1

Common Analysis Tools

1.1 Introduction

The electric machine consists of a stationary member called the stator and inside this stator is a rotating member called the rotor. The stator and rotor are generally constructed from conductive wire, iron (steel), and/or permanent magnets. For alternating current (ac) machines, the main focus of this text, the rotor is different for each type of machine, but the stators are essentially the same. This chapter introduces tools to analyze the currents and magnetic fields that flow through and about the stators and rotors of electric machines.

Since the beginning of analysis of machines, several basic tools have become more or less standard. These concepts are covered briefly in this chapter. Most are used in the analysis of the machines considered in this text. This chapter starts with phasors which is a complex-number means for analyzing steady-state ac variables and ends with two- and three-phase stator arrangements. These concepts have been covered by many authors but are necessary and warrant consideration in texts on the analysis of machines.

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1.2 Steady-State Phasor Calculations

We will deal with steady-state sinusoidal variables in this text and phasor analysis is very convenient for analyzing these variables. In the early 1900s, Charles Stienmetz set forth a method of analyzing the steady-state sinusoidal variables. This method has evolved over the years with different names, for example, vector analysis, sinor analysis, and now phasor analysis; however, depending on the area of application, the phasor may be slightly different. We will define it as used in the power and drives areas, which may differ somewhat from that taught in other courses.

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The phasor is established by expressing a steady-state sinusoidal variable as

$$F_s(t) = F_p \cos\theta_{ef} \tag{1.2-1}$$

where the s subscript is used here to denote sinusoidal quantities. In the following chapters, the s subscript will denote stator variables. The sinusoidal variations are expressed as cosines, capital letters are used to denote steady-state quantities, and F_p is the peak value of the sinusoidal variation. Here, F is just a placeholder for any quantity of interest. Generally, in circuit analysis, F will be V for voltage or I for current. For steady-state conditions, θ_{ef} may be written as

$$\theta_{ef}(t) = \omega_e t + \theta_{ef}(0) \tag{1.2-2}$$

where ω_e is the electrical angular velocity in rad/sec and $\theta_{ef}(0)$ is the time-zero position of the electrical variable. Substituting (1.2-2) into (1.2-1) yields

$$F_s(t) = F_p \cos\left[\omega_e t + \theta_{ef}(0)\right] \tag{1.2-3}$$

Now, Euler's formula is

$$e^{j\alpha} = \cos\alpha + j\sin\alpha \tag{1.2-4}$$

and since we are expressing the sinusoidal variation as a cosine, (1.2-3) may be written as

$$F_s(t) = \text{Re}\left\{F_p e^{j\left[\omega_e t + \theta_{ef}(0)\right]}\right\} \tag{1.2-5}$$

where Re is shorthand notation for the "real part of." Equations (1.2-3) and (1.2-5) are equivalent. Let us rewrite (1.2-5) as

$$F_s(t) = \text{Re}\left\{F_p e^{j\theta_{ef}(0)} e^{j\omega_e t}\right\} \tag{1.2-6}$$

We need to take a moment to define what is referred to as the root-mean-square (rms) of a sinusoidal variation. In particular, the rms value is defined as

$$F = \left(\frac{1}{T}\int_0^T F_s^2(t)dt\right)^{1/2} \tag{1.2-7}$$

where F is the rms value of $F_s(t)$ and T is the period of the sinusoidal variation. It is left to the reader to show that the rms value of (1.2-3) is $F_p/\sqrt{2}$. Therefore, we can express (1.2-6) as

$$F_s(t) = \text{Re}\left[\sqrt{2}Fe^{j\theta_{ef}(0)}e^{j\omega_e t}\right] \tag{1.2-8}$$

By definition, the phasor representing $F_s(t)$, which is denoted with a raised tilde, is

$$\tilde{F}_s = Fe^{j\theta_{ef}(0)} \tag{1.2-9}$$

which is a complex number. We see from (1.2-8) and (1.2-9) that if we consider the complex plane and rotate $\sqrt{2}\tilde{F}_s$ counterclockwise (ccw) at the angular velocity of the sinusoidal variable, the real projection is the instantaneous sinusoidal variable. We can stop the rotation and work only with the complex number. In sinusoidal steady state with a single source, the quantities of interest in a linear system will oscillate at the same frequency but with different magnitudes and relative phases. Phasor analysis keeps the amplitude and relative phase of sinusoidal quantities and eliminates the redundant information, frequency. Phasors of all like frequencies may be added by adding the real parts and imaginary parts of each phasor. We will use phasors extensively.

The reason for using the rms value as the magnitude of the phasor will be addressed later in this section. Equation (1.2-6) may now be written as

$$F_s(t) = \text{Re}\left[\sqrt{2}\tilde{F}_s e^{j\omega_e t}\right] \tag{1.2-10}$$

A shorthand notation for (1.2-9) is

$$\tilde{F}_s = F\underline{/\theta_{ef}(0)} \tag{1.2-11}$$

Equation (1.2-11) is commonly referred to as the *polar form* of the phasor. The *Cartesian* form is

$$\tilde{F}_s = F\cos\theta_{ef}(0) + jF\sin\theta_{ef}(0) \tag{1.2-12}$$

When using phasors to calculate steady-state voltages and currents, we think of the phasors as being stationary at $t = 0$; however, we know from (1.2-10) that a phasor is related to the instantaneous value of the sinusoidal quantity it represents. In other words, the real projection of the phasor $\tilde{F}_s$ rotating counterclockwise at ω_e is the instantaneous value of $F_s(t)/\sqrt{2}$. Thus, with $\theta_{ef}(0) = 0$ in (1.2-3)

$$F_s(t) = \sqrt{2}F\cos\omega_e t \tag{1.2-13}$$

the phasor representing (1.2-13) is

$$\tilde{F}_s = Fe^{j0} = F\underline{/0^\circ} = F + j0 \tag{1.2-14}$$

For

$$\begin{aligned} F_s(t) &= \sqrt{2}F\sin\omega_e t \\ &= \sqrt{2}F\cos(\omega_e t - 90^\circ) \end{aligned} \tag{1.2-15}$$

the phasor is

$$\tilde{F}_s = Fe^{-j\pi/2} = F\underline{/-90^\circ} = 0 - jF \tag{1.2-16}$$

We will use degrees and radians interchangeably when expressing phasors. Although there are several ways to arrive at (1.2-16) from (1.2-15), it is helpful to ask yourself where must the rotating phasor be positioned at time zero so that, when it rotates counterclockwise at ω_e, its real projection is $\left(1/\sqrt{2}\right)F_p \sin \omega_e t$? It follows that a phasor of amplitude F positioned at 90° represents $-\sqrt{2}F \sin \omega_e t$.

To summarize, a sinusoidal variation can be viewed as the real projection of a rotating line equal in magnitude to the positive peak value $\left(\sqrt{2}F\right)$ of the variation and rotating counterclockwise in the complex plane at the electrical angular velocity of the sinusoidal variation. Since we are in steady state and the electrical angular velocity is constant, we can stop the rotation at any time and view it as a fixed line. This fixed line is the phasor representation of the sinusoidal quantity depicted in phasor diagrams. A phasor diagram is shown in Fig. 1.A-1. Please understand that if we ran at ω_e in unison with the rotating $\sqrt{2}F$ line, it would appear as a constant to us.

In order to show the facility of the phasor in the analysis of steady-state performance of ac circuits and devices, we will consider the following circuit elements, a resistor with resistance, R, an inductor with inductance, L, and a capacitor with capacitance, C. Thus, using uppercase letters to indicate sinusoidal steady-state variables, the voltage across a resistance may be expressed in terms of the current flowing through it. That is, with I_R given as

$$I_R = \sqrt{2}I \cos\left[\omega_e t + \theta_{esi}(0)\right] \tag{1.2-17}$$

$$\begin{aligned} V_R &= RI_R \\ &= R\sqrt{2}I \cos\left[\omega_e t + \theta_{esi}(0)\right] \end{aligned} \tag{1.2-18}$$

In phasor form, the voltage across the resistor is in phase with the current through it as shown in Fig. 1.2-1 [$\theta_{esv}(0) = \theta_{esi}(0)$]. Thus,

$$\tilde{V}_R = R\tilde{I}_R \tag{1.2-19}$$

For the inductor

$$V_L = L\frac{dI_L}{dt} \tag{1.2-20}$$

where

$$I_L = \sqrt{2}I \cos\left[\omega_e t + \theta_{esi}(0)\right] \tag{1.2-21}$$

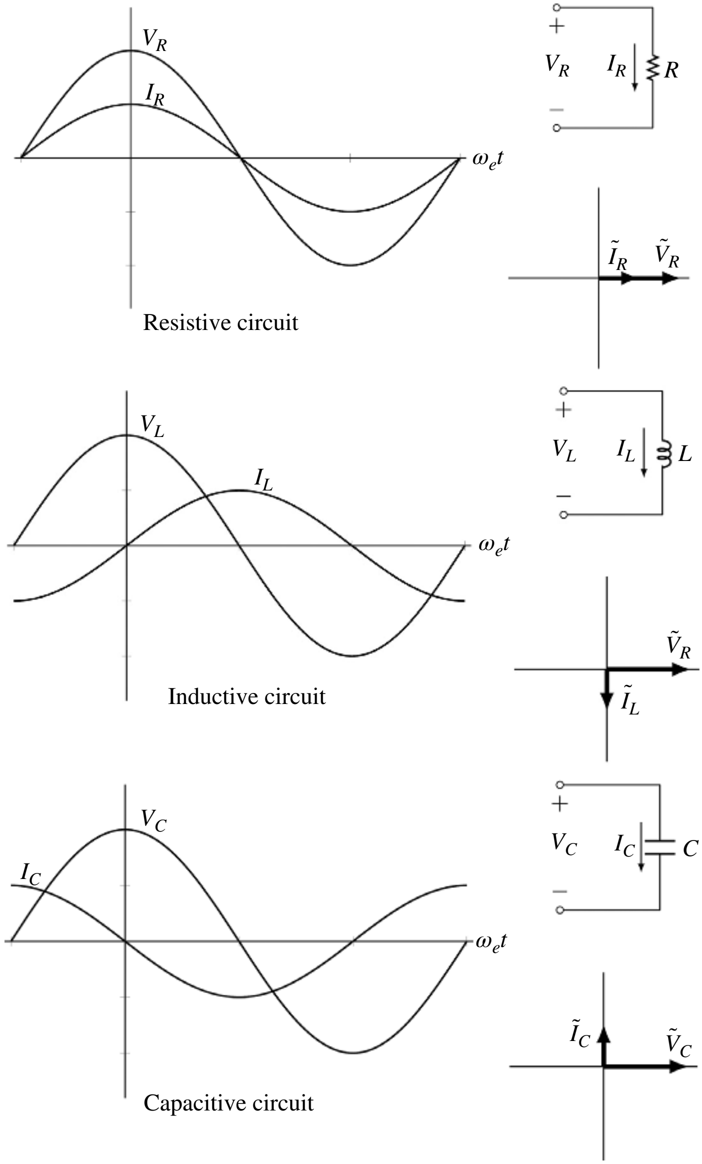

Figure 1.2-1 Waveforms of steady-state variables in resistive (R), inductive (L), and capacitive (C) circuits.

$$\frac{dI_L}{dt} = \omega_e \sqrt{2} I \cos\left[\omega_e t + \theta_{esi}(0) + \frac{1}{2}\pi\right] \tag{1.2-22}$$

Now

$$\begin{aligned}\tilde{V}_L &= \omega_e L\, e^{j\frac{\pi}{2}}\, \tilde{I}_L \\ &= j\omega_e L\, \tilde{I}_L\end{aligned} \tag{1.2-23}$$

with $\omega_e L = X_L$, which is referred to as the inductive reactance. The phasor form of (1.2-23) is

$$\tilde{V}_L = jX_L\, \tilde{I}_L \tag{1.2-24}$$

Thus, the voltage across the inductor leads the current through it by $\pi/2$. That is, the current through the inductor lags the voltage across it by $\pi/2$ $\left[\theta_{esv}(0) = \theta_{esi}(0) + \frac{1}{2}\pi\right]$. This is shown in Fig. 1.2-1.

For the capacitor

$$\begin{aligned}V_C &= \frac{1}{C}\int I_C\, dt \\ &= \frac{1}{\omega_e C}\sqrt{2} I \cos\left[\omega_e t + \theta_{esi}(0) - \frac{1}{2}\pi\right]\end{aligned} \tag{1.2-25}$$

Following the procedure used for the inductor, the phasor voltage across it becomes

$$\tilde{V}_C = -jX_C \tilde{I}_C \tag{1.2-26}$$

where $X_C = 1/\omega_e C$, the capacitive reactance. The voltage across the capacitor lags the current through it by $\pi/2$ $\left[\theta_{esv}(0) = \theta_{esi}(0) - \frac{1}{2}\pi\right]$, or the current through the capacitor leads the voltage across it by $\pi/2$. This is also shown in Fig. 1.2-1.

A series *RLC* circuit is shown in Fig. 1.2-2. From Fig. 1.2-2,

$$\tilde{V}_s = Z\tilde{I}_s \tag{1.2-27}$$

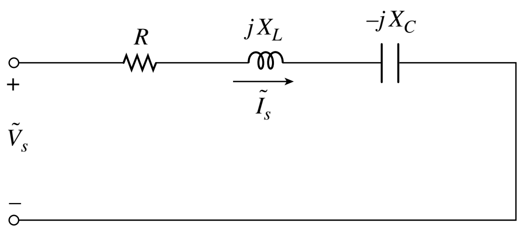

Figure 1.2-2 Phasor equivalent circuit for a series *RLC* circuit.

where $Z = R + j(X_L - X_C)$. We should be careful here. Some prefer to write (1.2-27) as $R + jX$ where X is $X_L + X_C$ and let X_C be negative. This is essentially a matter of choice and does not change the end result. We will deal primarily with X_L and not X_C, therefore, this will have little impact on our work; nevertheless, since some authors will use a negative X_C, we should make the reader aware of this difference.

It is appropriate to discuss the notation that will be used throughout the text. When an equation is written with the variables in lowercase letters, it is valid for transient and steady state. If the variables are written with uppercase letters, the equation is a function of time and valid for instantaneous steady-state conditions. Equation (1.2-27) is a phasor equation representing steady-state sinusoidal variables and are written in uppercase letters with an over tilde.

1.2.1 Power and Reactive Power

The instantaneous steady-state power is

$$\begin{aligned} P &= V_s I_s \\ &= \sqrt{2}V\cos\left[\omega_e t + \theta_{ev}(0)\right]\sqrt{2}I\cos\left[\omega_e t + \theta_{ei}(0)\right] \end{aligned} \tag{1.2-28}$$

where V and I are rms values. After some manipulation, we can write (1.2-28) as

$$P = VI\cos\left[\theta_{ev}(0) - \theta_{ei}(0)\right] + VI\cos\left[2\omega_e t + \theta_{ev}(0) + \theta_{ei}(0)\right] \tag{1.2-29}$$

The instantaneous steady-state power given by (1.2-29) varies about an average value at a frequency of $2\omega_e$. That is, the second term of (1.2-29) has a zero average value and the average power P_{ave} may be written as

$$P_{\text{ave}} = \left|\tilde{V}_s\right|\left|\tilde{I}_s\right|\cos\left[\theta_{ev}(0) - \theta_{ei}(0)\right] \tag{1.2-30}$$

where $\left|\tilde{V}_s\right|$ and $\left|\tilde{I}_s\right|$ are V and I, respectively, which are the magnitudes of the phasors (rms value), $\theta_{ev}(0) - \theta_{ei}(0)$ is referred to as the *power factor angle* ϕ_{pf}, and $\cos\left[\theta_{ev}(0) - \theta_{ei}(0)\right]$ is the *power factor*. Power is in watts. If current is assumed positive in the direction of voltage drop, then (1.2-30) is positive if power is consumed and negative if power is generated. It is interesting to point out that in going from (1.2-28) to (1.2-29), the coefficient of the two right-hand terms is $1/2\left(\sqrt{2}V\sqrt{2}I\right)$ or one half the product of the peak values of the sinusoidal variables. Therefore, it was considered more convenient to use the rms values for the phasors, whereupon average steady-state power could be calculated by the product of the magnitude of the voltage and current phasors as given by (1.2-30).

The reactive power is defined as

$$Q = \left|\tilde{V}_s\right|\left|\tilde{I}_s\right|\sin\left[\theta_{ev}(0) - \theta_{ei}(0)\right] \tag{1.2-31}$$

The units of Q are var (volt-ampere reactive). An inductance is said to absorb reactive power where the current lags the voltage by 90° and Q is positive. In the case of a capacitor, where the current leads the voltage by 90°, Q is supplied and is negative. Actually, Q is a measure of the interchange of energy supplied by the source that is stored in the electric (capacitor) and magnetic (inductor) fields. However, unlike instantaneous real power, the average value of instantaneous reactive power is zero.

We would like to minimize reactive power flow over the transmission lines in a power system. In other words, we would like to transmit only real power from the source to the load. The loads are generally inductive; therefore, capacitors are often placed in parallel with the load to interchange reactive power with the inductive load thus preventing the interchange current from flowing over the transmission line. This is often referred to as power factor correction since the transmission power factor approaches unity.

Example 1.A Phasor Analysis

The parameters of a series *RLC* circuit are $R = 6\ \Omega, L = 20$ mH, and $C = 1 \times 10^3\ \mu$F. The 60-Hz applied voltage is $V_s = 155.6 \cos \omega_e t$. Calculate $\tilde{I}_s$, P_{ave}, and Q and draw the phasor diagram. From the expression of V_s,

$$\tilde{V}_s = 110\underline{/0^\circ}\text{ V} \tag{1A-1}$$

Now, $\omega_e = 2\pi f = 2\pi \times 60 = 377$ rad/s and

$$\begin{aligned} Z &= R + j(X_L - X_C) \\ &= R + j\left(\omega_e L - \frac{1}{\omega_e C}\right) \\ &= 6 + j\left(377 \times 20 \times 10^{-3} - \frac{1}{377 \times 1 \times 10^{-3}}\right) = 7.73\underline{/39.1^\circ}\ \Omega \end{aligned} \tag{1A-2}$$

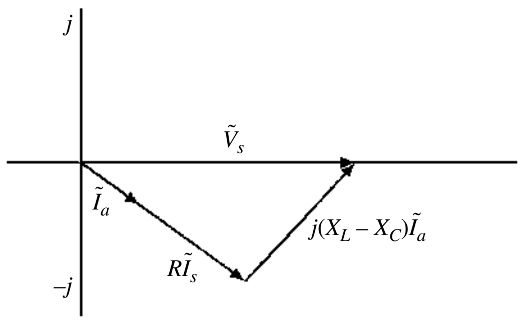

Figure 1.A-1 Phasor diagram.

$$\tilde{I}_s = \frac{\tilde{V}_s}{Z} = \frac{110\angle 0^\circ}{7.73\angle 39.1^\circ} = 14.2\angle -39.1^\circ \text{ A} \tag{1A-3}$$

$$P_{\text{ave}} = |\tilde{V}_s|\,|\tilde{I}_s| \cos\phi_{pf} \tag{1A-4}$$

where

$$\begin{aligned}\phi_{pf} &= \theta_{ev}(0) - \theta_{ei}(0)\\ &= 0 - (-39.1^\circ) = 39.1^\circ\end{aligned} \tag{1A-5}$$

$$\begin{aligned}P_{\text{ave}} &= 110 \times 14.2 \cos\ 39.1^\circ\\ &= 1212.2\text{ W}\end{aligned} \tag{1.A-6}$$

$$\begin{aligned}Q &= |\tilde{V}_s|\,|\tilde{I}_s| \sin\phi_{pf}\\ &= 110 \times 14.2 \sin 39.1^\circ = 985.1 \text{ vars}\end{aligned} \tag{1.A-7}$$

The phasor diagram is shown Fig. 1.A-1.

SP1.2-1. Express the instantaneous steady-state power for Example 1.A. [Substitute into (1.2-29)].

SP1.2-2. Redraw the phasor diagram shown in Fig. 1.A-1 showing $jX_L\tilde{I}_s$ and $-jX_C\tilde{I}_s$ as individual voltages. [Show $jX_L\tilde{I}_s$ and then from the terminus of $jX_L\tilde{I}_s$, show $-jX_C\tilde{I}_s$].

SP1.2-3. We know that $P_{\text{ave}} = |\tilde{I}_s|^2 R$, does $Q = |\tilde{I}_s|^2 X_L - |\tilde{I}_s|^2 X_C$? [Yes]

SP1.2-4. If $\tilde{V}_s = 1\angle 0^\circ$ V and $\tilde{I}_s = 1\angle 180^\circ$ A in the direction of the voltage drop, calculate Z and P_{ave}. Is power generated or consumed? [$(-1 + j0)$ ohms, 1 watt, generated]

SP1.2-5. Express the instantaneous power for 60-Hz voltage, $\tilde{V}_s = 1\angle 0^\circ$, applied to a resistive circuit, $\tilde{I}_s = 1\angle 0^\circ$. [$1 + \cos 754t$]

SP1.2-6. Repeat SP1.2-5 for (a) an inductance, $\tilde{I}_s = I_L\angle -90^\circ$ and (b) a capacitance, $\tilde{I}_s = I_C\angle 90^\circ$. [(a) $I_L \cos(754t - 90^\circ)$, (b) $I_C \cos(754t + 90^\circ)$]

1.3 Stationary Magnetically Linear Systems

Before analyzing electromagnetic systems with motion, it is helpful to start with stationary electromagnetic systems. A stationary, single winding electromagnetic system is shown in Fig. 1.3-1. A coiled, conductive wire is referred to as a winding. Usually the wire is wound (coiled) around some structure called the core. Each loop of the winding around the core is referred to as a turn. The core is typically made up of ferromagnetic material to guide the magnetic flux created by current flowing in the wire. Magnetic flux prefers to travel through materials of high permeability, a property of ferromagnetic materials. Here, we use N to represent the number of turns of the winding.

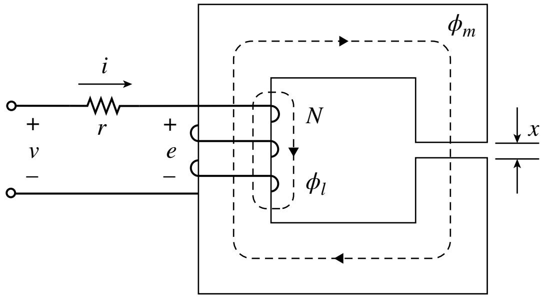

Figure 1.3-1 Single winding electromagnetic system.

In Fig. 1.3-1, ϕ_l is the leakage flux, which does not traverse the ferromagnetic core and ϕ_m is the magnetizing flux, which traverses the entire ferromagnetic core. Both ϕ_l and ϕ_m link N. The voltage equation is

$$v = ri + e \tag{1.3-1}$$

where $e = p\lambda$ (p is the operator d/dt). The resistor voltage term is due to Ohm's law. The induced voltage term due to the change of flux linkages is Faraday's law.

From Fig. 1.3-1, the flux linking the winding is

$$\phi = \phi_l + \phi_m \tag{1.3-2}$$

The magnetizing flux also travels across the slot in the core of width x. This slot is referred to as an air gap. In a structure such as this where the core is made of highly permeable material, the leakage flux, ϕ_l, is small and it generally makes up between 2 and 4% of the total flux linking the winding. Flux is said to link a winding if it travels through the turns of the winding. Both ϕ_l and ϕ_m link the winding. The total flux linked by the winding, called the flux linkage λ, is the flux through the winding multiplied by the number of turns of the winding

$$\begin{aligned} \lambda &= N\phi \\ &= N(\phi_l + \phi_m) \end{aligned} \tag{1.3-3}$$

Next, we define magnetic equivalent circuits. Magnetic equivalent circuits are a model for magnetic systems based on Maxwell's equations and ideas from electrical circuit models. In magnetic circuits, we think of flux as current, magnetomotive force (mmf) as voltage, and reluctance as resistance. Ohm's law for magnetic circuits becomes

$$\phi_l = \frac{Ni}{\mathcal{R}_l} \tag{1.3-4}$$

$$\phi_m = \frac{Ni}{\mathcal{R}_m} \tag{1.3-5}$$

where $\mathcal{R}$ is the reluctance and Ni is the mmf. The reluctance of the leakage path, $\mathcal{R}_l$, is large since a significant part of the path is in air. The reluctance of the ferromagnetic core and air gap may be calculated as

$$\begin{aligned} \mathcal{R}_m &= \mathcal{R}_i + \mathcal{R}_g \\ &= \frac{l_i}{\mu_r \mu_0 A_i} + \frac{x}{\mu_0 A_g} \end{aligned} \tag{1.3-6}$$

where $l_i(x)$ is the length of the iron path (gap) and $A_i(A_g)$ is the cross-sectional area of the core (gap), and μ_o is the permeability of free space (4π x 10^{-7} Wb/amp · m or H/m) and μ_r is the relative permeability. For air $\mu_r = 1$, for the ferromagnetic core, μ_r can be in the thousands, thus $\mathcal{R}_i < \mathcal{R}_g$. The equivalent magnetic circuit is shown in Fig. 1.3-2.

Substituting (1.3-4) and (1.3-5) into (1.3-3) yields

$$\begin{aligned} \lambda &= N\left(\frac{Ni}{\mathcal{R}_l} + \frac{Ni}{\mathcal{R}_m}\right) \\ &= \left(\frac{N^2}{\mathcal{R}_l} + \frac{N^2}{\mathcal{R}_m}\right) i \\ &= (L_l + L_m)\, i \end{aligned} \tag{1.3-7}$$

The self-inductance is $L_l + L_m$ where L_l and L_m are the leakage and magnetizing inductances, respectively. As seen above, inductance is defined as the relationship between current and flux linkage. The voltage equation given by (1.3-1) may now be written for a linear magnetic system as

$$v = ri + L\frac{di}{dt} \tag{1.3-8}$$

where

$$L = L_l + L_m \tag{1.3-9}$$

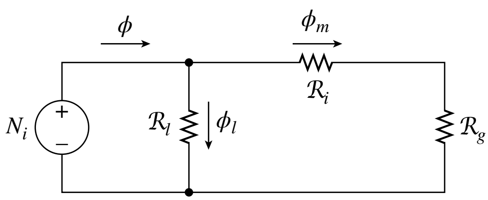

Figure 1.3-2 Magnetic equivalent circuit for the system shown in Fig. 1.3-1.

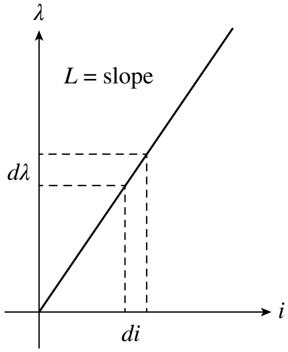

Figure 1.3-3 λi characteristic of a magnetically linear system.

Since we are assuming a linear magnetic system, core saturation and hysteresis are neglected and we have the λi plot shown in Fig. 1.3-3.

The magnetic equivalent circuit is similar to a resistive circuit. In particular, if we replace the magnetomotive force, mmf or Ni, with electromotive force, emf or voltage, and replace all reluctances, $\mathcal{R}$, with resistances R, then ϕ in Fig. 1.3-2 becomes the current i.

An important concept used in machine analysis is the idea of magnetic poles. The reader should have an intuition of north and south magnetic poles thanks to elementary physics classes and permanent magnets. Let us incorporate poles into our analysis of this stationary magnetically linear system. If fringing fields are neglected, the magnetizing flux, ϕ_m, travels uniformly across the air gap of Fig. 1.3-1. We can define magnetic north and south poles as the following: a north pole is a source of magnetic flux and a south pole is a sink for magnetic flux. To help determine pole assignment, place yourself on the member with the winding and where the positive flux enters the air gap is a north pole and where the positive flux enters the iron core is a south pole as shown in Fig. 1.3-4.

The concepts introduced in this chapter will be used throughout this text to analyze energy conversion systems. It is necessary to define quantities related to energy. The total energy stored in the field, W_f, may be expressed as

$$\begin{aligned} W_f &= \int eidt \\ &= \int id\lambda \end{aligned} \tag{1.3-10}$$

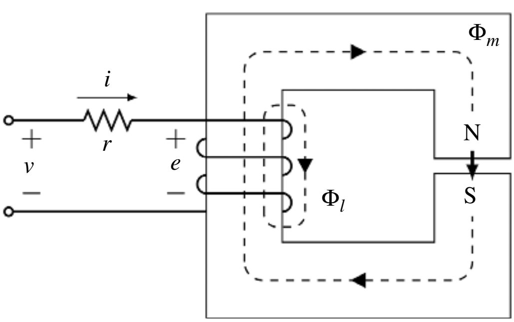

Figure 1.3-4 Repeat of Fig. 1.3-1 indicating north and south poles.

which is to the left of the plot shown in Fig. 1.3-3 for a given value of *i*. The area to the right of the plot is called the coenergy, W_c. It is expressed as

$$W_c = \int \lambda di \tag{1.3-11}$$

Coenergy is a quantity we define for analytical purposes. It is calculated from physical quantities, but has no direct physical meaning. Coenergy is convenient to formulate some expressions, for example, we can write from Fig. 1.3-3,

$$\lambda i = W_f + W_c \tag{1.3-12}$$

It should be clear that only for a magnetically linear system $W_f = W_c$.

1.3.1 Two-Winding Transformer

A two-winding transformer is shown in Fig.1.3-5. Here, we have mutual coupling between the two windings which we will take care of in a minute. Magnetic coupling refers to the situation where current through one winding creates flux which contributes to the flux linkage of another winding. Magnetic coupling is an essential aspect of transformers and electric machines. These devices may contain multiple windings that may be magnetically coupled. We analyze the two winding cases first. As always, we start with the winding voltage equations. The voltage equations are [1]

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} r_1 & 0 \\ 0 & r_2 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} + p \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} \tag{1.3-13}$$

where the first term on the right-hand side comes from Ohm's law and the second from Faraday's law.

In Fig. 1.3-5, r_1 and r_2 are resistances of the windings and $e_1 = p\lambda_1$ and $e_2 = p\lambda_2$ where $p = \frac{d}{dt}$.

The flux linkages λ_1 and λ_2 may be expressed as

$$\begin{aligned} \lambda_1 &= N_1 \phi_1 \\ &= N_1(\phi_{l1} + \phi_{m1} + \phi_{m2}) \\ &= \frac{N_1^2}{\mathcal{R}_{l1}} i_1 + \frac{N_1^2}{\mathcal{R}_m} i_1 + \frac{N_1 N_2}{\mathcal{R}_m} i_2 \end{aligned} \tag{1.3-14}$$

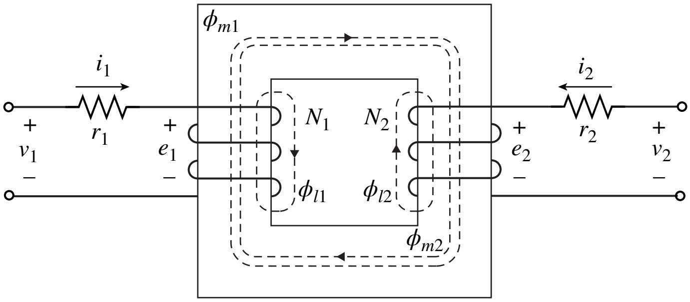

Figure 1.3-5 Two-winding transformer.

and

$$\begin{aligned}\lambda_2 &= N_2\phi_2 \\ &= N_2(\phi_{l2} + \phi_{m2} + \phi_{m1}) \\ &= \frac{N_2^2}{\mathcal{R}_{l2}} i_2 + \frac{N_2^2}{\mathcal{R}_m} i_2 + \frac{N_2N_1}{\mathcal{R}_m} i_1\end{aligned} \tag{1.3-15}$$

where $\phi_1(\phi_2)$ is the flux that flows through the winding with $N_1(N_2)$ turns. The self-inductance of the windings comes from the first two terms of (1.3-14) and (1.3-15). It would exist even if the other coil were not present. That is,

$$L_{11} = \frac{N_1^2}{\mathcal{R}_{l1}} + \frac{N_1^2}{\mathcal{R}_m} = L_{l1} + L_{m1} \tag{1.3-16}$$

For winding 2,

$$L_{22} = \frac{N_2^2}{\mathcal{R}_{l2}} + \frac{N_2^2}{\mathcal{R}_m} = L_{l2} + L_{m2} \tag{1.3-17}$$

It is clear that the self-inductances are independent of other windings. The coefficient of the last term of (1.3-14) and (1.3-15) is called the mutual inductance, that is,

$$L_{12} = L_{21} = \frac{N_1N_2}{\mathcal{R}_m} \tag{1.3-18}$$

Therefore,

$$\begin{bmatrix}\lambda_1 \\ \lambda_2\end{bmatrix} = \begin{bmatrix}L_{11} & L_{12} \\ L_{21} & L_{22}\end{bmatrix}\begin{bmatrix}i_1 \\ i_2\end{bmatrix} \tag{1.3-19}$$

The mutual inductance which is new to most of us can be positive or negative depending on the relative direction of ϕ_{m1} and ϕ_{m2}. In this case it is positive; if, however, the sense of winding 2 or the current i_2 is reversed, the mutual inductance would be negative.

The inductances L_{m1}, L_{m2}, L_{12}, and L_{21} have a common term $\mathcal{R}_m$. This allows us to write the flux linkages in terms of L_{m1} or L_{m2} and a turns ratio. We do this to create an electric circuit model of the two-winding transformer. If we write λ_1 and λ_2 in terms of $L_{m1}(L_{m2})$, we are referring voltages, currents, and flux linkages to winding 1 (winding 2). Referring to winding 1, λ_1 becomes

$$\lambda_1 = L_{l1}i_1 + L_{m1}\left(i_1 + \frac{N_2}{N_1}i_2\right) \tag{1.3-20}$$

Substituting

$$i_2' = \frac{N_2}{N_1}i_2 \tag{1.3-21}$$

we see that i_2' flowing in N_1 produces the same mmf as i_2 flowing in N_2. Now, in order to make $v_2'i_2' = v_2i_2$,

$$v_2' = \frac{N_1}{N_2}v_2 \tag{1.3-22}$$

Now since λ_2 is in volt·sec, λ_2' becomes

$$\lambda_2' = \frac{N_1}{N_2}\lambda_2 \tag{1.3-23}$$

and λ_1 and λ_2' become

$$\lambda_1 = L_{l1}i_1 + L_{m1}\left(i_1 + i_2'\right) \tag{1.3-24}$$

$$\lambda_2' = L_{l2}'i_2' + L_{m1}\left(i_1 + i_2'\right) \tag{1.3-25}$$

The voltage equations for all variables referred to winding 1 are

$$\begin{bmatrix} v_1 \\ v_2' \end{bmatrix} = \begin{bmatrix} r_1 & 0 \\ 0 & r_2' \end{bmatrix}\begin{bmatrix} i_1 \\ i_2' \end{bmatrix} + p\begin{bmatrix} \lambda_1 \\ \lambda_2' \end{bmatrix} \tag{1.3-26}$$

where

$$L_{l2}' = \left(\frac{N_1}{N_2}\right)^2 L_{l2} \tag{1.3-27}$$

$$r_2' = \left(\frac{N_1}{N_2}\right)^2 r_2 \tag{1.3-28}$$

These equations suggest the equivalent circuit given in Fig. 1.3-6.

With two windings, the total energy stored in the fields becomes

$$W_f = \int (e_1i_1 + e_2i_2)\,dt \tag{1.3-29}$$

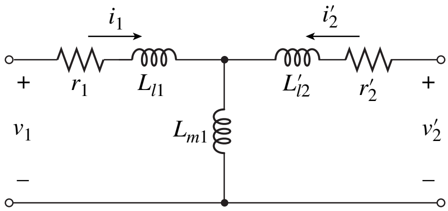

Figure 1.3-6 Transformer equivalent T circuit with winding 1 selected as reference winding.

or in terms of referred variables

$$\begin{aligned} W_f &= \int \left(e_1 i_1 + e_2' i_2'\right) dt \\ &= \int \left(i_1 d\lambda_i + i_2' d\lambda_2'\right) \end{aligned} \tag{1.3-30}$$

where the 2 variables are referred to N_1.
Now,

$$d\lambda_1 = L_{11} di_1 + L_{12}' di_2' \tag{1.3-31}$$

$$d\lambda_2' = L_{12}' di_1 + L_{22}' di_2' \tag{1.3-32}$$

where

$$L_{12}' = \frac{N_1}{N_2} L_{12} \tag{1.3-33}$$

$$L_{22}' = L_{l2}' + \left(\frac{N_1}{N_2}\right)^2 L_{m2} \tag{1.3-34}$$

we can evaluate (1.3-30) in two steps; first we will hold i_2' at zero, thus $di_2' = 0$, and allow i_1 to go from zero to i_1. Thus,

$$W_{f(1)} = \int_0^{i_1} L_{11} \xi d\xi = \frac{1}{2} L_{11} i_1^2 \tag{1.3-35}$$

where ξ is the dummy variable of integration. For the second step, we will hold i_1 at i_1 with $di_1 = 0$, and allow i_2' to go from zero to i_2'. Thus,

$$W_{f(2)} = \int_0^{i_2'} \left(L_{12}' i_1 d\xi + L_{22}' \xi d\xi\right) \tag{1.3-36}$$

The stored energy in the fields is

$$\begin{aligned} W_f &= W_{f(1)} + W_{f(2)} \\ &= \frac{1}{2} L_{11} i_1^2 + L_{12}' i_1 i_2' + \frac{1}{2} L_{22}' i_2'^2 \end{aligned} \tag{1.3-37}$$

It is clear that (1.3-37) includes the energy stored in the leakage inductances, which do not couple other fields also (1.3-37) is valid with or without the primes.

For multi-winding systems, (1.3-12) becomes

$$\sum_{j=1}^{J} i_j \lambda_j = W_f + W_c \tag{1.3-38}$$

Example 1.B Parameters of the Transformer Equivalent Circuit

It is instructive to illustrate the method of deriving an equivalent T circuit from open- and short-circuit measurements of the transformer. When winding 2 of the two-winding transformer shown in Fig 1.3-6 is open circuited and a 60-Hz voltage of 110 V (rms) is applied to winding 1, the average power supplied to winding 1 is 6.66 W. The measured current in winding 1 is 1.05 A (rms). Next, with winding 2 short-circuited, the current flowing in winding 1 is 2 A when the applied 60-Hz voltage is 30 V (rms). The average input power is 44 W. If we assume $L_{l1} = L'_{l2}$, an approximate equivalent T circuit can be determined from these measurements with winding 1 selected as the reference winding.

The average power supplied to winding 1 may be expressed from (1.2-30) as

$$P_1 = \left|\tilde{V}_1\right| \left|\tilde{I}_1\right| \cos \varphi_{pf} \tag{1B-1}$$

where

$$\phi_{pf} = \theta_{ev}(0) - \theta_{ei}(0) \tag{1B-2}$$

Here, $\tilde{V}_1$ and $\tilde{I}_1$ are phasors with the positive direction of $\tilde{I}_1$ taken in the direction of voltage drop, and $\theta_{ev}(0)$ and $\theta_{ei}(0)$ are the phase angles of $\tilde{V}_1$ and $\tilde{I}_1$, respectively. Solving for ϕ_{pf} during the open-circuit test, we have

$$\phi_{pf} = \cos^{-1} \frac{P_1}{\left|\tilde{V}_1\right| \left|\tilde{I}_1\right|} = \cos^{-1} \frac{6.66}{(110)(1.05)} = 86.7^\circ \tag{1B-3}$$

Although $\phi_{pf} = -86.7^\circ$ is also a legitimate solution of (1B-3), the positive value is taken since $\tilde{V}_1$ leads $\tilde{I}_1$ in an inductive circuit. With winding 2 open-circuited, the input impedance of winding 1 is

$$Z = \frac{\tilde{V}_1}{\tilde{I}_1} = r_1 + j(X_{l1} + X_{m1}) \tag{1B-4}$$

With $\tilde{V}_1$ as the reference phasor, $\tilde{V}_1 = 110\underline{/0^\circ}$, $\tilde{I}_1 = 1.05\underline{/-86.7^\circ}$. Thus,

$$r_1 + j(X_{l1} + X_{m1}) = \frac{110\underline{/0^\circ}}{1.05\underline{/-86.7^\circ}} = 6 + j104.6\,\Omega \tag{1B-5}$$

From (1B-5), $r_1 = 6\,\Omega$. We also see from (1B-5) that $X_{l1} + X_{m1} = 104.6\,\Omega$.

For the short-circuit test, we will assume that $\tilde{I}_1 = -\tilde{I}_2'$ since transformers are designed so that at rated frequency $X_{m1} \gg \left|r_2' + jX_{l2}'\right|$. Hence, using (1B-1) again,

$$\phi_{pf} = \cos^{-1}\frac{44}{(30)(2)} = 42.8^\circ \tag{1B-6}$$

In this case, the input impedance is $Z = \left(r_1 + r_2'\right) + j\left(X_{l1} + X_{l2}'\right)$. This may be determined as

$$Z = \frac{30\underline{/0^\circ}}{2\underline{/-42.8^\circ}} = 11 + j10.2\,\Omega \tag{1B-7}$$

Hence, $r_2' = 11 - r_1 = 5\,\Omega$ and, since it is assumed that $X_{l1} = X_{l2}'$, both are $10.2/2 = 5.1\ \Omega$. Therefore, $X_{m1} = 104.6 - 5.1 = 99.5\,\Omega$. In summary, $r_1 = 6\,\Omega$, $L_{l1} = 13.5$ mH, $L_{m1} = 263.9$ mH, $r_2' = 5\,\Omega$, and $L_{l2}' = 13.5$ mH. It is left to the reader to verify the conversion from X's to L's.

SP1.3-1. Show that the total field energy if a third winding is added to Fig. 1.3-5 is

$$W_f = \frac{1}{2}L_{11}i_1^2 + \frac{1}{2}L_{22}i_2^2 + \frac{1}{2}L_{33}i_3^2 + L_{12}i_1i_2 + L_{13}i_1i_3 + L_{23}i_2i_3$$

SP1.3-2. Draw the equivalent circuit for a three-winding transformer with all variables referred to winding 1.

SP1.3-3. Consider the transformer and parameters calculated in Example 1.B. Winding 2 is short-circuited and 12 V (dc) is applied to winding 1. Calculate the steady-state values of i_1 and i_2. Repeat with winding 2 open-circuited. [$I_1 = 2$ A and $I_2 = 0$ in both cases]

1.4 Winding Configurations

The previous sections analyzed basic electromagnetic structures. Now, we will introduce more complicated geometries useful for the construction of electric machines. Electric machines are configurations of windings and ferromagnetic material that create and guide magnetic fields. The magnetic fields interact to create forces that turn the rotor.

The stator windings are shown in Fig. 1.4-1. We can see this multiphase stator is very involved. In this section, we are going to consider the stator windings as simply as possible.

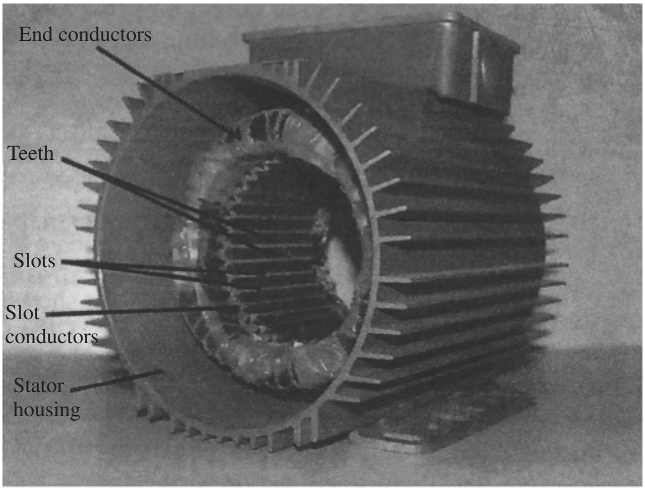

Figure 1.4-1 Stator windings of a multiphase machine.

To form the stator of an electric machine, conductive wire is wound in the slots of a steel structure. The number of turns or coils of the stator windings of most ac machines are distributed to approximate a space sinusoid as shown in Fig. 1.4-2. In Section 1.2, we used the "s" subscript to denote sinusoidal variables. In Fig. 1.4-2, we use the "s" subscript to denote stator or stationary. We will use this definition of s as a subscript or superscript for the remainder of the text. Also in Fig. 1.4-2, the "as" subscript denotes the variables associated with the a-phase of the stator. In some machines, great pains are taken to obtain a sinusoidal distribution of the stator windings to meet harmonic specifications. We attempt to distribute windings sinusoidally because sinusoidal currents through sinusoidally distributed windings create a constant amplitude rotating air-gap mmf. We use the terms rotating air-gap mmf and rotating magnetic field interchangeably. We will talk about the rotating air-gap mmf in detail in Chapter 2. With a constant amplitude rotating air-gap mmf, a constant power or torque is produced.

In Fig. 1.4-2, each winding segment $as_1 - as_1'$, $as_2 - as_2'$, $as_3 - as_3'$, and $as_4 - as_4'$ has nc_s coils for each $\otimes$ or $\odot$. Positive current is into the paper indicated by $\otimes$ and out of the paper at $\odot$. The current through the windings is alternating so the cross, $\otimes$, and $\odot$, will change; however, we are looking at an instant of time where positive current is in at as_1, as_2, as_3, and as_4.

If we follow the path of assumed positive current i_{as} flowing in the as winding, we see that current enters as_1, depicted by $\otimes$, to indicate that the assumed direction of positive current is down the length of the stator in an axial direction (into the paper). Current flows down the length of the stator, loops at the end, and flows

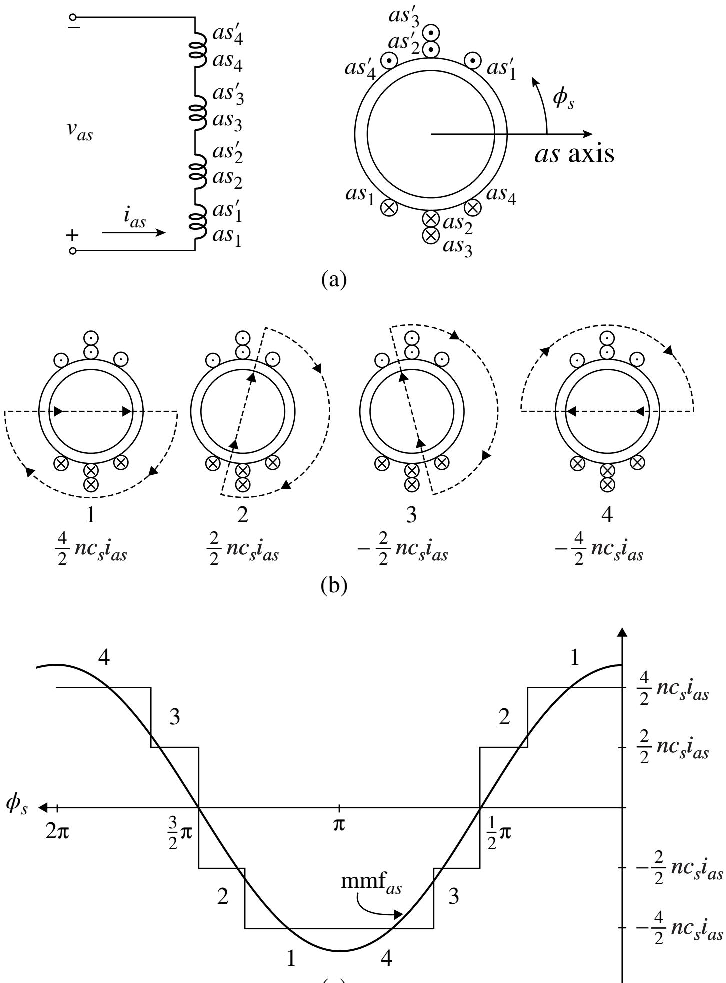

Figure 1.4-2 Elementary sinusoidally distributed windings. (a) Winding connections and distribution, (b) Ampere's law, and (c) mmf_{as}.

back down the length of the stator and out at as'_1, depicted by $\odot$. Note that as_1 and as'_1 are placed in stator slots that span π radians. This is referred to as the *winding pitch* of π radians which is characteristic of a two-pole machine. Now, as_1 around to as'_1 is referred to as a *coil* and as_1 or as'_1 is a *coil side*. In practice, a coil will contain more than one conductor. Current flows into as_1 in a conductor and out of as'_1 via the same conductor. The conductor, which is insulated, may then be looped back to as_1 and the winding of the conductor around the $as_1 - as'_1$ path repeated, thereby forming a coil with numerous turns. The number of conductors in a coil side tells us the number of turns in the coil, which is denoted as nc_s.

Once we have wound nc_s turns in the $as_1 - as_1'$ coil, we will take the same conductor and repeat this winding process to form the $as_2 - as_2'$ coil. We will assume that the same number of turns (nc_s) make up the $as_2 - as_2'$ coil as the $as_1 - as_1'$ coil and, similarly, for $as_3 - as_3'$ and $as_4 - as_4'$. We could have wound a different number of turns in each coil but we will assume that this was not done. Once the winding is wound, we can use the right-hand rule to give a meaning to the *as* axis shown in Fig. 1.4-2a. It is, by definition, the principal direction of the magnetic flux established by the assumed positive current flowing in the *as* winding. It is said to indicate the assumed positive direction of the magnetic axis of the *as* winding of this elementary sinusoidally distributed winding. The positive direction of the *as* axis reverses when i_{as} reverses.

Before getting into the mmf due to the current flowing in the winding let us consider the self-inductance of the winding. The rotor is round and the magnetizing flux established by this winding must cross the air gap twice and for positive current as shown in Fig. 1.4-2, the positive *as* axis is to the right. Since the air gap is uniform, the self-inductance is constant independent of rotor position. Therefore, the self-inductance is of the same form as given by (1.3-9). The difference is the leakage inductance makes up 5–15% of the self-inductance and the reluctance to the flux is dominated by the air gaps.

Ampere's law is

$$\oint \overline{H}\cdot dL = i \tag{1.4-1}$$

which says that the closed line integral of the mmf drops equals that current enclosed. For the instant shown in Fig. 1.4-2

$$\text{mmf}(0) + \text{mmf}(\pi) = N_s i_{as} \tag{1.4-2}$$

where one half of the mmf is dropped at $\phi_s = 0$ and one half at $\phi_s = \pi$. Since there is no point source of mmf, we will assume that rotor to stator is positive. Since the reluctance of air is much larger than iron, we will neglect the mmf drop in the iron and assume that the air gap is uniform, thus,

$$\text{mmf}(0) = \frac{N_s}{2} i_{as} \tag{1.4-3}$$

$$\text{mmf}(\pi) = -\frac{N_s}{2} i_{as} \tag{1.4-4}$$

Also the path of integration in Fig. 1.4-2 is 180°. Regardless of the type of rotor, the air gap is the same every 180° for the two-pole device. Following this same procedure for paths 2 through 4, we obtain the stepped plot of mmf_{as} as shown in Fig. 1.4-2c. The fundamental component of this stepped mmf is

$$\text{mmf}_{as} = \frac{N_s}{2} i_{as} \cos \phi_s \tag{1.4-5}$$

where N_s is the amplitude of the fundamental component of the Fourier transform of the winding distribution. For the winding distribution given in Fig. 1.4-2, N_s is 2.37 nc_s. Note, also that the sinusoidal distributed winding is denoted with ⊗ and ⊙ placed at the maximum winding density.

The winding shown in Fig. 1.4-2 is an approximation of a sinusoidal distributed winding. In the case of a large generator, pains would be taken to distribute the winding much closer to a sinusoidal winding to minimize the voltage harmonics. Nevertheless, Ampere's law would be performed the same. Also, for multiphase machines, windings of two- or three-phase may exist in the same slot.

Example 1.C Air-Gap mmf for a Uniformly Distributed Winding
Consider the uniformly distributed winding arrangement shown in Fig. 1.C-1. Each coil has nc_s turns and the current in each turn is i_{as} with the positive direction as shown. Follow the procedure used in Fig. 1.4-2 to establish the air-gap mmf.

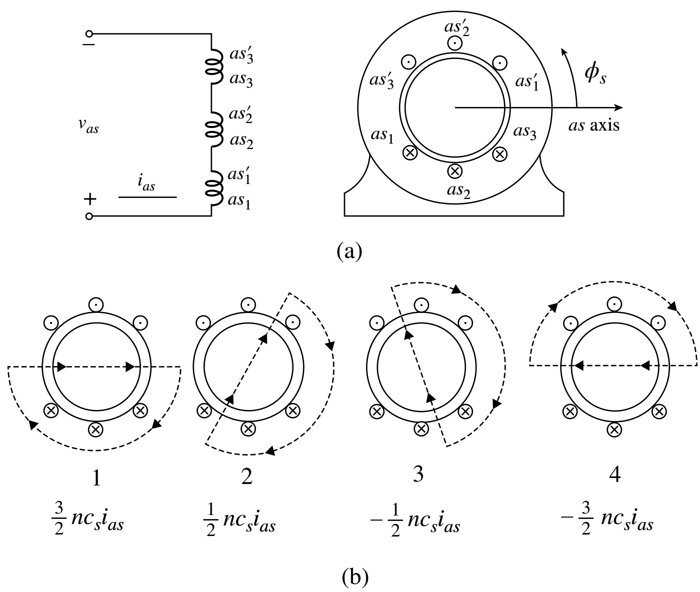

Figure 1.C-1 Elementary two-pole single-phase stator winding uniformly distributed. (a) Winding connections and distribution, (b) Ampere's law, and (c) mmf_{as}, here $N_s = 1.534\, nc_s$.

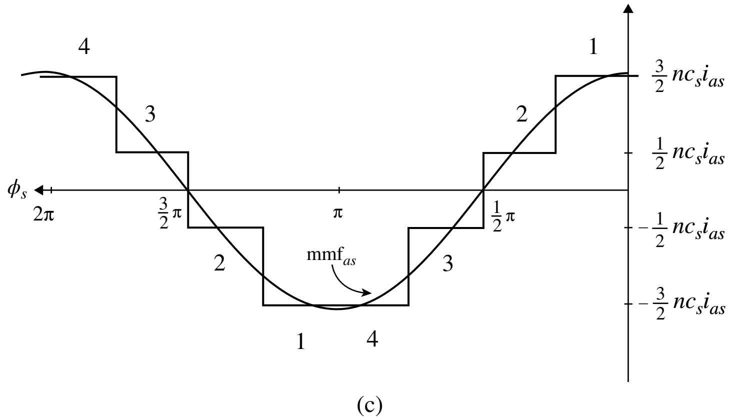

Figure 1.C-1 (Continued)

SP1.4-1. Express the sinusoidal approximation of mmf$_{as}$ if the assumed positive direction of i_{as} is reversed in Fig. 1.4-2. [mmf$_{as}$ = $-(1.4-5)$]

SP1.4-2. Assume that only the as_2 winding exists in Fig. 1.C-1a which has one turn. Sketch the air-gap mmf due to current i_1 flowing in this winding. [$\frac{1}{2}i_1$ for $-\frac{1}{2}\pi < \phi_s < \frac{1}{2}\pi$ and $-\frac{1}{2}i_1$ for $\frac{1}{2}\pi < \phi_s < \frac{2}{3}\pi$]

1.5 Two- and Three-Phase Stators

High-voltage transmission, most inverter-supplied electric drives, and the alternator of your car are examples of three-phase systems. Although two-phase systems are not common, a two-phase system is far less involved when it comes to machine analysis than its three-phase big sister. Fortunately, once the derivations have been set forth for a two-phase machine, the extension to a three-phase machine is straightforward and easily achieved. This section is devoted to the introduction of these multiphase systems.

By definition, a two-phase set of variables is balanced if the variables are equal-amplitude sinusoidal quantities in time quadrature (90° out of time phase). A three-phase set of variables is balanced if the sinusoidal variables are equal-amplitude quantities that are 120° out of time phase with each other.

1.5.1 Two-Phase Stator

In the broadest sense of the above definition, two-phase balanced sets may be expressed as

$$f_a(t) = \pm f\cos\theta_{ef} \tag{1.5-1}$$

$$f_b(t) = \pm f\sin\theta_{ef} \tag{1.5-2}$$

where

$$\theta_{ef}(t) = \int_0^t \omega_e(\xi)d\xi + \theta_{ef}(0) \tag{1.5-3}$$

In (1.5-1) and (1.5-2), $f_a(t)$ is the a-phase and $f_b(t)$ is the b-phase of voltage, current, or flux linkage. The amplitude f is assumed to be constant. In (1.5-3), ω_e is the electrical angular velocity and ξ is a dummy variable of integration. Equations (1.5-1) and (1.5-2) express four balanced two-phase sets. Like signs of (1.5-1) and (1.5-2) define balanced sets where $f_a(t)$ leads $f_b(t)$ by 90°, an ab sequence; for unlike signs $f_a(t)$ lags $f_b(t)$ by 90°, a ba sequence.

For steady-state balanced conditions, ω_e is constant and (1.5-3) becomes

$$\theta_{ef}(t) = \omega_e t + \theta_{ef}(0) \tag{1.5-4}$$

Whereupon (1.5-1) and (1.5-2) are written as

$$F_a(t) = \pm\sqrt{2}F\cos\left[\omega_e t + \theta_{ef}(0)\right] \tag{1.5-5}$$

$$F_b(t) = \pm\sqrt{2}F\sin\left[\omega_e t + \theta_{ef}(0)\right] \tag{1.5-6}$$

For like signs of (1.5-5) and (1.5-6), $\tilde{F}_a = j\tilde{F}_b$; for unlike signs, $\tilde{F}_a = -j\tilde{F}_b$.

Most large horsepower electric machines are three-phase and smaller household machines are two-phase machines powered from a single-phase source like a common wall outlet. For single-phase machines, a capacitor is generally connected in series with one of the windings (see Chapter 7). It is helpful to take a brief look at the stator winding arrangement of the two-phase machine, shown in Fig. 1.5-1. The windings are assumed to be identical in parameters and distribution. The displacement around the stator is denoted ϕ_s. The two windings are displaced 90° degrees from each other. The voltage equations may be written as

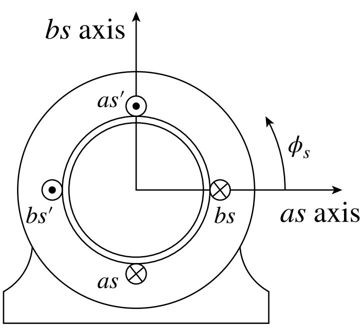

Figure 1.5-1 Elementary two-pole two-phase sinusoidally distributed stator windings.

$$v_{as} = r_s i_{as} + \frac{d\lambda_{as}}{dt} \tag{1.5-7}$$

$$v_{bs} = r_s i_{bs} + \frac{d\lambda_{bs}}{dt} \tag{1.5-8}$$

where the subscripts as and bs denote phase a of the stator and phase b of the stator, respectively. In matrix form

$$\mathbf{v}_{abs} = \mathbf{r}_s\mathbf{i}_{abs} + p\boldsymbol{\lambda}_{abs} \tag{1.5-9}$$

The stator has identical, sinusoidally distributed windings and the air gap is uniform.

The magnetic axes are orthogonal (thus $L_{asbs} = 0$) and the flux linkage equations may be written as

$$\begin{aligned}\lambda_{as} &= L_{asas} i_{as} \\ &= (L_{ls} + L_{ms}) i_{as}\end{aligned} \tag{1.5-10}$$

$$\begin{aligned}\lambda_{bs} &= L_{bsbs} i_{bs} \\ &= (L_{ls} + L_{ms}) i_{bs}\end{aligned} \tag{1.5-11}$$

where L_{ls} is the leakage inductance and L_{ms} is the magnetizing inductance of the stator windings. In matrix form

$$\begin{bmatrix} \lambda_{as} \\ \lambda_{bs} \end{bmatrix} = \begin{bmatrix} L_{ss} & 0 \\ 0 & L_{ss} \end{bmatrix} \begin{bmatrix} i_{as} \\ i_{bs} \end{bmatrix} \tag{1.5-12}$$

or

$$\boldsymbol{\lambda}_{abs} = \mathbf{L}_s \mathbf{i}_{abs} \tag{1.5-13}$$

where

$$\mathbf{L}_s = \begin{bmatrix} L_{ss} & 0 \\ 0 & L_{ss} \end{bmatrix} \tag{1.5-14}$$

and

$$L_{ss} = L_{ls} + L_{ms} \tag{1.5-15}$$

An important feature of multiphase systems is that the instantaneous power is constant for balanced operation. You are asked to show this in SP1.5-2. Recall that in a single-phase system the instantaneous power has an average value and a double frequency component.

1.5.2 Three-Phase Stator

The three-phase stator is shown in Fig. 1.5-2. A three-phase balanced set may be expressed as

$$f_a(t) = f \cos \theta_{ef} \tag{1.5-16}$$

$$f_b(t) = f \cos\left(\theta_{ef} - \frac{2}{3}\pi\right) \tag{1.5-17}$$

$$f_c(t) = f \cos\left(\theta_{ef} + \frac{2}{3}\pi\right) \tag{1.5-18}$$

where θ_{ef} is given by (1.5-3). This set is referred to as an *abc* sequence, since $f_a(t)$ leads $f_b(t)$ by 120° and $f_b(t)$ leads $f_c(t)$ by 120°. An *acb* sequence is obtained by interchanging $f_b(t)$ and $f_c(t)$, that is,

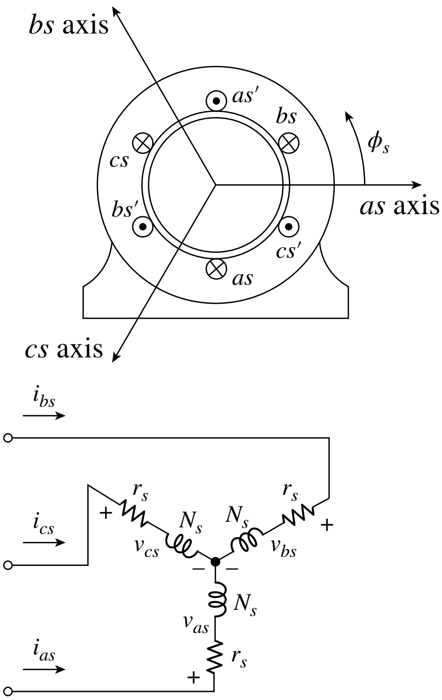

Figure 1.5-2 Elementary two-pole three-phase sinusoidally distributed stator windings.

$$f_a(t) = f \cos \theta_{ef} \tag{1.5-19}$$

$$f_b(t) = f \cos\left(\theta_{ef} + \frac{2}{3}\pi\right) \tag{1.5-20}$$

$$f_c(t) = f \cos\left(\theta_{ef} - \frac{2}{3}\pi\right) \tag{1.5-21}$$

For steady-state balanced conditions, the *abc* sequence may be written as

$$F_a(t) = \sqrt{2}F \cos\left[\omega_e t + \theta_{ef}(0)\right] \tag{1.5-22}$$

$$F_b(t) = \sqrt{2}F \cos\left[\omega_e t - \frac{2}{3}\pi + \theta_{ef}(0)\right] \tag{1.5-23}$$

$$F_c(t) = \sqrt{2}F \cos\left[\omega_e t + \frac{2}{3}\pi + \theta_{ef}(0)\right] \tag{1.5-24}$$

with $\tilde{F}_a = F\underline{/\theta_{ef}(0)}$, $\tilde{F}_b = F\underline{/\theta_{ef}(0) - \frac{2}{3}\pi}$, and $\tilde{F}_c = F\underline{/\theta_{ef}(0) + \frac{2}{3}\pi}$. For an *acb* sequence, $\tilde{F}_b$ and $\tilde{F}_c$ are interchanged.

A three-phase stator is shown in Fig. 1.5-2. Again, the stator has identical, sinusoidally distributed windings and the air gap is uniform. The magnetic axes of the windings are displaced 120° and the windings are often "wye-connected" as shown. The line-to-neutral voltage equations may be written as

$$v_{as} = r_s i_{as} + \frac{d\lambda_{as}}{dt} \tag{1.5-25}$$

$$v_{bs} = r_s i_{bs} + \frac{d\lambda_{bs}}{dt} \tag{1.5-26}$$

$$v_{cs} = r_s i_{cs} + \frac{d\lambda_{cs}}{dt} \tag{1.5-27}$$

where subscripts *as*, *bs*, and *cs* denote the three phases of the stator. In matrix form

$$\mathbf{v}_{abcs} = \mathbf{r}_s \mathbf{i}_{abcs} + p\boldsymbol{\lambda}_{abcs} \tag{1.5-28}$$

Since the windings are displaced 120° from each other, there is a mutual coupling between the stator windings. Let us assume that we can move the *bs* winding clockwise through the iron until it is "on top" of the *as* winding at $\phi_s = 0$. The coupling would be maximum positive. Now, assume we can rotate the *bs* winding counterclockwise back to $\phi_s = 120°$ where the mutual inductance between the *as* and *bs* windings can be approximated as

$$\begin{aligned} L_{asbs} &= L_{ms}\cos 120° \\ &= -\frac{1}{2}L_{ms} \end{aligned} \tag{1.5-29}$$

where L_{ms} is the magnetizing inductance of the stator windings. Following this same approach, we can express the flux-linkage matrix as

$$\begin{bmatrix} \lambda_{as} \\ \lambda_{bs} \\ \lambda_{cs} \end{bmatrix} = \begin{bmatrix} L_{ss} & -\frac{1}{2}L_{ms} & -\frac{1}{2}L_{ms} \\ -\frac{1}{2}L_{ms} & L_{ss} & -\frac{1}{2}L_{ms} \\ -\frac{1}{2}L_{ms} & -\frac{1}{2}L_{ms} & L_{ss} \end{bmatrix} \begin{bmatrix} i_{as} \\ i_{bs} \\ i_{cs} \end{bmatrix} \tag{1.5-30}$$

where

$$L_{ss} = L_{ls} + L_{ms} \tag{1.5-31}$$

Equation (1.5-30) may also be written as

$$\boldsymbol{\lambda}_{abcs} = \mathbf{L}_s \mathbf{i}_{abcs} \tag{1.5-32}$$

1.5.3 Line-to-Line Voltage

In the case of a three-phase stator as shown in Fig. 1.5-2, the voltage rating is generally given as line-to-line voltage. For example, $\tilde{V}_{ab}$ is

$$\tilde{V}_{ab} = \tilde{V}_{as} - \tilde{V}_{bs} \tag{1.5-33}$$

For an *abc* sequence

$$\begin{aligned}\tilde{V}_{ab} &= V_s\angle 0^\circ - V_s\angle -120^\circ \\ &= V_s(1 + j0) - V_s(-0.5 - j0.866) \\ &= \sqrt{3}V_s\angle 30^\circ\end{aligned} \tag{1.5-34}$$

$$\begin{aligned}\tilde{V}_{bc} &= V_s\angle -120^\circ - V_s\angle 120^\circ \\ &= V_s(-0.5 - j0.866) - V_s(-0.5 + j0.866) \\ &= \sqrt{3}V_s\angle -90^\circ\end{aligned} \tag{1.5-35}$$

$$\begin{aligned}\tilde{V}_{ca} &= V_s\angle 120^\circ - V_s\angle 0^\circ \\ &= V_s(-0.5 + j0.866) - V_s(1 + j0) \\ &= \sqrt{3}V_s\angle 150^\circ\end{aligned} \tag{1.5-36}$$

The magnitude of the line-to-line voltages is $\sqrt{3}$ times the phase voltages and shifted 30° ccw.

Example 1.D Voltage Equations for a Three-Wire System

A three-phase stator similar to that given in Fig. 1.5-2 is connected to a three-phase source as shown in Fig. 1.D-1. Assume the stator is symmetrical, that is, the windings have the same resistance and same number of turns and displaced 120°. The stator configuration could be that of induction or synchronous machine. The

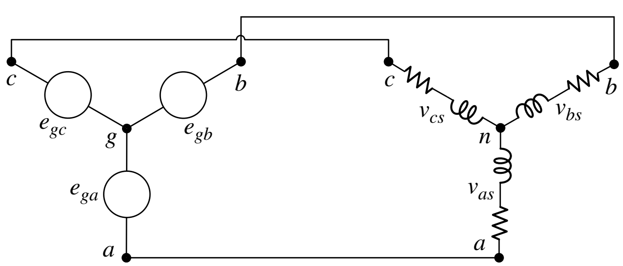

Figure 1.D-1 Three-phase source connected to symmetrical stator windings.

source voltages e_{ga}, e_{gb}, and e_{gc} may be of any form. Express v_{as}, v_{bs}, and v_{cs} in terms of e_{ga}, e_{gb}, and e_{gc}.

From Fig. 1.D-1, we can write

$$e_{ga} = v_{as} + v_{ng} \tag{1D-1}$$

$$e_{gb} = v_{bs} + v_{ng} \tag{1D-2}$$

$$e_{gc} = v_{cs} + v_{ng} \tag{1D-3}$$

Adding (1D-1) through (1D-3) yields

$$e_{ga} + e_{gb} + e_{gc} = v_{as} + v_{bs} + v_{cs} + 3v_{ng} \tag{1D-4}$$

Let us look at $v_{as} + v_{bs} + v_{cs}$. From (1.5-25) through (1.5-27)

$$v_{as} + v_{bs} + v_{cs} = r_s(i_{as} + i_{bs} + i_{cs}) + p(\lambda_{as} + \lambda_{bs} + \lambda_{cs}) \tag{1D-5}$$

In a three-wire, wye-connected stator, the sum of $i_{as} + i_{bs} + i_{cs}$ must be zero regardless of the form of the currents. Now from (1.5-30)

$$\lambda_{as} + \lambda_{bs} + \lambda_{cs} = L_{ss}(i_{as} + i_{bs} + i_{cs}) - L_{ms}(i_{as} + i_{bs} + i_{cs}) = 0 \tag{ID-6}$$

Thus,

$$v_{as} + v_{bs} + v_{cs} = 0 \tag{1D-7}$$

Will this be the case when we bring the rotor into play? We will find that for the electromechanical devices we will consider, it will be true. Substituting (1D-7) into (1D-4) yields

$$v_{ng} = \frac{1}{3}\left(e_{ga} + e_{gb} + e_{gc}\right) \tag{1D-8}$$

Going back to (1D-1) through (1D-3), we can write

$$\begin{aligned} v_{as} &= e_{ga} - v_{ng} \\ &= \frac{2}{3}e_{ga} - \frac{1}{3}\left(e_{gb} + e_{gc}\right) \end{aligned} \tag{1D-9}$$

$$\begin{aligned} v_{bs} &= e_{gb} - v_{ng} \\ &= \frac{2}{3}e_{gb} - \frac{1}{3}\left(e_{gc} + e_{ga}\right) \end{aligned} \tag{1D-10}$$

$$\begin{aligned} v_{cs} &= e_{gc} - v_{ng} \\ &= \frac{2}{3}e_{gc} - \frac{1}{3}\left(e_{ga} + e_{gb}\right) \end{aligned} \tag{1D-11}$$

We will make use of these equations when considering electric drives.

SP1.5-1. In Fig. 1.D-1, let $e_{ga} = 1$, $e_{gb} = 0$, and $e_{gc} = \cos\omega_e t$. Determine v_{as}, v_{bs}, and v_{cs}. $\left[\frac{2}{3} - \frac{1}{3}\cos\omega_e t;\ -\frac{1}{3} - \frac{1}{3}\cos\omega_e t;\ -\frac{1}{3} + \frac{2}{3}\cos\omega_e t\right]$. Note that $v_{as} + v_{bs} + v_{cs} = 0$.

SP1.5-2. In a two-phase system, let $V_a = \sqrt{2}V\cos[\omega_e t + \theta_{ev}(0)]$, $I_a = \sqrt{2}I\cos[\omega_e t + \theta_{ei}(0)]$, $V_b = \sqrt{2}V\sin[\omega_e t + \theta_{ev}(0)]$, and $I_b = \sqrt{2}I\sin[\omega_e t + \theta_{ei}(0)]$. Show that the total instantaneous power is $P = 2VI\cos[\theta_{ev}(0) - \theta_{ei}(0)]$.

SP1.5-3. Express the line-to-line voltages for an *acb* sequence. $[\tilde{V}_{ab} = \sqrt{3}V_s\underline{/-30^\circ}$, $\tilde{V}_{bc} = \sqrt{3}V_s\underline{/90^\circ}$, $\tilde{V}_{ca} = \sqrt{3}V_s\underline{/-150^\circ}]$

1.6 Problems

1 Derive (1.2-26).

2 Derive (1.3-25).

3 Consider Fig. 1.3-5. The negative terminal of winding 1 is connected to the positive terminal of winding 2 and 110 V (rms) is applied between the positive terminal of winding 1 to the negative terminal of winding 2. Express the input impedance.

4 During the open-circuit test performed in Example 1.B, the rms voltage across the open-circuit 2 winding was 34.8 V. Determine X_{m2} $(\omega_e L_{m2})$.

5 Show that (1.3-38) is true for the two-winding system given in Fig. 1.3-5.

6 Determine mmf for a winding distribution uniformly as shown in Fig. 1.6-1 where coils with nc_s turns are 30° apart.

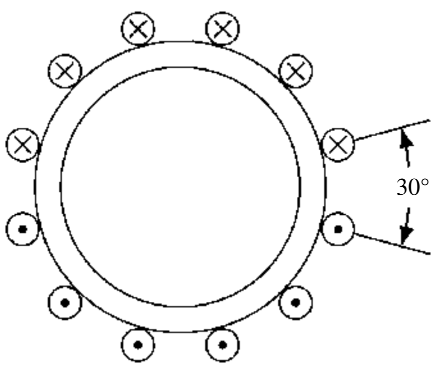

Figure 1.6-1 Uniform winding distribution.

7 Consider Example 1.D and Fig. 1.D-1. The load is symmetrical and the source voltages are given in Fig. 1.6-2. (a) Plot v_{ng}, v_{as}, v_{bs}, and v_{cs}. (b) Connect n to g in Fig. 1.D-1 and repeat part (a).

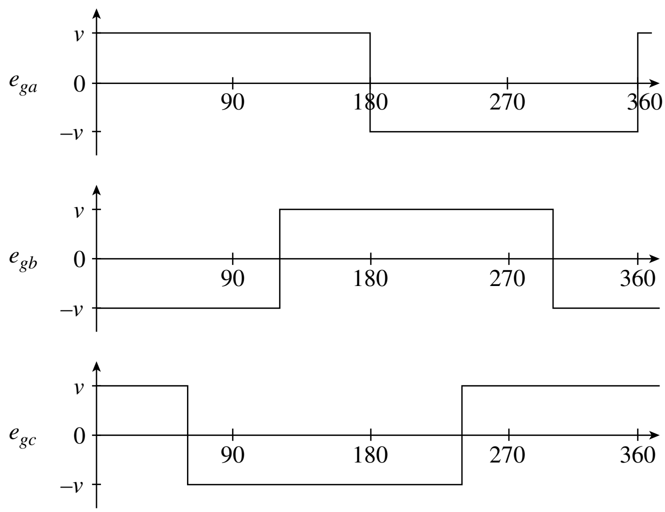

Figure 1.6-2 Waveforms of the source voltages of Fig. 1.D-1.

8 Assume that the direction of positive current is reversed in winding 2 of Fig. 1.3-5. Express (a) L_{12} in terms of N_1, N_2, and $\mathcal{R}_m$; (b) λ_1 and λ_2 in the form of (1.3-14) and (1.3-15); (c) λ_1 and λ_2' in the form of (1.3-24) and (1.3-25); and (d) v_1 and v_2' in the form of (1.3-26).

9 The parameters of a transformer are: $r_1 = r_2' = 10\,\Omega$, $L_{m1} = 300\,\text{mH}$, and $L_{l1} = L_{l2}' = 30\,\text{mH}$. A 10-V peak-to-peak 30-Hz sinusoidal voltage is applied to winding 1. Winding 2 is short-circuited. Assume $i_1 = -i_2'$. Calculate the phasor $\tilde{I}_1$ with $\tilde{V}_1$ at zero degrees.

10 A transformer with two windings has the following parameters: $r_1 = r_2 = 1\,\Omega$, $L_{m1} = 1$ H, $L_{l1} = L_{l2} = 0.01$ H, and $N_1 = N_2$. A $2 - \Omega$ load resistance R_L is connected across winding 2. $V_1 = 2\cos 400t$. (a) Calculate $\tilde{I}_1$. (b) Express I_1.

Reference

1 P. C. Krause, *Analysis of Electric Machinery*, McGraw-Hill Book Company, New York, 1986.