^2}$

Preview to Calculus

For Exercises 119–126, refer to the following:

In calculus, the difference quotient $\dfrac{f(x+h) - f(x)}{h}$ of a function f is used to find a new function f', called the *derivative of f*. To find f', we let h approach 0, $h \to 0$, in the difference quotient. For example, if $f(x) = x^2$, $\dfrac{f(x+h) - f(x)}{h} = 2x - h$, and allowing $h = 0$, we have $f'(x) = 2x$.

119. $f(x) = 2x^2 + 7x + 3$

120. $f(x) = 6x^2 + 5x + 1$

121. Given $f(x) = x^3 + x$, find $f'(x)$.

122. Given $f(x) = 6x + \sqrt{x}$, find $f'(x)$.

123. Given $f(x) = \dfrac{x-5}{x+3}$, find $f'(x)$.

124. Given $f(x) = \sqrt{\dfrac{x+7}{5-x}}$, find $f'(x)$.

125. $f(x) = 2\sqrt{x} + 1$

126. $f(x) = \dfrac{1}{x-2}$

1.2 Graphs of Functions; Piecewise-Defined Functions; Increasing and Decreasing Functions; Average Rate of Change

SKILLS OBJECTIVES

- Classify functions as even, odd, or neither.
- Determine whether functions are increasing, decreasing, or constant.
- Calculate the average rate of change of a function.
- Graph piecewise-defined functions.

CONCEPTUAL OBJECTIVES

- Identify common functions and understand that even functions have graphs that are symmetric about the *y*-axis and odd functions have graphs that are symmetric about the origin.
- Understand that intervals of increasing, decreasing, and constant correspond to the *x*-coordinates.
- Understand that the difference quotient is just another form of the average rate of change.
- Understand points of discontinuity and domain and range of piecewise-defined functions.

1.2.1 Recognizing and Classifying Functions

1.2.1 Skill Classify functions as even, odd, or neither.

1.2.1 Conceptual Identify common functions and understand that even functions have graphs that are symmetric about the *y*-axis and odd functions have graphs that are symmetric about the origin.

Common Functions The nine main functions you will read about in this section will constitute a "library" of functions that you should commit to memory. We will draw on this library of functions in the next section when graphing transformations are discussed. Several of these functions have been shown previously in this chapter, but now we will classify them specifically by name and identify properties that each function exhibits.

In Section 0.6, we discussed equations and graphs of lines. All lines (with the exception of vertical lines) pass the vertical line test and hence are classified as functions. Instead of the traditional notation of a line, $y = mx + b$, we use function notation and classify a function whose graph is a *line* as a *linear* function.

Linear Function

$f(x) = mx + b$ $\quad$ m and b are real numbers.

The domain of a linear function $f(x) = mx + b$ is the set of all real numbers $\mathbb{R}$. The graph of this function has slope m and y-intercept b.

Linear Function: $f(x) = mx + b$	**Slope: m**	**y-Intercept: b**
$f(x) = 2x - 7$	$m = 2$	$b = -7$
$f(x) = -x + 3$	$m = -1$	$b = 3$
$f(x) = x$	$m = 1$	$b = 0$
$f(x) = 5$	$m = 0$	$b = 5$

One special case of the linear function is the *constant function* ($m = 0$).

Constant Function

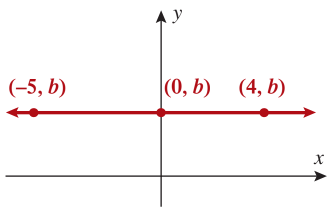
$f(x) = b$ $\quad\quad$ b is any real number.

The graph of a constant function $f(x) = b$ is a horizontal line. The y-intercept corresponds to the point $(0, b)$. The domain of a constant function is the set of all real numbers $\mathbb{R}$. The range, however, is a single value b. In other words, all x-values correspond to a single y-value.

Points that lie on the graph of a constant function $f(x) = b$ are

$(-5, b)$
$(-1, b)$
$(0, b)$
$(2, b)$
$(4, b)$
. . .
(x, b)

Domain: $(-\infty, \infty)$ $\quad$ Range: $[b, b]$ or $\{b\}$

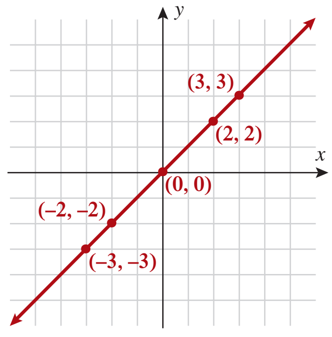


Another specific example of a linear function is the function having a slope of one ($m = 1$) and a y-intercept of zero ($b = 0$). This special case is called the *identity function.*

Identity Function

$$f(x) = x$$

Identity Function
Domain: $(-\infty, \infty)$ $\quad$ Range: $(-\infty, \infty)$



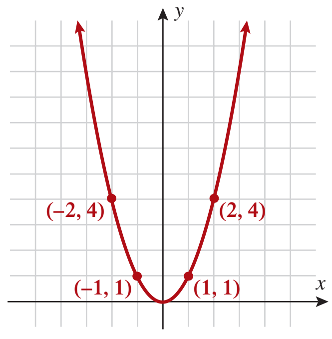
The graph of the identity function has the following properties: It passes through the origin, and every point that lies on the line has equal x- and y-coordinates. Both the domain and the range of the identity function are the set of all real numbers $\mathbb{R}$.

A function that squares the input is called the *square function.*

Square Function

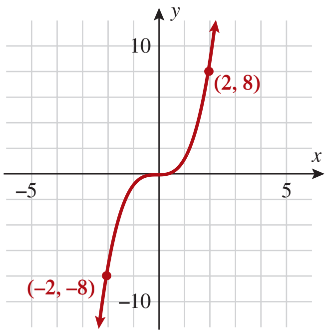
$$f(x) = x^2$$

Square Function
Domain: $(-\infty, \infty)$ $\quad$ Range: $[0, \infty)$



The graph of the square function is called a parabola and will be discussed in further detail in Chapters 2 and 9. The domain of the square function is the set of all real numbers $\mathbb{R}$. Because squaring a real number always yields a positive number or zero, the range of the square function is the set of all nonnegative numbers. Note that the intercept is the origin and the square function is symmetric about the y-axis. This graph is contained in quadrants I and II.

Cube Function
Domain: $(-\infty, \infty)$ Range: $(-\infty, \infty)$



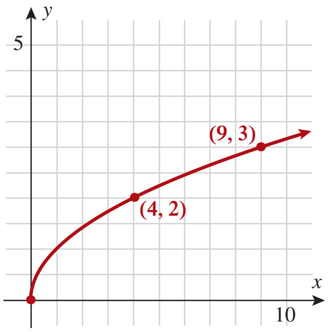
A function that cubes the input is called the *cube function.*

> **Cube Function**
>
> $$f(x) = x^3$$

The domain of the cube function is the set of all real numbers $\mathbb{R}$. Because cubing a negative number yields a negative number, cubing a positive number yields a positive number, and cubing 0 yields 0, the range of the cube function is also the set of all real numbers $\mathbb{R}$. Note that the only intercept is the origin and the cube function is symmetric about the origin. This graph extends only into quadrants I and III.

The next two functions are counterparts of the previous two functions: square root and cube root. When a function takes the square root of the input or the cube root of the input, the function is called the *square root function* or the *cube root function*, respectively.

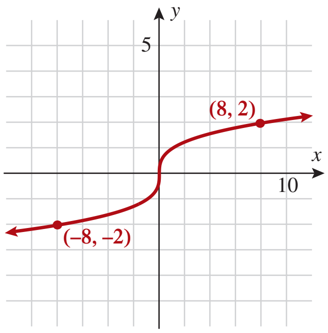
Square Root Function
Domain: $[0, \infty)$ Range: $[0, \infty)$



> **Square Root Function**
>
> $$f(x) = \sqrt{x} \quad \text{or} \quad f(x) = x^{1/2}$$

In Section 1.1, we found the domain to be $[0, \infty)$. The output of the function will be all real numbers greater than or equal to zero. Therefore, the range of the square root function is $[0, \infty)$. The graph of this function will be contained in quadrant I.

Cube Root Function
Domain: $(-\infty, \infty)$ Range: $(-\infty, \infty)$

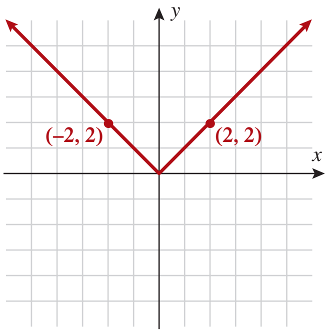


> **Cube Root Function**
>
> $$f(x) = \sqrt[3]{x} \quad \text{or} \quad f(x) = x^{1/3}$$

In Section 1.1, we stated the domain of the cube root function to be $(-\infty, \infty)$. We see by the graph that the range is also $(-\infty, \infty)$. This graph is contained in quadrants I and III and passes through the origin. This function is symmetric about the origin.

In Sections 0.3 and 0.4 you read about absolute value equations and inequalities. Now we shift our focus to the graph of the *absolute value function.*

Absolute Value Function
Domain: $(-\infty, \infty)$ Range: $[0, \infty)$



> **Absolute Value Function**
>
> $$f(x) = |x|$$

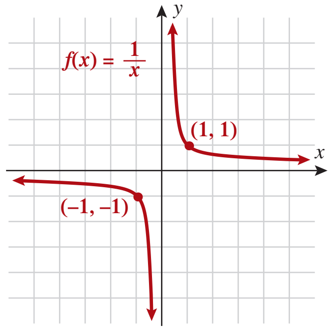
Some points that are on the graph of the absolute value function are $(-1, 1)$, $(0, 0)$, and $(1, 1)$. The domain of the absolute value function is the set of all real numbers $\mathbb{R}$, yet the range is the set of nonnegative real numbers. The graph of this function is symmetric with respect to the y-axis and is contained in quadrants I and II.

A function whose output is the reciprocal of its input is called the *reciprocal function.*

Reciprocal Function

$$f(x) = \frac{1}{x} \qquad x \neq 0$$

Reciprocal Function
Domain: $(-\infty, 0) \cup (0, \infty)$
Range: $(-\infty, 0) \cup (0, \infty)$



The only restriction on the domain of the reciprocal function is that $x \neq 0$. Therefore, we say the domain is the set of all real numbers excluding zero. The graph of the reciprocal function illustrates that its range is also the set of all real numbers except zero. Note that the reciprocal function is symmetric with respect to the origin and is contained in quadrants I and III.

Even and Odd Functions

Of the nine functions discussed above, several have similar properties of symmetry. The constant function, square function, and absolute value function are all symmetric with respect to the y-axis. The identity function, cube function, cube root function, and reciprocal function are all symmetric with respect to the origin. The term **even** is used to describe functions that are symmetric with respect to the y-axis, or vertical axis, and the term **odd** is used to describe functions that are symmetric with respect to the origin. Recall from Section 0.5 that symmetry can be determined both graphically and algebraically. The following box summarizes the graphic and algebraic characteristics of even and odd functions.

Even and Odd Functions

Function	Symmetric with Respect to	On Replacing x with $-x$
Even	y-axis, or vertical axis	$f(-x) = f(x)$
Odd	origin	$f(-x) = -f(x)$

The algebraic method for determining symmetry with respect to the y-axis, or vertical axis, is to substitute $-x$ for x. If the result is an equivalent equation, the function is symmetric with respect to the y-axis. Some examples of even functions are $f(x) = b$, $f(x) = x^2$, $f(x) = x^4$, and $f(x) = |x|$. In any of these equations, if $-x$ is substituted for x, the result is the same; that is, $f(-x) = f(x)$. Also note that, with the exception of the absolute value function, these examples are all even-degree polynomial equations. All constant functions are degree zero and are even functions.

Concept Check

Classify the functions $f(x) = x^{2n}$ and $g(x) = x^{(2n+1)}$ where n is a positive integer (1, 2, 3, . . .) as Even, Odd, or Neither.

Answer: $f(x)$ is even; $g(x)$ is odd.

The algebraic method for determining symmetry with respect to the origin is to substitute $-x$ for x. If the result is the negative of the original function, that is, if $f(-x) = -f(x)$, then the function is symmetric with respect to the origin and, hence, classified as an odd function. Examples of odd functions are $f(x) = x$, $f(x) = x^3$, $f(x) = x^5$, and $f(x) = x^{1/3}$. In any of these functions, if $-x$ is substituted for x, the result is the negative of the original function. Note that with the exception of the cube root function, these equations are odd-degree polynomials.

Be careful, though, because functions that are combinations of even- and odd-degree polynomials can turn out to be neither even nor odd, as we will see in Example 1.

Video **EXAMPLE 1** | **Determining Whether a Function Is Even, Odd, or Neither**

Determine whether the functions are even, odd, or neither.

a. $f(x) = x^2 - 3$ **b.** $g(x) = x^5 + x^3$ **c.** $h(x) = x^2 - x$

Solution (a)

Original function.	$f(x) = x^2 - 3$
Replace x with $-x$.	$f(-x) = (-x)^2 - 3$
Simplify.	$f(-x) = x^2 - 3 = f(x)$

Because $f(-x) = f(x)$, we say that $f(x)$ is an *even* function.

Solution (b)

Original function.	$g(x) = x^5 + x^3$
Replace x with $-x$.	$g(-x) = (-x)^5 + (-x)^3$
Simplify.	$g(-x) = -x^5 - x^3 = -(x^5 + x^3) = -g(x)$

Because $g(-x) = -g(x)$, we say that $g(x)$ is an *odd* function.

Solution (c)

Original function.	$h(x) = x^2 - x$
Replace x with $-x$.	$h(-x) = (-x)^2 - (-x)$
Simplify.	$h(-x) = x^2 + x$

$h(-x)$ is neither $-h(x)$ nor $h(x)$; therefore the function $h(x)$ is neither even nor odd.

In parts (a), (b), and (c), we classified these functions as even, odd, or neither, using the algebraic test. Look back at them now and reflect on whether these classifications agree with your intuition. In part (a), we combined two functions: the square function and the constant function. Both of these functions are even, and adding even functions yields another even function. In part (b), we combined two odd functions: the fifth-power function and the cube function. Both of these functions are odd, and adding two odd functions yields another odd function. In part (c), we combined two functions: the square function and the identity function. The square function is even, and the identity function is odd. In this part, combining an even function with an odd function yields a function that is neither even nor odd and, hence, has no symmetry with respect to the vertical axis or the origin.

Your Turn Classify the functions as even, odd, or neither.

a. $f(x) = |x| + 4$ **b.** $f(x) = x^3 - 1$

Answer

a. even

b. neither

1.2.2 Increasing and Decreasing Functions

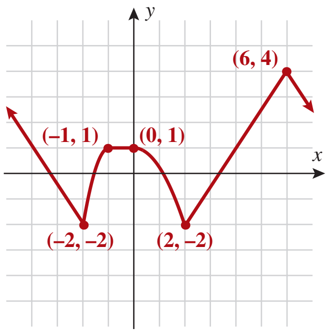
1.2.2 Skill Determine whether functions are increasing, decreasing, or constant.

1.2.2 Conceptual Understand that intervals of increasing, decreasing, and constant correspond to the x-coordinates.

Graphs are read from **left to right**. If we start at the left side of the graph and trace the red curve with our pen, we see that the function values (values in the vertical direction) are decreasing until arriving at the point (−2, −2). Then, the function values increase until arriving at the point (−1, 1). The values then remain constant ($y = 1$) between the points (−1, 1) and (0, 1). Proceeding beyond the point (0, 1), the function values decrease again until the point (2, −2). Beyond the point (2, −2), the function values increase again until the point (6, 4). Finally, the function values decrease and continue to do so.

When specifying a function as increasing, decreasing, or constant, the **intervals are classified according to the *x*-coordinate**. For instance, in this graph, we say the function is increasing when x is between $x = -2$ and $x = -1$ and again when x is between $x = 2$ and $x = 6$. The graph is classified as decreasing when x is less than −2 and again when x is between 0 and 2 and again when x is greater than 6. The graph is classified as constant when x is between −1 and 0. In interval notation, this is summarized as

Decreasing	**Increasing**	**Constant**
$(-\infty, -2) \cup (0, 2) \cup (6, \infty)$	$(-2, -1) \cup (2, 6)$	$(-1, 0)$



An algebraic test for determining whether a function is increasing, decreasing, or constant is to compare the value $f(x)$ of the function for particular points in the intervals.

Increasing, Decreasing, and Constant Functions

1. A function f is **increasing** on an open interval I if for any x_1 and x_2 in I, where $x_1 < x_2$, then $f(x_1) < f(x_2)$.
2. A function f is **decreasing** on an open interval I if for any x_1 and x_2 in I, where $x_1 < x_2$, then $f(x_1) > f(x_2)$.
3. A function f is **constant** on an open interval I if for any x_1 and x_2 in I, then $f(x_1) = f(x_2)$.

STUDY TIP

- **Graphs are read from left to right.**
- **Intervals correspond to the *x*-coordinates.**

In addition to classifying a function as increasing, decreasing, or constant, we can determine the domain and range of a function by inspecting its graph from left to right:

- The domain is the set of all x-values (from left to right) where the function is defined.
- The range is the set of all y-values (from bottom to top) that the graph of the function corresponds to.
- A solid dot on the left or right end of a graph indicates that the graph terminates there and the point is included in the graph.
- An open dot indicates that the graph terminates there and the point is not included in the graph.
- Unless a dot is present, it is assumed that a graph continues indefinitely in the same direction. (An arrow is used in some books to indicate direction.)

Concept Check

TRUE OR FALSE An even function has both increasing and decreasing intervals, but an odd function only has one or the other.

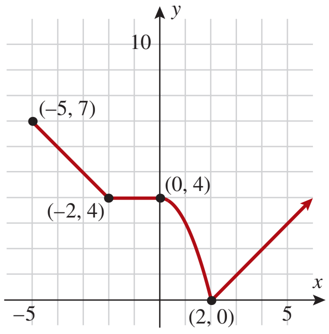
Answer: True

Video EXAMPLE 2 | Finding Intervals When a Function Is Increasing or Decreasing

Given the graph of a function:

a. State the domain and range of the function.

b. Find the intervals when the function is increasing, decreasing, or constant.



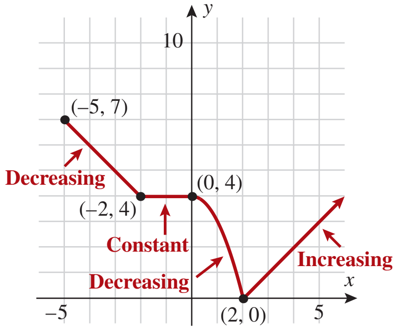
Solution (a)

Domain: $[-5, \infty)$ Range: $[0, \infty)$

Solution (b)

Reading the graph from **left to right**, we see that the graph

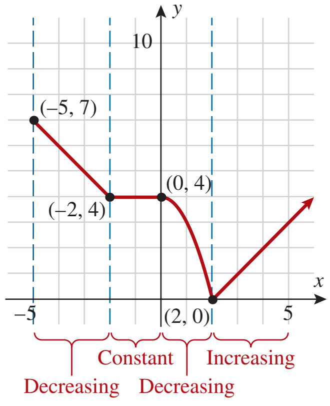
- decreases from the point $(-5, 7)$ to the point $(-2, 4)$.
- is constant from the point $(-2, 4)$ to the point $(0, 4)$.
- decreases from the point $(0, 4)$ to the point $(2, 0)$.
- increases from the point $(2, 0)$ on.



The intervals of increasing and decreasing correspond to the ***x*-coordinates**.

We say that this function is

- increasing on the interval $(2, \infty)$.
- decreasing on the interval $(-5, -2) \cup (0, 2)$.
- constant on the interval $(-2, 0)$.



Note: The intervals of increasing or decreasing are defined on *open* intervals. This should not be confused with the domain. For example, the point $x = -5$ is included in the domain of the function but not in the interval where the function is classified as decreasing.

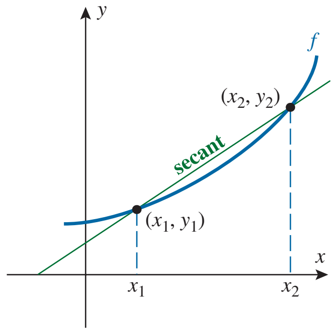
1.2.3 Average Rate of Change

1.2.3 Skill Calculate the average rate of change of a function.

1.2.3 Conceptual Understand that the difference quotient is just another form of the average rate of change.

How do we know *how much* a function is increasing or decreasing? For example, is the price of a stock slightly increasing or is it doubling every week? One way we determine how much a function is increasing or decreasing is by calculating its *average rate of change*.

Let (x_1, y_1) and (x_2, y_2) be two points that lie on the graph of a function f. Draw the line that passes through these two points (x_1, y_1) and (x_2, y_2). This line is called a **secant line**.

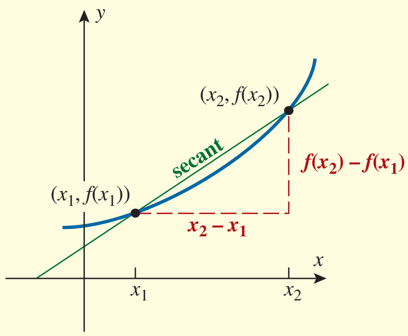


Note that the slope of the secant line is given by $m = \frac{y_2 - y_1}{x_2 - x_1}$, and recall that the slope of a line is the rate of change of that line. The **slope of the secant line** is used to represent the *average rate of change* of the function.

Average Rate of Change

Let $(x_1, f(x_1))$ and $(x_2, f(x_2))$ be two distinct points, $(x_1 \neq x_2)$, on the graph of the function f. The **average rate of change** of f between x_1 and x_2 is given by

$$\textbf{Average rate of change} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$



Video EXAMPLE 3 | Average Rate of Change

Find the average rate of change of $f(x) = x^4$ from:

a. $x = -1$ to $x = 0$ **b.** $x = 0$ to $x = 1$ **c.** $x = 1$ to $x = 2$

Solution (a)

Write the average rate of change formula. $\frac{f(x_2) - f(x_1)}{x_2 - x_1}$

Let $x_1 = -1$ and $x_2 = 0$. $= \frac{f(0) - f(-1)}{0 - (-1)}$

Substitute $f(-1) = (-1)^4 = 1$ and $f(0) = 0^4 = 0$. $= \frac{0 - 1}{0 - (-1)}$

Simplify. $= \boxed{-1}$

Solution (b)

Write the average rate of change formula. $\frac{f(x_2) - f(x_1)}{x_2 - x_1}$

Let $x_1 = 0$ and $x_2 = 1$. $= \frac{f(1) - f(0)}{1 - 0}$

Substitute $f(0) = 0^4 = 0$ and $f(1) = (1)^4 = 1$. $= \frac{1 - 0}{1 - 0}$

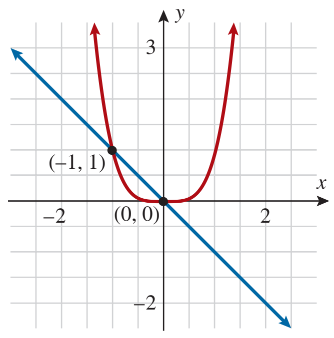
Simplify. $= \boxed{1}$

Solution (c)

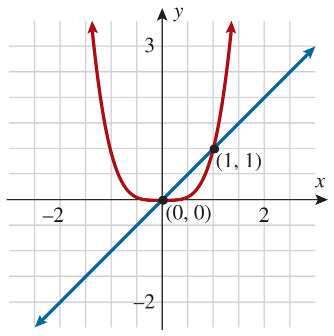
Write the average rate of change formula.	$\frac{f(x_2) - f(x_1)}{x_2 - x_1}$
Let $x_1 = 1$ and $x_2 = 2$.	$= \frac{f(2) - f(1)}{2 - 1}$
Substitute $f(1) = 1^4 = 1$ and $f(2) = (2)^4 = 16$.	$= \frac{16 - 1}{2 - 1}$
Simplify.	$= \boxed{15}$

Graphical Interpretation: Slope of the Secant Line

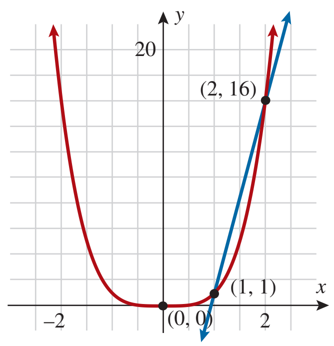
a. Average rate of change of f from $x = -1$ to $x = 0$: Decreasing at a rate of 1



b. Average rate of change of f from $x = 0$ to $x = 1$: Increasing at a rate of 1



c. Average rate of change of f from $x = 1$ to $x = 2$: Increasing at a rate of 15



Your Turn Find the average rate of change of $f(x) = x^2$ from:

a. $x = -2$ to $x = 0$ **b.** $x = 0$ to $x = 2$

Answer

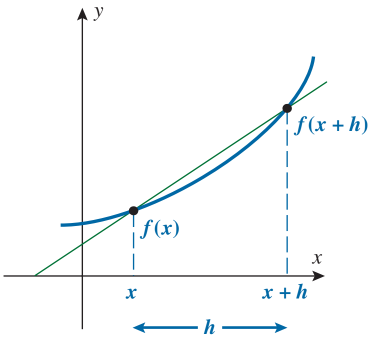
a. -2

b. 2

The average rate of change can also be written in terms of the difference quotient.

Words	**Math**
Let the difference between x_1 and x_2 be h.	$x_2 - x_1 = h$
Solve for x_2.	$x_2 = x_1 + h$
Substitute $x_2 - x_1 = h$ into the denominator and $x_2 = x_1 + h$ into the numerator of the average rate of change.	Average rate of change $= \dfrac{f(x_2) - f(x_1)}{x_2 - x_1}$ $= \dfrac{f(x_1 + h) - f(x_1)}{h}$
Let $x_1 = x$.	$= \boxed{\dfrac{f(x+h) - f(x)}{h}}$

When written in this form, the average rate of change is called the **difference quotient**.



Difference Quotient

The expression $\dfrac{f(x+h) - f(x)}{h}$, where $h \neq 0$, is called the **difference quotient**.

The difference quotient is more meaningful when h is small. In calculus, the difference quotient is used to define a derivative.

EXAMPLE 4 | Calculating the Difference Quotient

Calculate the difference quotient for the function $f(x) = 2x^2 + 1$.

Solution

Find $f(x + h)$.

$$f(x+h) = 2(x+h)^2 + 1$$
$$= 2(x^2 + 2xh + h^2) + 1$$
$$= 2x^2 + 4xh + 2h^2 + 1$$

Find the difference quotient.

$$\frac{f(x+h) - f(x)}{h} = \frac{\overbrace{2x^2 + 4xh + 2h^2 + 1}^{f(x+h)} - \overbrace{(2x^2 + 1)}^{f(x)}}{h}$$

Simplify.

$$\frac{f(x+h) - f(x)}{h} = \frac{\cancel{2x^2} + 4xh + 2h^2 \cancel{+1} - \cancel{2x^2} \cancel{-1}}{h}$$

$$\frac{f(x+h) - f(x)}{h} = \frac{4xh + 2h^2}{h}$$

Factor the numerator.

$$\frac{f(x+h) - f(x)}{h} = \frac{h(4x + 2h)}{h}$$

Cancel (divide out) the common h.

$$\frac{f(x+h) - f(x)}{h} = \boxed{4x + 2h} \quad h \neq 0$$

Your Turn Calculate the difference quotient for the function $f(x) = -x^2 + 2$.

Answer

$$\frac{f(x+h) - f(x)}{h} = -2x - h$$

STUDY TIP

Use brackets or parentheses around $f(x)$ to avoid forgetting to distribute the negative sign:

$$\frac{f(x+h) - [f(x)]}{h}$$

Concept Check

Find the difference quotient for the line $f(x) = mx + b$.

Answer: m

EXAMPLE 5 | Evaluating the Difference Quotient

For the function $f(x) = x^2 - x$, find $\frac{f(x+h) - f(x)}{h}$

Solution

Use placeholder notation for the function $f(x) = x^2 - x$. $\quad f(\square) = (\square)^2 - (\square)$

Calculate $f(x+h)$. $\quad f(x+h) = (x+h)^2 - (x+h)$

Write the difference quotient. $\quad \frac{f(x+h) - f(x)}{h}$

Let $f(x+h) = (x+h)^2 - (x+h)$ and $f(x) = x^2 - x$.

$$\frac{f(x+h) - f(x)}{h} = \frac{\overbrace{[(x+h)^2 - (x+h)]}^{f(x+h)} - \overbrace{[x^2 - x]}^{f(x)}}{h} \qquad h \neq 0$$

Eliminate the parentheses inside the first set of brackets. $\quad = \frac{[x^2 + 2xh + h^2 - x - h] - [x^2 - x]}{h}$

Eliminate the brackets in the numerator. $\quad = \frac{x^2 + 2xh + h^2 - x - h - x^2 + x}{h}$

Combine like terms. $\quad = \frac{2xh + h^2 - h}{h}$

Factor the numerator. $\quad = \frac{h(2x + h - 1)}{h}$

Divide out the common factor, h. $\quad = \boxed{2x + h - 1} \quad h \neq 0$

Your Turn Evaluate the difference quotient for $f(x) = x^2 - 1$.

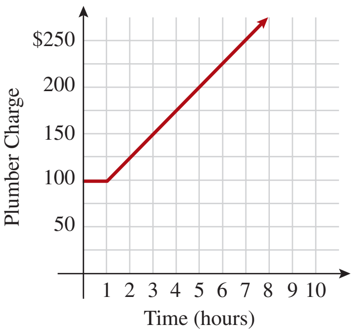
Answer

$2x + h$

1.2.4 Piecewise-Defined Functions

1.2.4 Skill Graph piecewise-defined functions.

1.2.4 Conceptual Understand points of discontinuity and domain and range of piecewise-defined functions.



Most of the functions that we have seen in this text are functions defined by polynomials. Sometimes the need arises to define functions in terms of *pieces.* For example, most plumbers charge a flat fee for a house call and then an additional hourly rate for the job. For instance, if a particular plumber charges \$100 to drive out to your house and work for 1 hour and then an additional \$25 an hour for every additional hour they work on your job, we would define this function in pieces. If we let h be the number of hours worked, then the charge is defined as

$$\text{Plumbing charge} = \begin{cases} 100 & h \leq 1 \\ 100 + 25(h-1) & h > 1 \end{cases}$$

If we were to graph this function, we would see that there is 1 hour that is constant and after that the function continually increases.

The next example is a piecewise-defined function given in terms of pieces of functions from our "library of functions". Because the function is defined in terms of pieces of other functions, we draw the graph of each individual function, and then for each function, we darken the piece corresponding to its part of the domain.

EXAMPLE 6 | Graphing Piecewise-Defined Functions

Graph the piecewise-defined function, and state the domain, range, and intervals when the function is increasing, decreasing, or constant.

$$G(x) = \begin{cases} x^2 & x < -1 \\ 1 & -1 \le x \le 1 \\ x & x > 1 \end{cases}$$

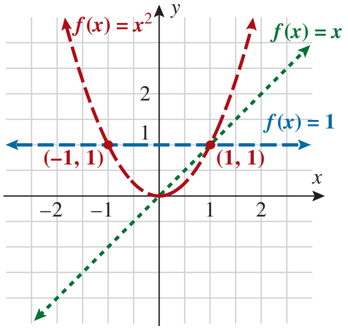
Solution

Graph each of the functions on the same plane.

Square function:
$f(x) = x^2$

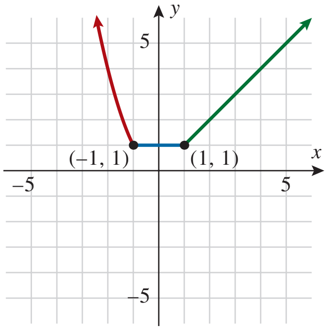
Constant function:
$f(x) = 1$

Identity function:
$f(x) = x$



The points to focus on in particular are the x-values where the pieces change over, that is, $x = -1$ and $x = 1$.

Let's now investigate each piece. When $x < -1$, this function is defined by the square function, $f(x) = x^2$, so darken that particular function to the left of $x = -1$. When $-1 \le x \le 1$, the function is defined by the constant function, $f(x) = 1$, so darken that particular function between the x values of -1 and 1. When $x > 1$, the function is defined by the identity function, $f(x) = x$, so darken that function to the right of $x = 1$. Erase everything that is not darkened, and the resulting graph of the piecewise-defined function is given on the right.



This function is defined for all real values of x, so the domain of this function is the set of all real numbers. The values that this function yields in the vertical direction are all real numbers greater than or equal to 1. Hence, the range of this function is $[1, \infty)$. The intervals of increasing, decreasing, and constant are as follows:

Decreasing: $(-\infty, -1)$

Constant: $(-1, 1)$

Increasing: $(1, \infty)$

The term **continuous** implies that there are no holes or jumps and that the graph can be drawn without picking up your pencil. A function that does have holes or jumps and cannot be drawn in one motion without picking up your pencil is classified as **discontinuous**, and the points where the holes or jumps occur are called *points of discontinuity.*

The previous example illustrates a *continuous* piecewise-defined function. At the $x = -1$ junction, the square function and constant function both pass through the point $(-1, 1)$. At the $x = 1$ junction, the constant function and the identity function both pass through the point $(1, 1)$. Since the graph of this piecewise-defined function has no holes or jumps, we classify it as a continuous function.

The next example illustrates a *discontinuous* piecewise-defined function.

Video EXAMPLE 7 | Graphing a Discontinuous Piecewise-Defined Function

Graph the piecewise-defined function, and state the intervals where the function is increasing, decreasing, or constant, along with the domain and range.

$$f(x)=\begin{cases} 1-x & x<0 \\ x & 0\le x<2 \\ -1 & x>2 \end{cases}$$

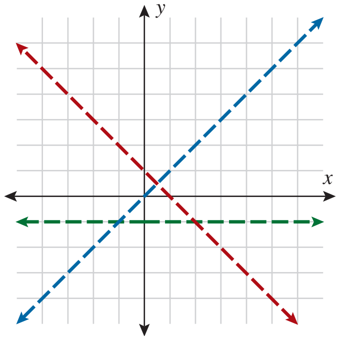
Solution

Graph these functions on the same plane.

Linear function:
$f(x) = 1 - x$

Identity function:
$f(x) = x$

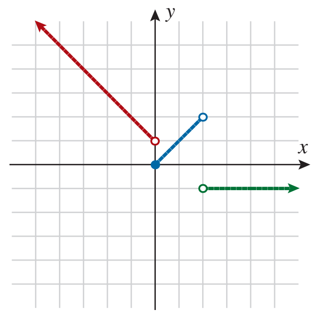
Constant function:
$f(x) = -1$



Darken the piecewise-defined function on the graph. For all values less than zero **($x < 0$)**, the function is defined by the **linear function**. Note the use of an open circle, indicating up to but not including $x = 0$. For values **$0 \le x < 2$**, the function is defined by the **identity function**.

The circle is filled in at the left endpoint, $x = 0$. An open circle is used at $x = 2$. For all values greater than 2, **$x > 2$**, the function is defined by the **constant function**. Because this interval does not include the point $x = 2$, an open circle is used.

At what intervals is the function increasing, decreasing, or constant? Remember that the intervals correspond to the x-values.



Decreasing: **$(-\infty, 0)$**   Increasing: **$(0, 2)$**   Constant: **$(2, \infty)$**

The function is defined for all values of x except $x = 2$.

Domain: $(-\infty, 2) \cup (2, \infty)$

The output of this function (vertical direction) takes on the y-values $y \ge 0$ and the additional single value $y = -1$.

Range: $[-1, -1] \cup [0, \infty)$ or $\{-1\} \cup [0, \infty)$

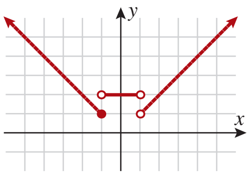
We mentioned earlier that a discontinuous function has a graph that exhibits holes or jumps. In this example, the point $x = 0$ corresponds to a jump because you would have to pick up your pencil to continue drawing the graph. The point $x = 2$ corresponds to both a hole and a jump. The hole indicates that the function is not defined at that point, and there is still a jump because the identity function and the constant function do not meet at the same y-value at $x = 2$.

Your Turn Graph the piecewise-defined function, and state the intervals where the function is increasing, decreasing, or constant, along with the domain and range.

$$f(x)=\begin{cases} -x & x\le -1 \\ 2 & -1<x<1 \\ x & x>1 \end{cases}$$

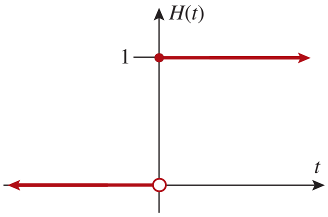
Answer

Increasing: $(1, \infty)$
Decreasing: $(-\infty, -1)$
Constant: $(-1, 1)$
Domain:
$(-\infty, 1) \cup (1, \infty)$
Range: $[1, \infty)$



Piecewise-defined functions whose "pieces" are constants are called **step functions**. The reason for this name is that the graph of a step function looks like steps of a staircase. A common step function used in engineering is the **Heaviside step function** (also called the **unit step function**):

$$H(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases}$$



This function is used in signal processing to represent a signal that turns on at some time and stays on indefinitely.

Concept Check

State the domain, range, and any points of discontinuity for the Heaviside function.

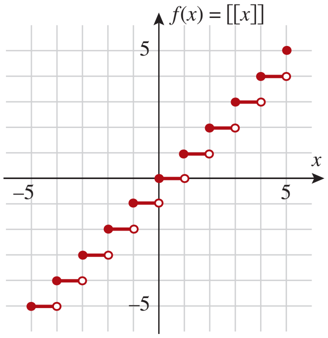
Answer: Domain: $(-\infty, \infty)$
Range: $[0] \cup [1]$
Point of Discontinuity: $x = 0$

A common step function used in business applications is the *greatest integer function.*

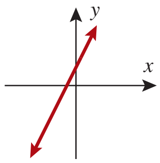
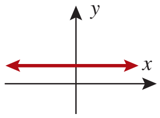
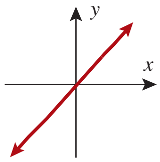
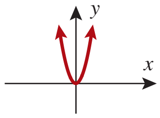
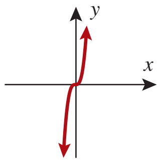
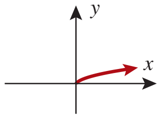
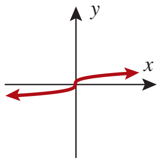
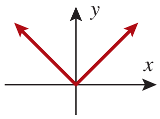
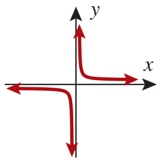
Greatest Integer Function

$f(x) = [[x]] =$ greatest integer less than or equal to x.

x	1.0	1.3	1.5	1.7	1.9	2.0
$f(x) = [[x]]$	1	1	1	1	1	2



Section 1.2 Summary

Name	Function	Domain	Range	Graph	Even/Odd
Linear	$f(x) = mx + b$, $m \neq 0$	$(-\infty, \infty)$	$(-\infty, \infty)$		Neither (unless $y = x$)
Constant	$f(x) = c$	$(-\infty, \infty)$	$[c, c]$ or $\{c\}$		Even
Identity	$f(x) = x$	$(-\infty, \infty)$	$(-\infty, \infty)$		Odd
Square	$f(x) = x^2$	$(-\infty, \infty)$	$[0, \infty)$		Even
Cube	$f(x) = x^3$	$(-\infty, \infty)$	$(-\infty, \infty)$		Odd
Square Root	$f(x) = \sqrt{x}$	$[0, \infty)$	$[0, \infty)$		Neither
Cube Root	$f(x) = \sqrt[3]{x}$	$(-\infty, \infty)$	$(-\infty, \infty)$		Odd
Absolute Value	$f(x) = \|x\|$	$(-\infty, \infty)$	$[0, \infty)$		Even
Reciprocal	$f(x) = \frac{1}{x}$	$(-\infty, 0) \cup (0, \infty)$	$(-\infty, 0) \cup (0, \infty)$		Odd

Domain and Range of a Function

- Implied domain: Exclude any values that lead to the function being undefined (dividing by zero) or imaginary outputs (square root of a negative real number).
- Inspect the graph to determine the set of all inputs (domain) and the set of all outputs (range).

Finding Intervals Where a Function Is Increasing, Decreasing, or Constant

- Increasing: Graph of function rises from left to right.
- Decreasing: Graph of function falls from left to right.
- Constant: Graph of function does not change height from left to right.

Average Rate of Change $\dfrac{f(x_2) - f(x_1)}{x_2 - x_1}$ $\quad x_1 \neq x_2$

Difference Quotient $\dfrac{f(x+h) - f(x)}{h}$ $\quad h \neq 0$

Piecewise-Defined Functions

- Continuous: You can draw the graph of a function without picking up the pencil.
- Discontinuous: Graph has holes and/or jumps.

Section 1.2 Exercises

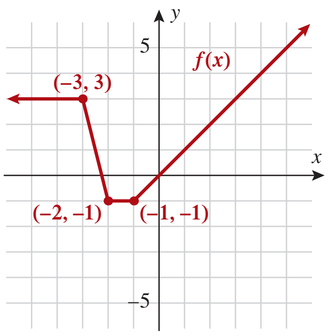
Skills

In Exercises 1–16, determine whether the function is even, odd, or neither.

1. $h(x) = x^2 + 2x$
2. $G(x) = 2x^4 + 3x^3$
3. $h(x) = x^{1/3} - x$
4. $g(x) = x^{-1} + x$
5. $f(x) = |x| + 5$
6. $f(x) = |x| + x^2$
7. $f(x) = |x|$
8. $f(x) = |x^3|$
9. $G(t) = |t - 3|$
10. $g(t) = |t + 2|$
11. $G(t) = \sqrt{t - 3}$
12. $f(x) = \sqrt{2 - x}$
13. $g(x) = \sqrt{x^2 + x}$
14. $f(x) = \sqrt{x^2 + 2}$
15. $h(x) = \dfrac{1}{x} + 3$
16. $h(x) = \dfrac{1}{x} - 2x$

In Exercises 17–28, state the (a) domain, (b) range, and (c) *x*-interval(s) where the function is increasing, decreasing, or constant. Find the values of (d) $f(0)$, (e) $f(-2)$, and (f) $f(2)$.

17.



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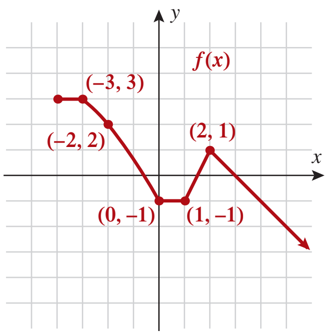

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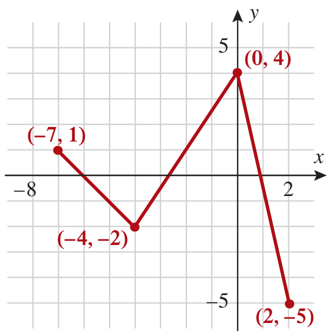

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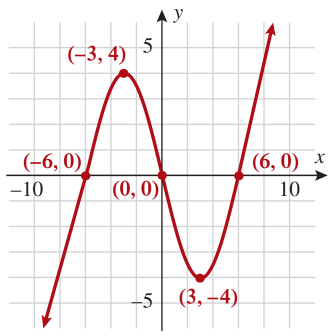

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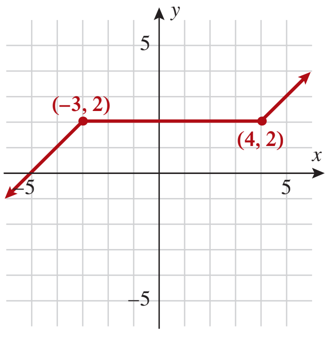

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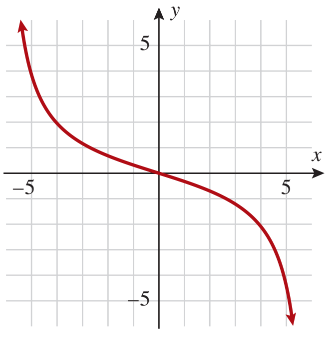

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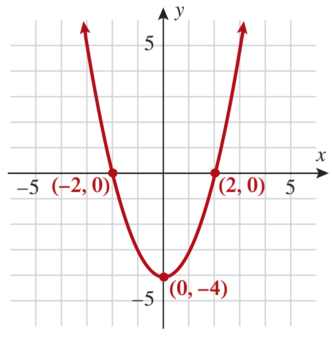

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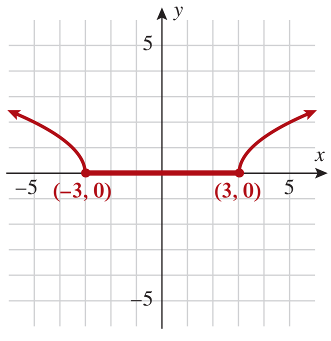

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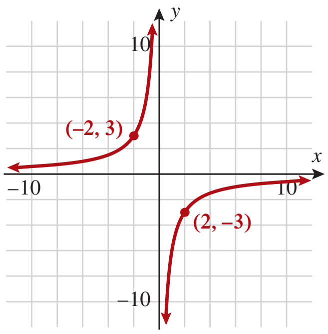

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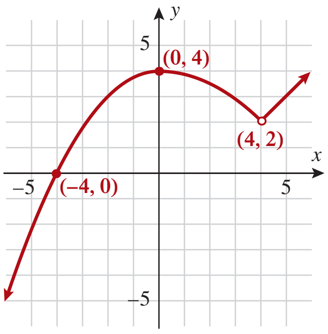

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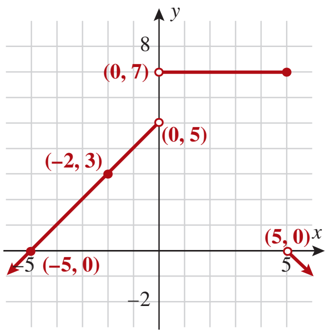

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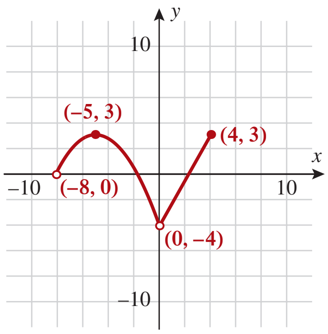


In Exercises 29–48, find the difference quotient $\frac{f(x+h)-f(x)}{h}$ for each function.

29. $f(x) = x^2 - x$

30. $f(x) = x^2 + 2x$

31. $f(x) = 3x + x^2$

32. $f(x) = 5x - x^2$

33. $f(x) = x^2 - 3x + 2$

34. $f(x) = x^2 - 2x + 5$

35. $f(x) = -3x^2 + 5x - 4$

36. $f(x) = -4x^2 + 2x - 3$

37. $f(x) = \frac{1}{2x}$

38. $f(x) = \frac{1}{x+3}$

39. $f(x) = \sqrt{x} + 1$

40. $f(x) = \sqrt{x-2}$

41. $f(x) = x^3 + x^2$

42. $f(x) = (x-1)^4$

43. $f(x) = \frac{2}{x-2}$

44. $f(x) = \frac{x+5}{x-7}$

45. $f(x) = \sqrt{1-2x}$

46. $f(x) = \sqrt{x^2+x+1}$

47. $f(x) = \frac{4}{\sqrt{x}}$

48. $f(x) = \sqrt{\frac{x}{x+1}}$

In Exercises 49–56, find the average rate of change of the function from $x = 1$ to $x = 3$.

49. $f(x) = x^3$

50. $f(x) = \frac{1}{x}$

51. $f(x) = |x|$

52. $f(x) = 2x$

53. $f(x) = 1 - 2x$

54. $f(x) = 9 - x^2$

55. $f(x) = |5 - 2x|$

56. $f(x) = \sqrt{x^2 - 1}$

In Exercises 57–64, find the average rate of change of the function over the given interval.

57. $f(x) = 2x^2$ from $x = -\frac{5}{4}$ to $x = -\frac{3}{4}$

58. $f(x) = \sqrt{x+1}$ from $x = -0.36$ to $x = -0.19$

59. $f(x) = -x^3$ from $x = \sqrt[3]{5}$ to $x = 2\sqrt[3]{5}$

60. $f(x) = |x - 7|$ from $x = 1.25$ to $x = 3.75$

61. $f(x) = x^2 - \frac{1}{2}$ from $x = \frac{2}{3}$ to $x = \frac{4}{5}$

62. $f(x) = |x^2 + 6|$ from $x = \sqrt{2}$ to $x = 3\sqrt{2}$

63. $f(x) = \sqrt{5 - 3x}$ from $x = \frac{4}{3}$ to $x = \frac{5}{3}$

64. $f(x) = 25x^3$ from $x = \frac{1}{5}$ to $x = \frac{3}{5}$

In Exercises 65–92, graph the piecewise-defined functions. State the domain and range in interval notation. Determine the intervals where the function is increasing, decreasing, or constant.

65. $f(x) = \begin{cases} x+1 & x < 2 \\ x-1 & x \geq 2 \end{cases}$

66. $f(x) = \begin{cases} -2x & x < 1 \\ 3x-2 & x \geq 1 \end{cases}$

67. $f(x) = \begin{cases} x & x < 2 \\ 2 & x \geq 2 \end{cases}$

68. $f(x) = \begin{cases} -x & x < -1 \\ -1 & x \geq -1 \end{cases}$

69. $f(x) = \begin{cases} 1 & x < -1 \\ x^2 & x \geq -1 \end{cases}$

70. $f(x) = \begin{cases} x^2 & x < 2 \\ 4 & x \geq 2 \end{cases}$

71. $f(x) = \begin{cases} x & x < 0 \\ x^2 & x \geq 0 \end{cases}$

72. $f(x) = \begin{cases} -x & x \leq 0 \\ x^2 & x > 0 \end{cases}$

73. $f(x) = \begin{cases} -x+2 & x < 1 \\ x^2 & x \geq 1 \end{cases}$

74. $f(x) = \begin{cases} 2+x & x \leq -1 \\ x^2 & x > -1 \end{cases}$

75. $f(x) = \begin{cases} 5-2x & x < 2 \\ 3x-2 & x > 2 \end{cases}$

76. $f(x) = \begin{cases} 3 - \frac{1}{2}x & x < -2 \\ 4 + \frac{3}{2}x & x > -2 \end{cases}$

77. $G(x) = \begin{cases} -1 & x < -1 \\ x & -1 \leq x \leq 3 \\ 3 & x > 3 \end{cases}$

78. $G(x) = \begin{cases} -1 & x < -1 \\ x & -1 < x < 3 \\ 3 & x > 3 \end{cases}$

79. $G(t) = \begin{cases} 1 & t < 1 \\ t^2 & 1 \leq t \leq 2 \\ 4 & t > 2 \end{cases}$

80. $G(t) = \begin{cases} 1 & t < 1 \\ t^2 & 1 < t < 2 \\ 4 & t > 2 \end{cases}$

81. $f(x) = \begin{cases} -x-1 & x < -2 \\ x+1 & -2 < x < 1 \\ -x+1 & x \geq 1 \end{cases}$

82. $f(x) = \begin{cases} -x-1 & x \leq -2 \\ x+1 & -2 < x < 1 \\ -x+1 & x > 1 \end{cases}$

83. $G(x) = \begin{cases} 0 & x < 0 \\ \sqrt{x} & x \geq 0 \end{cases}$

84. $G(x) = \begin{cases} 1 & x < 1 \\ \sqrt[3]{x} & x > 1 \end{cases}$

85. $G(x) = \begin{cases} 0 & x = 0 \\ \frac{1}{x} & x \neq 0 \end{cases}$

86. $G(x) = \begin{cases} 0 & x = 0 \\ -\frac{1}{x} & x \neq 0 \end{cases}$

87. $G(x) = \begin{cases} -\sqrt[3]{x} & x \leq -1 \\ x & -1 < x < 1 \\ -\sqrt{x} & x > 1 \end{cases}$

88. $G(x) = \begin{cases} -\sqrt[3]{x} & x < -1 \\ x & -1 \leq x < 1 \\ \sqrt{x} & x > 1 \end{cases}$

89. $f(x) = \begin{cases} x+3 & x \leq -2 \\ |x| & -2 < x < 2 \\ x^2 & x \geq 2 \end{cases}$

90. $f(x) = \begin{cases} |x| & x < -1 \\ 1 & -1 < x < 1 \\ |x| & x > 1 \end{cases}$

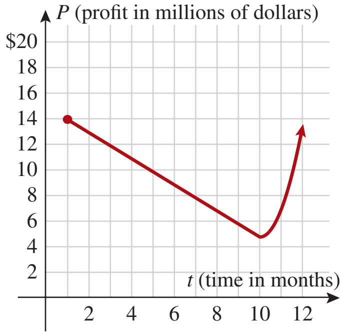
91. $f(x) = \begin{cases} x & x \leq -1 \\ x^3 & -1 < x < 1 \\ x^2 & x > 1 \end{cases}$

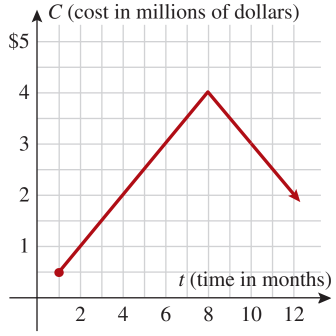
92. $f(x) = \begin{cases} x^2 & x \leq -1 \\ x^3 & -1 < x < 1 \\ x & x \geq 1 \end{cases}$

Applications

For Exercises 93 and 94, refer to the following:

A manufacturer determines that their *profit* and *cost* functions over one year are represented by the following graphs.





93. Business. Find the intervals on which profit is increasing, decreasing, and constant.

94. Business. Find the intervals on which cost is increasing, decreasing, and constant.

95. Budget: Costs. The Kappa Kappa Gamma sorority decides to order custom-made T-shirts for its *Kappa Krush* mixer with the Sigma Alpha Epsilon fraternity. If the sorority orders 50 or fewer T-shirts, the cost is $10 per shirt. If it orders more than 50 but less than or equal to 100, the cost is $9 per shirt. If it orders more than 100, the cost is $8 per shirt. Find the cost function $C(x)$ as a function of the number of T-shirts x ordered.

96. Budget: Costs. The marching band at a university is ordering some additional uniforms to replace existing uniforms that are worn out. If the band orders 50 or fewer, the cost is $176.12 per uniform. If it orders more than 50 but less than or equal to 100, the cost is $159.73 per uniform. Find the cost function $C(x)$ as a function of the number of new uniforms x ordered.

97. Budget: Costs. The Richmond rowing club is planning to enter the *Head of the Charles* race in Boston and is trying to figure out how much money to raise. The entry fee is $250 per boat for the first 10 boats and $175 for each additional boat. Find the cost function $C(x)$ as a function of the number of boats x the club enters.

98. Home Improvement. A contractor needs to buy sprinkler heads for an upcoming project. At a local home improvement store, the sprinkler heads are $10 each but the store offers a contractor's discount. When they are six or more, the cost per sprinkler drops to $8.50. Write a cost function $C(x)$ as a function of the number of sprinkler heads x the contractor purchases.

99. Subscriptions. MidState CrossFit has an unlimited monthly membership plan that costs $139 per person. When there are multiple members from the same household, the additional members will receive a 10% discount off their plan. Find a function $C(x)$ that determines the amount a household with x members will pay.

100. Subscriptions. Rework Exercise 99 if the gym also offers a 20% discount for active duty military. If the gym in Exercises 99 also offers a 20% military discount for military families, Find a function $C(x)$ that determines the amount a military family with x members will pay.

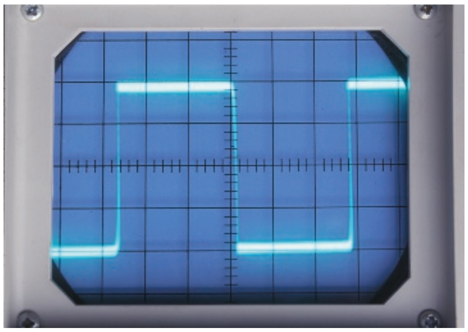
101. Postage Rates. The following table corresponds to first-class postage rates for the U.S. Postal Service. Write a piecewise-defined function in terms of the greatest integer function that models this cost of mailing flat envelopes first class.

Weight Less Than (Ounces)	First-Class Rate (Flat Envelopes)
1	$1.00
2	1.20
3	1.40
4	1.60
5	1.80
6	2.00
7	2.20
8	2.40
9	2.60
10	2.80
11	3.00
12	3.20
13	3.40

102. Postage Rates. The following table corresponds to first-class postage rates for the U.S. Postal Service. Write a piecewise-defined function in terms of the greatest integer function that models this cost of mailing parcels first class.

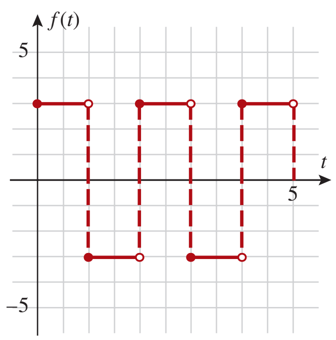
Weight Less Than (Ounces)	Parcels
1	$2.19
2	2.42
3	2.65
4	2.88
5	3.11
6	3.34
7	3.57
8	3.80
9	4.03
10	4.26
11	4.49
12	4.72
13	4.95

A square wave is a waveform used in electronic circuit testing and signal processing. A square wave alternates regularly and instantaneously between two levels.

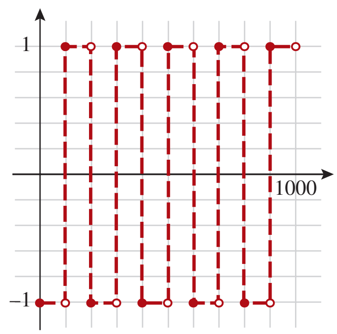


sciencephotos/Alamy Stock Photo

103. Electronics: Signals. Write a step function $f(t)$ that represents the following square wave.



104. Electronics: Signals. Write a step function $f(x)$ that represents the following square wave, where x represents frequency in Hz.



For Exercises 105 and 106, refer to the following table:

Global Carbon Emissions from Fossil Fuel Burning

Year	Millions of Tons of Carbon
1995	6700
2000	7200
2005	7300
2010	7000
2015	6600

105. Carbon Emissions. Find the average rate of change in global carbon emissions from fossil fuel burning. Is the function increasing or decreasing over that period of time?

a. From 1995 to 2005
b. From 2005 to 2015

106. Carbon Emissions. Find the average rate of change in global carbon emissions from fossil fuel burning. Is the function increasing or decreasing over that period of time?

a. From 2000 to 2005
b. From 2005 to 2010

For Exercises 107 and 108, use the following information:

The height (in feet) of a falling object with an initial velocity of 48 feet per second launched straight upward from the ground is given by $h(t) = -16t^2 + 48t$, where t is time (in seconds).

107. Falling Objects. What is the average rate of change of the height as a function of time from $t = 1$ to $t = 2$?

108. Falling Objects. What is the average rate of change of the height as a function of time from $t = 1$ to $t = 3$?

For Exercises 109 and 110, refer to the following:

An analysis of sales indicates that demand for a product during a calendar year (not a leap year) is modeled by

$$d(t) = 3\sqrt{t^2 + 1} - 2.75t$$

where d is demand in thousands of units and t is the day of the year and $t = 1$ represents January 1.

109. Economics. Find the average rate of change of the demand of the product over the first quarter.

110. Economics. Find the average rate of change of the demand of the product over the fourth quarter.

Catch the Mistake

In Exercises 111 and 112, explain the mistake that is made.

111. The cost to get into a high school football game is \$5 for students ages 6-17 and \$7 for adults 18 and over. Kids 5 and under get in for free. Write a function describing the cost of the admission as a function of age in years.

Solution $C(x) = \begin{cases} 0 & 0 \le x \le 5 \\ 5x & 6 \le x \le 17 \\ 7x & x \ge 18 \end{cases}$

This is incorrect. What mistake was made?

112. Most money market accounts pay a higher interest with a higher principal. If the credit union is offering 2% on accounts with less than or equal to \$10,000 and 4% on the additional money over \$10,000, write the interest function $I(x)$ that represents the interest earned on an account as a function of dollars in the account.

Solution $I(x) = \begin{cases} 0.02x & x \le 10{,}000 \\ 0.02(10{,}000) + 0.04x & x > 10{,}000 \end{cases}$

This is incorrect. What mistake was made?

Conceptual

In Exercises 113 and 114, determine whether each statement is true or false.

113. If an odd function has an interval where the function is increasing, then it also has to have an interval where the function is decreasing.

114. If an even function has an interval where the function is increasing, then it also has to have an interval where the function is decreasing.

In Exercises 115 and 116, for *a* and b real numbers, can the function given ever be a continuous function? If so, specify the values for *a* and *b* that would make it so.

115. $f(x) = \begin{cases} ax & x \le 2 \\ bx^2 & x > 2 \end{cases}$

116. $f(x) = \begin{cases} -\frac{1}{x} & x < a \\ \frac{1}{x} & x \ge a \end{cases}$

Challenge

In Exercises 117 and 118, find the values of *a* and *b* that make *f* continuous.

117. $f(x) = \begin{cases} -x^2 - 10x - 13 & x \le -2 \\ ax + b & -2 < x < 1 \\ \sqrt{x - 1} - 9 & x \le 1 \end{cases}$

118. $f(x) = \begin{cases} -2x - a + 2b & x \le -2 \\ \sqrt{x + a} & -2 < x \le 2 \\ x^2 - 4x + a + 4 & x > 2 \end{cases}$

Preview to Calculus

For Exercises 119–124, refer to the following:

In calculus, the difference quotient $\frac{f(x+h) - f(x)}{h}$ of a function f is used to find the derivative f' of f, by allowing h to approach zero, $h \to 0$. Find the derivative of the following functions.

119. $f(x) = k$, where k is a constant

120. $f(x) = mx + b$, where m and b are constants, $m \neq 0$

121. $f(x) = ax^2 + bx + c$, where a, b, and c are constants, $a \neq 0$

122. $f(x) = \begin{cases} 7 & x < 0 \\ 2 - 3x & 0 < x < 4 \\ x^2 + 4x - 6 & x > 4 \end{cases}$

123. $f(x) = \begin{cases} -x + 4 & x \le 1 \\ 5 & x > 1 \end{cases}$

124. $f(x) = \begin{cases} \frac{1}{2}x + 5 & x < 5 \\ 5 - 2x^2 & x > 5 \end{cases}$

1.3 Graphing Techniques: Transformations

SKILLS OBJECTIVES

- Sketch the graph of a function using horizontal and vertical shifting of common functions.
- Sketch the graph of a function by reflecting a common function about the *x*-axis or *y*-axis.
- Sketch the graph of a function by stretching or compressing a common function.

CONCEPTUAL OBJECTIVES

- Understand why a shift in the argument inside the function corresponds to a horizontal shift and a shift outside the function corresponds to a vertical shift.
- Understand why a negative argument inside the function corresponds to a reflection about the *y*-axis and a negative outside the function corresponds to a reflection about the *x*-axis.
- Understand the difference between rigid and nonrigid transformations.

1.3.1 Horizontal and Vertical Shifts

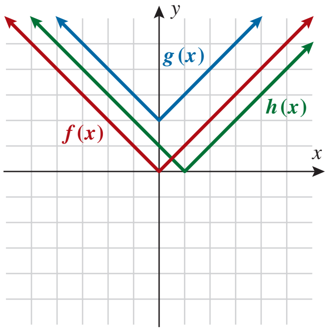
1.3.1 **Skill** Sketch the graph of a function using horizontal and vertical shifting of common functions.

1.3.1 **Conceptual** Understand why a shift in the argument inside the function corresponds to a horizontal shift and a shift outside the function corresponds to a vertical shift.

The focus of the previous section was to learn the graphs that correspond to particular functions such as identity, square, cube, square root, cube root, absolute value, and reciprocal. Therefore, at this point, you should be able to recognize and generate the graphs of $y = x$,

$y = x^2$, $y = x^3$, $y = \sqrt{x}$, $y = \sqrt[3]{x}$, $y = |x|$, and $y = \frac{1}{x}$. In this section, we will discuss how to sketch the graphs of functions that are very simple modifications of these functions. For instance, a common function may be shifted (horizontally or vertically), reflected, or stretched (or compressed). Collectively, these techniques are called **transformations**.

Let's take the absolute value function as an example. The graph of $f(x) = |x|$ was given in the last section. Now look at two examples that are much like this function: $g(x) = |x| + 2$ and $h(x) = |x - 1|$. Graphing these functions by point-plotting yields



x	$f(x)$
−2	2
−1	1
0	0
1	1
2	2

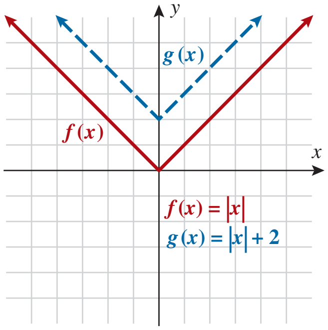
x	$g(x)$
−2	4
−1	3
0	2
1	3
2	4

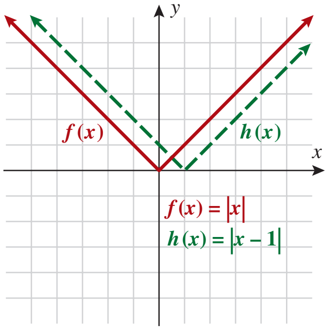
x	$h(x)$
−2	3
−1	2
0	1
1	0
2	1

Instead of point-plotting the function $g(x) = |x| + 2$, we could have started with the function $f(x) = |x|$ and shifted the entire graph *up* two units. Similarly, we could have generated the graph of the function $h(x) = |x - 1|$ by shifting the function $f(x) = |x|$ to the *right* one unit. In both cases, the base or starting function is $f(x) = |x|$. Why did we go up for $g(x)$ and to the right for $h(x)$?

Note that we could rewrite the functions $g(x)$ and $h(x)$ in terms of $f(x)$:

$$g(x) = |x| + 2 = f(x) + 2$$
$$h(x) = |x - 1| = f(x - 1)$$





In the case of $g(x)$, the shift (+2) occurs "outside" the function—that is, outside the parentheses showing the argument. Therefore, the output for $g(x)$ is two more than the typical output for $f(x)$. Because the output corresponds to the vertical axis, this results in a shift *upward* of two units. In general, shifts that occur *outside* the function correspond to a *vertical* shift corresponding to the sign of the shift. For instance, had the function been $G(x) = |x| - 2$, this graph would have started with the graph of the function $f(x)$ and shifted down two units.

In the case of $h(x)$, the shift occurs "inside" the function—that is, inside the parentheses showing the argument. Note that the point (0, 0) that lies on the graph of $f(x)$ was shifted to the point (1, 0) on the graph of the function $h(x)$. The y-value remained the same, but the x-value shifted to the right one unit. Similarly, the points (−1, 1) and (1, 1) were shifted to the points (0, 1) and (2, 1), respectively. In general, shifts that occur *inside* the function correspond to a *horizontal* shift opposite the sign. In this case, the graph of the function $h(x) = |x - 1|$ shifted the graph of the function $f(x)$ to the right one unit. If, instead, we had the function $H(x) = |x + 1|$, this graph would have started with the graph of the function $f(x)$ and shifted to the left one unit.

STUDY TIP

Shifts outside the function are vertical shifts *with* the sign.

Up (+)
Down (−)

Vertical Shifts

Assuming that c is a positive constant,

To Graph	**Shift the Graph of $f(x)$**
$f(x) + c$	c units upward
$f(x) - c$	c units downward

Adding or subtracting a constant **outside** the function corresponds to a **vertical** shift that goes **with the sign**.

Horizontal Shifts

Assuming that c is a positive constant,

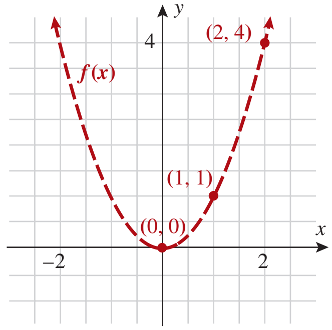
To Graph	**Shift the Graph of $f(x)$**
$f(x + c)$	c units to the left
$f(x - c)$	c units to the right

Adding or subtracting a constant **inside** the function corresponds to a **horizontal** shift that goes **opposite the sign**.

EXAMPLE 1 | Horizontal and Vertical Shifts

Sketch the graphs of the given functions using horizontal and vertical shifts:

a. $g(x) = x^2 - 1$ **b.** $H(x) = (x + 1)^2$

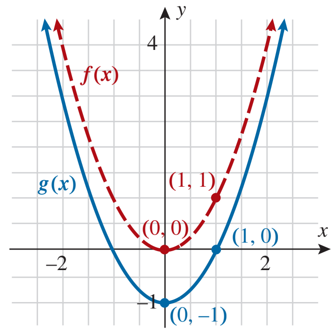


Solution

In both cases, the function to start with is $f(x) = x^2$.

a. $g(x) = x^2 - 1$ can be rewritten as $g(x) = f(x) - 1$.

1. The shift (one unit) occurs *outside* of the function. Therefore, we expect a vertical shift that goes with the sign.
2. Since the sign is *negative*, this corresponds to a *downward* shift.
3. Shifting the graph of the function $f(x) = x^2$ down one unit yields the graph of $g(x) = x^2 - 1$.



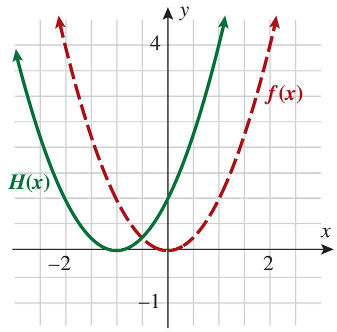
STUDY TIP

Shifts inside the function are horizontal shifts *opposite* the sign.

Left (+)
Right (−)

b. $H(x) = (x + 1)^2$ can be rewritten as $H(x) = f(x + 1)$.

1. The shift (one unit) occurs *inside* of the function. Therefore, we expect a horizontal shift that goes *opposite* the sign.
2. Since the sign is *positive*, this corresponds to a shift to the *left*.
3. Shifting the graph of the function $f(x) = x^2$ to the left one unit yields the graph of $H(x) = (x + 1)^2$.

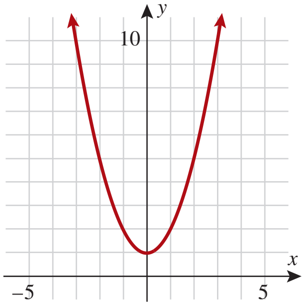


Your Turn Sketch the graphs of the given functions using horizontal and vertical shifts.

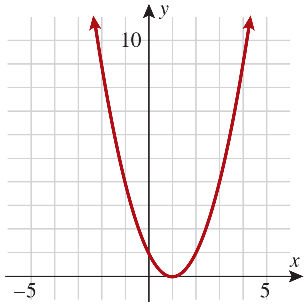
a. $g(x) = x^2 + 1$ **b.** $H(x) = (x - 1)^2$

Answer

a.



b.



It is important to note that the domain and range of the resulting function can be thought of as also being shifted. Shifts in the domain correspond to horizontal shifts, and shifts in the range correspond to vertical shifts.

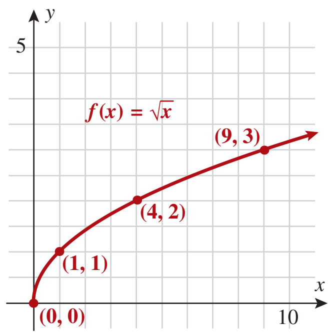
Video

EXAMPLE 2 | Horizontal and Vertical Shifts and Changes in the Domain and Range

Graph the functions using translations and state the domain and range of each function.

a. $g(x) = \sqrt{x+1}$

b. $G(x) = \sqrt{x} - 2$



Solution

In both cases, the function to start with is $\boldsymbol{f(x) = \sqrt{x}}$.

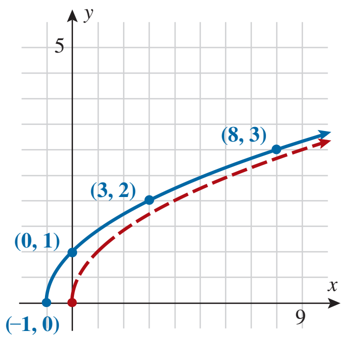
Domain: $[0, \infty)$

Range: $[0, \infty)$

a. $\boldsymbol{g(x) = \sqrt{x+1}}$ can be rewritten as $g(x) = f(x + 1)$.

1. The shift (one unit) is *inside* the function, which corresponds to a *horizontal* shift *opposite the sign.*

2. Shifting the graph of $\boldsymbol{f(x) = \sqrt{x}}$ to the *left* one unit yields the graph of $\boldsymbol{g(x) = \sqrt{x+1}}$. Notice that the point **(0, 0)**, which lies on the graph of $f(x)$, gets shifted to the point **(−1, 0)** on the graph of $g(x)$.



Although the original function $f(x) = \sqrt{x}$ had an implicit restriction on the domain: $[0, \infty)$, the function $g(x) = \sqrt{x+1}$ has the implicit restriction that $x \geq -1$. We see that the output or range of $g(x)$ is the same as the output of the original function $f(x)$.

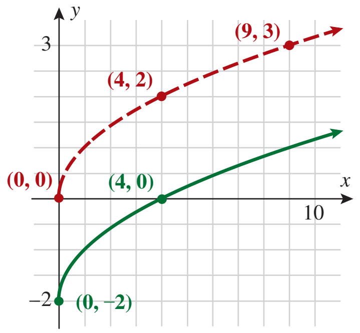
Domain: $[-1, \infty)$ **Range: $[0, \infty)$**

b. $G(x) = \sqrt{x} - 2$ can be rewritten as

$G(x) = f(x) - 2$.

1. The shift (two units) is *outside* the function, which corresponds to a *vertical* shift *with the sign.*

2. The graph of $G(x) = \sqrt{x} - 2$ is found by shifting $f(x) = \sqrt{x}$ down two units. Note that the point **(0, 0)**, which lies on the graph of $f(x)$, gets shifted to the point **(0, −2)** on the graph of $G(x)$.



The original function $f(x) = \sqrt{x}$ has an implicit restriction on the domain: $[0, \infty)$. The function $G(x) = \sqrt{x} - 2$ also has the implicit restriction that $x \geq 0$. The output or range of $G(x)$ is always two units less than the output of the original function $f(x)$.

Domain: $[0, \infty)$ **Range: $[-2, \infty)$**

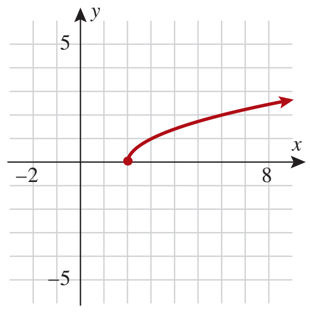
Your Turn Sketch the graph of the functions using shifts and state the domain and range.

a. $G(x) = \sqrt{x-2}$

b. $h(x) = |x| + 1$

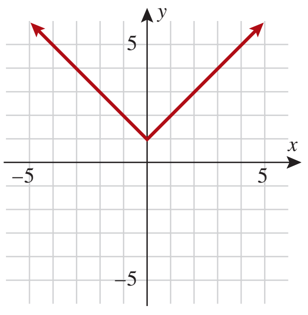
Answer

a. $G(x) = \sqrt{x-2}$



Domain: $[2, \infty)$ Range: $[0, \infty)$

b. $h(x) = |x| + 1$



Domain: $[-\infty, \infty)$ Range: $[1, \infty)$

Concept Check

For the functions $F(x) = \sqrt{x - a}$ and $G(x) = \sqrt{x} - a$ where $a > 0$ explain the shifts on $y = \sqrt{x}$.

Answer:

$F(x)$ is the graph of $y = \sqrt{x}$ shifted a units to the right. $G(x)$ is the graph of $y = \sqrt{x}$ shifted a units down.

The previous examples have involved graphing functions by shifting a known function either in the horizontal or vertical direction. Let us now look at combinations of horizontal and vertical shifts.

Video EXAMPLE 3 | Combining Horizontal and Vertical Shifts

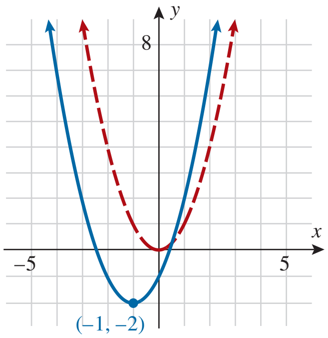
Sketch the graph of the function $F(x) = (x + 1)^2 - 2$. State the domain and range of F.

Solution

The base function is $y = x^2$.

1. The shift (one unit) is *inside* the function, so it represents a *horizontal* shift *opposite the sign.*
2. The -2 shift is *outside* the function, which represents a *vertical* shift *with the sign.*
3. Therefore, we shift the graph of $y = x^2$ to the left one unit and down two units. For instance, the point **(0, 0)** on the graph of $y = x^2$ shifts to the point **(−1, −2)** on the graph of $F(x) = (x + 1)^2 - 2$.

Domain: $(-\infty, \infty)$ Range: $[-2, \infty)$



Your Turn Sketch the graph of the function $f(x) = |x - 2| + 1$. State the domain and range of f.

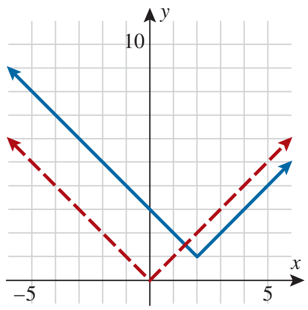
Answer

$f(x) = |x - 2| + 1$

$f(x) = |x|$

Domain: $(-\infty, \infty)$

Range: $[1, \infty)$



All of the previous transformation examples involve starting with a common function and shifting the function in either the horizontal or the vertical direction (or a combination of both). Now, let's investigate *reflections* of functions about the x-axis or y-axis.

1.3.2 Reflection About the Axes

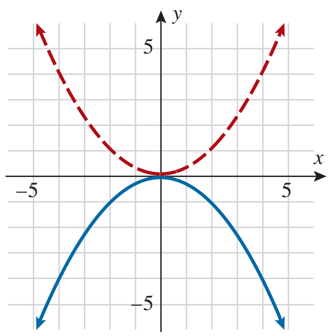
1.3.2 Skill Sketch the graph of a function by reflecting a common function about the x-axis or y-axis.

1.3.2 Conceptual Understand why a negative argument inside the function corresponds to a reflection about the y-axis and a negative outside the function corresponds to a reflection about the x-axis.

To sketch the graphs of $f(x) = x^2$ and $g(x) = -x^2$, start by first listing points that are on each of the graphs and then connecting the points with smooth curves.

x	$f(x)$
−2	4
−1	1
0	0
1	1
2	4

x	$g(x)$
−2	−4
−1	−1
0	0
1	−1
2	−4

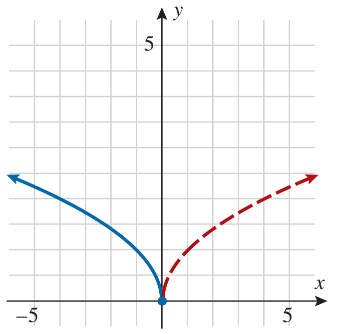


Note that if the graph of $f(x) = x^2$ is reflected about the x-axis, the result is the graph of $g(x) = -x^2$. Also note that the function $g(x)$ can be written as the negative of the function $f(x)$; that is, $g(x) = -f(x)$. In general, **reflection about the x-axis** is produced by multiplying a function by -1.

Let's now investigate reflection about the y-axis. To sketch the graphs of $f(x) = \sqrt{x}$ and $g(x) = \sqrt{-x}$ start by listing points that are on each of the graphs and then connecting the points with smooth curves.

x	$f(x)$
0	0
1	1
4	2
9	3

x	$g(x)$
−9	3
−4	2
−1	1
0	0



Note that if the graph of $f(x) = \sqrt{x}$ is reflected about the y-axis, the result is the graph of $g(x) = \sqrt{-x}$. Also note that the function $g(x)$ can be written as $g(x) = f(-x)$. In general, **reflection about the y-axis** is produced by replacing x with $-x$ in the function. Notice that the domain of f is $[0, \infty)$, whereas the domain of g is $(-\infty, 0]$.

Reflection About the Axes

The graph of $-f(x)$ is obtained by reflecting the graph of $f(x)$ about the **x-axis**.
The graph of $f(-x)$ is obtained by reflecting the graph of $f(x)$ about the **y-axis**.

EXAMPLE 4 | Sketching the Graph of a Function Using Both Shifts and Reflections

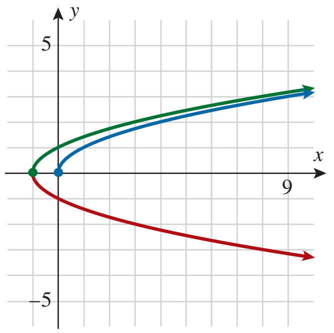
Sketch the graph of the function $G(x) = -\sqrt{x + 1}$.

Solution

Start with the square root function. $f(x) = \sqrt{x}$

Shift the graph of $f(x)$ to the left one unit to arrive at the graph of $f(x + 1)$. $f(x + 1) = \sqrt{x + 1}$

Reflect the graph of $f(x + 1)$ about the x-axis to arrive at the graph of $-f(x + 1)$. $-f(x + 1) = -\sqrt{x + 1}$



Video **EXAMPLE 5 | Sketching the Graph of a Function Using Both Shifts and Reflections**

Sketch the graph of the function $f(x) = \sqrt{2-x} + 1$.

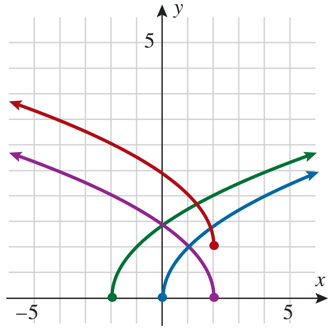
Solution

Start with the square root function. $g(x) = \sqrt{x}$

Shift the graph of $g(x)$ to the left two units to arrive at the graph of $g(x+2)$. $g(x+2) = \sqrt{x+2}$

Reflect the graph of $g(x+2)$ about the y-axis to arrive at the graph of $g(-x+2)$. $g(-x+2) = \sqrt{-x+2}$

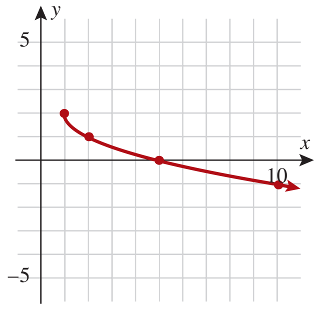
Shift the graph $g(-x+2)$ up one unit to arrive at the graph of $g(-x+2)+1$. $g(-x+2)+1 = \sqrt{2-x}+1$



Your Turn Use shifts and reflections to sketch the graph of the function $f(x) = -\sqrt{x-1} + 2$. State the domain and range of $f(x)$.

Answer

Domain: $[1, \infty)$
Range: $(-\infty, 2]$



Concept Check

For any even function $f(x)$, describe the graph of $f(-x)$.

Answer: The graph of $f(-x)$ is identical to the graph of $f(x)$ because for even functions $f(-x) = f(x)$.

Look back at the order in which transformations were performed in Example 5: horizontal shift, reflection, and then vertical shift. Let us consider an alternate order of transformations.

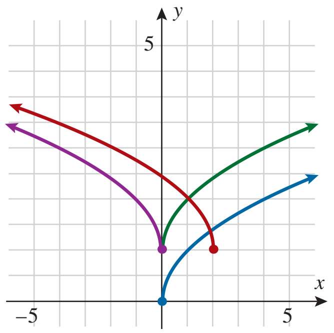
Words	Math
Start with the square root function.	$g(x) = \sqrt{x}$
Shift the graph of $g(x)$ up one unit to arrive at the graph of $g(x)+1$.	$g(x)+1 = \sqrt{x}+1$
Reflect the graph of $g(x)+1$ about the y-axis to arrive at the graph of $g(-x)+1$.	$g(-x)+1 = \sqrt{-x}+1$

Replace x with $x-2$, which corresponds to a shift of the graph of $g(-x)+1$ to the right two units to arrive at the graph of $g[-(x-2)]+1$.

In the last step, we replaced x with $x-2$, which required us to think ahead knowing the desired result was $2-x$ inside the radical. To avoid any possible confusion, follow this order of transformations:

1. Horizontal shifts: $f(x \pm c)$
2. Reflection: $f(-x)$ and/or $-f(x)$
3. Vertical shifts: $f(x) \pm c$

$g(-x+2)+1 = \sqrt{2-x}+1$



1.3.3 Stretching and Compressing

1.3.3 Skill Sketch the graph of a function by stretching or compressing a common function.

1.3.3 Conceptual Understand the difference between rigid and nonrigid transformations.

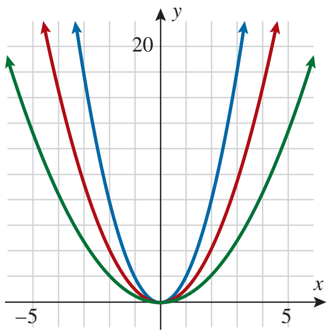
Horizontal shifts, vertical shifts, and reflections change only the position of the graph in the Cartesian plane, leaving the basic shape of the graph unchanged. These transformations (shifts and reflections) are called **rigid transformations** because they alter only the *position*. **Nonrigid transformations**, on the other hand, distort the *shape* of the original graph. We now consider *stretching* and *compressing* of graphs in both the vertical and the horizontal direction.

A vertical stretch or compression of a graph occurs when the function is multiplied by a positive constant. See, for example, the graphs of the functions $f(x)=x^2$, $g(x)=2f(x)=2x^2$, and $h(x)=\frac{1}{2}f(x)=\frac{1}{2}x^2$. Depending on if the constant is larger than 1 or smaller than 1 will determine whether it corresponds to a stretch (expansion) or compression (contraction) in the vertical direction.

x	$f(x)$
−2	4
−1	1
0	0
1	1
2	4

x	$g(x)$
−2	8
−1	2
0	0
1	2
2	8

x	$h(x)$
−2	2
−1	$\frac{1}{2}$
0	0
1	$\frac{1}{2}$
2	2



Note that when the function $f(x)=x^2$ is multiplied by 2, so that $g(x)=2f(x)=2x^2$, the result is a graph stretched in the vertical direction. When the function $f(x)=x^2$ is multiplied by $\frac{1}{2}$, so that $h(x)=\frac{1}{2}f(x)=\frac{1}{2}x^2$, the result is a graph that is compressed in the vertical direction.

Vertical Stretching and Vertical Compressing of Graphs

The graph of $cf(x)$ is found by:

- **Vertically stretching** the graph of $f(x)$ if $c > 1$
- **Vertically compressing** the graph of $f(x)$ if $0 < c < 1$

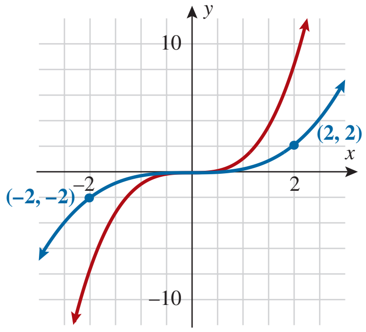
Note: c is any positive real number.

EXAMPLE 6 | Vertically Stretching and Compressing Graphs

Graph the function $h(x) = \frac{1}{4}x^3$.

Solution

1. Start with the cube function. $f(x) = x^3$
2. Vertical compression is expected because $\frac{1}{4}$ is less than 1. $h(x) = \frac{1}{4}x^3$
3. Determine a few points that lie on the graph of h.
   $(0, 0)$ $(2, 2)$ $(-2, -2)$



Conversely, if the argument x of a function f is multiplied by a positive real number c, then the result is a *horizontal* stretch of the graph of f if $0 < c < 1$. If $c > 1$, then the result is a *horizontal* compression of the graph of f.

Horizontal Stretching and Horizontal Compressing of Graphs

The graph of $f(cx)$ is found by:

- **Horizontally stretching** the graph of $f(x)$ — if $0 < c < 1$
- **Horizontally compressing** the graph of $f(x)$ — if $c > 1$

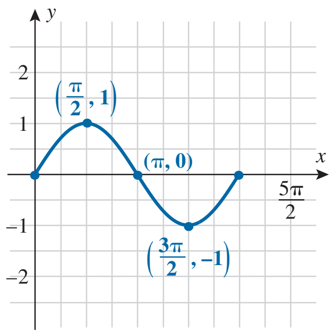
Note: c is any positive real number.

Video EXAMPLE 7 | Vertically Stretching and Horizontally Compressing Graphs

Given the graph of $f(x)$, graph:

a. $2f(x)$

b. $f(2x)$

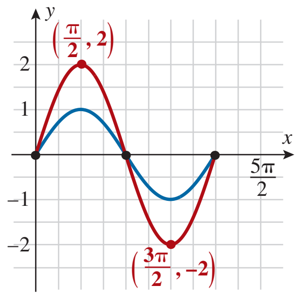


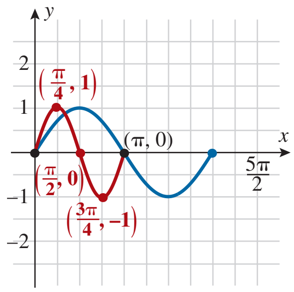
Solution (a)

Since the function is multiplied (on the outside) by 2, the result is that each ***y*-value** of $f(x)$ is ***multiplied* by 2**, which corresponds to vertical stretching.

Solution (b)

Since the argument of the function is multiplied (on the inside) by 2, the result is that each ***x*-value** of $f(x)$ is **divided by 2**, which corresponds to horizontal compression.

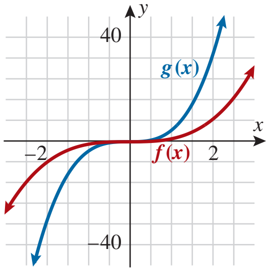




Your Turn Graph the function $g(x) = 4x^3$.

Answer

Stretching of the graph $f(x) = x^3$.



Concept Check

Describe where the graphs of $f(x)$ and $a \cdot f(x)$ intersect.

Answer: Only at the points when $f(x) = 0$ (x-intercepts)

Video EXAMPLE 8 | Sketching the Graph of a Function Using Multiple Transformations

Sketch the graph of the function $H(x) = -2(x-3)^2$.

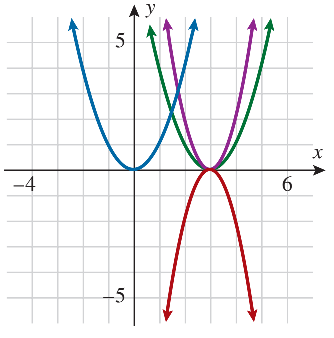
Solution

Start with the square function. $f(x) = x^2$

Shift the graph of $f(x)$ to the right three units to arrive at the graph of $f(x-3)$. $f(x-3) = (x-3)^2$

Vertically stretch the graph of $f(x-3)$ by a factor of 2 to arrive at the graph of $2f(x-3)$. $2f(x-3) = 2(x-3)^2$

Reflect the graph $2f(x-3)$ about the x-axis to arrive at the graph of $-2f(x-3)$. $-2f(x-3) = -2(x-3)^2$



In Example 8, we followed the same "inside out" approach with the functions to determine the order for the transformations: horizontal shift, vertical stretch, and reflection.

Section 1.3 Summary

Transformation	To graph the function . . .	Draw the graph of f and then . . .	Description
Horizontal shifts	$f(x+c)$ $f(x-c)$	Shift the graph of f to the left c units. Shift the graph of f to the right c units.	Replace x by $x+c$. Replace x by $x-c$.
Vertical shifts	$f(x)+c$ $f(x)-c$	Shift the graph of f up c units. Shift the graph of f down c units.	Add c to $f(x)$. Subtract c from $f(x)$.
Reflection about the x-axis	$-f(x)$	Reflect the graph of f about the x-axis.	Multiply $f(x)$ by -1.
Reflection about the y-axis	$f(-x)$	Reflect the graph of f about the y-axis.	Replace x by $-x$.
Vertical stretch	$cf(x)$, where $c>1$	Vertically stretch the graph of f.	Multiply $f(x)$ by c.
Vertical compression	$cf(x)$, where $0<c<1$	Vertically compress the graph of f.	Multiply $f(x)$ by c.
Horizontal stretch	$f(cx)$, where $0<c<1$	Horizontally stretch the graph of f.	Replace x by cx.
Horizontal compression	$f(cx)$, where $c>1$	Horizontally compress the graph of f.	Replace x by cx.

Section 1.3 Exercises

Skills

In Exercises 1–6, write the function whose graph is the graph of $y=|x|$ but is transformed accordingly.

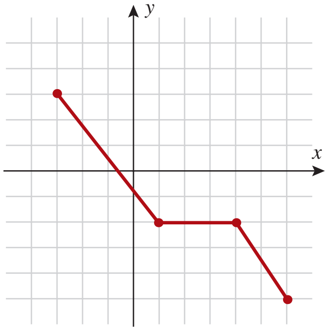
1. Shifted up three units
2. Shifted to the left four units
3. Reflected about the y-axis
4. Reflected about the x-axis
5. Vertically stretched by a factor of 3
6. Vertically compressed by a factor of 3

In Exercises 7–12, write the function whose graph is the graph of $y=x^3$ but is transformed accordingly.

7. Shifted down four units
8. Shifted to the right three units
9. Shifted up three units and to the left one unit
10. Reflected about the x-axis
11. Reflected about the y-axis
12. Reflected about both the x-axis and the y-axis

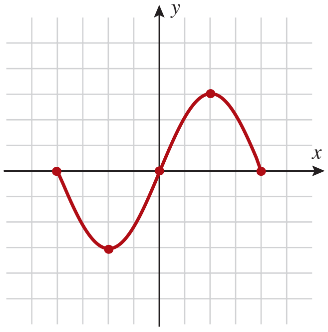
In Exercises 13–36, use the given graph to sketch the graph of the indicated functions.

13.



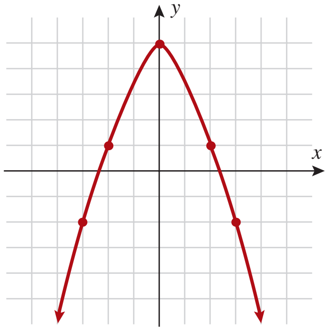
a. $y=f(x-2)$
b. $y=f(x)-2$

14.



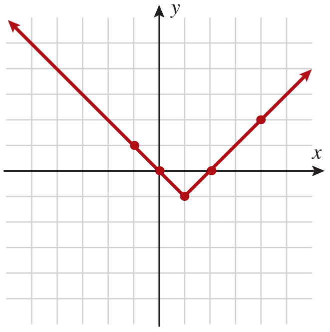
a. $y=f(x+2)$
b. $y=f(x)+2$

15.



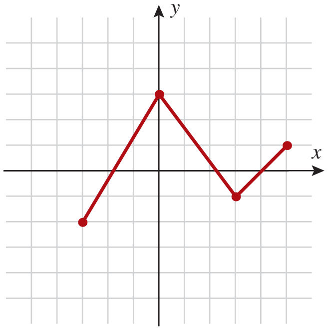
a. $y=f(x)-3$
b. $y=f(x-3)$

16.



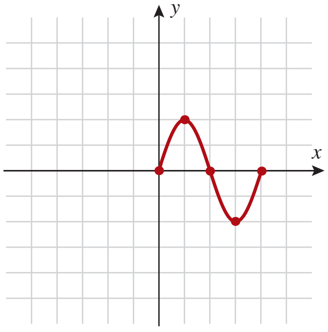
a. $y=f(x)+3$
b. $y=f(x+3)$

17.



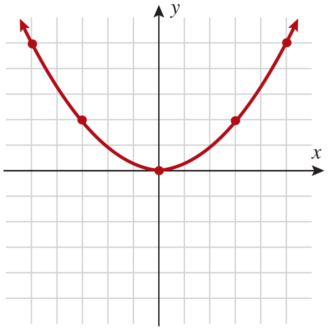
a. $y = -f(x)$
b. $y = f(-x)$

18.



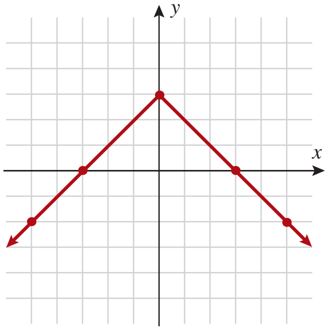
a. $y = -f(x)$
b. $y = f(-x)$

19.



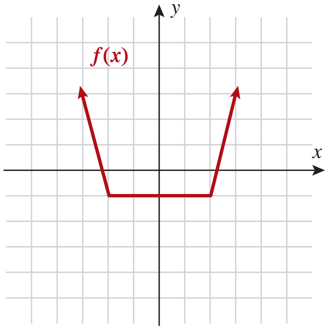
a. $y = 2f(x)$
b. $y = f(2x)$

20.

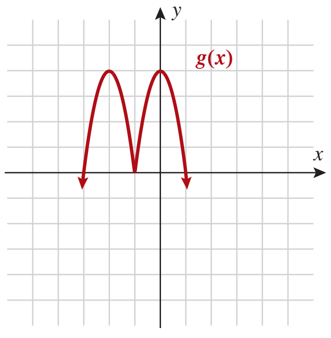


a. $y = 2f(x)$
b. $y = f(2x)$

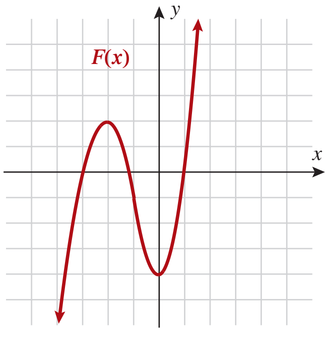
21. $y = f(x-2) - 3$
22. $y = f(x+1) - 2$
23. $y = -f(x-1) + 2$
24. $y = -2f(x) + 1$



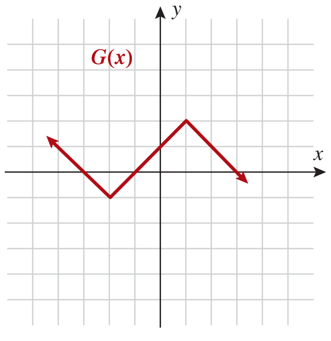
25. $y = -\frac{1}{2}g(x)$
26. $y = \frac{1}{4}g(-x)$
27. $y = -g(2x)$
28. $y = -g(\frac{1}{2}x)$



29. $y = \frac{1}{2}F(x-1) + 2$
30. $y = \frac{1}{2}F(-x)$
31. $y = -F(1-x)$
32. $y = -F(x-2) - 1$



33. $y = 2G(x+1) - 4$
34. $y = 2G(-x) + 1$
35. $y = -2G(x-1) + 3$
36. $y = -G(x-2) - 1$



In Exercises 37–64, graph the function using transformations.

37. $y = x^2 - 2$
38. $y = x^2 + 3$
39. $y = (x+1)^2$
40. $y = (x-2)^2$
41. $y = (x-3)^2 + 2$
42. $y = (x+2)^2 + 1$
43. $y = -(1-x)^2$
44. $y = -(x+2)^2$
45. $y = |-x|$
46. $y = -|x|$
47. $y = -|x+2| - 1$
48. $y = |1-x| + 2$
49. $y = 2x^2 + 1$
50. $y = 2|x| + 1$
51. $y = -\sqrt{x-2}$
52. $y = \sqrt{2-x}$
53. $y = -\sqrt{2+x} - 1$
54. $y = \sqrt{2-x} + 3$
55. $y = \sqrt[3]{x-1} + 2$
56. $y = \sqrt[3]{x+2} - 1$
57. $f(x) = \frac{1}{x} + 2$
58. $f(x) = \frac{1}{x+2}$
59. $y = \frac{1}{x+3} + 2$
60. $y = \frac{1}{3-x}$
61. $y = 2 - \frac{1}{x+2}$
62. $y = 2 - \frac{1}{1-x}$
63. $y = 5\sqrt{-x}$
64. $y = -\frac{1}{5}\sqrt{x}$

In Exercises 65–70, transform the function into the form $f(x) = c(x-h)^2 + k$, where c, k, and h are constants, by completing the square. Use graph-shifting techniques to graph the function.

65. $y = x^2 - 6x + 11$
66. $f(x) = x^2 + 2x - 2$
67. $f(x) = -x^2 - 2x$
68. $f(x) = -x^2 + 6x - 7$
69. $f(x) = 2x^2 - 8x + 3$
70. $f(x) = 3x^2 - 6x + 5$

Applications

71. Salary. A manager hires an employee at a rate of \$15 per hour. Write the function $f(x)$ that describes the current salary of the employee as a function of the number of hours worked per week, x. After a year, the manager decides to award the employee a raise equivalent to paying them for an additional 5 hours per week. Write the function $g(x)$ that describes the salary of the employee after the raise.

72. Profit. The profit associated with St. Augustine sod in Florida is typically $P(x) = -x^2 + 14{,}000x - 48{,}700{,}000$, where x is the number of pallets sold per year in a normal year. In rainy years Sod King gives away 10 free pallets per year. Write the function that describes the profit of x pallets of sod in rainy years.

73. Taxes. Every year in the United States each working American typically pays in taxes a percentage of their earnings (minus the standard deduction). Karen's 2020 taxes were calculated based on the formula $T(x) = 0.24(x - 12{,}200)$. That year the standard deduction was \$12,200 and her tax bracket paid 24% in taxes. Write the function that will determine her 2021 taxes, assuming she receives a raise that will place her in the 32% bracket and the standard deduction remains the same.

74. Medication. The amount of medication that an infant requires is typically a function of the baby's weight. The number of milliliters of an antiseizure medication A is given by $A(x) = \sqrt{x} + 2$, where x is the weight of the infant in ounces. In emergencies there is often not enough time to weigh the infant, so nurses have to estimate the baby's weight. What is the function that represents the actual amount of medication the infant is given if their weight is overestimated by 3 ounces?

75. Profit. A company that started in 1900 has made a profit corresponding to $P(t) = t^3 - t^2 + t - 1$, where P is the profit in dollars and t is the year (with $t = 0$ corresponding to 1950). Write the profit function with $t = 0$ corresponding to the year 2000.

76. Profit. For the company in Exercise 75, write the profit function with $t = 0$ corresponding to the year 2010.

For Exercises 77 and 78, refer to the following:

Body Surface Area (BSA) is used in physiology and medicine for many clinical purposes. BSA can be modeled by the function

$$\text{BSA} = \sqrt{\frac{wh}{3600}}$$

where w is weight in kilograms and h is height in centimeters. Since BSA depends on weight and height, it is often thought of as a function of both weight and height. However, for an individual adult height is generally considered constant; thus BSA can be thought of as a function of weight alone.

77. Health/Medicine. (a) If an adult female is 162 centimeters tall, find her BSA as a function of weight. (b) If she loses 3 kilograms, find a function that represents her new BSA.

78. Health/Medicine. (a) If an adult male is 180 centimeters tall, find his BSA as a function of weight. (b) If he gains 5 kilograms, find a function that represents his new BSA.

Catch the Mistake

In Exercises 79–82, explain the mistake that is made.

79. Describe a procedure for graphing the function $f(x) = \sqrt{x - 3} + 2$.

Solution

a. Start with the function $f(x) = \sqrt{x}$.

b. Shift the function to the left three units.

c. Shift the function up two units.

This is incorrect. What mistake was made?

80. Describe a procedure for graphing the function $f(x) = -\sqrt{x + 2} - 3$.

Solution

a. Start with the function $f(x) = \sqrt{x}$.

b. Shift the function to the left two units.

c. Reflect the function about the y-axis.

d. Shift the function down three units.

This is incorrect. What mistake was made?

81. Describe a procedure for graphing the function $f(x) = |3 - x| + 1$.

Solution

a. Start with the function $f(x) = |x|$.

b. Reflect the function about the y-axis.

c. Shift the function to the left three units.

d. Shift the function up one unit.

This is incorrect. What mistake was made?

82. Describe a procedure for graphing the function $f(x) = -2x^2 + 1$.

Solution

a. Start with the function $f(x) = x^2$.

b. Reflect the function about the y-axis.

c. Shift the function up one unit.

d. Expand in the vertical direction by a factor of 2.

This is incorrect. What mistake was made?

Conceptual

In Exercises 83–88, determine whether each statement is true or false.

83. The graph of $y = |-x|$ is the same as the graph of $y = |x|$.

84. The graph of $y = \sqrt{-x}$ is the same as the graph of $y = \sqrt{x}$.

85. If the graph of an odd function is reflected about the x-axis and then the y-axis, the result is the graph of the original odd function.

86. If the graph of $y = \frac{1}{x}$ is reflected about the x-axis, it produces the same graph as if it had been reflected about the y-axis.

87. If f is a function and $c > 1$ is a constant, then the graph of $-cf$ is a reflection about the x-axis of a vertical stretch of the graph of f.

88. If a and b are positive constants and f is a function, then the graph of $f(x + a) + b$ is obtained by shifting the graph of f to the right a units and then shifting this graph up b units.

Challenge

89. The point (a, b) lies on the graph of the function $y = f(x)$. What point is guaranteed to lie on the graph of $f(x - 3) + 2$?

90. The point (a, b) lies on the graph of the function $y = f(x)$. What point is guaranteed to lie on the graph of $-f(-x) + 1$?

91. The point (a, b) lies on the graph of the function $y = f(x)$. What point is guaranteed to lie on the graph of $2f(x + 1) - 1$?

92. The point (a, b) lies on the graph of the function $y = f(x)$. What point is guaranteed to lie on the graph of $-2f(x - 3) + 4$?

Preview to Calculus

For Exercises 93–100, refer to the following:

In calculus, the difference quotient $\frac{f(x+h) - f(x)}{h}$ of a function f is used to find the derivative f' of f, by letting h approach 0, $h \to 0$. Find the derivatives of f and g.

93. Horizontal Shift. $f(x) = x^2$, $g(x) = (x - 1)^2$. How are the graphs of g' and f' related?

94. Horizontal Shift. $f(x) = \sqrt{x}$, $g(x) = \sqrt{x + 5}$. How are the graphs of g' and f' related?

95. Vertical Shift. $f(x) = 2x$, $g(x) = 2x + 7$. How are the graphs of g' and f' related?

96. Vertical Shift. $f(x) = x^3$, $g(x) = x^3 - 4$. How are the graphs of g' and f' related?

97. Reflection. $f(x) = (x + 1)^2$, $g(x) = -(x + 1)^2$. How are the graphs of g' and f' related?

98. Stretch. $f(x) = x^2$, $g(x) = 3x^2$. How are the graphs of g' and f' related?

99. Compression. $f(x) = \sqrt{x}$, $g(x) = \frac{1}{4}\sqrt{x}$. How are the graphs of g' and f' related?

100. Reflection. $f(x) = (x - 2)^3$, $g(x) = (2 - x)^3$. How are the graphs of g' and f' related?

1.4 Operations on Functions and Composition of Functions

SKILLS OBJECTIVES

- Add, subtract, multiply, and divide functions.
- Evaluate composite functions and determine the corresponding domains.

CONCEPTUAL OBJECTIVES

- Understand domain restrictions when dividing functions.
- Realize that the domain of a composition of functions excludes values that are not in the domain of the inside function.

Two different functions can be combined using mathematical operations such as addition, subtraction, multiplication, and division. Also, there is an operation on functions called composition, which can be thought of as a function of a function. When we combine functions, we do so algebraically. Special attention must be paid to the domain and range of the combined functions.

1.4.1 Adding, Subtracting, Multiplying, and Dividing Functions

1.4.1 Skill Add, subtract, multiply, and divide functions.

1.4.1 Conceptual Understand domain restrictions when dividing functions.

Two functions can be added, subtracted, and multiplied. The resulting function domain is therefore the intersection of the domains of the two functions. However, for division, any value of x (input) that makes the denominator equal to zero must be eliminated from the domain.

Function	Notation	Domain
Sum	$(f + g)(x) = f(x) + g(x)$	$\{\text{domain of } f\} \cap \{\text{domain of } g\}$
Difference	$(f - g)(x) = f(x) - g(x)$	$\{\text{domain of } f\} \cap \{\text{domain of } g\}$
Product	$(f \cdot g)(x) = f(x) \cdot g(x)$	$\{\text{domain of } f\} \cap \{\text{domain of } g\}$
Quotient	$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$	$\{\text{domain of } f\} \cap \{\text{domain of } g\} \cap \{g(x) \neq 0\}$

We can think of this in the following way: Any number that is in the domain of *both* functions is in the domain of the combined function. The exception to this is the quotient function, which also eliminates values that make the denominator equal to zero.

EXAMPLE 1 | Operations on Functions: Determining Domains of New Functions

For the functions $f(x) = \sqrt{x - 1}$ and $g(x) = \sqrt{4 - x}$, determine the sum function, difference function, product function, and quotient function. State the domain of these four new functions.

Solution

Sum function: $f(x) + g(x) = \sqrt{x - 1} + \sqrt{4 - x}$

Difference function: $f(x) - g(x) = \sqrt{x - 1} - \sqrt{4 - x}$

Product function: $f(x) \cdot g(x) = \sqrt{x - 1} \cdot \sqrt{4 - x}$

$$= \sqrt{(x - 1)(4 - x)} = \sqrt{-x^2 + 5x - 4}$$

Quotient function: $\frac{f(x)}{g(x)} = \frac{\sqrt{x - 1}}{\sqrt{4 - x}} = \sqrt{\frac{x - 1}{4 - x}}$

The domain of the square root function is determined by setting the argument under the radical greater than or equal to zero.

Domain of $f(x)$: $[1, \infty)$

Domain of $g(x)$: $(-\infty, 4]$

The domain of the sum, difference, and product functions is

$$[1, \infty) \cap (-\infty, 4] = [1, 4]$$

The quotient function has the additional constraint that the denominator cannot be zero. This implies that $x \neq 4$, so the domain of the quotient function is $[1, 4)$.

Your Turn Given the function $f(x) = \sqrt{x + 3}$ and $g(x) = \sqrt{1 - x}$, find $(f + g)(x)$ and state its domain.

Answer

$(f + g)(x) = \sqrt{x + 3} + \sqrt{1 - x}$

Domain: $[-3, 1]$

Concept Check

Given $f(x) = \sqrt{x+1}$ and $g(x) = \frac{1}{x}$, find the domain of $\frac{g(x)}{f(x)}$.

Answer: $(-1, 0) \cup (0, \infty)$

Video

EXAMPLE 2 | Quotient Function and Domain Restrictions

Given the functions $F(x) = \sqrt{x}$ and $G(x) = |x - 3|$, find the quotient function, $\left(\frac{F}{G}\right)(x)$, and state its domain.

Solution

The quotient function is written as

$$\left(\frac{F}{G}\right)(x) = \frac{F(x)}{G(x)} = \frac{\sqrt{x}}{|x-3|}$$

Domain of $F(x)$: $[0, \infty)$ Domain of $G(x)$: $(-\infty, \infty)$

The real numbers that are in both the domain for $F(x)$ and the domain for $G(x)$ are represented by the intersection $[0, \infty) \cap (-\infty, \infty) = [0, \infty)$. Also, the denominator of the quotient function is equal to zero when $x = 3$, so we must eliminate this value from the domain.

Domain of $\left(\frac{F}{x}\right)(x)$: $[0, 3) \cup (3, \infty)$

Your Turn For the functions given in Example 2, determine the quotient function $\left(\frac{G}{F}\right)(x)$, and state its domain.

Answer

$$\left(\frac{G}{F}\right)(x) = \frac{G(x)}{F(x)} = \frac{|x-3|}{\sqrt{x}}$$

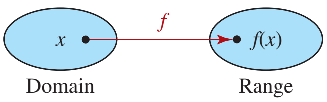
Domain: $(0, \infty)$

1.4.2 Composition of Functions

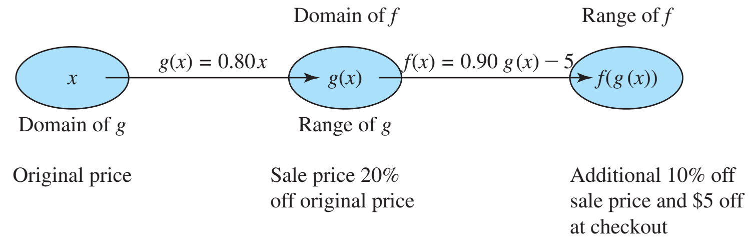
1.4.2 Skill Evaluate composite functions and determine the corresponding domains.

1.4.2 Conceptual Realize that the domain of a composition of functions excludes values that are not in the domain of the inside.

Recall that a function maps every element in the domain to exactly one corresponding element in the range:



Suppose there is a sales rack of clothes in a department store. Let x correspond to the original price of each item on the rack. These clothes have recently been marked down 20%. Therefore, the function $g(x) = 0.80x$ represents the current sale price of each item. You have been invited to a special sale that lets you take 10% off the current sale price and an additional \$5 off every item at checkout. The function $f(g(x)) = 0.90g(x) - 5$ determines the checkout price. Note that the output of the function g is the input of the function f as shown in the figure.



This is an example of a **composition of functions**, when the output of one function is the input of another function. It is commonly referred to as a function of a function.

An algebraic example of this is the function $y = \sqrt{x^2 - 2}$. Suppose we let $g(x) = x^2 - 2$ and $f(x) = \sqrt{x}$. Recall that the independent variable in function notation is a placeholder. Since $f(\square) = \sqrt{(\square)}$, then $f(g(x)) = \sqrt{(g(x))}$. Substituting the expression for $g(x)$, we find $f(g(x)) = \sqrt{x^2 - 2}$. The function $y = \sqrt{x^2 - 2}$ is said to be a composite function, $y = f(g(x))$.

Note that the domain of $g(x)$ is the set of all real numbers, and the domain of $f(x)$ is the set of all nonnegative numbers. The domain of a composite function is the set of all x such that $g(x)$ is in the domain of f. For instance, in the composite function $y = f(g(x))$, we know that the allowable inputs into f are all numbers greater than or equal to zero. Therefore, we restrict the outputs of $g(x) \geq 0$ and find the corresponding x-values. Those x-values are the only allowable inputs and constitute the domain of the composite function $y = f(g(x))$.

The symbol that represents composition of functions is a small open circle; thus $(f \circ g)(x) = f(g(x))$, which is read aloud as "f of g." It is important not to confuse this with the multiplication sign: $(f \cdot g)(x) = f(x)g(x)$.

Caution

$f \circ g \neq f \cdot g$

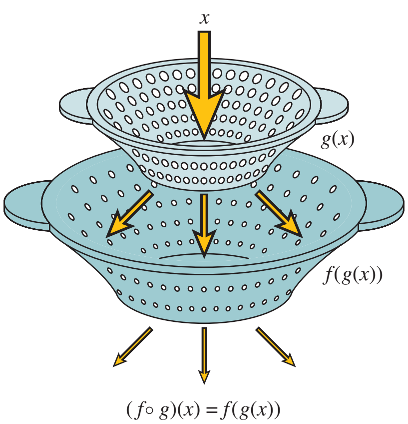
Composition of Functions

Given two functions f and g, there are two **composite functions** that can be formed.

Notation	Words	Definition	Domain
$f \circ g$	f composed with g	$f(g(x))$	The set of all real numbers x in the domain of g such that $g(x)$ is also in the domain of f.
$g \circ f$	g composed with f	$g(f(x))$	The set of all real numbers x in the domain of f such that $f(x)$ is also in the domain of g.

It is important to realize that there are two "filters" that allow certain values of x into the domain of $(f \circ g)(x) = f(g(x))$. The first filter is $g(x)$. If x is not in the domain of $g(x)$, it cannot be in the domain of $(f \circ g)(x) = f(g(x))$. Of those values for x that are in the domain of $g(x)$, only some pass through because we restrict the output of $g(x)$ to values that are allowable as input into f. This adds an additional filter.

The domain of $f \circ g$ is always a subset of the domain of g, and the range of $f \circ g$ is always a subset of the range of f.



STUDY TIP

Order is important:
$(f \circ g)(x) = f(g(x))$
$(g \circ f)(x) = g(f(x)$

STUDY TIP

The domain of $f \circ g$ is always a subset of the domain of g, and the range of $f \circ g$ is always a subset of the range of f.

Video EXAMPLE 3 | Finding a Composite Function

Given the functions $f(x) = x^2 + 1$ and $g(x) = x - 3$, find $(f \circ g)(x)$.

Solution

Write $f(x)$ using placeholder notation. $f(\square) = (\square)^2 + 1$

Express the composite function $f \circ g$. $f(g(x)) = (g(x))^2 + 1$

Substitute $g(x) = x - 3$ into f. $f(g(x)) = (x - 3)^2 + 1$

Eliminate the parentheses on the right side. $f(g(x)) = x^2 - 6x + 10$

$$\boxed{(f \circ g)(x) = f(g(x)) = x^2 - 6x + 10}$$

Your Turn Given the functions in Example 3, find $(g \circ f)(x)$.

Answer

$g \circ f = g(f(x)) = x^2 - 2$

Video EXAMPLE 4 | Determining the Domain of a Composite Function

Given the functions $f(x) = \dfrac{1}{x - 1}$ and $g(x) = \dfrac{1}{x}$, determine $f \circ g$, and state its domain.

Solution

Write $f(x)$ using placeholder notation. $f(\square) = \dfrac{1}{(\square) - 1}$

Express the composite function $f \circ g$. $f(g(x)) = \dfrac{1}{g(x) - 1}$

Substitute $g(x) = \dfrac{1}{x}$ into f. $f(g(x)) = \dfrac{1}{\frac{1}{x} - 1}$

Multiply the right side by $\dfrac{x}{x}$. $f(g(x)) = \dfrac{1}{\frac{1}{x} - 1} \cdot \dfrac{x}{x} = \dfrac{x}{1 - x}$

$$\boxed{(f \circ g) = f(g(x)) = \frac{x}{1 - x}}$$

What is the domain of $(f \circ g)(x) = f(g(x))$? By inspecting the final result of $f(g(x))$, we see that the denominator is zero when $x = 1$. Therefore, $x \neq 1$. Are there any other values for x that are not allowed? The function $g(x)$ has the domain $x \neq 0$; therefore we must also exclude zero.

The domain of $(f \circ g)(x) = f(g(x))$ excludes $x = 0$ and $x = 1$ or, in interval notation,

$$\boxed{(-\infty, 0) \cup (0, 1) \cup (1, \infty)}$$

Caution

The domain of the composite function cannot always be determined by examining the final form of $f \circ g$.

Your Turn For the functions f and g given in Example 4, determine the composite function $g \circ f$ and state its domain.

Answer

$g(f(x)) = x - 1$. Domain of $g \circ f$ is $x \neq 1$, or in interval notation, $(-\infty, 1) \cup (1, \infty)$.

The domain of the composite function cannot always be determined by examining the final form of $f \circ g$.

Concept Check

For the functions $f(x) = x - a$ and $g(x) = \frac{1}{x+a}$ find $g(f(x))$ and state its domain.

Answer: $g(f(x)) = \frac{1}{x}$ and domain is all real numbers except $x = 0$.

EXAMPLE 5 | Determining the Domain of a Composite Function (Without Finding the Composite Function)

Let $f(x) = \frac{1}{x-2}$ and $g(x) = \sqrt{x+3}$. Find the domain of $f(g(x))$. Do not find the composite function.

Solution

Find the domain of g. $[-3, \infty)$

Find the range of g. $[0, \infty)$

In $f(g(x))$, the output of g becomes the input for f. Since the domain of f is the set of all real numbers except 2, we eliminate any values of x in the domain of g that correspond to $g(x) = 2$.

Let $g(x) = 2$. $\sqrt{x+3} = 2$

Square both sides. $x + 3 = 4$

Solve for x. $x = 1$

Eliminate $x = 1$ from the domain of g, $[-3, \infty)$.

State the domain of $f(g(x))$. $\boxed{[-3, 1) \cup (1, \infty)}$

Video EXAMPLE 6 | Evaluating a Composite Function

Given the functions $f(x) = x^2 - 7$ and $g(x) = 5 - x^2$, evaluate:

a. $f(g(1))$ **b.** $f(g(-2))$ **c.** $g(f(3))$ **d.** $g(f(-4))$

Solution

One way of evaluating these composite functions is to calculate the two individual composites in terms of x: $f(g(x))$ and $g(f(x))$. Once those functions are known, the values can be substituted for x and evaluated.

Another way of proceeding is as follows:

a. Write the desired quantity. $f(g(1))$

Find the value of the inner function g. $g(1) = 5 - 1^2 = 4$

Substitute $g(1) = 4$ into f. $f(g(1)) = f(4)$

Evaluate $f(4)$. $f(4) = 4^2 - 7 = 9$

$\boxed{f(g(1)) = 9}$

b. Write the desired quantity. $f(g(-2))$

Find the value of the inner function g. $g(-2) = 5 - (-2)^2 = 1$

Substitute $g(-2) = 1$ into f. $f(g(-2)) = f(1)$

Evaluate $f(1)$. $f(1) = 1^2 - 7 = -6$

$\boxed{f(g(-2)) = -6}$

c. Write the desired quantity. $g(f(3))$

Find the value of the inner function f. $f(3) = 3^2 - 7 = 2$

Substitute $f(3) = 2$ into g. $g(f(3)) = g(2)$

Evaluate $g(2)$. $g(2) = 5 - 2^2 = 1$

$$\boxed{g(f(3)) = 1}$$

d. Write the desired quantity. $g(f(-4))$

Find the value of the inner function f. $f(-4) = (-4)^2 - 7 = 9$

Substitute $f(-4) = 9$ into g. $g(f(-4)) = g(9)$

Evaluate $g(9)$. $g(9) = 5 - 9^2 = -76$

$$\boxed{g(f(-4)) = -76}$$

Your Turn Given the functions $f(x) = x^3 - 3$ and $g(x) = 1 + x^3$, evaluate $f(g(1))$ and $g(f(1))$.

Answer

$f(g(1)) = 5$ and $g(f(1)) = -7$

Application Problems

Recall the example at the beginning of this chapter regarding the clothes that are on sale. Often, real-world applications are modeled with composite functions. In the clothes example, x is the original price of each item. The first function maps its input (original price) to an output (sale price). The second function maps its input (sale price) to an output (checkout price). Example 7 is another real-world application of composite functions.

Three temperature scales are commonly used:

- The degree Celsius (°C) scale
  - This scale was devised by dividing the range between the freezing (0°C) and boiling (100°C) points of pure water at sea level into 100 equal parts. This scale is used in science and is one of the standards of the "metric" (SI) system of measurements.
- The Kelvin (K) temperature scale
  - This scale shifts the Celsius scale down so that the zero point is equal to absolute zero (about −273.15°C), a hypothetical temperature at which there is a complete absence of heat energy.
  - Temperatures on this scale are called **kelvins**, *not* degrees kelvin, and kelvin is not capitalized. The symbol for the kelvin is K.
- The degree Fahrenheit (°F) scale
  - This scale evolved over time and is still widely used mainly in the United States, although Celsius is the preferred "metric" scale.
  - With respect to pure water at sea level, the **degrees Fahrenheit** are gauged by the spread from 32°F (freezing) to 212°F (boiling).

The equations that relate these temperature scales are

$$F = \frac{9}{5}C + 32 \qquad C = K - 273.15$$

EXAMPLE 7 | **Applications Involving Composite Functions**

Determine degrees Fahrenheit as a function of kelvins.

Solution

Degrees Fahrenheit is a function of degrees Celsius. $F = \frac{9}{5}C + 32$

Now substitute $C = K - 273.15$ into the equation for F. $F = \frac{9}{5}(K - 273.15) + 32$

Simplify. $F = \frac{9}{5}K - 491.67 + 32$

$$\boxed{F = \frac{9}{5}K - 459.67}$$

Section 1.4 Summary

Operations on Functions

Function	Notation
Sum	$(f + g)(x) = f(x) + g(x)$
Difference	$(f - g)(x) = f(x) - g(x)$
Product	$(f \cdot g)(x) = f(x) \cdot g(x)$
Quotient	$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$ $g(x) \neq 0$

The domain of the sum, difference, and product functions is the intersection of the domains, or common domain shared by both f and g. The domain of the quotient function is also the intersection of the domain shared by both f and g with an additional restriction that $g(x) \neq 0$.

Composition of Functions

$$(f \circ g)(x) = f(g(x))$$

The domain restrictions cannot always be determined simply by inspecting the final form of $f(g(x))$. Rather, the domain of the composite function is a subset of the domain of $g(x)$. Values of x must be eliminated if their corresponding values of $g(x)$ are not in the domain of f.

Section 1.4 Exercises

Skills

In Exercises 1–10, given the functions f and g, find $f + g$, $f - g$, $f \cdot g$, and $\frac{f}{g}$, and state the domain of each.

1. $f(x) = 2x + 1$, $g(x) = 1 - x$
2. $f(x) = 3x + 2$, $g(x) = 2x - 4$
3. $f(x) = 2x^2 - x$, $g(x) = x^2 - 4$
4. $f(x) = 3x + 2$, $g(x) = x^2 - 25$
5. $f(x) = \frac{1}{x}$, $g(x) = x$
6. $f(x) = \frac{2x + 3}{x - 4}$, $g(x) = \frac{x - 4}{3x + 2}$
7. $f(x) = \sqrt{x}$, $g(x) = 2\sqrt{x}$
8. $f(x) = \sqrt{x - 1}$, $g(x) = 2x^2$
9. $f(x) = \sqrt{4 - x}$, $g(x) = \sqrt{x + 3}$
10. $f(x) = \sqrt{1 - 2x}$, $g(x) = \frac{1}{x}$

In Exercises 11–22, for the given functions f and g, find the composite functions $f \cdot g$ and $g \cdot f$, and state their domains.

11. $f(x) = 3x - 1$, $g(x) = x + 2$
12. $f(x) = 2x - 7$, $g(x) = 5 - x$
13. $f(x) = 2x + 1$, $g(x) = x^2 - 3$
14. $f(x) = x^2 - 1$, $g(x) = 2 - x$
15. $f(x) = \frac{1}{x - 1}$, $g(x) = x + 2$
16. $f(x) = \frac{2}{x - 3}$, $g(x) = 2 + x$
17. $f(x) = |x|$, $g(x) = \frac{1}{x - 1}$
18. $f(x) = |x - 1|$, $g(x) = \frac{1}{x}$

19. $f(x) = \sqrt{x-1}$, $g(x) = x + 5$

20. $f(x) = \sqrt{2-x}$, $g(x) = x^2 + 2$

21. $f(x) = x^3 + 4$, $g(x) = (x-4)^{1/3}$

22. $f(x) = \sqrt[3]{x^2 - 1}$, $g(x) = x^{2/3} + 1$

In Exercises 23–40, evaluate the functions for the specified values, if possible.

$$f(x) = x^2 + 10 \qquad g(x) = \sqrt{x-1}$$

23. $(f+g)(2)$
24. $(f+g)(10)$
25. $(f-g)(2)$
26. $(f-g)(5)$
27. $(f \circ g)(4)$
28. $(f \circ g)(5)$
29. $\left(\frac{f}{g}\right)(10)$
30. $\left(\frac{f}{g}\right)(2)$
31. $f(g(2))$
32. $f(g(1))$
33. $g(f(-3))$
34. $g(f(4))$
35. $f(g(0))$
36. $g(f(0))$
37. $f(g(-3))$
38. $g(f(\sqrt{7}))$
39. $(f \circ g)(4)$
40. $(g \circ f)(-3)$

In Exercises 41–52, evaluate $f(g(1))$ and $g(f(2))$, if possible.

41. $f(x) = \frac{1}{x}$, $g(x) = 2x + 1$
42. $f(x) = x^2 + 1$, $g(x) = \frac{1}{2-x}$
43. $f(x) = \sqrt{1-x}$, $g(x) = x^2 + 2$
44. $f(x) = \sqrt{3-x}$, $g(x) = x^2 + 1$
45. $f(x) = \frac{1}{|x-1|}$, $g(x) = x + 3$
46. $f(x) = \frac{1}{x}$, $g(x) = |2x - 3|$
47. $f(x) = \sqrt{x-1}$, $g(x) = x^2 + 5$
48. $f(x) = \sqrt[3]{x-3}$, $g(x) = \frac{1}{x-3}$
49. $f(x) = \frac{1}{x^2-3}$, $g(x) = \sqrt{x-3}$
50. $f(x) = \frac{x}{2-x}$, $g(x) = 4 - x^2$
51. $f(x) = (x-1)^{1/3}$, $g(x) = x^2 + 2x + 1$
52. $f(x) = (1-x^2)^{1/2}$, $g(x) = (x-3)^{1/3}$

In Exercises 53–62, show that $f(g(x)) = x$ and $g(f(x)) = x$.

53. $f(x) = 2x + 1$, $g(x) = \frac{x-1}{2}$
54. $f(x) = \frac{x-2}{3}$, $g(x) = 3x + 2$
55. $f(x) = \sqrt{x-1}$, $g(x) = x^2 + 1$ for $x \geq 1$
56. $f(x) = 2 - x^2$ $g(x) = \sqrt{2-x}$ for $x \leq 2$
57. $f(x) = \frac{1}{x}$, $g(x) = \frac{1}{x}$ for $x \neq 0$
58. $f(x) = (5-x)^{1/3}$, $g(x) = 5 - x^3$
59. $f(x) = 4x^2 - 9$, $g(x) = \frac{\sqrt{x+9}}{2}$ for $x \geq 0$
60. $f(x) = \sqrt[3]{8x-1}$, $g(x) = \frac{x^3+1}{8}$
61. $f(x) = \frac{1}{x-1}$, $g(x) = \frac{x+1}{x}$ for $x \neq 0, x \neq 1$
62. $f(x) = \sqrt{25-x^2}$, $g(x) = \sqrt{25-x^2}$ for $0 \leq x \leq 5$

In Exercises 63–68, write the function as a composite of two functions f and g. (There is more than one correct answer.)

63. $f(g(x)) = 2(3x-1)^2 + 5(3x-1)$
64. $f(g(x)) = \frac{1}{1+x^2}$
65. $f(g(x)) = \frac{2}{|x-3|}$
66. $f(g(x)) = \sqrt{1-x^2}$
67. $f(g(x)) = \frac{3}{\sqrt{x+1} - 2}$
68. $f(g(x)) = \frac{\sqrt{x}}{3\sqrt{x} + 2}$

Applications

Exercises 69 and 70 depend on the relationship between degrees Fahrenheit, degrees Celsius, and kelvins:

$$F = \frac{9}{5}C + 32 \qquad C = K - 273.15$$

69. **Temperature.** Write a composite function that converts kelvins into degrees Fahrenheit.
70. **Temperature.** Convert the following degrees Fahrenheit to kelvins: 32°F and 212°F.

71. Market Price. Typical supply and demand relationships state that as the number of units for sale increases, the market price decreases. Assume that the market price p and the number of units for sale x are related by the demand equation:

$$p = 3000 - \frac{1}{2}x$$

Assume that the cost $C(x)$ of producing x items is governed by the equation

$$C(x) = 2000 + 10x$$

and the revenue $R(x)$ generated by selling x units is governed by

$$R(x) = 100x$$

a. Write the cost as a function of price p.

b. Write the revenue as a function of price p.

c. Write the profit as a function of price p.

72. Market Price. Typical supply and demand relationships state that as the number of units for sale increases, the market price decreases. Assume that the market price p and the number of units for sale x are related by the demand equation:

$$p = 10{,}000 - \frac{1}{4}x$$

Assume that the cost $C(x)$ of producing x items is governed by the equation

$$C(x) = 30{,}000 + 5x$$

and the revenue $R(x)$ generated by selling x units is governed by

$$R(x) = 1000x$$

a. Write the cost as a function of price p.

b. Write the revenue as a function of price p.

c. Write the profit as a function of price p.

In Exercises 73 and 74, refer to the following:

The regular price for a pair of Nikes is x dollars. You have three coupon codes for a website selling the shoes represented by $f(x) = x - 10$, $g(x) = 0.75x$, and $h(x) = 0.9x$. As you check out, you realize that the site calculates your discount based on the order you type in the codes.

73. Shopping. The website only allows you to input two codes per order, so you decide to use codes f and g.

a. Find $(f \circ g)(x)$. Describe what your answer means in terms of the price of the shoes.

b. Find $(g \circ f)(x)$. Describe what your answer means in terms of the price of the shoes.

c. Compare your answers from part (a) and part (b). Which coupon should you type in first, f or g?

74. Shopping. Your friend likes your shoes and wants to order a similar pair, but they can only find codes g and h.

a. Find $(g \circ h)(x)$. Describe what your answer means in terms of the price of the shoes.

b. Find $(h \circ g)(x)$. Describe what your answer means in terms of the price of the shoes.

c. Compare your answers from part (a) and part (b). Which coupon should they type in first, g or h?

In Exercises 75 and 76, refer to the following:

Surveys performed immediately following an accidental oil spill at sea indicate the oil moved outward from the source of the spill in a nearly circular pattern. The radius of the oil spill r measured in miles is a function of time t measured in days from the start of the spill, while the area of the oil spill is a function of radius, that is, the function $A(r)$.

75. Environment: Oil Spill. If the radius of the oil spill is given by

$$r(t) = 10t - 0.2t^2$$

and the area of the oil spill is given by

$$A(r) = \pi r^2$$

a. Find a function that gives the area of the oil spill in terms of the number of days since the start of the spill, $A(r(t))$.

b. Find the area of the oil spill to the nearest square mile 7 days after the start of the spill.

76. Environment: Oil Spill. If the radius of the oil spill is given by

$$r(t) = 8t - 0.1t^2$$

and the area of the oil spill is given by

$$A(r) = \pi r^2$$

a. Find a function that gives the area of the oil spill in terms of the number of days since the start of the spill, $A(r(t))$.

b. Find the area of the oil spill to the nearest square mile 5 days after the start of the spill.

77. Environment: Oil Spill. An oil spill makes a circular pattern around a ship such that the radius in feet grows as a function of time in hours $r(t) = 150\sqrt{t}$. Find the area of the spill as a function of time.

78. Pool Volume. A 20 foot by 10 foot rectangular pool has been built. If 50 cubic feet of water is pumped into the pool per hour, write the water-level height (feet) as a function of time (hours).

79. Fireworks. A family is watching a fireworks display. If the family is 2 miles from where the fireworks are being launched and the fireworks travel vertically, what is the distance between the family and the fireworks as a function of height above ground?

80. Real Estate. Laura decided to put her spare bedroom on Airbnb for extra income. She charges \$75 per night for someone to rent her room and, each night it is rented, she pays 6% commission to Airbnb.

a. Write a function $R(n)$ that represents the revenue she will make as a function of the number of nights her house is rented n.

b. Laura's mortgage payment is \$1700 per month. What is the minimum number of days she would have to rent her spare bedroom in a month in order to cover her entire mortgage payment that month?

Catch the Mistake

In Exercises 81–86, for the functions $f(x) = x + 2$ and $g(x) = x^2 - 4$, find the indicated function and state its domain. Explain the mistake that is made in each problem.

81. $\frac{g}{f}$

Solution

$$\frac{g(x)}{f(x)} = \frac{x^2 - 4}{x + 2} = \frac{(x-2)(x+2)}{x+2} = x - 2$$

Domain: $(-\infty, \infty)$

This is incorrect. What mistake was made?

82. $\frac{f}{g}$

Solution

$$\frac{f(x)}{g(x)} = \frac{x+2}{x^2-4} = \frac{x+2}{(x-2)(x+2)} = \frac{1}{x-2} = \frac{1}{x-2}$$

Domain: $(-\infty, 2) \cup (2, \infty)$

This is incorrect. What mistake was made?

83. $f \circ g$

Solution

$$f \circ g = f(x)g(x) = (x+2)(x^2-4) = x^3 + 2x^2 - 4x - 8$$

Domain: $(-\infty, \infty)$

This is incorrect. What mistake was made?

84. $f(x) - g(x)$

Solution

$$f(x) - g(x) = x + 2 - x^2 - 4 = -x^2 + x - 2$$

Domain: $(-\infty, \infty)$

This is incorrect. What mistake was made?

85. $(f+g)(2)$

Solution

$$(f+g)(2) = (x + 2 + x^2 - 4)(2) = (x^2 + x - 2)(2) = 2x^2 + 2x - 4$$

Domain: $(-\infty, \infty)$

This is incorrect. What mistake was made?

86. Given the function $f(x) = x^2 + 7$ and $g(x) = \sqrt{x-3}$, find $f \circ g$, and state the domain.

Solution

$$f \circ g = f(g(x)) = (\sqrt{x-3})^2 + 7 = f(g(x)) = x - 3 + 7 = x - 4$$

Domain: $(-\infty, \infty)$

This is incorrect. What mistake was made?

Conceptual

In Exercises 87–90, determine whether each statement is true or false.

87. When adding, subtracting, multiplying, or dividing two functions, the domain of the resulting function is the union of the domains of the individual functions.

88. For any functions f and g, $f(g(x)) = g(f(x))$ for all values of x that are in the domain of both f and g.

89. For any functions f and g, $(f \circ g)(x)$ exists for all values of x that are in the domain of $g(x)$, provided the range of g is a subset of the domain of f.

90. The domain of a composite function can be found by inspection, without knowledge of the domain of the individual functions.

Challenge

91. For the functions $f(x) = x + a$ and $g(x) = \frac{1}{x-a}$, find $g \circ f$ and state its domain.

92. For the functions $f(x) = ax^2 + bx + c$ and $g(x) = \frac{1}{x-c}$, find $g \circ f$ and state its domain.

93. For the functions $f(x) = \sqrt{x+a}$ and $g(x) = x^2 - a$, find$g \circ f$ and state its domain.

94. For the functions $f(x) = \frac{1}{x^a}$ and $g(x) = \frac{1}{x^b}$, find $g \circ f$ and state its domain. Assume $a > 1$ and $b > 1$.

Preview to Calculus

For Exercises 95–100, refer to the following:

In calculus, the difference quotient $\frac{f(x+h) - f(x)}{h}$ of a function f is used to find the derivative f' of f by letting h approach 0, $h \to 0$.

95. Addition. Find the derivatives of $F(x) = x$, $G(x) = x^2$, and $H(x) = (F + G)(x) = x + x^2$. What do you observe?

96. Subtraction. Find the derivatives of $F(x) = \sqrt{x}$, $G(x) = x^3 + 1$, and $H(x) = (F - G)(x) = \sqrt{x} - x^3 - 1$. What do you observe?

97. Multiplication. Find the derivatives of $F(x) = 5$, $G(x) = \sqrt{x-1}$, and $H(x) = (FG)(x) = 5\sqrt{x-1}$. What do you observe?

98. Division. Find the derivatives of $F(x) = x$, $G(x) = \sqrt{x+1}$, and $H(x) = \left(\frac{F}{G}\right)(x) = \frac{x}{\sqrt{x+1}}$ What do you observe?

99. Composition. Find the derivatives of $F(x) = x + 2$, $G(x) = x^2 - 4$, and $H(x) = (F \circ G)(x) = x^2 - 2$. What do you observe?

100. Composition. Find the derivatives of $F(x) = x + 2$, $G(x) = x^2 - 4$, and $H(x) = (G \circ F)(x) = x^2 + 4$. What do you observe?

1.5 One-to-One Functions and Inverse Functions

SKILLS OBJECTIVES

- Determine whether a function is a one-to-one function.
- Verify that two functions are inverses of one another.
- Graph the inverse function given the graph of the function.
- Find the inverse of a function.

CONCEPTUAL OBJECTIVES

- Understand why a function that passes the horizontal line test is one-to-one.
- Visualize the relationships between the domain and range of a function and the domain and range of its inverse.
- Understand why functions and their inverses are symmetric about $y = x$.
- Understand why a function has to be one-to-one in order for its inverse to exist.

Every human being has a blood type, and every human being has a DNA sequence. These are examples of functions, where a person is the input and the output is blood type or DNA sequence. These relationships are classified as functions because each person can have one and only one blood type or DNA strand. The difference between these functions is that many people have the same blood type, but DNA is unique to each individual. Can we map backwards? For instance, if you know the blood type, do you know specifically which person it came from? No, but, if you know the DNA sequence, you know exactly to which person it corresponds. When a function has a one-to-one correspondence, like the DNA example, then mapping backwards is possible. The map back is called the *inverse function.*

1.5.1 Determine Whether a Function Is One-to-One

1.5.1 Skill Determine whether a function is a one-to-one function.

1.5.1 Conceptual Understand why a function that passes the horizontal line test is one-to-one.

In Section 1.1, we defined a function as a relationship that maps an input (contained in the domain) to exactly one output (found in the range). Algebraically, each value for x can correspond to only a single value for y. Recall the square, identity, absolute value, and reciprocal functions from our library of functions in Section 1.2.

All of the graphs of these functions satisfy the vertical line test. Although the square function and the absolute value function map each value of x to exactly one value for y, these two functions map two values of x to the same value for y. For example, $(-1, 1)$ and $(1, 1)$ lie on both graphs. The identity and reciprocal functions, on the other hand, map each x to a single value for y, and no two x-values map to the same y-value. These two functions are examples of *one-to-one functions.*

One-to-One Function

A function $f(x)$ is **one-to-one** if no two elements in the domain correspond to the same element in the range; that is,

$$\text{if } x_1 \neq x_2, \text{ then } f(x_1) \neq f(x_2).$$

In other words, it is one-to-one if no two inputs map to the same output.

EXAMPLE 1 | Determining Whether a Function Defined as a Set of Points Is a One-to-One Function

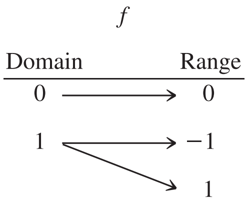
For each of the three relations, determine whether the relation is a function. If it is a function, determine whether it is a one-to-one function.

$$f = \{(0, 0), (1, 1), (1, -1)\}$$

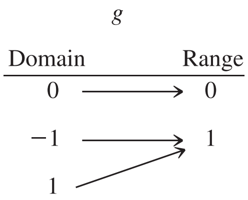
$$g = \{(-1, 1), (0, 0), (1, 1)\}$$

$$h = \{(-1, -1), (0, 0), (1, 1)\}$$

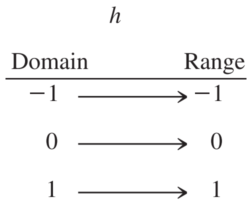
Solution



f is not a function.



g is a function but not one-to-one.



h is a one-to-one function.

Just as there is a graphical test for functions, the vertical line test, there is a graphical test for one-to-one functions, the *horizontal line test.* Note that a horizontal line can be drawn on the square and absolute value functions so that it intersects the graph of each function at two points. The identity and reciprocal functions, however, will intersect a horizontal line in at most only one point. This leads us to the horizontal line test for one-to-one functions.

Horizontal Line Test

If every horizontal line intersects the graph of a function in at most one point, then the function is classified as a one-to-one function.

Concept Check

Draw a horizontal line at $y = 4$ on the graph of $y = x^2$. What are the two points of intersection? Explain why the equation $y = x^2$ cannot be a one-to-one function.

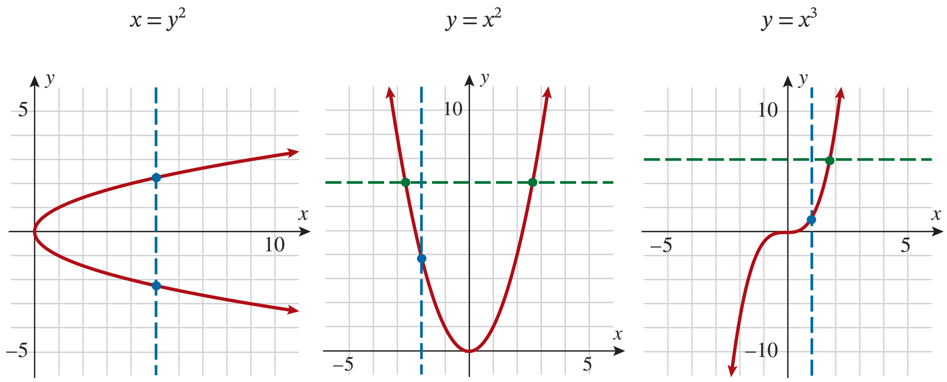
Answer: $(-2, 4)$ and $(2, 4)$: Two different x values map to the same y value, so this graph fails the horizontal line test.

Video EXAMPLE 2 | Using the Horizontal Line Test to Determine Whether a Function Is One-to-One

For each of the three relations, determine whether the relation is a function. If it is a function, determine whether it is a one-to-one function. Assume that x is the independent variable and y is the dependent variable.

$$x = y^2 \qquad y = x^2 \qquad y = x^3$$

Solution



Not a function	Function but not one-to-one	One-to-one function
(fails vertical line test)	(passes vertical line test but fails horizontal line test)	(passes both horizontal and vertical line tests)

Your Turn Determine whether each of the functions is a one-to-one function.

a. $f(x) = x + 2$ **b.** $f(x) = x^2 + 1$

Answer

a. yes

b. no

Another way of writing the definition of a one-to-one function is:

$$\text{If } f(x_1) = f(x_2), \text{ then } x_1 = x_2.$$

In the Your Turn following Example 2, we found (using the horizontal line test) that $f(x) = x + 2$ is a one-to-one function but that $f(x) = x^2 + 1$ is not a one-to-one function. We can also use this alternative definition to determine algebraically whether a function is one-to-one.

Words	**Math**
State the function.	$f(x) = x + 2$
Let there be two real numbers, x_1 and x_2, such that $f(x_1) = f(x_2)$.	$x_1 + 2 = x_2 + 2$
Subtract 2 from both sides of the equation.	$x_1 = x_2$

$f(x) = x + 2$ is a one-to-one function.

Words	**Math**
State the function.	$f(x) = x^2 + 1$
Let there be two real numbers, x_1 and x_2, such that $f(x_1) = f(x_2)$.	$x_1^2 + 1 = x_2^2 + 1$
Subtract 1 from both sides of the equation.	$x_1^2 = x_2^2$
Solve for x_1.	$x_1 = \pm x_2$

$f(x) = x^2 + 2$ is *not* a one-to-one function.

Video

EXAMPLE 3 | Determining Algebraically Whether a Function Is One-to-One

Determine algebraically whether the following functions are one-to-one:

a. $f(x) = 5x^3 - 2$ **b.** $f(x) = |x + 1|$

Solution (a)

Find $f(x_1)$ and $f(x_2)$.	$f(x_1) = 5x_1^3 - 2$ and $f(x_2) = 5x_2^3 - 2$
Let $f(x_1) = f(x_2)$.	$5x_1^3 - 2 = 5x_2^3 - 2$
Add 2 to both sides of the equation.	$5x_1^3 = 5x_2^3$
Divide both sides of the equation by 5.	$x_1^3 = x_2^3$
Take the cube root of both sides of the equation.	$(x_1^3)^{1/3} = (x_2^3)^{1/3}$
Simplify.	$x_1 = x_2$

$f(x) = 5x^3 - 2$ is a one-to-one function.

Solution (b)

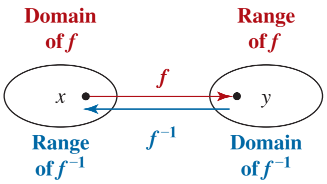
Find $f(x_1)$ and $f(x_2)$.	$f(x_1) = \lvert x_1 + 1\rvert$ and $f(x_2) = \lvert x_2 + 1\rvert$
Let $f(x_1) = f(x_2)$.	$\lvert x_1 + 1\rvert = \lvert x_2 + 1\rvert$
Solve the absolute value equation.	$(x_1 + 1) = (x_2 + 1)$ or $(x_1 + 1) = -(x_2 + 1)$
	$x_1 = x_2$ or $x_1 = -x_2 - 2$

$f(x) = |x + 1|$ is **not** a one-to-one function.

1.5.2 Inverse Functions

1.5.2 Skill Verify that two functions are inverses of one another.

1.5.2 Conceptual Visualize the relationships between the domain and range of a function and the domain and range of its inverse.

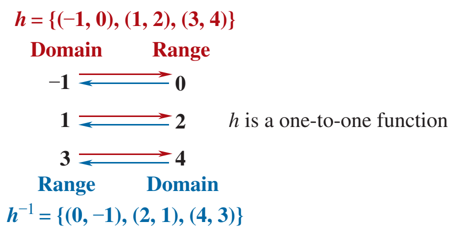


If a function is one-to-one, then the function maps each x to exactly one y, and no two x-values map to the same y-value. This implies that there is a one-to-one correspondence between the inputs (domain) and outputs (range) of a one-to-one function $f(x)$. In the special case of a one-to-one function, it would be possible to map from the output (range of f) back to the input (domain of f), and this mapping would also be a function. The function that maps the output back to the input of a function f is called the **inverse function** and is denoted $f^{-1}(x)$.

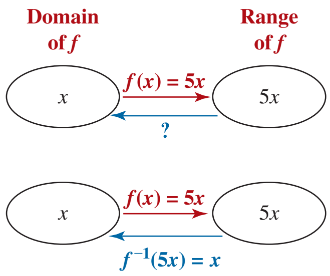
A one-to-one function f maps every x in the domain to a unique and distinct corresponding y in the range. Therefore, the inverse function f^{-1} maps every y back to a unique and distinct x.

The function notations $\boldsymbol{f(x) = y}$ and $\boldsymbol{f^{-1}(y) = x}$ indicate that if the point $\boldsymbol{(x, y)}$ satisfies the function, then the point $\boldsymbol{(y, x)}$ satisfies the inverse function.

For example, let the function $h(x) = \{(-1, 0), (1, 2), (3, 4)\}$.



The inverse function undoes whatever the function does. For example, if $f(x) = 5x$, then the function f maps any value x in the domain to a value $5x$ in the range. If we want to map backwards or undo the $5x$, we develop a function called the inverse function that takes $5x$ as input and maps back to x as output. The inverse function is $f^{-1}(x) = \frac{1}{5}x$. Note that if we input $5x$ into the inverse function, the output is x: $f^{-1}(5x) = \frac{1}{5}(5x) = x$.



Inverse Function

If f and g denote two one-to-one functions such that

$$f(g(x)) = x \text{ for every } x \text{ in the domain of } g$$

and

$$g(f(x)) = x \text{ for every } x \text{ in the domain of } f,$$

then g is the **inverse** of the function f. The function g is denoted by f^{-1} (read "f-inverse").

Note: f^{-1} is used to denote the inverse of f. The -1 is not used as an exponent and, therefore, does not represent the reciprocal of f: $\frac{1}{f}$.

Caution

$f^{-1} \neq \frac{1}{f}$

Two properties hold true relating one-to-one functions to their inverses: (1) the range of the function is the domain of the inverse, and the range of the inverse is the domain of the function, and (2) the composite function that results with a function and its inverse (and vice versa) is the identity function x.

Domain of f = range of f^{-1} and range of f = domain of f^{-1}

$$f^{-1}(f(x)) = x \quad \text{and} \quad f(f^{-1}(x)) = x$$

EXAMPLE 4 | Verifying Inverse Functions

Verify that $f^{-1}(x) = \frac{1}{2}x - 2$ is the inverse of $f(x) = 2x + 4$.

Solution

Show that $f^{-1}(f(x)) = x$ and $f(f^{-1}(x)) = x$.

Write f^{-1} using placeholder notation. $\quad f^{-1}(\square) = \frac{1}{2}(\square) - 2$

Substitute $f(x) = 2x + 4$ into f^{-1}. $\quad f^{-1}(f(x)) = \frac{1}{2}(2x + 4) - 2$

Simplify. $\quad f^{-1}(f(x)) = x + 2 - 2 = x$

$$f^{-1}(f(x)) = x$$

Write f using placeholder notation. $\quad f(\square) = 2(\square) + 4$

Substitute $f^{-1}(x) = \frac{1}{2}x - 2$ into f. $\quad f(f^{-1}(x)) = 2\left(\frac{1}{2}x - 2\right) + 4$

Simplify. $\quad f(f^{-1}(x)) = x - 4 + 4 = x$

$$f(f^{-1}(x)) = x$$

Note the relationship between the domain and range of f and f^{-1}.

	Domain	Range
$f(x) = 2x + 4$	$(-\infty, \infty)$	$(-\infty, \infty)$
$f^{-1}(x) = \frac{1}{2}x - 2$	$(-\infty, \infty)$	$(-\infty, \infty)$

Concept Check

If a one-to-one function f has domain $[a, \infty)$ and range $[b, \infty)$ then what are the domain and range of its inverse, f^{-1}?

Answer: The inverse function, f^{-1}, has domain $[b, \infty)$ and range $[a, \infty)$.

Video **EXAMPLE 5 | Verifying Inverse Functions with Domain Restrictions**

Verify that $f^{-1}(x) = x^2$, for $x \geq 0$, is the inverse of $f(x) = \sqrt{x}$.

Solution

Show that $f^{-1}(f(x)) = x$ and $f(f^{-1}(x)) = x$.

Write f^{-1} using placeholder notation. $\quad f^{-1}(\square) = (\square)^2$

Substitute $f(x) = \sqrt{x}$ into f^{-1}. $\quad f^{-1}(f(x)) = (\sqrt{x})^2 = x$

$f^{-1}(f(x)) = x$ for $x \geq 0$

Write f using placeholder notation. $\quad f(\square) = \sqrt{(\square)}$

Substitute $f^{-1}(x) = x^2, x \geq 0$ into f. $\quad f(f^{-1}(x)) = \sqrt{x^2} = x, \quad x \geq 0$

$f(f^{-1}(x)) = x$ for $x \geq 0$

	Domain	Range
$f(x) = \sqrt{x}$	$[0, \infty)$	$[0, \infty)$
$f^{-1}(x) = x^2, x \geq 0$	$[0, \infty)$	$[0, \infty)$

1.5.3 Graphical Interpretation of Inverse Functions

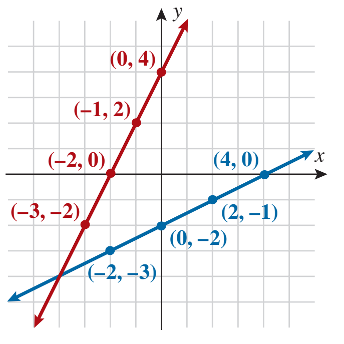
1.5.3 Skill Graph the inverse function given the graph of the function.

1.5.3 Conceptual Understand why functions and their inverses are symmetric about $y = x$.

In Example 4, we showed that $f^{-1}(x) = \frac{1}{2}x - 2$ is the inverse of $f(x) = 2x + 4$. Let's now investigate the graphs that correspond to the function f and its inverse f^{-1}.

$f(x)$	
x	y
−3	−2
−2	0
−1	2
0	4

$f^{-1}(x)$	
x	y
−2	−3
0	−2
2	−1
4	0



STUDY TIP

If the point (a, b) is on the function, then the point (b, a) is on the inverse. Notice the interchanging of the x- and y-coordinates.

Note that the point $(-3, -2)$ lies on the function and the point $(-2, -3)$ lies on the inverse. In fact, every point (a, b) that lies on the function corresponds to a point (b, a) that lies on the inverse.

Draw the line $y = x$ on the graph. In general, the point (b, a) on the inverse $f^{-1}(x)$ is the reflection (about $y = x$) of the point (a, b) on the function $f(x)$.

In general, if the point (a, b) is on the graph of a function, then the point (b, a) is on the graph of its inverse.

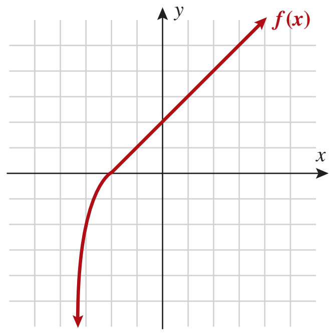
Concept Check

If the point (b, a) lies on the graph of a one-to-one function f, then what point must lie on the graph of its inverse, f^{-1}?

Answer: (a, b)

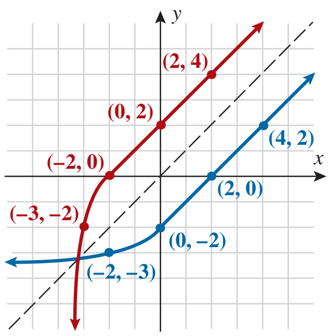
Video EXAMPLE 6 | Graphing the Inverse Function

Given the graph of the function $f(x)$, plot the graph of its inverse $f^{-1}(x)$.

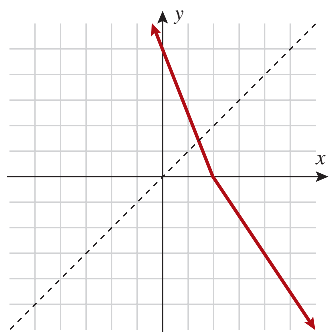


Solution

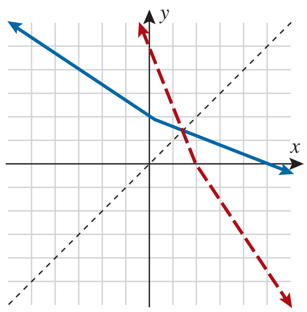
Because the points **(−3, −2)**, **(−2, 0)**, **(0, 2)**, and **(2, 4)** lie on the graph of ***f***, then the points **(−2, −3)**, **(0, −2)**, **(2, 0)**, and **(4, 2)** lie on the graph of $\boldsymbol{f^{-1}}$.



Your Turn Given the graph of a function f, plot the inverse function.



Answer



EXAMPLE 7 | Using the Graph of a Function to Determine Characteristics of Its Inverse

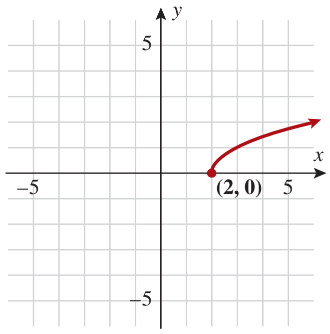
Use the graph of the function $f(x)$ to determine the following:

a. Find $f(2)$.

b. Find x that corresponds to $f(x) = 1$.

c. Find $f^{-1}(0)$.

d. Find $f^{-1}(1)$.



Solution

a. The point (2,0) lies on the graph of f. $f(2) = 0$

b. The point (3,1) lies on the graph of f. $f(3) = 1$

c. Since the point (2,0) lies on the graph of f, then the point (0,2) lies on the graph of f^{-1}. $f^{-1}(0) = 2$

d. Since the point (3,1) lies on the graph of f, then the point (1,3) lies on the graph of f^{-1}. $f^{-1}(1) = 3$

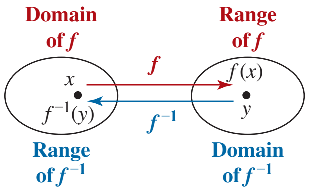
We have developed the definition of an inverse function and described properties of inverses. At this point, you should be able to determine whether two functions are inverses of one another. Let's turn our attention to another problem: How do you find the inverse of a function?

1.5.4 Finding the Inverse Function

1.5.4 **Skill** Find the inverse of a function.

1.5.4 **Conceptual** Understand why a function has to be one-to-one in order for its inverse to exist.

If the point (a, b) lies on the graph of a function, then the point (b, a) lies on the graph of the inverse function. The symmetry about the line $y = x$ tells us that the roles of x and y interchange. Therefore, if we start with every point (x, y) that lies on the graph of a function, then every point (y, x) lies on the graph of its inverse. Algebraically, this corresponds to interchanging x and y. Finding the inverse of a finite set of ordered pairs is easy: simply interchange the x- and y-coordinates. Earlier, we found that if $h(x) = \{(-1, 0), (1, 2), (3, 4)\}$, then $h^{-1}(x) = \{(0, -1), (2, 1), (4, 3)\}$. But how do we find the inverse of a function defined by an equation?



Recall the mapping relationship if f is a one-to-one function. This relationship implies that $\mathbf{f(x) = y}$ and $\mathbf{f^{-1}(y) = x}$. Let's use these two identities to find the inverse. Now consider the function defined by $f(x) = 3x - 1$. To find f^{-1}, we let $\mathbf{f(x) = y}$, which yields $y = 3x - 1$. Solve for the variable x: $x = \frac{1}{3}y + \frac{1}{3}$.

Recall that $\mathbf{f^{-1}(y) = x}$, so we have found the inverse to be $f^{-1}(y) = \frac{1}{3}y + \frac{1}{3}$. It is customary to write the independent variable as x, so we write the inverse as $f^{-1}(x) = \frac{1}{3}x + \frac{1}{3}$. Now that we have found the inverse, let's confirm that the properties $f^{-1}(f(x)) = x$ and $f(f^{-1}(x)) = x$ hold.

$$f(f^{-1}(x)) = 3\left(\frac{1}{3}x + \frac{1}{3}\right) - 1 = x + 1 - 1 = x$$

$$f^{-1}(f(x)) = \frac{1}{3}(3x - 1) + \frac{1}{3} = x - \frac{1}{3} + \frac{1}{3} = x$$

Finding the Inverse of a Function

Let f be a one-to-one function. Then the following procedure can be used to find the inverse function f^{-1} if the inverse exists.

Step	Procedure	Example
1	Let $y = f(x)$.	$f(x) = -3x + 5$ $y = -3x + 5$
2	Solve the resulting equation for x in terms of y (if possible).	$3x = -y + 5$ $x = -\frac{1}{3}y + \frac{5}{3}$
3	Let $x = f^{-1}(y)$.	$f^{-1}(y) = -\frac{1}{3}y + \frac{5}{3}$
4	Let $y = x$ (interchange x and y).	$f^{-1}(x) = -\frac{1}{3}x + \frac{5}{3}$

The same result is found if we first interchange x and y and then solve for y in terms of x.

Step	Procedure	Example
1	Let $y = f(x)$.	$f(x) = -3x + 5$ $y = -3x + 5$
2	Interchange x and y.	$x = -3y + 5$
3	Solve for y in terms of x.	$3y = -x + 5$ $y = -\frac{1}{3}x + \frac{5}{3}$
4	Let $y = f^{-1}(x)$	$f^{-1}(x) = -\frac{1}{3}x + \frac{5}{3}$

Note the following:

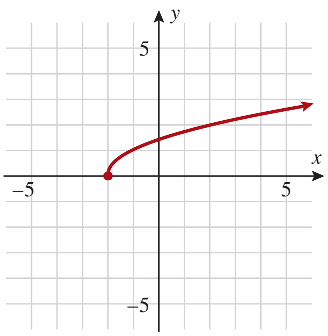
- Verify first that a function is one-to-one prior to finding an inverse (if it is not one-to-one, then the inverse does not exist).
- State the domain restrictions on the inverse function. The domain of f is the range of f^{-1} and vice versa.
- To verify that you have found the inverse, show that $f(f^{-1}(x)) = x$ for all x in the domain of f^{-1} and $f^{-1}(f(x)) = x$ for all x in the domain of f.

EXAMPLE 8 | The Inverse of a Square Root Function

Find the inverse of the function $f(x) = \sqrt{x+2}$. State the domain and range of both f and f^{-1}.

Solution

$f(x)$ is a one-to-one function because it passes the horizontal line test.



STEP 1 Let $y = f(x)$. $\quad y = \sqrt{x+2}$

STEP 2 Interchange x and y. $\quad x = \sqrt{y+2}$

STEP 3 Solve for y.

Square both sides of the equation. $x^2 = y + 2$

Subtract 2 from both sides. $x^2 - 2 = y$ or $y = x^2 - 2$

STEP 4 Let $y = f^{-1}(x)$. $f^{-1}(x) = x^2 - 2$

Note any domain restrictions. (State the domain and range of both f and f^{-1}.)

f: Domain: $[-2, \infty)$ Range: $[0, \infty)$

f^{-1}: Domain: $[0, \infty)$ Range: $[-2, \infty)$

The inverse of $f(x) = \sqrt{x+2}$ is $\boxed{f^{-1}(x) = x^2 - 2 \text{ for } x \geq 0}$.

Check

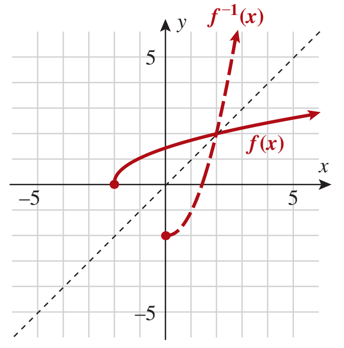
$f^{-1}(f(x)) = x$ for all x in the domain of f.

$$f^{-1}(f(x)) = (\sqrt{x+2})^2 - 2 = x + 2 - 2 \text{ for } x \geq -2 = x$$

$f(f^{-1}(x)) = x$ for all x in the domain of f^{-1}.

$$f(f^{-1}(x)) = \sqrt{(x^2 - 2) + 2} = \sqrt{x^2} \text{ for } x \geq 0 = x$$

Note that the function $f(x) = \sqrt{x+2}$ and its inverse $f^{-1}(x) = x^2 - 2$ for $x \geq 0$ are symmetric about the line $y = x$.



Your Turn Find the inverse of the given function. State the domain and range of the inverse function.

a. $f(x) = 7x - 3$ **b.** $g(x) = \sqrt{x-1}$

Answer

a. $f^{-1}(x) = \dfrac{x+3}{7}$

Domain: $(-\infty, \infty)$, Range: $(-\infty, \infty)$

b. $g^{-1}(x) = x^2 + 1$

Domain: $[0, \infty)$, Range: $[1, \infty)$

STUDY TIP

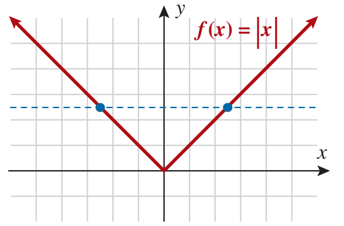
Had we ignored the domain and range in Example 8, we would have found the inverse function to be the square function $f(x) = x^2 - 2$, which is not a one-to-one function. It is only when we restrict the domain of the square function that we get a one-to-one function.

EXAMPLE 9 | A Function That Does Not Have an Inverse Function

Find the inverse of the function $f(x) = |x|$ if it exists.

Solution

The function $f(x) = |x|$ fails the horizontal line test and therefore is not a one-to-one function. Because f is not a one-to-one function, its inverse function does not exist.



Video EXAMPLE 10 | Finding the Inverse Function

The function $f(x) \dfrac{2}{x+3}$, $x \neq -3$ is a one-to-one function. Find its inverse.

Solution

STEP 1 Let $y = f(x)$. $\quad y = \dfrac{2}{x+3}$

STEP 2 Interchange x and y. $\quad x = \dfrac{2}{y+3}$

STEP 3 Solve for y.

Multiply the equation by $(y+3)$. $\quad x(y+3) = 2$

Eliminate the parentheses. $\quad xy + 3x = 2$

Subtract $3x$ from both sides. $\quad xy = -3x + 2$

Divide the equation by x. $\quad y = \dfrac{-3x+2}{x} = -3 + \dfrac{2}{x}$

STEP 4 Let $y = f^{-1}(x)$. $\quad f^{-1}(x) = -3 + \dfrac{2}{x}$

Note any domain restrictions on $f^{-1}(x)$. $\quad x \neq 0$

The inverse of the function $f(x) = \dfrac{2}{x+3}$, $x \neq -3$, is $\boxed{f^{-1}(x) = -3 + \frac{2}{x}, x \neq 0}$.

Check

$$f^{-1}(f(x)) = -3 + \frac{2}{\left(\frac{2}{x+3}\right)} = -3 + (x+3) = x, x \neq -3$$

$$f(f^{-1}(x)) = \frac{2}{\left(-3 + \frac{2}{x}\right) + 3} = \frac{2}{\left(\frac{2}{x}\right)} = x, x \neq 0$$

Your Turn The function $f(x) = \dfrac{4}{x-1}$, $x \neq 1$, is a one-to-one function. Find its inverse.

Answer $f^{-1}(x) = 1 + \dfrac{4}{x}$, $x \neq 0$

Note in Example 10 that the domain of f is $(-\infty, -3) \cup (-3, \infty)$ and the domain of f^{-1} is $(-\infty, 0) \cup (0, \infty)$. Therefore, we know that the range of f is $(-\infty, 0) \cup (0, \infty)$, and the range of f^{-1} is $(-\infty, -3) \cup (-3, \infty)$.

STUDY TIP

The range of a function is equal to the domain of its inverse function.

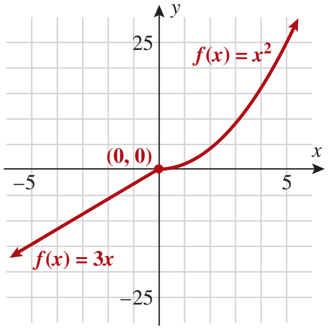
Video EXAMPLE 11 | Finding the Inverse of a Piecewise-Defined Function

The function $f(x) = \begin{cases} 3x & x < 0 \\ x^2 & x \geq 0 \end{cases}$ is a one-to-one function. Find its inverse.

Solution

From the graph of f we can make a table with corresponding domain and range values.

Domain of f	Range of f
$(-\infty, 0)$	$(-\infty, 0)$
$[0, \infty)$	$[0, \infty)$



From this information we can also list domain and range values for f^{-1}.

Domain of f^{-1}	Range of f^{-1}
$(-\infty, 0)$	$(-\infty, 0)$
$[0, \infty)$	$[0, \infty)$

$f(x) = 3x$ on $(-\infty, 0)$; find $f^{-1}(x)$ on $(-\infty, 0)$.

STEP 1 Let $y = f(x)$. $\quad y = 3x$

STEP 2 Solve for x in terms of y. $\quad x = 3y$

STEP 3 Solve for y. $\quad y = \frac{1}{3}x$

STEP 4 Let $y = f^{-1}(x)$. $\quad f^{-1}(x) = \frac{1}{3}x$ on $(-\infty, 0)$

$f(x) = x^2$ on $[0, \infty)$; find $f^{-1}(x)$ on $[0, \infty)$.

STEP 1 Let $y = f(x)$. $\quad y = x^2$

STEP 2 Solve for x in terms of y. $\quad x = y^2$

STEP 3 Solve for y. $\quad y = \pm\sqrt{x}$

STEP 4 Let $y = f^{-1}(x)$. $\quad f^{-1}(x) = \pm\sqrt{x}$

STEP 5 The range of f^{-1} is $[0, \infty)$ $\quad f^{-1}(x) = \sqrt{x}$

Combining the two pieces yields a piecewise-defined inverse function.

$$f^{-1}(x) = \begin{cases} \frac{1}{3}x & x < 0 \\ \sqrt{x} & x \ge 0 \end{cases}$$

Concept Check

Explain why you cannot find the inverse of $f(x) = x^2$ without restricting the domain.

Answer: $f(x) = x^2$ is not a one-to-one function.

Section 1.5 Summary

One-to-One Functions

Each input in the domain corresponds to exactly one output in the range, and no two inputs map to the same output. There are three ways to test a function to determine whether it is a one-to-one function.

1. Discrete points: For the set of all points (a, b) verify that no y-values are repeated.
2. Algebraic equations: Let $f(x_1) = f(x_2)$; if it can be shown that $x_1 = x_2$, then the function is one-to-one.
3. Graphs: Use the horizontal line test; if any horizontal line intersects the graph of the function in more than one point, then the function is not one-to-one.

Properties of Inverse Functions

1. If f is a one-to-one function, then f^{-1} exists.
2. Domain and range
   - Domain of f = range of f^{-1}
   - Domain of f^{-1} = range of f
3. Composition of inverse functions
   - $f^{-1}(f(x)) = x$ for all x in the domain of f.
   - $f(f^{-1}(x)) = x$ for all x in the domain of f^{-1}.
4. The graphs of f and f^{-1} are symmetric with respect to the line $y = x$.

Procedure for Finding the Inverse of a Function

1. Let $y = f(x)$.
2. Interchange x and y.
3. Solve for y.
4. Let $y = f^{-1}(x)$.

Section 1.5 Exercises

Skills

In Exercises 1–10, determine whether the given relation is a function. If it is a function, determine whether it is a one-to-one function.

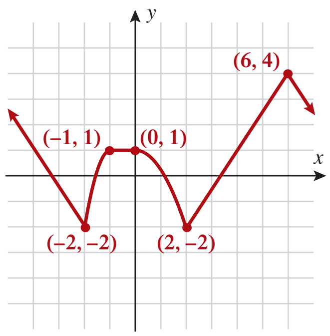
1. $\{(0, 0), (9, -3), (4, -2), (4, 2), (9, 3)\}$

2. $\{(0, 1), (1, 1), (2, 1), (3, 1)\}$

3. $\{(0, 1), (1, 0), (2, 1), (-2, 1), (5, 4), (-3, 4)\}$

4. $\{(0, 0), (-1, -1), (-2, -8), (1, 1), (2, 8)\}$

5.



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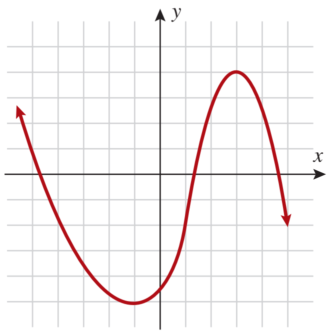

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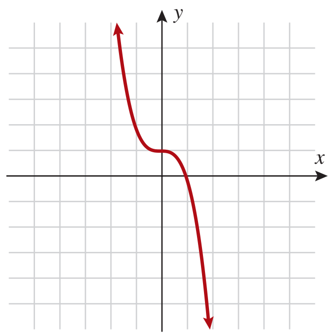

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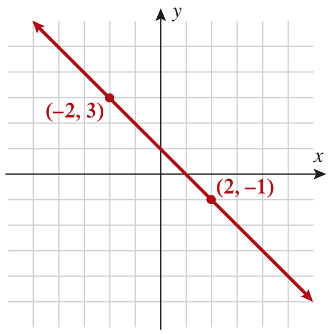

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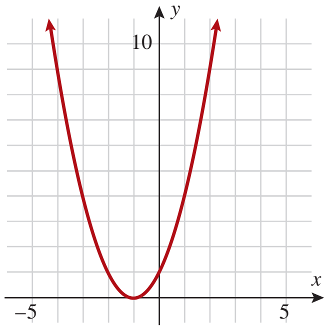

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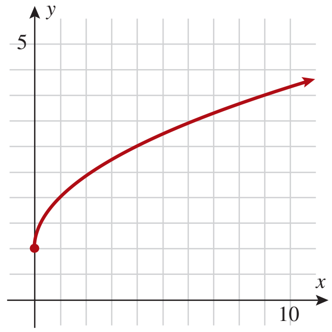


In Exercises 11–18, determine algebraically and graphically whether the function is one-to-one.

11. $f(x) = |x - 3|$

12. $f(x) = (x - 2)^2 + 1$

13. $f(x) = \dfrac{1}{x - 1}$

14. $f(x) = \sqrt[3]{x}$

15. $f(x) = x^2 - 4$

16. $f(x) = \sqrt{x + 1}$

17. $f(x) = x^3 - 1$

18. $f(x) = \dfrac{1}{x + 2}$

In Exercises 19–32, verify that the function $f^{-1}(x)$ is the inverse of $f(x)$ by showing that $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$. Graph $f(x)$ and $f^{-1}(x)$ on the same axes to show the symmetry about the line $y = x$.

19. $f(x) = 2x + 1; f^{-1}(x) = \dfrac{x - 1}{2}$

20. $f(x) = \dfrac{x - 2}{3}; f^{-1}(x) = 3x + 2$

21. $f(x) = -3x + 5; g(x) = \dfrac{x - 5}{-3}$

22. $f(x) = \dfrac{1 - x}{4}; g(x) = 1 - 4x$

23. $f(x) = x^3 + 7; g(x) = \sqrt[3]{x - 7}$

24. $f(x) = \sqrt[3]{x} + 2; g(x) = (x - 2)^3$

25. $f(x) = \sqrt{x - 1}, x \geq 1; f^{-1}(x) = x^2 + 1, x \geq 0$

26. $f(x) = 2 - x^2, x \geq 0; f^{-1}(x) = \sqrt{2 - x}, x \geq 2$

27. $f(x) = \dfrac{1}{x}; f^{-1}(x) = \dfrac{1}{x}, x \neq 0$

28. $f(x) = (5 - x)^{1/3}; f^{-1}(x) = 5 - x^3$

29. $f(x) = \dfrac{1}{2x + 6}, x \neq -3; f^{-1}(x) = \dfrac{1}{2x} - 3, x \neq 0$

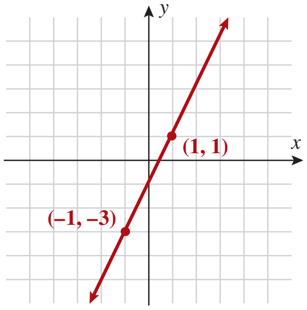
30. $f(x) = \dfrac{3}{4 - x}, x \neq 4; f^{-1}(x) = 4 - \dfrac{3}{x}, x \neq 0$

31. $f(x) = \dfrac{x + 3}{x + 4}, x \neq -4; f^{-1}(x) = \dfrac{3 - 4x}{x - 1}, x \neq 1$

32. $f(x) = \dfrac{x - 5}{3 - x}, x \neq 3; f^{-1}(x) = \dfrac{3x + 5}{x + 1}, x \neq -1$

In Exercises 33–40, graph the inverse of the one-to-one function that is given.

33.



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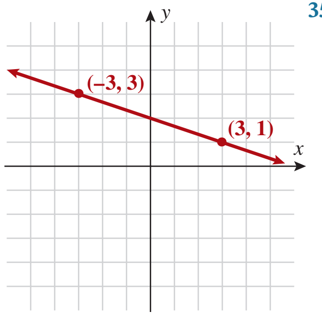

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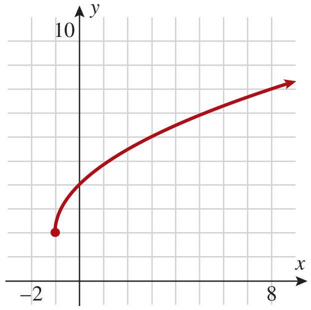

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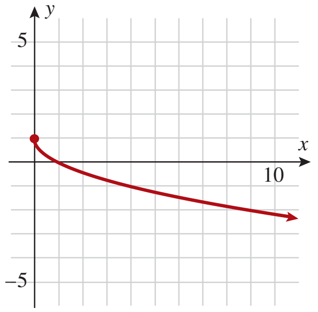

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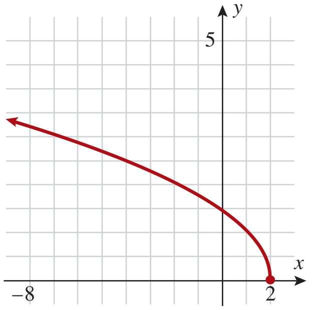

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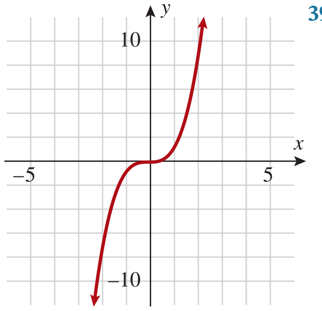

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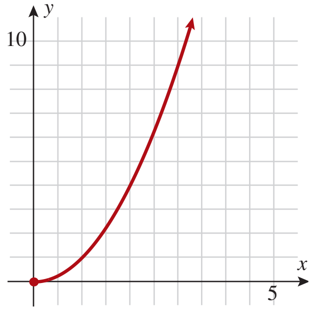

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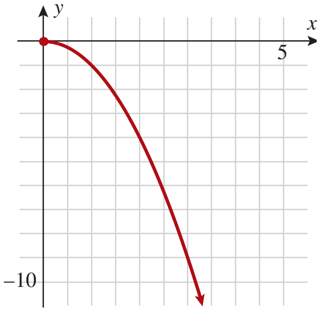


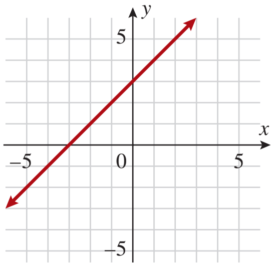
In Exercises 41–44, use the graph of *f* to determine the following.

41. **a.** Find $f(2)$.

b. Find the x that corresponds to $f(x) = 0$.

c. Find $f^{-1}(0)$.

d. Find the x that corresponds to $f^{-1}(x) = 0$.

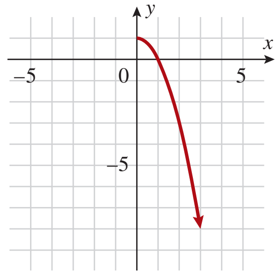


42. **a.** Find $f(2)$.

b. Find the x that corresponds to $f(x) = 0$.

c. Find $f^{-1}(-3)$.

d. Find the x that corresponds to $f^{-1}(x) = 0$.

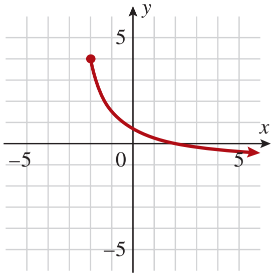


43. **a.** Find $f(-2)$.

b. Find the x that corresponds to $f(x) = 0$.

c. Find $f^{-1}(0)$.

d. Find the x that corresponds to $f^{-1}(x) = 4$.

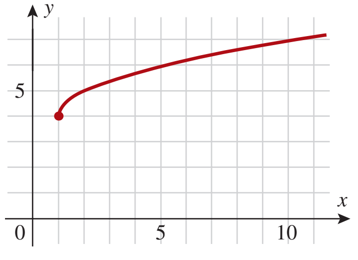


44. **a.** Find $f(5)$.

b. Find the x that corresponds to $f(x) = 4$.

c. Find $f^{-1}(5)$.

d. Find the x that corresponds to $f^{-1}(x) = 1$.



In Exercises 45–64, the function *f* is one-to-one. Find its inverse, and check your answer. State the domain and range of both *f* and f^{-1}.

45. $f(x) = -3x + 2$

46. $f(x) = 2x + 3$

47. $f(x) = x^3 + 1$

48. $f(x) = x^3 - 1$

49. $f(x) = \sqrt{x - 3}$

50. $f(x) = \sqrt{3 - x}$

51. $f(x) = x^2 - 1, x \geq 0$

52. $f(x) = 2x^2 + 1, x \geq 0$

53. $f(x) = (x + 2)^2 - 3, x \geq -2$

54. $f(x) = (x - 3)^2 - 2, x \geq 3$

55. $f(x) = \frac{2}{x}$

56. $f(x) = -\frac{3}{x}$

57. $f(x) = \frac{2}{3 - x}$

58. $f(x) = \frac{7}{x + 2}$

59. $f(x) = \frac{7x + 1}{5 - 2}$

60. $f(x) = \frac{2x + 5}{7 + x}$

61. $f(x) = \frac{1}{\sqrt{x}}$

62. $f(x) = \frac{x}{\sqrt{x + 1}}$

63. $f(x) = \sqrt{\frac{x + 1}{x - 2}}$

64. $f(x) = \sqrt{x^2 - 1}, x \geq 1$

In Exercises 65–70, graph the piecewise-defined function to determine whether it is a one-to-one function. If it is a one-to-one function, find its inverse.

65. $G(x) = \begin{cases} 0 & x < 0 \\ \sqrt{x} & x \geq 0 \end{cases}$

66. $G(x) = \begin{cases} \frac{1}{x} & x < 0 \\ \sqrt{x} & x \geq 0 \end{cases}$

67. $f(x) = \begin{cases} \sqrt[3]{x} & x \leq -1 \\ x^2 + 2x & -1 < x \leq 1 \\ \sqrt{x} + 2 & x > 1 \end{cases}$

68. $f(x) = \begin{cases} -x & x < -2 \\ \sqrt{4 - x^2} & -2 \leq x \leq 0 \\ -\frac{1}{x} & x > 0 \end{cases}$

69. $f(x) = \begin{cases} x & x \leq -1 \\ x^3 & -1 < x < 1 \\ x & x \geq 1 \end{cases}$

70. $f(x) = \begin{cases} x + 3 & x \leq -2 \\ |x| & -2 < x < 2 \\ x^2 & x \geq 2 \end{cases}$

Applications

71. Temperature. The equation used to convert from degrees Celsius to degrees Fahrenheit is $f(x) = \frac{9}{5}x + 32$. Determine the inverse function $f^{-1}(x)$. What does the inverse function represent?

72. Measurement. The equation used to convert from feet to inches is $f(x) = 12x$. Determine the inverse function $f^{-1}(x)$. What does the inverse function represent?

73. Budget. The Richmond rowing club is planning to enter the Head of the Charles race in Boston and is trying to figure out how much money to raise. The entry fee is $250 per boat for the first 10 boats and $175 for each additional boat. Find the cost function $C(x)$ as a function of the number of boats the club enters x. Find the inverse function that will yield how many boats the club can enter as a function of how much money it will raise.

74. New Car. Mrs. Fluharty bought a new car for $25,000 in 2020. Each year it loses $1500 in value due to depreciation. Find the function $V(x)$ for the value of the car as a function of the years she has owned the car. Let x = the number of years after 2020 that she has owned the car. Suppose she wants to sell the car after its value reaches a certain amount. Find the inverse function she can use to determine how long she can own the car before the value reaches that amount.

75. Salary. Mr. Garay is an Uber driver. He brings in $30 per hour from fares, and the weekly number of hours per week x varies. He has to pay 30% of his fares in commission to Uber. In addition, Mr. Garay expects to lose approximately $350 per week to gas, car cleanings, insurance, maintenance, and parking costs. Write a function $E(x)$ to predict Mr. Garay's expected wages each week. Find the inverse function $E^{-1}(x)$. What does the inverse function tell you?

76. Salary. A grocery store pays a cashier $15 per hour for the first 40 hours per week and time and a half for overtime. Write a piecewise-defined function that represents a cashier's weekly earnings $E(x)$ as a function of the number of hours worked x. Find the inverse function $E^{-1}(x)$. What does the inverse function help the cashier determine?

In Exercises 77–80, refer to the following:

By analyzing available empirical data it was determined that during an illness a patient's body temperature fluctuated during one 24-hour period according to the function

$$T(t) = 0.0003(t - 24)^3 + 101.70$$

where T represents that patient's temperature in degrees Fahrenheit and t represents the time of day in hours measured from 12:00 A.M. (midnight).

77. Health/Medicine. Find the domain and range of the function $T(t)$.

78. Health/Medicine. Find time as a function of temperature, that is, the inverse function $t(T)$.

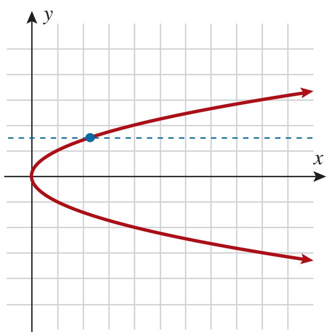
79. Health/Medicine. Find the domain and range of the function $t(T)$ found in Exercise 78.

80. Health/Medicine. At what time, to the nearest hour, was the patient's temperature 99.5°F?

Catch the Mistake

In Exercises 81 and 82, explain the mistake that is made.

81. Is $x = y^2$ a one-to-one function?



Solution

Yes, this graph represents a one-to-one function because it passes the horizontal line test.

This is incorrect. What mistake was made?

82. Given the function $f(x) = \sqrt{x - 2}$, find the inverse function $f^{-1}(x)$, and state the domain restrictions on $f^{-1}(x)$.

Solution

STEP 1 Let $y = f(x)$. $\quad y = \sqrt{x - 2}$

STEP 2 Interchange x and y. $\quad x = \sqrt{y - 2}$

STEP 3 Solve for y. $\quad y = x^2 + 2$

STEP 4 Let $f^{-1}(x) = y$. $\quad f^{-1}(x) = x^2 + 2$

STEP 5 Domain restrictions: $f(x) = \sqrt{x - 2}$ has the domain restriction that $x \geq 2$.

The inverse of $f(x) = \sqrt{x - 2}$ is $f^{-1}(x) = x^2 + 2$.

The domain of $f^{-1}(x)$ is $x \geq 2$.

This is incorrect. What mistake was made?

Conceptual

In Exercises 83–86, determine whether each statement is true or false.

83. Every even function is a one-to-one function.

84. Every odd function is a one-to-one function.

85. It is not possible that $f = f^{-1}$.

86. A function f has an inverse. If the function lies in quadrant II, then its inverse lies in quadrant IV.

87. If $(0, b)$ is the y-intercept of a one-to-one function f, what is the x-intercept of the inverse f^{-1}?

88. If $(a, 0)$ is the x-intercept of a one-to-one function f, what is the y-intercept of the inverse f^{-1}?

Challenge

89. The unit circle is not a function. If we restrict ourselves to the semicircle that lies in quadrants I and II, the graph represents a function, but it is not a one-to-one function. If we further restrict ourselves to the quarter circle lying in quadrant I, the graph does represent a one-to-one function. Determine the equations of both the one-to-one function and its inverse. State the domain and range of both.

90. Find the inverse of $f(x) = \frac{c}{x}$, $c \neq 0$.

91. Under what conditions is the linear function $f(x) = mx + b$ a one-to-one function?

92. Assuming that the conditions found in Exercise 91 are met, determine the inverse of the linear function.

93. Determine the value of a that makes $f(x) = \dfrac{x-2}{x^2 - a}$ a one-to-one function. Determine $f^{-1}(x)$ and its domain.

94. The point (a, b) lies on the graph of the one-to-one function $y = f(x)$. What other points are guaranteed to lie on the graph of $y = f^{-1}(x)$?

Preview to Calculus

For Exercises 95–100, refer to the following:

In calculus, the difference quotient $\dfrac{f(x+h) - f(x)}{h}$ of a function f is used to find the derivative f' of f, by allowing h to approach zero, $h \to 0$. The derivative of the inverse function $(f^{-1})'$ can be found using the formula

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$

provided that the denominator is not 0 and both f and f^{-1} are differentiable. For the following one-to-one function, find (a) f^{-1}, (b) f', (c) $(f^{-1})'$, and (d) verify the formula above. For (b) and (c), use the difference quotient.

95. $f(x) = 2x + 1$

96. $f(x) = x^2, x > 0$

97. $f(x) = \frac{1}{x}, x > 0$

98. $f(x) = \sqrt{x}, x > 0$

99. $f(x) = \sqrt{x+2}, x > {}^{-}2$

100. $f(x) = \frac{1}{x+1}, x > {}^{-}1$

Chapter 1 Review

Section	Concept	Key Ideas/Formulas
1.1	**Functions**	
	Relations and functions	All functions are relations, but not all relations are functions.
	Functions defined by equations	A vertical line can intersect a function in at most one point.
	Function notation	*Placeholder notation:* $f(x) = 3x^2 - 6x + 2 \quad f(\square) = 3(\square)^2 - 6(\square) + 2$ *Difference quotient*: $\frac{f(x+h) - f(x)}{h}; h \neq 0$
	Domain of a function	Are there any restrictions on x?
1.2	**Graphs of functions; piecewise-defined functions; increasing and decreasing functions; average rate of change**	
	Recognizing and classifying functions	*Common functions* $f(x) = mx + b, f(x) = x, f(x) = x^2,$ $f(x) = x^3, f(x) = \sqrt{x}, f(x) = \sqrt[3]{x},$ $f(x) = \lvert x \rvert, f(x) = \frac{1}{x}$ *Even and odd functions* Even: Symmetry about y-axis: $f(-x) = f(x)$ Odd: Symmetry about origin: $f(-x) = -f(x)$
	Increasing and decreasing functions	• Increasing: rises (left to right) • Decreasing: falls (left to right)
	Average rate of change	$\frac{f(x_2) - f(x_1)}{x_2 - x_1} \quad x_1 \neq x_2$
	Piecewise-defined functions	Points of discontinuity
1.3	**Graphing techniques: Transformations**	Shift the graph of $f(x)$.
	Horizontal and vertical shifts	$f(x + c)$ c units to the left, $c > 0$ $f(x - c)$ c units to the right, $c > 0$ $f(x) + c$ c units upward, $c > 0$ $f(x) - c$ c units downward, $c > 0$
	Reflection about the axes	$-f(x)$ Reflection about the x-axis $f(-x)$ Reflection about the y-axis
	Stretching and compressing	$cf(x)$ if $c > 1$; stretch vertically $cf(x)$ if $0 < c < 1$; compress vertically $f(cx)$ if $c > 1$; compress horizontally $f(cx)$ if $0 < c < 1$; stretch horizontally
1.4	**Operations on functions and composition of functions**	
	Adding, subtracting, multiplying, and dividing functions	$(f + g)(x) = f(x) + g(x)$ $(f - g)(x) = f(x) - g(x)$ $(f \cdot g)(x) = f(x) \cdot g(x)$ The domain of the resulting function is the intersection of the individual domains. $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}, \quad g(x) \neq 0$ The domain of the quotient is the intersection of the domains of f and g, and any points when $g(x) = 0$ must be eliminated.
	Composition of functions	$(f \circ g)(x) = f(g(x))$ The domain of the composite function is a subset of the domain of $g(x)$. Values for x must be eliminated if their corresponding values $g(x)$ are not in the domain of f.

(*Continued*)

Section	Concept	Key Ideas/Formulas
1.5	**One-to-one functions and inverse functions**	
	Determine whether a function is one-to-one	• No two x-values map to the same y-value. If $f(x_1) = f(x_2)$, then $x_1 = x_2$. • A horizontal line may intersect a one-to-one function in at most one point.
	Inverse functions	• Only one-to-one functions have inverses. • $f^{-1}(f(x)) = x$ and $f(f^{-1}(x)) = x$. • Domain of f = range of f^{-1}. Range of f = domain of f^{-1}.
	Graphical interpretation of inverse functions	• The graph of a function and its inverse are symmetric about the line $y = x$. • If the point (a, b) lies on the graph of a function, then the point (b, a) lies on the graph of its inverse.
	Finding the inverse function	1. Let $y = f(x)$. 2. Interchange x and y. 3. Solve for y. 4. Let $y = f^{-1}(x)$.

Chapter 1 Review Exercises

1.1 Functions

In Exercises 1–8, determine whether each relation is a function. Assume that the coordinate pair (x, y) represents independent variable x and dependent variable y.

1. $\{(-2, 3), (1, -3), (0, 4), (2, 6)\}$

2. $\{(4, 7), (2, 6), (3, 8), (1, 7)\}$

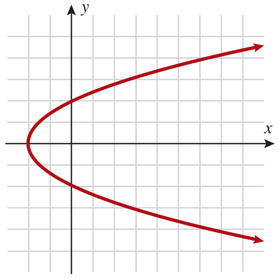
3. $x^2 + y^2 = 36$

4. $x = 4$

5. $y = |x + 2|$

6. $y = \sqrt{x}$

7.



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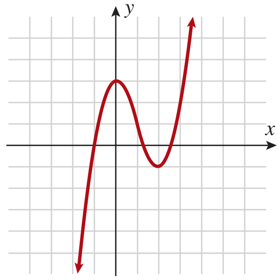
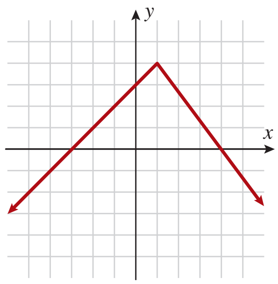


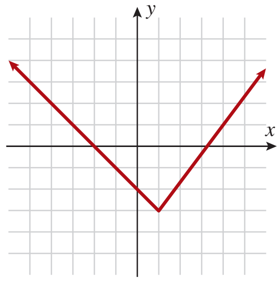
In Exercises 9–12, use the graphs of the functions to find:

9.



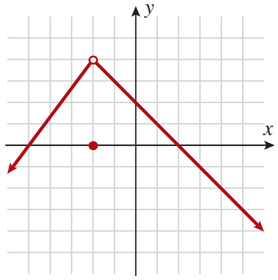
a. $f(-1)$ **b.** $f(1)$
c. x, where $f(x) = 0$

10.



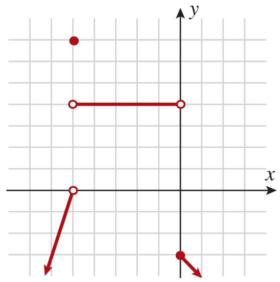
a. $f(-4)$ **b.** $f(0)$
c. x, where $f(x) = 0$

11.



a. $f(-2)$ **b.** $f(4)$
c. x, where $f(x) = 0$

12.



a. $f(-5)$ **b.** $f(0)$
c. x, where $f(x) = 0$

In Exercises 13–20, evaluate the given quantities using the following three functions.

$f(x) = 4x - 7$ $F(t) = t^2 + 4t - 3$ $g(x) = |x^2 + 2x + 4|$

13. $f(3)$

14. $F(4)$

15. $f(-7) \cdot g(3)$

16. $\dfrac{F(0)}{g(0)}$

17. $\dfrac{f(2) - F(2)}{g(0)}$

18. $f(3 + h)$

19. $\dfrac{f(3 + h) - f(3)}{h}$

20. $\dfrac{F(t + h) - F(t)}{h}$

In Exercises 21–26, find the domain of the given function. Express the domain in interval notation.

21. $f(x) = -3x - 4$

22. $g(x) = x^2 - 2x + 6$

23. $h(x) = \dfrac{1}{x + 4}$

24. $F(x) = \dfrac{7}{x^2 + 3}$

25. $G(x) = \sqrt{x - 4}$

26. $H(x) = \dfrac{1}{\sqrt{2x - 6}}$

Challenge

27. If $f(x) = \dfrac{D}{x^2-16}$, $f(4)$ and $f(-4)$ are undefined, and $f(5) = 2$, find D.

28. Construct a function that is undefined at $x = -3$ and $x = 2$ such that the point $(0, -4)$ lies on the graph of the function.

1.2 Graphs of Functions

In Exercises 29–32, determine whether the function is even, odd, or neither.

29. $h(x) = x^3 - 7x$

30. $f(x) = x^4 + 3x^2$

31. $f(x) = \dfrac{1}{x^3} + 3x$

32. $f(x) = \dfrac{1}{x^2} + 3x^4 + |x|$

In Exercises 33–36, use the graph of the functions to find:

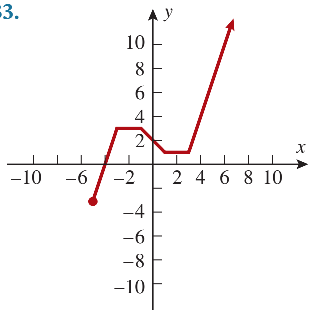
a. Domain

b. Range

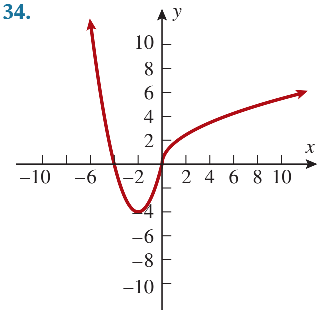
c. Intervals on which the function is increasing, decreasing, or constant.

Find the values of **(d)** $f(0)$, **(e)** $f(-3)$, and **(f)** $f(3)$.

33.



34.



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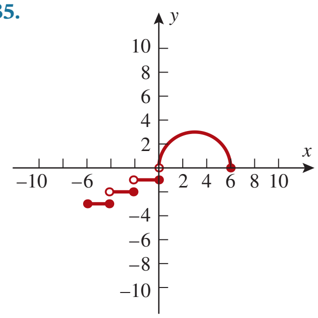

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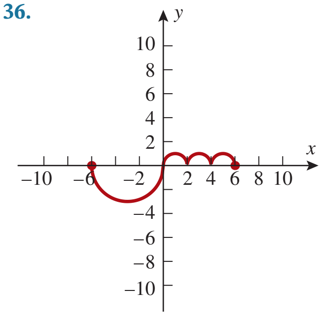


In Exercises 37-40, find the difference quotient $\dfrac{f(x + h) - f(x)}{h}$ for each function.

37. $f(x) = x^3 - 1$

38. $f(x) = \dfrac{x - 1}{x + 2}$

39. $f(x) = x + \dfrac{1}{x}$

40. $f(x) = \sqrt{\dfrac{x}{x + 1}}$

41. Find the average rate of change of $f(x) = 4 - x^2$ from $x = 0$ to $x = 2$.

42. Find the average rate of change of $f(x) = |2x - 1|$ from $x = 1$ to $x = 5$.

In Exercises 43–46, graph the piecewise-defined function. State the domain and range in interval notation.

43. $F(x) = \begin{cases} x^2 & x < 0 \\ 2 & x \ge 0 \end{cases}$

44. $f(x) = \begin{cases} -2x - 3 & x \le 0 \\ 4 & 0 < x \le 1 \\ x^2 + 4 & x > 1 \end{cases}$

45. $f(x) = \begin{cases} x^2 & x \le 0 \\ -\sqrt{x} & 0 < x \le 1 \\ |x + 2| & x > 1 \end{cases}$

46. $F(x) = \begin{cases} x^2 & x < 0 \\ x^3 & 0 < x < 1 \\ -|x| - 1 & x \ge 1 \end{cases}$

Applications

47. Housing Cost. In 2001 the market value of a house was \$135,000; in 2006 the market price of the same house was \$280,000. What is the average rate of change of the market price as a function of the time (in years), where $t = 0$ corresponds to 2001.

48. Digital TV Conversion. A newspaper reported that by February 2009, only 38% of the urban population was ready for the conversion to digital TV. Ten weeks later, the newspaper reported that 64% of the population was prepared for the broadcasting change. Find the average rate of change of the population percent as a function of the time (in weeks).

1.3 Graphing Techniques: Transformations

In Exercises 49–54, graph the following functions using graphing aids.

49. $y = -(x - 2)^2 + 4$

50. $y = |-x + 5| - 7$

51. $y = \sqrt[3]{x - 3} + 2$

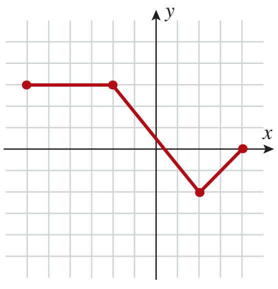
52. $y = \dfrac{1}{x - 2} - 4$

53. $y = -\frac{1}{2}x^3$

54. $y = 2x^2 + 3$

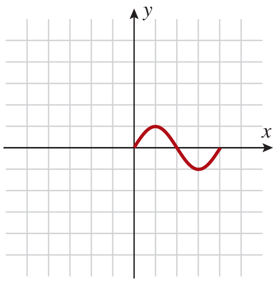
In Exercises 55–58, use the given graph to graph the following:

55.



$y = f(x - 2)$

56.



$y = 3f(x)$

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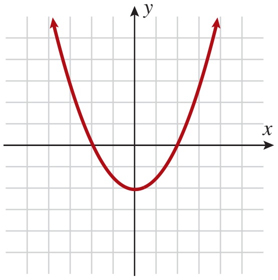
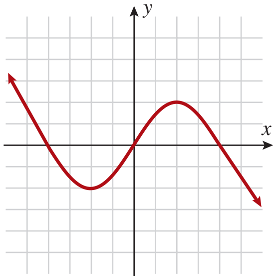


$y = -2f(x)$

58.



$y = f(x) + 3$

In Exercises 59–64, write the function whose graph is the graph of $y = \sqrt{x}$, but is transformed accordingly, and state the domain of the resulting function.

59. Shifted to the left three units

60. Shifted down four units

61. Shifted to the right two units and up three units

62. Reflected about the y-axis

63. Stretched by a factor of 5 and shifted down six units

64. Compressed by a factor of 2 and shifted up three units

In Exercises 65–66, transform the function into the form $f(x) = c(x - h)^2 + k$ by completing the square and graph the resulting function using transformations.

65. $y = x^2 + 4x - 8$

66. $y = 2x^2 + 6x - 5$

1.4 Operations on Functions and Composition of Functions

In Exercises 67–72, given the functions g and h, find $g + h$, $g - h$, $g \cdot h$, and $\dfrac{g}{h}$, and state the domain.

67. $g(x) = -3x - 4$
$h(x) = x - 3$

68. $g(x) = 2x + 3$
$h(x) = x^2 + 6$

69. $g(x) = \dfrac{1}{x^2}$
$h(x) = \sqrt{x}$

70. $g(x) = \dfrac{x + 3}{2x - 4}$
$h(x) = \dfrac{3x - 1}{x - 2}$

71. $g(x) = \sqrt{x - 4}$
$h(x) = \sqrt{2x + 1}$

72. $g(x) = x^2 - 4$
$h(x) = x + 2$

In Exercises 73–78, for the given functions f and g, find the composite functions $f \circ g$ and $g \circ f$, and state the domains.

73. $f(x) = 3x - 4$
$g(x) = 2x + 1$

74. $f(x) = x^3 + 2x - 1$
$g(x) = x + 3$

75. $f(x) = \dfrac{2}{x + 3}$
$g(x) = \dfrac{1}{4 - x}$

76. $f(x) = \sqrt{2x^2 - 5}$
$g(x) = \sqrt{x + 6}$

77. $f(x) = \sqrt{x - 5}$
$g(x) = x^2 - 4$

78. $f(x) = \dfrac{1}{\sqrt{x}}$
$g(x) = \dfrac{1}{x^2 - 4}$

In Exercises 79–84, evaluate $f(g(3))$ and $g(f(-1))$, if possible.

79. $f(x) = 4x^2 - 3x + 2$
$g(x) = 6x - 3$

80. $f(x) = \sqrt{4 - x}$
$g(x) = x^2 + 5$

81. $f(x) = \dfrac{x}{|2x - 3|}$
$g(x) = |5x + 2|$

82. $f(x) = \dfrac{1}{x - 1}$
$g(x) = x^2 - 1$

83. $f(x) = x^2 - x + 10$
$g(x) = \sqrt[3]{x - 4}$

84. $f(x) = \dfrac{4}{x^2 - 2}$
$g(x) = \dfrac{1}{x^2 - 9}$

In Exercises 85–88, write the function as a composite $f(g(x))$ of two functions f and g.

85. $h(x) = 3(x - 2)^2 + 4(x - 2) + 7$

86. $h(x) = \dfrac{\sqrt[3]{x}}{1 - \sqrt[3]{x}}$

87. $h(x) = \dfrac{1}{\sqrt{x^2 + 7}}$

88. $h(x) = \sqrt{|3x + 4|}$

Applications

89. Rain. A rain drop hitting a lake makes a circular ripple. If the radius, in inches, grows as a function of time, in minutes, $r(t) = 25\sqrt{t+2}$, find the area of the ripple as a function of time.

90. Geometry. Let the area of a rectangle be given by $42 = l \cdot w$, and let the perimeter be $36 = 2 \cdot l + 2 \cdot w$. Express the perimeter in terms of w.

1.5 One-to-One Functions and Inverse Functions

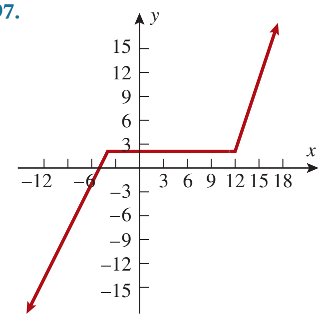
In Exercises 91–100, determine whether the given function is a one-to-one function.

91. $\{(-2, 0), (4, 5), (3, 7)\}$

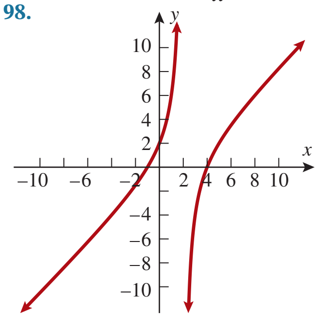
92. $\{(-8, -6), (-4, 2), (0, 3), (2, -8), (7, 4)\}$

93. $y = \sqrt{x}$ **94.** $y = x^2$ **95.** $f(x) = x^3$ **96.** $f(x) = \frac{1}{x^2}$

97.



98.



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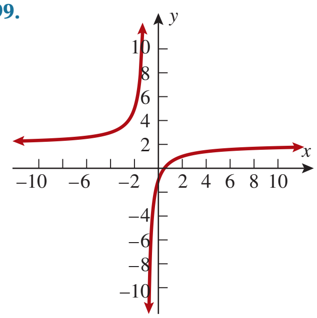
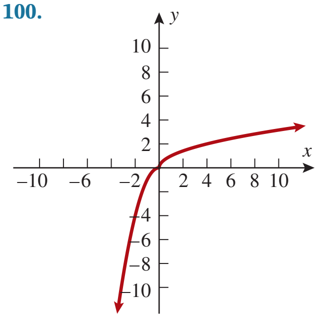


100.



In Exercises 101–104, verify that the function $f^{-1}(x)$ is the inverse of $f(x)$ by showing that $f(f^{-1}(x)) = x$. Graph $f(x)$ and $f^{-1}(x)$ on the same graph and show the symmetry about the line $y = x$.

101. $f(x) = 3x + 4; f^{-1}(x) = \frac{x-4}{3}$

102. $f(x) = \frac{1}{4x-7}; f^{-1}(x) = \frac{1+7x}{4x}$

103. $f(x) = \sqrt{x+4}; f^{-1}(x) = x^2 - 4 \quad x \geq 0$

104. $f(x) = \frac{x+2}{x-7}; f^{-1}(x) = \frac{7x+2}{x-1}$

In Exercises 105–110, the function f is one-to-one. Find its inverse and check your answer. State the domain and range of both f and f^{-1}.

105. $f(x) = 2x + 1$

106. $f(x) = x^5 + 2$

107. $f(x) = \sqrt{x+4}$

108. $f(x) = (x+4)^2 + 3 \quad x \geq -4$

109. $f(x) = \frac{x+6}{x+3}$

110. $f(x) = 2\sqrt[3]{x-5} - 8$

Applications

111. Salary. A pharmaceutical salesperson makes $98,000 base salary a year plus 11% of the total products sold. Write a function $S(x)$ that represents their yearly salary as a function of the total dollars worth of products sold x. Find $S^{-1}(x)$. What does this inverse function tell you?

112. Volume. Express the volume V of a rectangular box that has a square base of length s and is 3 feet high as a function of the square length. Find V^{-1}. If a certain volume is desired, what does the inverse tell you?

Chapter 1 Practice Test

In Exercises 1–3, assuming that x represents the independent variable and y represents the dependent variable, classify the relationships as:

a. not a function **b.** a function but not one-to-one

c. a one-to-one function

1. $f(x) = |2x + 3|$ **2.** $x = y^2 + 2$ **3.** $y = \sqrt[3]{x+1}$

In Exercises 4–9, use $f(x) = \sqrt{x-2}$ and $g(x) = x^2 + 11$, and determine the desired quantity or expression. In the case of an expression, state the domain.

4. $f(11) - 2g(-1)$ **5.** $\left(\frac{f}{g}\right)(x)$ **6.** $\left(\frac{g}{f}\right)(x)$

7. $g(f(x))$ **8.** $(f+g)(6)$ **9.** $f(g(\sqrt{7}))$

In Exercises 10–12, determine whether the function is odd, even, or neither.

10. $f(x) = |x| - x^2$ **11.** $f(x) = 9x^3 + 5x - 3$ **12.** $f(x) = \frac{2}{x}$

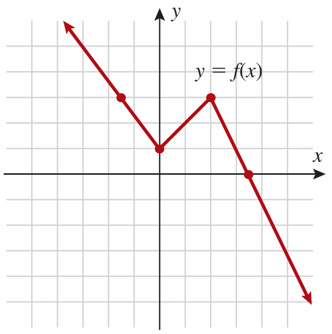
In Exercises 13–15, graph the functions. State the domain and range of each function.

13. $f(x) = -\sqrt{x-3} + 2$ **14.** $f(x) = -2(x-1)^2$

15. $f(x) = \begin{cases} -x & x < -1 \\ 1 & -1 < x < 2 \\ x^2 & x \geq 2 \end{cases}$

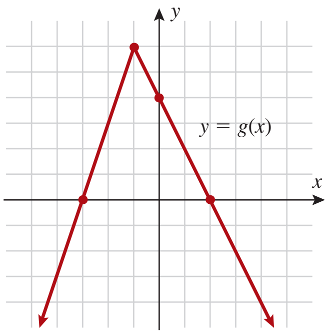
In Exercises 16–18, use the graphs of the function to find:

16.



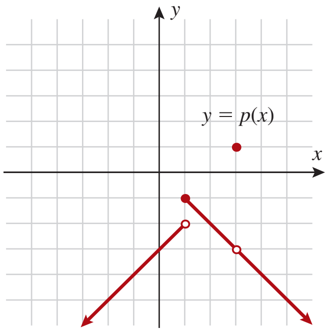
a. $f(3)$
b. $f(0)$
c. $f(-4)$
d. x, where $f(x) = 3$
e. x, where $f(x) = 0$

17.



a. $g(3)$
b. $g(0)$
c. $g(-4)$
d. x, where $g(x) = 0$

18.



a. $p(0)$
b. x, where $p(x) = 0$
c. $p(1)$
d. $p(3)$

In Exercises 19–20, find $\frac{f(x+h) - f(x)}{h}$ for:

19. $f(x) = 3x^2 - 4x + 1$ **20.** $f(x) = x^3 - \frac{1}{\sqrt{x}}$

In Exercises 21–22, find the average rate of change of the given functions.

21. $f(x) = 64 - 16x^2$ for $x = 0$ to $x = 2$

22. $f(x) = \sqrt{x-1}$ for $x = 2$ to $x = 10$

In Exercises 23–26, given the function f, find the inverse if it exists. State the domain and range of both f and f^{-1}.

23. $f(x) = \sqrt{x-5}$

24. $f(x) = x^2 + 5$

25. $f(x) = \frac{2x+1}{5-x}$

26. $f(x) = \begin{cases} -x & x \leq 0 \\ -x^2 & x > 0 \end{cases}$

27. What domain restriction can be made so that $f(x) = x^2$ has an inverse?

28. If the point $(-2, 5)$ lies on the graph of a function, what point lies on the graph of its inverse function?

29. Pressure. A mini-submarine descends at a rate of 5 feet per second. The pressure on the submarine structure is a linear function of the depth. When the submarine is on the surface, the pressure is 10 pounds per square inch, and when it is 100 feet under water, the pressure is 28 pounds per square inch. Write a function that describes the pressure P as a function of the time t in seconds.

30. Geometry. Both the volume V and the surface area S of a sphere are functions of the radius R. Write the volume as a function of the surface area.

31. Circles. If a quarter circle is drawn by tracing the unit circle in quadrant III, what does the inverse of that function look like? Where is it located?

32. Sprinkler. A sprinkler head malfunctions at midfield in a football field. The puddle of water forms a circular pattern around the sprinkler head with a radius in yards that grows as a function of time, in hours: $r(t) = 10\sqrt{t}$. When will the puddle reach the sidelines? (A football field is 30 yards from sideline to sideline.)

33. Cost of Admission. The cost to get into a high school football game is \$5 for students age 6-17 and \$7 for adults 18 and over. Kids 5 and under get in for free. Write a function describing the cost of the admission as a function of age in years.

34. Temperature and CO_2 Emissions. The following table shows average yearly temperature in degrees Fahrenheit (°F) and carbon dioxide emissions in parts per million (ppm) for Mauna Loa, Hawaii. Scientists discovered that both temperature and CO_2 emissions are linear functions of the time in years since 2000. Write a function that describes the temperature T as a function of the CO_2 emissions x. Use this function to determine the temperature when CO_2 emissions reach the level of 375 ppm.

Year	2000	2005
Temperature (°F)	45.86	46.23
CO_2 emissions (ppm)	369.4	379.7