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Background

1.1 Nyquist's Theorem and Noise Temperature

1.1.1 Nyquist's Theorem

Conduction electrons in a physical resistor at nonzero temperature are in continual thermal motion, and at any instant this motion induces a voltage v across the terminals of the resistor. Because the motion is random, the induced voltage averages to zero, $\langle v(t)\rangle = 0$, but the average squared voltage is nonzero, $\langle v^2(t)\rangle \neq 0$, and therefore electrical power can be extracted from the resistor. This phenomenon was first measured by Johnson [1] and was explained by Nyquist [2]. The entire field of thermal noise measurement rests on Nyquist's theorem,[1] which relates the mean square voltage across a resistor due to thermal motion of its electrons to the physical temperature of the resistor,

$$\langle v^2(f)\rangle df = 4R(f)\frac{hdf}{\left(e^{hf/k_BT} - 1\right)} \tag{1.1}$$

where $v(f)$ is the voltage in the differential frequency interval df centered at frequency f, $R(f)$ is the real part of the impedance at f, h is Planck's constant, and k_B is Boltzmann's constant.

For microwave frequencies, it is convenient to cast Eq. (1.1) in the form of an equation for the power available from the resistor,

$$\langle P_{avail}(f)\rangle = \frac{hf}{e^{hf/(k_BT)} - 1}\Delta f \tag{1.2a}$$

$$\langle p_{avail}(f)\rangle = \frac{hf}{e^{hf/(k_BT)} - 1} \tag{1.2b}$$

where $P_{avail}(f)$ is the available power in the interval Δf centered at f, and $p_{avail}(f)$ is the spectral available power density. In general, we will use upper case P to refer to

1 This is one of two fundamental Nyquist theorems that underpin entire fields (or subfields) of study, the other being his (and Shannon's) sampling theorem.

Precision Measurement of Microwave Thermal Noise, First Edition. James Randa.

power and lower-case p to refer to spectral power density. The brackets in Eqs. (1.2) indicate an ensemble or time average (assumed to be the same).

Nyquist's original derivation relied on an analysis of propagation modes in a lossless transmission line. He first derived the classical result (those were the early days of quantum mechanics), which assumed that the total energy per degree of freedom was equal to $k_B T$. That led to

$$\langle v^2(f)\rangle df \approx 4R(f)k_B T\Delta f \tag{1.3a}$$

or

$$\langle P_{avail}(f)\rangle \approx k_B T\Delta f \tag{1.3b}$$

He then noted that if instead of $k_B T$ the total energy per degree of freedom was $hf/(e^{hf/kT}-1)$ then the result was Eq. (1.1), and thus Eqs. (1.2). We have taken the liberty of using the approximately equal sign in Eqs. (1.3) in order to indicate that the true result is given by Eqs. (1.1) and (1.2).

Nyquist's treatment and result are reminiscent of the problem of black-body radiation from a heated object, and the thermal noise in an electrical circuit is in fact the one-dimensional version of black-body radiation. As in the black-body radiation case, the classical result, given by Eqs. (1.3), is plagued by the "ultraviolet catastrophe," the fact that the total power available is infinite when one integrates over all frequencies. The quantum factors provide the necessary damping at high frequency keeping the total energy available finite. A modern, full-quantum (i.e. second-quantized) treatment of Nyquist's theorem can be found in [3]. An interesting property that emerges in the full quantum treatment is that an auxiliary field (the noise field) is actually *required* for a linear two-port in order for the quantum commutation relations to be consistent (unless the $\boldsymbol{S}$ matrix is the identity matrix or the temperature is zero).

1.1.2 Limits and Numbers

It is instructive to consider the general behavior of the function in Eqs. (1.2). Figure 1.1a,b plots the available power spectral density as a function of frequency on a logarithmic and a linear scale, respectively, for different values of the physical temperature. There is a broad plateau that extends up to high frequency, where the spectral power density drops off precipitously. The low-frequency behavior is given by expanding Eqs. (1.2) for small f,

$$\langle P_{avail}\rangle \approx k_B T\Delta f\left[1-\frac{hf}{(2k_B T)}\right] \approx k_B T\Delta f \tag{1.4}$$

which is a constant, independent of frequency, depending only on the physical temperature. Furthermore, that constant is very small; even for a temperature of 10 000 K, the power density in a 1 MHz bandwidth is only 0.138 pW.

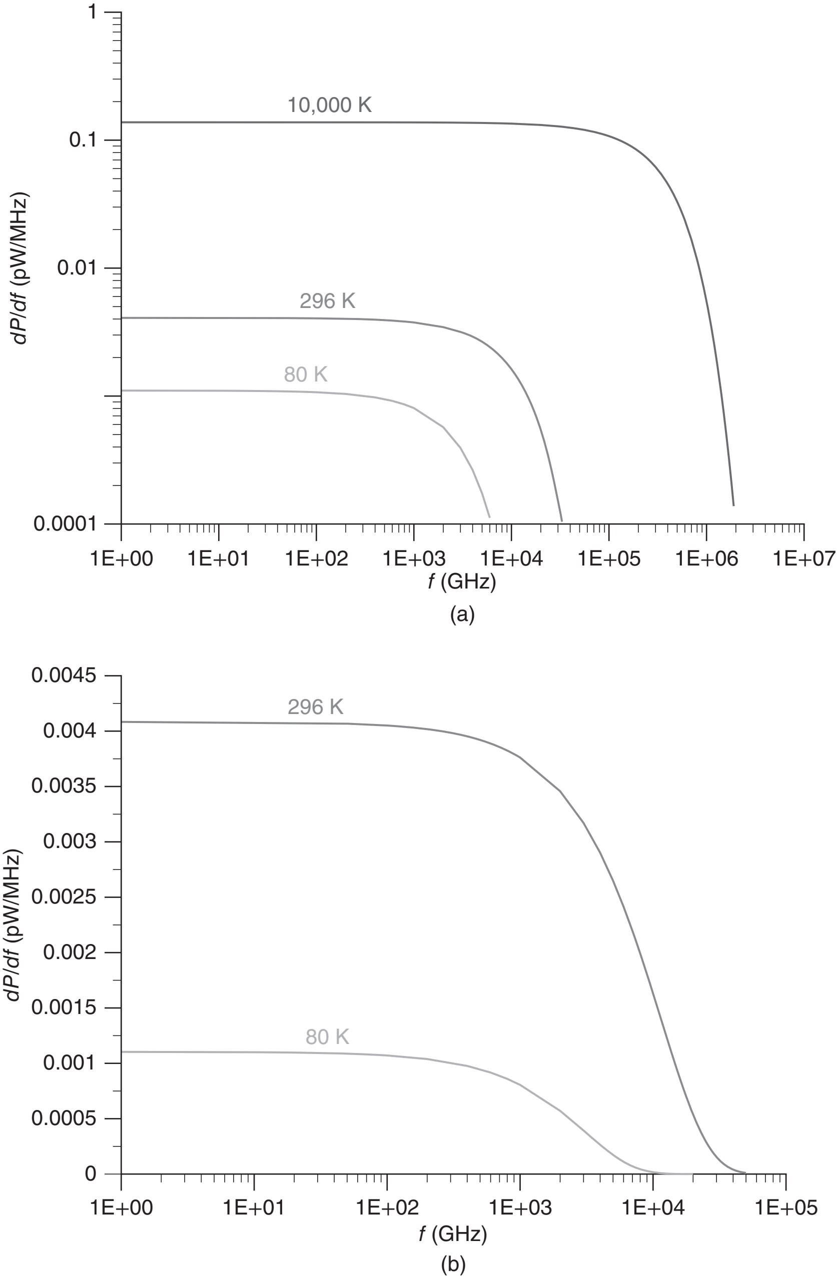

Figure 1.1 Available power spectral density as a function of frequency; (a) logarithmic scale, (b) linear scale.

The high-frequency behavior of the available power density is dominated by the exponential in the denominator, which drives the power rapidly to zero once $hf/(k_B T)$ becomes sizable. The "knee" in the graphs, where the behavior transitions from the low-frequency constant to the high-frequency damping, occurs at around $f(\text{GHz}) \approx 20\times\, T(\text{K})$. This transition occurs when quantum effects become important, which is governed by the value of $h/k_B = 0.04799\,\text{K/GHz}$. Thus departures from the simple constant behavior of the available power become important for very high frequency and/or very low temperature. For example, at 290 K it is a 1% effect at 116 GHz; at 100 K it is a 1% effect at 40 GHz and a 0.1% effect at 4 GHz; at 30 K and 40 GHz it is a 6.4% effect (about 0.26 dB).

1.1.3 Definition of Noise Temperature

Equations (1.2) relate the available noise power from a passive device to its physical temperature. But microwave circuitry involves more than passive components. It is therefore convenient to define a "noise temperature" for active devices. Many variations have been suggested (and used), but there are two principal ways to do this [4]. The first is to use Eq. (1.2b) and define the noise temperature to be the physical temperature of a passive device that would result in the observed available power density. We will refer to this definition as the "equivalent-physical-temperature" definition,

$$\langle p_{avail}(f)\rangle \equiv \frac{hf}{e^{hf/k_B T_{noise}} - 1} \tag{1.5a}$$

The average in Eq. (1.5a) is taken over the frequency interval Δf, centered at f. This definition is popular in the remote-sensing community, where the received power is used to measure the physical temperature of the object under observation. This definition has the appealing property that for a passive object or device, the noise temperature is simply the physical temperature. Inverting Eq. (1.5a) to obtain the equations for T_{noise} yields a rather complicated expression, a point to which we shall return when considering amplifier noise measurements in Chapter 4.

The other common choice [5], which we adopt here, is to define the noise temperature as the available power spectral density divided by the Boltzmann constant times the frequency interval, which we will call the "Power Definition,"

$$\langle p_{avail}(f)\rangle \equiv k_B T_{noise} \tag{1.5b}$$

With this definition, the noise temperature is just a surrogate for the noise power spectral density, which makes this the natural choice when dealing with microwave circuits. With the power definition, the noise temperature of a passive device or object is only approximately equal to the physical temperature,

$$T_{noise} = \frac{1}{k_B}\left[\frac{hf}{e^{\frac{hf}{k_B T_{phys}}} - 1}\right] \approx T_{phys} \tag{1.6}$$

The approximation of Eq. (1.6) is known as the Rayleigh–Jeans approximation.

Due to the approximate equality in Eq. (1.6), there is *usually* little difference "in every-day life" between the power definition and the equivalent-physical-temperature definition. However, in precision measurements it is not uncommon to encounter a combination of high frequency, low temperature, and high precision that requires a specific choice of definition. Since this book deals with microwave precision noise measurements, we adopt the power definition, Eq. (1.5b), for the noise temperature. The considerations of Section 1.1.2 above explain when the high-frequency, low-temperature corrections become important, and thus when the distinction between different noise-temperature definition starts to matter.

1.1.4 Excess Noise Ratio and T_0

In dealing with noise temperatures and powers of greatly differing magnitudes, it is sometimes useful to define a decibel quantity for noise temperature. The quantity that is commonly used is the Excess Noise Ratio (*ENR*), defined by

$$ENR \equiv 10log_{10}\left(\frac{T_{noise} - T_0}{T_0}\right) \tag{1.7}$$

where the reference temperature T_0 is taken to be $T_0 = 290\,\mathrm{K}$. Just to be clear, we treat T_0 as a noise-temperature constant; it is not a noise temperature that corresponds to a physical temperature by virtue of Eq. (1.6). It is just a number.

Some variants of the *ENR* definition can be found either in the literature or in casual use. Some practitioners use the power delivered to a matched (i.e. reflectionless) load in place of the available power implicit in the T_{noise} in Eq. (1.7). This has the effect of introducing a mismatch factor (see Section 1.2.3) into the first term in the numerator of Eq. (1.7). There is also the question [6] of whether the reference temperature should be 290 K, or should it be the noise temperature of a passive load at a physical temperature of 290 K. We take it to be $T_0 = 290\,\mathrm{K}$, just for simplicity. We are venturing into the hair-splitting realm here, but in precision measurements sometimes hairs must be split.

1.2 Microwave Networks

1.2.1 Notation

Noise measurements are just a type of power measurements, and in precision noise measurements it is imperative to carefully account for all sources and flows of power. Therefore, before delving into noise measurements, it is necessary to review some basic microwave network theory and to establish the conventions

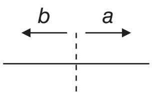

Figure 1.2 Waves on a lossless transmission line.

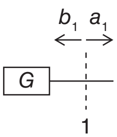

Figure 1.3 A linear one-port.

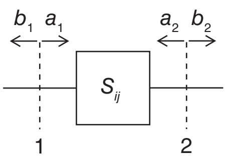

Figure 1.4 A linear two-port.

and notation that will be used in this work. Many books cover microwave circuit theory in some detail, including [7–12]. Here we just summarize the results that will be used. Our approach is similar to that of [13], with appropriate generalizations to include active devices. Unless specifically stated, we assume that all reference planes are in lossless transmission lines which support only a single propagation mode at the frequencies considered. We will typically use a and b to refer to the amplitudes of the traveling waves propagating in the two directions on a transmission line, with the normalization such that the spectral power density of the wave is given by the magnitude of its amplitude squared, $|a|^2$ or $|b|^2$. The net power spectral density delivered to the right in Figure 1.2 is thus given by

$$p_{del} = |a|^2 - |b|^2 \tag{1.8}$$

A linear one-port, as shown in Figure 1.3, is then described by

$$a_1 = \Gamma_G b_1 + c_G \tag{1.9}$$

where Γ_G is the reflection coefficient of the one-port G, and c_G is the wave emanating from G due to its intrinsic noise. A linear two-port, as depicted in Figure 1.4, is described by

$$\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \tag{1.10}$$

where c_1 and c_2 are the waves due to the intrinsic noise sources in the two-port.

1.2.2 Noise Correlation Matrix and Bosma's Theorem

In terms of the wave amplitudes introduced above, the noise correlation matrix of a two-port device is defined as

$$\boldsymbol{N} \equiv \begin{pmatrix} \langle |b_1|^2 \rangle & \langle b_1 b_2^* \rangle \\ \langle b_1^* b_2 \rangle & \langle |b_2|^2 \rangle \end{pmatrix} \tag{1.11}$$

This describes the noise properties of the device in a circuit, where there will in general be incident noise waves $\boldsymbol{a}$. If we are interested in the intrinsic properties of the device itself, we want to know the noise correlation matrix for the case in which there are no incident waves. That is given by the *intrinsic noise correlation matrix*,

$$\hat{\boldsymbol{N}} \equiv \begin{pmatrix} \langle |c_1|^2 \rangle & \langle c_1 c_2^* \rangle \\ \langle c_1^* c_2 \rangle & \langle |c_2|^2 \rangle \end{pmatrix} \tag{1.12}$$

In this book, we will be interested in properties of devices themselves, rather than in their use in circuits, and so we will deal primarily with the intrinsic noise matrix of Eq. (1.12). The exception will occur in Chapter 8, where we analyze multiport amplifiers.

An important result for the intrinsic noise matrix of a passive device is Bosma's theorem [12]. This states that for a passive device at noise temperature T_a the intrinsic noise-correlation matrix is related to the scattering matrix $\boldsymbol{S}$ by the equation

$$\widehat{\boldsymbol{N}} = k_B T_a(\boldsymbol{I} - \boldsymbol{S}\boldsymbol{S}^{\dagger}) \tag{1.13}$$

where $\boldsymbol{I}$ is the identity matrix. This means that the noise properties of a passive device are entirely determined by its $\boldsymbol{S}$ matrix (and its temperature). Although it is used mostly for two-ports, Eq. (1.13) applies for any number of ports. For a passive two port, the elements of the intrinsic noise-correlation matrix are given explicitly by

$$\begin{aligned}
\left\langle |c_1|^2 \right\rangle &= k_B T_a \left(1 - \left\lceil S_{11}{}^2 \right\rceil - |S_{12}|^2\right) \\
\left\langle c_1 c_2^* \right\rangle &= -k_B T_a \left(S_{11}S_{21}^* + S_{12}S_{22}^*\right) \\
\left\langle c_2 c_1^* \right\rangle &= -k_B T_a \left(S_{21}S_{11}^* + S_{22}S_{12}^*\right) \\
\left\langle |c_2|^2 \right\rangle &= k_B T_a \left(1 - \left\lceil S_{22}{}^2 \right\rceil - |S_{21}|^2\right)
\end{aligned} \tag{1.14}$$

1.2.3 Power Ratios

The power delivered to a load L, Figure 1.5, is given by $|a_1|^2 - |b_1|^2$. If we use $b_1 = \Gamma_L a_1$, we can write

$$p_{1,del} = |a_1|^2 - |b_1|^2 = |a_1|^2\left(1 - |\Gamma_L|^2\right) \tag{1.15}$$

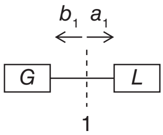

Figure 1.5 Power delivered to a load.

For the *available* power from a source G, we refer to Eq. (1.9) and Figure 1.5 and write the expression for the power delivered from the source to the load L,

$$a_1 = \Gamma_G b_1 + c_G = \Gamma_G \Gamma_L a_1 + c_G = \frac{c_G}{1 - \Gamma_G \Gamma_L}$$

$$p_{1,del} = |a_1|^2 - |b_1|^2 = \left(1 - |\Gamma_L|^2\right)|a_1|^2 = \frac{|c_G|^2}{|1 - \Gamma_G\Gamma_L|^2}\left(1 - |\Gamma_L|^2\right) \tag{1.16}$$

This delivered power has a maximum for $\Gamma_L = \Gamma_G^*$, whence

$$p_{1,avail} = \frac{|c_G|^2}{\left(1 - |\Gamma_G|^2\right)} \tag{1.17}$$

Referring to the definition of noise temperature in Eq. (1.5b), we obtain

$$\left\langle |c_G|^2 \right\rangle = \left(1 - |\Gamma_G|^2\right) k_B T_G \tag{1.18}$$

This relates the magnitude of the intrinsic noise wave to the noise temperature of the one-port device, and due to our choice of noise-temperature definition, it is exactly true for both active and passive devices.

In tracking the power flow in a microwave circuit, three ubiquitous quantities are the mismatch factor, the efficiency, and the available power ratios. The mismatch factor is the fraction of the available power that is actually delivered across a reference plane. Referring to Figure 1.5, the mismatch factor at plane 1 is defined as $M_1 \equiv p_{1,\,del}/p_{1,\,avail}$. From Eqs. (1.16) and (1.17) for the delivered and available powers respectively, we have

$$M_1 \equiv \frac{p_{1,del}}{p_{1,avail}} = \frac{\left(1-|\Gamma_L|^2\right)\left(1-|\Gamma_G|^2\right)}{|1-\Gamma_L\Gamma_G|^2} \tag{1.19}$$

The efficiency is a property of a two-port that measures the fraction of the power that is delivered to the two-port which is delivered at the output of the device. It is also referred to as the delivered gain. Referring to Figure 1.6, $\eta_{21} \equiv p_{2,\,del}/p_{1,\,del}$. In terms of the relevant scattering parameters and reflection coefficients, we can write (after some algebra)

$$\eta_{21} \equiv \frac{p_{2,del}}{p_{1,del}} = \frac{|S_{21}|^2\left(1-|\Gamma_L|^2\right)}{|1-\Gamma_L S_{22}|^2\left(1-|\Gamma_{SL}|^2\right)} \tag{1.20}$$

where Γ_{SL} is the reflection coefficient at plane 1 from the S–L combination. If we substitute the explicit expression for Γ_{SL} into Eq. (1.21), we obtain

$$\eta_{21} = \frac{|S_{21}|^2\left(1-|\Gamma_L|^2\right)}{|1-\Gamma_L S_{22}|^2 - |(S_{12}S_{21}-S_{11}S_{22})\Gamma_L + S_{11}|^2} \tag{1.21}$$

Note that the efficiency depends only on the S parameters and the reflection coefficient of the load; it is independent of the source.

Besides the efficiency, another important quantity when dealing with a two-port is the available power ratio (or available gain) α_{21}. If we refer to Figure 1.6, α_{21} is defined as the ratio of the available power at plane 2 to the available power at plane 1, disregarding any intrinsic noise sources in the load or the two-port, $c_L = c_{1,\,S} = c_{2,\,S} = 0$,

$$\alpha_{21} \equiv \frac{p_{2,avail}}{p_{1,avail}}(c_L = c_{1\cdot S} = c_{2,S} = 0) \tag{1.22}$$

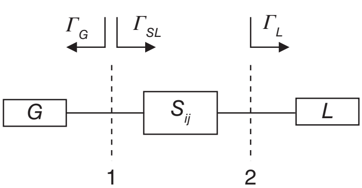

Figure 1.6 A two-port between source G and load L.

where $c_{1,S}$ and $c_{2,S}$ are the intrinsic sources of the two-port as defined by Eq. (1.10). In terms of the reflection coefficients and S parameters, Eq. (1.22) can be written as

$$\alpha_{21} = \frac{|S_{21}|^2 \left(1 - |\Gamma_G|^2\right)}{|1 - \Gamma_G S_{11}|^2 \left(1 - |\Gamma_{GS}|^2\right)} \tag{1.23}$$

where Γ_{GS} is the reflection coefficient from the G–S combination at plane 2,

$$\Gamma_{GS} = S_{22} + \frac{S_{21} S_{12} \Gamma_G}{1 - \Gamma_G S_{11}} \tag{1.24}$$

In Figure 1.6 (and in subsequent figures) the arrows indicate that the reflection coefficient is looking into and reflected from the direction of the arrow.

Whereas the efficiency was independent of the source reflection coefficient, the available power ratio is independent of the reflection coefficient of the load. Note that the available power ratio is also the *available gain*.

1.2.4 Noise-Temperature Translation Through a Passive Device

A situation that often occurs in analyzing noise measurements is depicted in Figure 1.7. The noise temperature is known at reference plane 1, and one needs to know the noise temperature at reference plane 2, on the other side of a passive device at noise temperature T_a. (Note that we specify a *noise* temperature of T_a, not a physical temperature of T_a, cf. Eq. (1.6).) The device can be any passive, linear two-port, such as an attenuator, adaptor, or section of lossy line. Because the noise temperature is defined in terms of the available power, Eq. (1.5b), it is possible to compute T_2 in terms of T_1 and T_a from the available power ratio α of the passive device. The fact that it is a linear device, plus the definition of the available power ratio mean that we can write

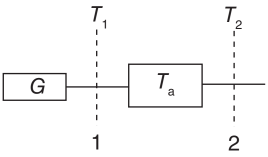

Figure 1.7 Configuration for computing T_2, knowing T_1.

$$p_{2,avail} = \alpha_{21} p_{1,avail} + f_0(T_a) \tag{1.25}$$

where f_0 is some function of T_a but does not depend on $p_{2,avail}$. From Eq. (1.5b) it then follows that

$$T_2 = \alpha_{21} T_1 + f(T_a) \tag{1.26}$$

where $f_0 = k_B f$. When $T_1 = T_a$, we must have $T_2 = T_a$, and therefore

$$T_a = \alpha_{21} T_a + f(T_a), \quad f(T_a) = (1 - \alpha_{21}) T_a \tag{1.27}$$

Combining Eqs. (1.27) and (1.28) yields the desired relationship

$$T_2 = \alpha_{21} T_1 + (1 - \alpha_{21}) T_a \tag{1.28}$$

Equation (1.28) is a fundamental result that is used extensively in analyzing noise properties of microwave circuits.

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