Chapter 1

Introduction

1.1 Fluids as a State of Matter

A standard dictionary definition of a fluid is

> *a substance whose particles can move about*
> *with freedom – a liquid or gas.*

Whilst this formulation encapsulates our general concept of a fluid, it is not entirely satisfactory as a scientific basis for the understanding of such materials. More formally within the context of *fluid mechanics* the fluid is seen as an isotropic, locally homogeneous, macroscopic material whose particles are free to move within the constraints established by the dynamical laws of continuum physics. The requirement that the fluid be a continuum implies that if a volume of fluid is successively subdivided into smaller elements, each element will remain structurally similar to its parent, and that this process of subdivision can be carried out down to infinitesimal volumes. Under these conditions several useful macroscopic concepts may be defined:

Fluid particle a fictitious particle fixed within the fluid continuum and moving with the velocity of the flow, and representing an average over a large number of microscopic particles.

Fluid point fixed in the fluid moving with the flow velocity. A fluid particle is always situated at the same fluid point.

Infinitesimal volume within the continuum of the fluid, and large compared with microscopic scales, but small compared with macroscopic ones.

Introductory Fluid Mechanics for Physicists and Mathematicians, First Edition. Geoffrey J. Pert.

In fact of course the fluid is not a continuum in the strict mathematical sense used above. The fluid is made up of discrete microscopic particles, namely molecules, which are distributed randomly with a distribution of velocities characteristic of the fluid in thermal equilibrium, typically given by the Maxwell–Boltzmann distribution in a gas. Fortunately, at the densities at which most experiments are conducted, the intermolecule separation is extremely small and very much less than the laboratory scale. It is therefore possible to average over small volumes which contain a very large number of particles, yet are very small on the laboratory scale, and allow us to recover the continuum approximation. In this manner we obtain terms which characterise the fluid as a bulk material. Typical of these average quantities are:

Density number or mass of particles per unit volume.

Temperature average energy of the random motion per particle in thermal equilibrium.

Pressure average momentum flow associated with the random motion per unit area.

Flow velocity mean velocity of the molecules averaging out the random motion.

The role of collisions amongst the particles plays an important role in defining irreversibility through the loss of correlation between the particles. Particles collide on average after a distance equal to the mean free path, and time after the collision interval. Since fluid mechanics assumes the fluid particles are in thermal equilibrium and randomly distributed, this condition requires that spatial and temporal averages be taken to include a large number of collisions, i.e. the laboratory-scale length is large compared with the mean free path and time to the collision interval. In practice this is not normally a restrictive condition. The effects of the collisions on fluid transport (momentum and energy) are thereby averaged over the thermal distribution to yield bulk properties of the material, namely viscosity and thermal conduction respectively. Consequently (ideal) fluid motion without viscosity or thermal conduction is dissipationless, entropy generation being due to viscosity and thermal conduction.

Within the continuum theory it is implicitly assumed that locally the fluid is in thermal equilibrium, although the temperature may vary globally through the flow. As a result the thermodynamics of bulk materials may be applied locally in the flow to calculate the pressure from the density and temperature (say) using the equation of state of the fluid. More generally the quantities and relations of equilibrium thermodynamics may be applied in the flow.

The flow of a basic fluid may be calculated using Newtonian mechanics, classical thermodynamics and the values of viscosity and thermal conductivity. From the above discussion, the conditions under which this theory may be applied are:

> *Laboratory-scale lengths must be large compared with the intermolecule separation and mean free path.*
> *Characteristic laboratory times must be large compared with the collision interval.*
> *The fluid must locally be in thermal equilibrium.*

The theory may be readily extended to relativistic mechanics and also to include additional dissipative terms, e.g. due to radiation. However, under normal laboratory conditions these are not required. Astrophysical systems provide examples of flows where more general approaches may be required.

Provided the above conditions are met, it is relatively straightforward to show that the fluid dynamical equations (to be obtained later) may be directly derived from the governing kinetic theory of the molecules.

1.2 The Fundamental Equations for Flow of a Dissipationless Fluid

The basic equations of fluid mechanics stem from simple concepts of conservation applied to mass, momentum and energy. These are completed by the thermodynamic equation of state of the material, in which the flow is to be calculated. The equations are, of course, complemented by the boundary conditions in an appropriate form, depending on the nature of the problem. In any problem, we seek to find five variables: three velocity components ($\mathbf{v}$) and two thermodynamic state variables, e.g. density ρ and pressure p, as functions of space $\mathbf{r}$ and time t. In many problems the actual number of variables required is reduced, either by symmetry to a restricted number of spatial dimensions or by a specified thermodynamic state, e.g. constant entropy or constant temperature. The problem is often further simplified by the restriction to *steady flow*, when there is no time variation.

Initially we will consider only dissipationless or *ideal* flow where the entropy of a fluid particle remains constant, i.e. viscosity and thermal conduction are neglected, deferring the treatment of flows in which viscosity plays a role until later; many important systems are treatable within the inviscid limit. We may quite generally identify two different conditions of flow involving the entropy of the fluid: *adiabatic flow* when the specific entropy of a fluid particle is constant in time; and *isentropic flow* where the specific entropy of each fluid particle has the same initial value. Many flows are both isentropic and adiabatic, e.g. the ideal steady flow of a fluid, whose specific entropy on entry is everywhere constant.

The basic equations may be formulated in two complementary ways:

- In the frame of the laboratory−the Eulerian frame. In this frame the co-ordinates are fixed in space and time. The derivatives used are the usual partial derivatives
$$\left.\frac{\partial}{\partial t}\right|_{\mathbf{r}} \quad \text{and} \quad \nabla\big|_t$$
- In the frame of the moving particle−the Lagrangian frame. In this system the spatial variation seen by the particle due to its motion is absorbed into the time derivative
$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla \tag{1.1}$$
  This system is often easier to set up, but becomes more complicated when the dissipative terms, viscosity and thermal conduction, are important.

However, the two systems are entirely equivalent and each may be easily derived from the other. They may also be used mixed if required. The actual choice of which to use will depend on the nature of the problem.

1.3 Lagrangian Frame

The Lagrangian frame of reference considers the fluid from the point of view of an observer on a fluid particle. Since many methods of calculation in computational fluid mechanics use the Lagrangian approach, we give a brief formal introduction to these methods. A fluid particle may be conveniently identified by a co-ordinate set, which is fixed on the particle, namely $\boldsymbol{\Lambda} = (\lambda, \mu, \nu)$, i.e. a triad of numbers. For example, these may be the initial position of the particle $\mathbf{r}_0 = (x_0, y_0, z_0)$. The position, velocity and thermodynamic state of the particle are therefore functions of time alone. Conceptually this leads to a simple set of kinematic and dynamic relations governing the motion of the particle, namely
$$\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = \mathbf{v} \qquad \text{and} \qquad \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \frac{\mathbf{F}}{m} \tag{1.2}$$

where $\mathbf{F}$ is the force acting on and m the mass of the fluid particle. The particle has a finite size expressed by the increments in the Lagrangian co-ordinates $\delta\Lambda_i = (\delta\lambda, \delta\mu, \delta\nu)$, and whose volume is given by the Jacobian

$$\delta V = \frac{\partial(x,y,z)}{\partial(\lambda,\mu,\nu)}\,\delta\lambda\,\delta\mu\,\delta\nu \tag{1.3}$$

which can be expressed as[1]

$$\begin{aligned} J = \frac{\partial(x,y,z)}{\partial(\lambda,\mu,\nu)} &= \frac{1}{N!}\left\{\varepsilon_{ijk}\,\varepsilon_{lmm}\,\frac{\partial x_i}{\partial\Lambda_l}\,\frac{\partial x_j}{\partial\Lambda_m}\,\frac{\partial x_k}{\partial\Lambda_n}\right\} \\ &= \left[\frac{\partial\mathbf{r}}{\partial\Lambda_1}\wedge\frac{\partial\mathbf{r}}{\partial\Lambda_2}\right]\cdot\frac{\partial\mathbf{r}}{\partial\Lambda_3} \end{aligned} \tag{1.4}$$

where ε_{ijk} is the perturbation symbol[2]

$$\varepsilon_{ijk} = \begin{cases} 1 & \text{if } (i \neq j \neq k) \text{ are in the sequence (1,2,3)} \\ -1 & \text{if } (i \neq j \neq k) \text{ are in the sequence (1,3,2)} \\ 0 & \text{otherwise} \end{cases} \tag{1.5}$$

Spatial derivatives of quantities associated with the fluid particles, e.g. thermodynamic variables, are directly calculated in a Lagrangian framework. The gradient of a scalar quantity $f(\lambda,\mu,\nu)$, which is defined on the fluid particle, is often required in the inertial Eulerian frame. Such terms are obtained by the use of the total differential for the variable f and Cramer's rule to solve

---

[1] We make extensive use of the index notation for vectors, where the vector is represented by a general component in a Cartesian co-ordinate system

$$\mathbf{A} \equiv (A_x,\, A_y,\, A_z) \equiv A_i \qquad \text{where} \quad i = 1,\, 2,\, 3$$

Sums over the indices are represented by Einstein's repeated index summation notation. Thus for example a scalar product is

$$\mathbf{A}\cdot\mathbf{B} = A_i\,B_i \equiv \sum_{i=1}^{N} A_i\,B_i$$

A repeated index indicates summation of that index over the full range of values N. The summation rule also applies to the elements of matrices.

[2] The expansion of the determinant of an $N \times N$ matrix $A = a_{ij}$ may be written as

$$\det A = \varepsilon_{ijk}a_{i1}\,a_{j2}\,a_{k3} = \varepsilon_{lmn}\,a_{1l}\,a_{2m}\,a_{3n} = \frac{1}{N!}\,\varepsilon_{ijk}\,\varepsilon_{lmn}a_{il}\,a_{jm}\,a_{kn}$$

the divisor $N!$ appearing because the first index l may be chosen in N different ways, the second m in $N-1$ ways, etc.

the resulting set of simultaneous equations.[3] Using the subscript notation and Einstein's repeated index summation rule gives

$$\nabla f\big|_i \equiv \frac{\partial f}{\partial x_i} = \frac{1}{N!\,J}\left\{\varepsilon_{ijk}\,\varepsilon_{lmn}\,\frac{\partial f}{\partial \Lambda_l}\,\frac{\partial x_j}{\partial \Lambda_m}\,\frac{\partial x_k}{\partial \Lambda_n}\right\} \tag{1.7}$$

where $N = 3$ is the dimensionality. The calculation of vector operators in Eulerian space, grad, div and curl, follows directly.

Taking the time derivative of the Jacobian, remembering that $\mathbf{v}_i = \mathrm{d}\mathbf{r}_i/\mathrm{d}t$ and using equation (1.7) we obtain

$$\frac{\mathrm{d}J}{\mathrm{d}t} = J\,\nabla\cdot\mathbf{v} \tag{1.8}$$

Since the mass δm of the particle is constant,

$$\rho\,J = \delta m/\delta\lambda\,\delta\mu\,\delta\nu = \rho_0\,J_0 \tag{1.9}$$

where the initial density is ρ_0 and the Jacobian J_0. This is the Lagrangian form for the conservation of mass. The specific volume of the particle is clearly related to the Jacobian through

$$V = \frac{1}{\rho} = \frac{J}{J_0}V_0$$

and making use of equation (1.8) we obtain the more familiar form of the Lagrangian mass conservation equation

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} + \rho\nabla\cdot\mathbf{v} = 0$$

1.3.1 Conservation of Mass

This equation may be derived in a more direct manner by considering the change in volume of a fluid particle ΔV with constant mass $\Delta m = \rho\Delta V$ and

---

[3] Cramer's rule solves the non-singular set of simultaneous equations

$$a_{ij}\,x_i = b_j$$

by forming a set of determinants for the matrix elements $D = \det a_{ij}$ and those formed by progressively replacing the ith column by the column 'vector' b_j, namely

$$D_i = \begin{vmatrix} a_{i'j} & \text{when} & i' \neq i \\ b_j & & i' = i \end{vmatrix}$$

The solution is then

$$x_i = D_i/D \tag{1.6}$$

surface ΔS as it moves through the fluid. In a time δt the volume increases by

$$\begin{aligned}\delta(\Delta V) &= \int_{\Delta S} \mathbf{v} \cdot \mathrm{d}\mathbf{S}\, \delta t \\ &= \int_{\Delta V} \nabla \cdot \mathbf{v}\, \mathrm{d}V\, \delta t\end{aligned}$$

by Gauss's theorem. As ΔV is small, the rate of dilation is

$$\dot{\Theta} = \frac{1}{\Delta V} \lim_{\delta t \to 0} \left\{ \frac{\delta(\Delta V)}{\delta t} \right\} = \nabla \cdot \mathbf{v} = -\frac{1}{\rho} \frac{\mathrm{d}\rho}{\mathrm{d}t}$$

since the mass of the particle is constant. Hence we obtain

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} + \rho \nabla \cdot \mathbf{v} = 0 \tag{1.10}$$

If the density of the fluid particle remains constant, i.e. the fluid is incompressible,

$$\nabla \cdot \mathbf{v} = 0 \tag{1.11}$$

and the rate of dilation is zero, or alternatively the volume of a fluid element is constant.

1.3.2 Conservation of Momentum–Euler's Equation

We consider the change of momentum of the fluid particle as a result of the forces applied to it. The total force is due to the pressure exerted inwards over the surface of the particle

$$-\int_{\Delta S} p\, \mathrm{d}\mathbf{S} = -\int_{\Delta V} \nabla p\, \mathrm{d}V \approx -\nabla p\, \Delta V$$

and gravity $\Delta m\, \mathbf{g}$ where $\mathbf{g}$ is the acceleration due to gravity. Hence using Newton's second law of motion we obtain Euler's equation:

$$\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = -\frac{1}{\rho} \nabla p + \mathbf{g} \tag{1.12}$$

The preceding equation for flow in an inertial frame must be modified to include the Coriolis and centrifugal forces in a rotating frame

$$\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} + 2\, \boldsymbol{\Omega} \wedge \mathbf{v} + \boldsymbol{\Omega} \wedge (\boldsymbol{\Omega} \wedge \mathbf{r}) = -\frac{1}{\rho} \nabla p + \mathbf{g} \tag{1.13}$$

where $\boldsymbol{\Omega}$ is the angular velocity of rotation.

1.3.3 Conservation of Angular Momentum

Angular momentum is not frequently used in fluid mechanics. However, as in mechanics, it must be conserved in the absence of external torques. We may obtain the relationship governing its variation directly by taking the vector product of the radius vector with Euler's equation (1.12). Thus the angular momentum per unit mass of a fluid particle is

$$\frac{\mathrm{d}(\mathbf{r} \wedge \mathbf{v})}{\mathrm{d}t} = \mathbf{r} \wedge \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = -\frac{1}{\rho}\, \mathbf{r} \wedge \nabla p \; \mathbf{r} \wedge \mathbf{g} \tag{1.14}$$

1.3.4 Conservation of Energy

Since the fluid is dissipationless, the energy equation takes the particularly simple form of the first law of thermodynamics for an adiabatic change:

$$\frac{\mathrm{d}\epsilon}{\mathrm{d}t} = \frac{p}{\rho^2}\frac{\mathrm{d}\rho}{\mathrm{d}t} = -\frac{p}{\rho}\, \nabla \cdot \mathbf{v} \tag{1.15}$$

where ϵ is the specific internal energy (per unit mass).

1.3.5 Conservation of Entropy

If dissipation in the fluid is negligible, i.e. the flow of an *ideal fluid*, the entropy of a fluid element is constant. Therefore

$$\frac{\mathrm{d}s}{\mathrm{d}t} = 0 \tag{1.16}$$

where s is the specific entropy (per unit mass).

1.4 Eulerian Frame

We turn now to the equations in Eulerian form.

1.4.1 Conservation of Mass−Equation of Continuity

Consider a fluid of density ρ moving with velocity $\mathbf{v}$ in a closed volume V stationary in the Eulerian frame with bounding surface S. The mass of fluid enclosed in V is $\int_V \rho \, \mathrm{d}V$. Thus the rate of increase of mass in V is

$$\int_V \frac{\partial \rho}{\partial t}\, \mathrm{d}V$$

This mass gain must be balanced by a mass flow rate into V through S. Since the mass flow rate out through an element $\mathrm{d}\mathbf{S}$ is $\rho\mathbf{v}\cdot\mathrm{d}\mathbf{S}$, we obtain

$$\begin{aligned}\int_V \frac{\partial\rho}{\partial t}\mathrm{d}V &= -\int_S \rho\,\mathbf{v}\cdot\mathrm{d}\mathbf{S} \\ &= -\int_V \nabla\cdot(\rho\,\mathbf{v})\,\mathrm{d}V \qquad (1.17)\end{aligned}$$

Hence, since V is arbitrary, the integrands must be equal, i.e.

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\,\mathbf{v}) = 0 \qquad (1.18)$$

namely, the equation of continuity.

This equation has the characteristic form of a conservation equation, i.e.

$$\begin{aligned}\frac{\partial}{\partial t}[\text{Quantity per unit volume}] \quad &+ \quad \text{div}[\text{Flux of quantity}] \\ &= \quad [\text{Input per unit volume per unit time}] \\ \{ \quad &= \quad 0 \text{ in this case}\}\end{aligned}$$

where the flux is the quantity flowing per unit time through unit area normal to the flow (see Appendix 10.A). In this case the flux $\mathbf{j} = \rho\,\mathbf{v}$.

1.4.2 Conservation of Momentum

Momentum introduces a complication in that momentum itself is a vector. The momentum flux is a tensor. We therefore work in Cartesian components using the general notation, i.e. u_i for the ith component ($i = 1, 2, 3\,[x, y, z]$). We also use the Einstein summation convention for a repeated index, namely $a_i\,b_i = \sum_{i=1}^{3} a_i b_i = \mathbf{a}\cdot\mathbf{b}$. The total momentum in V is thus $\int_V \rho\,v_i\,\mathrm{d}V$, and the flow of momentum leaving through $\mathrm{d}\mathbf{S}$ is $\rho\,v_i\,v_j\,\mathrm{d}S_j$. The sources of momentum in V are the forces: the internal force due to the hydrostatic pressure $-p\,\mathrm{d}S_i$ (minus sign since pressure acts inwards) and the external force due to gravity ρg_i per unit volume.

The momentum balance equation for V is thus

$$\begin{aligned}\int_V \frac{\partial(\rho\,v_i)}{\partial t}\mathrm{d}V &= -\int_S \rho\,v_i\,v_j\,\mathrm{d}S_j - \int_S p\,\mathrm{d}S_i + \int_V \rho\,g_i\,\mathrm{d}V \\ &= \int_V \rho\,g_i - \frac{\partial}{\partial x_j}\left[\rho\,v_i\,v_j + p\,\delta_{ij}\right]\mathrm{d}V \qquad (1.19)\end{aligned}$$

where δ_{ij} is the Kronecker delta.

Hence, as before, since V is arbitrary

$$\frac{\partial(\rho\,v_i)}{\partial t} + \frac{\partial(\rho\,v_i\,v_j + p\,\delta_{ij})}{\partial x_j} = \rho\,g_i \qquad (1.20)$$

This has the general form noted above since $\partial a_j / \partial x_j = \nabla \cdot \mathbf{a}$. The momentum flux $\Gamma_{ij} = \rho\, v_i\, v_j + p\, \delta_{ij}$ includes the internal force, in this case pressure alone. By Newton's second law of motion, this force corresponds to an impulse which transfers momentum within the fluid body, but is conserved overall, as one part of the fluid exerts an equal and opposite force (and therefore momentum transfer) on another. The external force, gravity, corresponds to a source term, which is not conserved.

1.4.3 Conservation of Angular Momentum

Angular momentum obeys a conservation law in the absence of external torques. However, the form is a little more difficult to establish than for linear momentum. As with linear momentum we expect that the angular momentum flux will be a second-order tensor. We may obtain the relations directly by considering the conservation of angular momentum in an arbitrary volume V with surface S. The rate of change of the total angular momentum in V must be balanced by the flow of angular momentum through the surface S due to transport and to the torques exerted on the fluid at S by the pressure and internally by any volume force. Thus

$$\frac{\partial}{\partial t} \int_V \rho\, (\mathbf{r} \wedge \mathbf{v})\, \mathrm{d}V = -\int_S [\rho\, \mathbf{r} \wedge \mathbf{v}]\;\mathbf{v} \cdot \mathrm{d}\mathbf{S} - \int_S p\, \mathbf{r} \wedge\, \mathrm{d}\mathbf{S} + \int_V \rho\, \mathbf{r} \wedge \mathbf{g}\, \mathrm{d}V$$

As before, noting that V is arbitrary and using Gauss's theorem,[4] we obtain

$$\frac{\partial}{\partial t}[\rho\, \mathbf{r} \wedge \mathbf{v}] + \frac{\partial}{\partial x_j}[\rho\, (\mathbf{r} \wedge \mathbf{v})_i\; v_j] + \mathbf{r} \wedge \nabla p = \mathbf{r} \wedge (\rho\, \mathbf{g}) \tag{1.21}$$

which may also be obtained by taking the vector product of $\mathbf{r}$ with equation (1.20). This is clearly not in conservation law form due to the pressure term. This can, however, be written as the divergence of a second-order tensor as follows:

$$\mathbf{r} \wedge \nabla p = \varepsilon_{ijk}\, x_j\, \frac{\partial p}{\partial x_k} = -\varepsilon_{ijk}\, \frac{\partial}{\partial x_j}\,(p\, x_k)$$

where ε_{ijk} is the perturbation symbol defined in equation (1.5).

---

[4] This result is obtained from the vector identities

$$\nabla \cdot (\mathbf{A} \wedge \mathbf{B}) = \mathbf{B} \cdot \nabla \wedge \mathbf{A} = \mathbf{A} \cdot \nabla \wedge \mathbf{B} \qquad \text{and} \qquad \nabla \wedge (\phi\, \mathbf{A}) = \phi\, \nabla \wedge \mathbf{A} - \mathbf{A} \wedge \nabla \phi$$

Hence using Gauss's theorem when $\mathbf{a}$ is an arbitrary constant vector

$$\mathbf{a} \cdot \int p\, \mathbf{r} \wedge \mathrm{d}\mathbf{S} = \int \mathbf{a} \wedge (p\mathbf{r}) \cdot \mathrm{d}\mathbf{S} = \int \nabla \cdot (\mathbf{a} \wedge p\mathbf{r})\; \mathrm{d}V = \mathbf{a} \cdot \int \mathbf{r} \wedge \nabla p\, \mathrm{d}V$$

from which, since $\mathbf{a}$ is arbitrary, equation (1.21) follows.

Hence the total angular momentum flux is

$$\Xi_{ij} = \rho v_i \left(\varepsilon_{jk\ell}\, x_k\, v_\ell\right) - \left(\varepsilon_{ijk}\, p\, x_k\right)$$

1.4.4 Conservation of Energy

The total energy per unit mass of the fluid includes both internal energy ϵ and kinetic energy $\frac{1}{2}v^2$. The work done on the fluid is due to the pressure force on the surface and to gravity. Thus

$$\int_V \frac{\partial}{\partial t}\left[\rho\left(\epsilon + \frac{1}{2}v^2\right)\right] \mathrm{d}V = -\int_S \left[\rho\left(\epsilon + \frac{1}{2}v^2\right)\right] \mathbf{v}\cdot \mathrm{d}\mathbf{S}$$

$$-\int_S p\,\mathbf{v}\cdot\mathrm{d}\mathbf{S} + \int_V \rho\,\mathbf{g}\cdot\mathbf{v}\,\mathrm{d}V + \int_V W\mathrm{d}V$$

where W is the energy deposited by external sources per unit volume per unit time. Hence we obtain by the use of Gauss's theorem and the arbitrary nature of the volume V, as before,

$$\frac{\partial}{\partial t}\left[\rho\left(\epsilon + \frac{1}{2}v^2\right)\right] + \nabla\cdot\left[\rho\left(h + \frac{1}{2}v^2\right)\mathbf{v}\right] = W + \rho\,\mathbf{g}\cdot\mathbf{v} \tag{1.22}$$

where $h = \epsilon + p/\rho$ is the specific enthalpy (per unit mass). The energy flux is thus $(h + \frac{1}{2}v^2)\,\mathbf{v}$ and includes in the enthalpy, h, a term for the work done by one section of the fluid on another corresponding to the internal forces. The work done by the external force is not conserved.

Since gravity is a conservative force, which is constant in time, we may include the gravitational field in overall fluid energy $(\epsilon + \frac{1}{2}v^2 + U)$ per unit mass where $\mathbf{g} = -\nabla U$ defines the gravitational potential. Using the equation of continuity (1.18) and the time invariance of the gravitational field, U, we obtain

$$\frac{\partial}{\partial t}\left[\rho\left(\epsilon + \frac{1}{2}v^2 + U\right)\right] + \nabla\cdot\left[\rho\left(h + \frac{1}{2}v^2 + U\right)\mathbf{v}\right] = W \tag{1.23}$$

1.4.5 Conservation of Entropy

In an ideal fluid, where entropy is conserved, entropy may also be written as a conservation law. From equations (1.16) and (1.18) we obtain

$$\frac{\partial}{\partial t}(\rho\, s) + \nabla\cdot(\rho\, s\,\mathbf{v}) = 0 \tag{1.24}$$

1.5 Hydrostatics

Consider the situation where the fluid is at rest. It follows from Euler's equation (1.12) (or directly) that the pressure force and gravity must balance, i.e.

$$\nabla p = \rho \mathbf{g} \tag{1.25}$$

Since gravity only acts in the vertical direction (z, measured upwards) this equation takes the simple form when the density is constant

$$p = -\rho \, g \, z + \text{const}$$

$\rho \, g \, z$ is known as the hydrostatic head.

If the thermodynamic condition of the fluid is predetermined, the pressure and density are related by the appropriate equation of state and are both functions of the vertical height only. The fluid is therefore stratified. We may identify two important cases.

1.5.1 Isothermal Fluid−Thermal and Mechanical Equilibrium

If the fluid is isothermal, i.e. the temperature T is constant everywhere, the system is in thermal equilibrium. The equilibrium condition is written in terms of the thermodynamic potential per unit mass Φ given by

$$\mathrm{d}\Phi = -s \, \mathrm{d}T + V \, \mathrm{d}p$$

where s is the specific entropy (per unit mass) and $V = 1/\rho$ the specific volume. Hence equation (1.25) integrates to

$$\Phi + gz = \text{const} \tag{1.26}$$

throughout the fluid.

This result is recognised as the standard result from thermodynamics for a system in thermal equilibrium in an external field.

1.5.2 Adiabatic Fluid−Lapse Rate

If the fluid is isentropic, the specific entropy of the fluid is everywhere constant. Since the enthalpy is

$$\mathrm{d}h = T \, \mathrm{d}s + V \, \mathrm{d}p$$

equation (1.25) takes the form

$$h + gz = \text{const} \tag{1.27}$$

Alternatively, making use of the thermodynamic relations

$$\left.\frac{\partial T}{\partial p}\right|_s = \left.\frac{\partial V}{\partial s}\right|_p = \left.\frac{\partial T}{\partial s}\right|_p \left.\frac{\partial V}{\partial T}\right|_p = \frac{\alpha\, V\, T}{c_p} \tag{1.28}$$

where α the coefficient of volume expansion and c_p the heat content at constant pressure (per unit mass),

$$\alpha = \frac{1}{V}\left.\frac{\partial V}{\partial T}\right|_p \quad \text{and} \quad c_p = \left.\frac{\partial h}{\partial T}\right|_p = T\left.\frac{\partial s}{\partial T}\right|_p \tag{1.29}$$

The temperature varies as a function of the height alone

$$\frac{\mathrm{d}T}{\mathrm{d}z} = \left.\frac{\partial T}{\partial p}\right|_s \frac{\mathrm{d}p}{\mathrm{d}z} = \frac{\alpha\, V\, T}{C_p}\,\frac{\mathrm{d}p}{\mathrm{d}z} = -\frac{\alpha\, T}{c_p}\, g \tag{1.30}$$

The rate at which the temperature decreases with height in the atmosphere, namely $\Gamma = -\mathrm{d}T/\mathrm{d}z$, and the corresponding density and pressure changes, are known as the adiabatic lapse rate. A very simple alternative derivation of this result is useful. Starting from the second $T\mathrm{d}S$ equation of thermodynamics we may write directly that

$$T\mathrm{d}s = c_p\mathrm{d}T - T\left.\frac{\partial V}{\partial T}\right|_p \mathrm{d}p = 0 \tag{1.31}$$

since $\mathrm{d}s = 0$ for an adiabatic change. Hence using equation (1.25) we obtain equation (1.30).

For a perfect gas obeying the ideal gas laws, $\alpha = 1/T$ and the lapse rate takes the particularly simple form $\Gamma = g/c_p$.

In atmospheric physics several distinct lapse rates are identified:

1. **Dry adiabatic lapse rate** is the rate of decrease of temperature with height of dry (unsaturated) air under adiabatic conditions. Since air is approximately a perfect gas the lapse rate is given by $g/c_p \approx 9.8\,\mathrm{K/km}$, where in dry air $c_p = 1004$ J/kg/K and $g = 9.81\,\mathrm{m/s^2}$. The dry adiabatic lapse rate is independent of the height.

2. **The moist saturated adiabatic lapse rate** is the rate at which the air temperature decreases as it rises when maintained at its dew point (i.e. saturated with water vapour). It is significantly smaller than the dry lapse rate due to the latent heat released by the water vapour as it condenses, forming a liquid cloud, and thereby raising the temperature. Unlike the dry lapse rate the saturated value varies with height, typically having a value of about $5\,\mathrm{K/km}$.

An approximate expression for the moist adiabatic lapse rate may be derived by including the latent heat released as the air rises by modifying equation (1.31) to include the condensation

$$T\,\mathrm{d}s = c_p\,\mathrm{d}T - T\frac{\partial V}{\partial T}\bigg|_p \mathrm{d}p + L\,\mathrm{d}r = 0 \tag{1.32}$$

where r is the *specific humidity* or *mixing ratio* (mass ratio of water vapour to dry air) and $L = 2.453 \times 10^6$ J/kg the latent heat of vaporisation.

The variation in the mixing ratio as the temperature and pressure change may be approximately calculated from the application of Dalton's law of partial pressures

$$r = \epsilon\frac{p_v}{(p - p_v)} \approx \epsilon\frac{p_v}{p} \tag{1.33}$$

where p_v is the water vapour partial pressure and $\epsilon = R_a/R_v = 287/462$ is the ratio of the gas constants for air and water vapour respectively, which can be expressed in terms of the ratio of molecular masses, $\epsilon = M_v/M_a = 18.015/28.964 = 0.622$. The saturated water vapour pressure p_v is given by the Clausius−Clapeyron equation

$$\frac{\mathrm{d}p_v}{\mathrm{d}T} = \frac{L}{(V_v - V_{lT})} \approx \frac{Lp_v}{R_vT^2} \tag{1.34}$$

where the specific volume of the vapour V_v is much larger than that of the liquid V_l, and the vapour behaves as an ideal gas. Hence we obtain

$$\begin{aligned}\frac{1}{r}\frac{\mathrm{d}r}{\mathrm{d}z} &\approx \frac{1}{p_v}\frac{\mathrm{d}p_v}{\mathrm{d}z} - \frac{1}{p}\frac{\mathrm{d}p}{\mathrm{d}z}\\ &\approx \frac{L}{R_vT^2}\frac{\mathrm{d}T}{\mathrm{d}z} + \frac{1}{p}\rho g\\ &= -\frac{L}{R_vT^2}\Gamma + \frac{1}{R_aT}g \end{aligned}\tag{1.35}$$

Substituting for $\mathrm{d}p/\mathrm{d}z$ and $\mathrm{d}r/\mathrm{d}z$ we obtain an approximate expression for the lapse rate of air saturated with water vapour

$$\Gamma = \frac{g}{c_p}\frac{\left[1 + \dfrac{L\,r}{R_a\,T}\right]}{\left[1 + \dfrac{L^2\,r\,\epsilon}{c_p\,R_a\,T^2}\right]} \tag{1.36}$$

The mixing ratio r varies with temperature as the saturated vapour pressure given by the integral of equation (1.34). Since the latent heat is almost constant

$$\ln\left(\frac{p_v}{p_0}\right) = \frac{L}{R_v}\left(\frac{1}{T_0} - \frac{1}{T}\right)$$

where (p_0, T_0) is a suitably chosen initial (saturated) condition, e.g. at the triple point of water where $p_0 = 611.73$ Pa, $T_0 = 273.16\,\mathrm{K}$. This equation taken with equation (1.33) may be used to calculate the *dew point*: that is, the temperature to which a given parcel of air (with known specific humidity and pressure) must be cooled before it starts to condense. However, tabular and graphical representations are used to make this a much easier task in practice.

3. **Environmental lapse rate** is the measured rate of decrease of temperature with height in the atmosphere.

1.5.3 Stability of an Equilibrium Configuration

Although the system may be in mechanical equilibrium, it may not be stable. If the configuration is unstable, *convection* currents are set up within the fluid, which tend to mix the fluid to establish a uniform temperature.

We may derive the condition for the equilibrium to be stable by considering the effect of a small displacement of a fluid element ξ in the direction z. The system is stable if its response is to restore the perturbation to its original position. Thus let us suppose that the fluid element has a specific volume $V(p, s)$ at its equilibrium position z and let us suppose that the specific volume changes adiabatically in response to a pressure change to p' at $(z + \xi)$ to $V(p', s)$.[5] The fluid will displace an equal volume of fluid whose pressure p' and entropy s' correspond to the equilibrium values at $(z + \xi)$. If the displaced fluid element is heavier than the one it displaces, it will tend to sink and the equilibrium will be restored. Thus a necessary condition for stability is

$$V(p', s') - V(p', s) \approx \left.\frac{\partial V}{\partial s}\right|_p \frac{\mathrm{d}s}{\mathrm{d}z}\,\xi = \frac{\alpha\,T}{\rho\,c_p}\frac{\mathrm{d}s}{\mathrm{d}z}\,\xi > 0 \tag{1.37}$$

where we have made use of the thermodynamic relations as before, equation (1.28).

Since the constants in the above inequality are all positive, the condition for stability reduces to

$$\frac{\mathrm{d}s}{\mathrm{d}z} > 0 \tag{1.38}$$

that is, the entropy increases with height.

The limiting case of stability, that the entropy is constant with height, corresponds to the adiabatic lapse rate, equation (1.30). Applying these results to the atmosphere, we can see that it is unstable if the temperature falls less

---

[5] We may imagine that the change takes place sufficiently slowly to be reversible with viscosity and thermal conduction negligible.

rapidly than the adiabatic lapse rate going to higher altitudes, i.e. if the environmental lapse rate is greater than the adiabatic one

$$\frac{\mathrm{d}T}{\mathrm{d}z} > \left.\frac{\mathrm{d}T}{\mathrm{d}z}\right|_{\mathrm{lapse}} = -\frac{\alpha\, T}{c_p}\, g \tag{1.39}$$

In practice convection tends to reduce temperature gradients to the adiabatic lapse rate where the atmosphere is marginally stable. The adiabatic lapse rate may therefore be used to give an approximation to the variation of temperature and pressure with altitude.

The relationship of the adiabatic and the environmental lapse rates plays an important role in determining the generation of upward thermals. If a parcel of air is unsaturated and rises, being unstable it ascends, cooling at the dry lapse rate, until the dew point is reached and water vapour starts to condense. This is approximately the level of the cloud base. At this point the lapse rate is decreased to the moist saturated vapour rate, causing the parcel to ascend more rapidly and leading to the formation of rain. In an extreme case the rapidly rising stream of air leads to the formation of a characteristic thunder cloud.

1.6 Streamlines

A streamline is a line whose tangent is everywhere parallel to the flow. In Cartesian co-ordinates its equation is

$$\frac{\mathrm{d}x}{v_x} = \frac{\mathrm{d}y}{v_y} = \frac{\mathrm{d}z}{v_z} \tag{1.40}$$

In steady flow, but *not* in non-steady flow, the streamlines are the particle paths (streaklines). The surface of a body immersed in a flow must be a streamline, since there is no flow through it. In streamlined flow around a body, the neighbouring streamlines closely parallel the surface from entry to exit. This is in contrast to the flow around a non-streamlined body, where the streamline touching the surface may separate and leave the neighbourhood of the body.

A closely related concept is the tube of flow, namely a region of flow whose walls are streamlines, and thus parallel to the flow. Consequently no flow can take place through the wall of the tube. The total flux of any quantity through any cross-section of a tube of flow is therefore constant.

1.7 Bernoulli's Equation: Weak Form

Bernoulli's equation is a direct consequence of the equations for a dissipationless fluid. It has a simple form, which makes it suitable for many applications, particularly for order of magnitude estimates.

In steady, dissipationless flow, the flow at any point on a streamline satisfies the following simple relation:

$$h + \frac{1}{2}v^2 + U = \text{const} \tag{1.41}$$

If the fluid is incompressible, $\rho = \text{const}$, then the enthalpy h is replaced by p/ρ. Under the above conditions this equation is known as the *weak form of Bernoulli's equation.* Other forms will be seen to occur under different conditions.

The proof follows from Euler's equation (1.12) as follows. In steady flow $\partial/\partial t \equiv 0$, and thus

$$\begin{aligned} (\mathbf{v} \cdot \nabla)\mathbf{v} &= -\frac{1}{\rho}\nabla p - \nabla U \\ \nabla(1/2v^2) - \mathbf{v} \wedge (\nabla \wedge \mathbf{v}) &= -\nabla(h + U) \end{aligned} \tag{1.42}$$

since the fluid is dissipationless, the entropy change $\mathrm{d}s = 0$ and $\mathrm{d}h = (1/\rho)\,\mathrm{d}p$. Hence integrating along the streamline and noting that $\mathbf{v} \wedge (\nabla \wedge \mathbf{v})$ is perpendicular to $\mathbf{v}$, we obtain equation (1.41).

Alternatively, if we consider a narrow tube of flow surrounding a streamline of cross-section δS, normal to the flow, then in steady flow the total mass flow $\rho\, v\, \delta S = \text{const}$, the total energy flow

$$\rho\, v\, (h + 1/2v^2 + U)\, \delta S = \text{const}$$

and we again obtain Bernoulli's equation (1.41).

The two proofs are based on momentum and energy flow respectively−a consequence of the fact that in ideal flow the equation of state has only one independent thermodynamic variable, because, due to the dissipationless nature of the flow, the entropy of a fluid particles is constant.

Bernoulli's equation enables the definition of a useful quantity expressing the total energy available in the flow, which is the value of the constant along the streamline in equation (1.41):

$$\frac{1}{2}v_{\max}^2 = h + \frac{1}{2}v^2 + U \tag{1.43}$$

$v_{\max}$ being the limit speed, i.e. the velocity that the fluid would acquire if both the pressure and gravitational potential were zero.

1.8 Polytropic Gases

A useful representation of many real gases is the polytropic gas, whose equation of state is the ideal gas law, and whose adiabatic equation of state is the familiar expression

$$\frac{p}{\rho^\gamma} = \text{const} \tag{1.44}$$

where γ is the ratio of specific heats, or adiabatic index. A large number of gases behave as polytropic gases with an appropriate value of the adiabatic index γ. For these materials the specific internal energy and specific enthalpy are given by

$$\epsilon = \frac{1}{(\gamma-1)}\frac{p}{\rho} \qquad \text{and} \qquad h = \frac{\gamma}{(\gamma-1)}\frac{p}{\rho} \tag{1.45}$$

For future reference the specific entropy (per unit mass) is given by

$$s = c_V \ln\left(\frac{p}{\rho^\gamma}\right) + s_0 \tag{1.46}$$

where c_V is the specific heat per unit mass.

The limit speed in polytropic gases can be expressed in terms of the critical velocity, when the flow velocity equals the local sound speed $c = \sqrt{\partial p/\partial\rho|_s} = \sqrt{\gamma p/\rho}$. From Bernoulli's equation (1.41) we obtain

$$v_* = c_* = \sqrt{\frac{(\gamma-1)}{(\gamma+1)}}\, v_{\max} \tag{1.47}$$

Another set of quantities, which are often useful in calculations with polytropic gases, is the stagnation sound speed, pressure and density. These are specified by the condition that the flow velocity is zero. They are generally defined by the stagnation sound speed

$$c_0 = \sqrt{\frac{(\gamma-1)}{2}}\, v_{\max} \tag{1.48}$$

with p_0 and ρ_0 obtained through the equation of state in the form

$$\frac{c}{c_0} = \left(\frac{p}{p_0}\right)^{(\gamma-1)/2\gamma} = \left(\frac{\rho}{\rho_0}\right)^{(\gamma-1)/2} = \left\{\frac{1+(\gamma-1){M_0}^2/2}{1+(\gamma-1)M^2/2}\right\}^{1/2} \tag{1.49}$$

where $M = v/c$ is the Mach number.

The other class of material, which is important in dissipationless flows, is incompressible, where the density of a fluid particle is constant, $\rho = \text{const}$. Liquids are the obvious examples of this condition. However, as we shall see, gases also behave in this way when their flow speed is much less than the sound speed (subsonic flows). For incompressible flow the integral

$$\frac{1}{\rho}\int \mathrm{d}p = \int \frac{\mathrm{d}p}{\rho} = \frac{p}{\rho}$$

is used in applications such as Bernoulli's equation. We can therefore replace $h \Rightarrow p/\rho$ in these cases.

1.8.1 Applications of Bernoulli's Theorem

1.8.1.1 Vena Contracta

We consider the flow from a reservoir containing an incompressible fluid through a small hole in one of the walls of area S_2. The pressure in the fluid in the reservoir far from the hole is approximately constant p_1 and the flow velocity $v_1 \approx 0$. Bernoulli's equation for the flow speed through the hole v_2 where the pressure is atmospheric, p_2, is

$$\frac{1}{2}\cancelto{0}{v_1^2} + \frac{p_1}{\rho} = \frac{1}{2}v_2^2 + \frac{p_2}{\rho} \tag{1.50}$$

However, in steady flow, the momentum flux through the hole must balance that in the reservoir. Thus the momentum balance in direction i is given by

$$\int_S (\rho\, v_i\, v_j + p\, \delta_{ij})\, \mathrm{d}S_j = \int_{(S-S_2)} (\rho\, v_i\, v_j + p)\, \mathrm{d}S_j + \int_{S_2} (\rho\, v_i\, v_{2x} + p_2)\, \mathrm{d}S_x = 0$$

where S is the surface area of the reservoir including the hole and x is the direction of the normal to the area of the hole. We assume that the pressure over the wall (excluding the hole) is approximately constant and equal to p_1. By symmetry we may assume that the components in the y and z directions cancel. Similarly the integral over the inner surface contains area elements which cancel except over the projection of the hole on to the internal surface. Therefore

$$\int_{S_2} (\rho \cancelto{0}{v_1^2} + p)\, \mathrm{d}S = \left(\rho \overline{v_{2_x}{}^2} + p_2\right)\, S_2 \tag{1.51}$$

where $\overline{v_{2_x}{}^2}$ is the mean square velocity at the hole in the direction normal to its area x. Comparing equation (1.50) with equation (1.51) we see that $\overline{v_{2_x}{}^2} \not\approx {v_2}^2$. This is due to flow near the wall, where the velocity is non-parallel through the hole. As a result, after leaving the hole the flow continues to converge, reaching a minimum cross-section when all the streamlines are approximately parallel, and the velocity is approximately ${v_2}^2 \approx {v_{2_x}}^2$. The minimum area–vena contracta–is thus approximately $S_{\min} \approx \frac{1}{2} S_2$. In fact experiment gives a value of $S_{\min} \approx 0.624\, S_2$. The difference is accounted for mainly by pressure variations in the fluid near the hole.

If a tube is inserted into the fluid–Borda's mouthpiece–so that the fluid enters the exiting flow well away from the wall, the pressure and velocity in the neighbourhood of the entry correspond to the uniform symmetric value, as assumed above. As a result the area reduction is found to be very nearly the predicted value $\frac{1}{2}$.

1.8.1.2 Flow of gas along a pipe of varying cross-section

Gas obeying the polytropic equation of state moves steadily along a pipe of decreasing cross-section from a reservoir at pressure p_0 and density ρ_0 to an exit at pressure p_1. We assume that the speed of flow across a cross−section is constant−the hydraulic approximation. Since the flow is adiabatic we may apply Bernoulli's equation to the flow

$$h + \frac{1}{2}\, v^2 = \frac{\gamma}{(\gamma - 1)}\, \frac{p}{\rho} + \frac{1}{2}\, v^2 = \frac{\gamma}{(\gamma - 1)}\, \frac{p_0}{\rho_0}$$

where the initial speed v_0 is assumed to be very small, i.e. stagnation.

Introducing the sound speed and noting that the adiabatic equation of state (1.44) is appropriate,

$$c = \surd\left(\frac{\gamma\, p}{\rho}\right) = c_0 \left(\frac{\rho}{\rho_0}\right)^{(\gamma-1)/2}$$

Bernoulli's equation can be rewritten as

$$\frac{1}{2} v^2 = \frac{1}{(\gamma - 1)}\, (c_0^2 - c^2)$$

The mass flux is therefore

$$j = \rho v = \surd\left[\frac{2}{(\gamma - 1)}\right]\, \rho_0 \left(\frac{c^2}{c_0^2}\right)^{1/(\gamma-1)} \surd\left(c_0^2 - c^2\right)$$

Differentiating j with respect to c^2 we find a turning point at $c_0^2 - c^2 = 1/2(\gamma - 1)\, c^2$. Since $j \to 0$ as $v \to 0$ and $c \to c_0$, this turning point must be a maximum. At the maximum, the flow speed equals the sound speed (sonic flow):

$$v = v_* = c = c_* = \sqrt{\frac{2}{\gamma + 1}}\, c_0 \tag{1.52}$$

the critical speed.

Since the initial velocity $v_0 \approx 0$ the critical pressure and density at the sonic point are

$$p_* = \left[\frac{2}{(\gamma + 1)}\right]^{\gamma/(\gamma-1)} p_0 \qquad \text{and} \qquad \rho_* = \left[\frac{2}{(\gamma + 1)}\right]^{1/(\gamma+1)} \rho_0 \tag{1.53}$$

Figure 1.1 shows the characteristic parameters of the flow plotted as fractions of those at the critical point, where the flow velocity equals the sound speed, together with the local Mach number $M = v/c$.

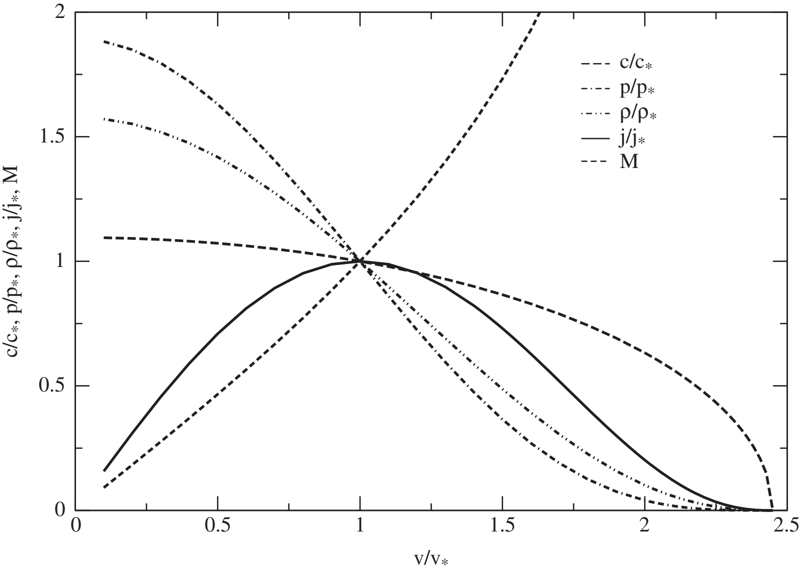

Figure 1.1: Characteristic parameters of the flow through a convergent/divergent nozzle.

Provided the external pressure p_{ext} exceeds the critical pressure p_*, the flow in the pipe is determined by the conditions at the exit, $p_1 = p_{ext}$.

The discharge (total mass flow) jS must remain constant, whilst the flux j cannot increase beyond the critical value, even if the cross-section of the pipe continues to decrease. Therefore the flux must adjust itself to be a maximum at the minimum cross-section, i.e. at the exit $p_1 = p_*$ when the external pressure p_{ext} is less than p_*. The discharge is then

$$\rho_* v_* S_{\min}$$

If the external pressure p_{ext} is less than the critical p_*, the flow is said to be *choked*, and there must be an additional expansion external to the pipe. In a uniformly converging pipe, the flow cannot become supersonic even if the external pressure is very low. To achieve a supersonic flow it is necessary to allow the flow to expand after the sonic point so that the increasing speed can be accommodated by a decreasing flux. This is accomplished by terminating the converging section by a throat of minimum cross-section followed by a diverging section. Such a pipe is a *de Laval nozzle*, and may be used to produce a supersonic jet of gas. The discharge in such a nozzle is determined by the critical flux at the throat.

The nozzle with fixed walls is an example of a tube of flow, since in each case the boundary condition of no flow through the wall must be upheld. The flow parameters along such a tube must therefore be identical to those derived above. In particular, in the neighbourhood of the sonic point, where $M = 1$, the flux is nearly constant even though the flow speed changes. The cross-section of the tube of flow is therefore nearly constant in this locality. Hence, within the ideal (dissipationless) flow approximation, the flow can neither expand nor contract transversely in the transonic region where $M \approx 1$ −behaviour which Busemann called a *streampipe*. This result turns out to be important in the design of aircraft at near sonic speeds (see Section 12.5.1).

Case study 1.I Munroe Effect−Shaped Charge Explosive

The collapse of lined cavities has long been known to produce a high-velocity jet and a slower moving slug from the liner material. The jet can penetrate steel plate. Originally used in mining it was developed as an anti-tank weapon during the Second World War. The basic theory of the effect is relatively simple (Birkhoff *et al.*, 1948). A wedge-shaped block of explosive, lined with a thin metal layer (liner), is detonated from the apex end. The detonation wave moving through the explosive causes an inward implosion. The metal layer is fluidised by the intense pressure generated by the detonation, and driven towards the axis with velocity v_0. As a result the apex moves along the axis as the detonation proceeds through the block, Figure 1.2(a). To an observer moving with the apex it appears that material flows steadily down the arms and leaves as a jet (forward) and slug (backward) along the axis, Figure 1.2(b). Provided the detonation moves with constant velocity through the block u_0, the flow in this frame of reference is steady. Therefore we may apply Bernoulli's theorem to the collision of the two streams from each arm, and thus to the incoming and outgoing flows. Following the impulsive pressure pulse immediately after the detonation, the pressure falls rapidly and the fluid moves freely along the arms, so that the pressure on both the fluid in the arms and that moving along the axis is approximately constant on impact at the apex. Hence it follows from Bernoulli's theorem that the flow speeds in both axial flows, backwards (slug) and forward (jet), are equal to that of the incoming flows v' in the apex frame. The velocities of the slug and jet in the laboratory frame are then obtained by transforming back from the apex frame.

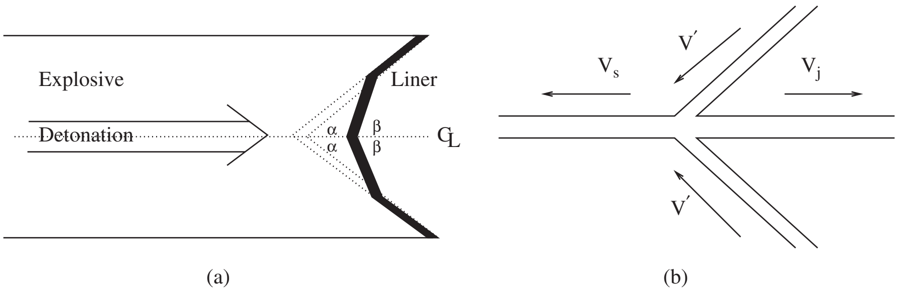

Figure 1.2: The geometrical arrangement of the flow system for the shaped charge detonation with a wedge: (a) shows the arrangement in the laboratory frame, and (b) that seen by an observer moving with the apex.

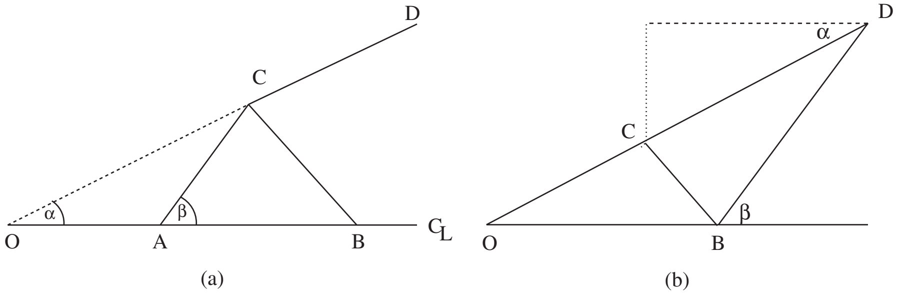

Figure 1.3: The geometrical arrangement of the flow system for the shaped charge detonation with a wedge in the laboratory frame. OCD is the original position of the liner. OAB is the axis, i.e. the path of the apex. $AC = v'(t' - t)$ is the position of the liner at time t, apex position A. $BC = v_0(t' - t)$ is the path of the liner particles from t to t', when they reach the apex. Thus the apex moves from A to B in time t to t' and $AB = u(t' - t)$. In time t to t' the detonation has moved from C to D and $CD = u_0(t' - t)/\cos(\alpha)$, and the liner lies along BD. The relevant angles are $\angle OCA = \angle BDC = (\beta - \alpha)$, $\angle ACB = \angle BCD = \angle CBD = [\pi - (\beta - \alpha)]/2$ and $\angle ABC = [\pi - (\beta + \alpha)]/2$, and the dynamics described by $\triangle ABC$ and $\triangle ABD$. (a) Detonation at C at time t. (b) Detonation at D at time t'.

The analysis of the problem in the laboratory frame involves the geometry of the flow. Let α be the half angle of the wedge, and β the angle of the flow the fluidised liner makes with the centre line, Figure 1.3. We assume that in the laboratory frame, the fluidised liner velocity bisects the initial and accelerated planes of the layer. In Figure 1.3(a) we see the situation where the detonation has reached the point C at time t. The line AC represents the line of the fluidised layer at this time, A being at the apex at time t. AC also represents the line of the flow in the apex frame. The fluid itself moves along the line BC bisecting the angle $\angle ACD$ in the laboratory frame, reaching the axis at time t'. Thus AB is the movement of the apex in time $t' - t$, namely $u\,(t' - t)$, and BC the flow of the fluid over the same time $v_0\,(t' - t)$. Since $\angle ACB = [\pi - (\beta - \alpha)]/2$ and $\angle BAC = \beta$, the velocity of the apex

$$u = v_0 \frac{\cos[\frac{1}{2}\,(\beta - \alpha)]}{\sin\beta}$$

Furthermore, since the velocity of the fluid along the arms in the apex frame $\mathbf{v}' = \mathbf{v}_0 - \mathbf{u}$, we see that it is given by the third side of the $\triangle ABC$, namely

$$v' = v_0 \frac{\cos[\frac{1}{2}\,(\beta + \alpha)]}{\sin\beta}$$

since $\angle ABC = [\pi - (\beta + \alpha)]/2$. Alternatively AC is the line of flow in the apex frame for fluid starting at t and arriving at the apex at t', thus $AC = v'\,(t' - t)$ and we obtain the same result for v'.

It now remains to calculate the value of β from the speed of the detonation through the block, u_0. Referring to Figure 1.3(b), the detonation reaches point D at time t'.

$\triangle BCD$ is isosceles since $\angle BCD = \angle CBD = [\pi - (\beta - \alpha)]/2$, so that

$$\sin\left[\frac{1}{2}(\beta - \alpha)\right] = \frac{v_0 \cos\alpha}{2u_0} \qquad \text{and} \qquad \beta = \alpha + 2 \arcsin\left(\frac{v_0 \cos\alpha}{2u_0}\right)$$

In the apex frame, the component of momentum along the axis after collision must balance that before

$$m_s v_s - m_j v_j = m_0\, v' \cos\beta \tag{1.54}$$

where m_s, m_j and $m_0 = m_s + m_j$ are the masses of the slug, jet and liner respectively, and the angle of the incoming flow to the axis is β. The laboratory velocities of the slug and the jet respectively are given by the transformation of the velocities $\mp v'$ from the apex frame back into the laboratory frame:

$$v_s = u - v' \qquad \text{and} \qquad v_j = u + v' \tag{1.55}$$

Thus we obtain the velocity of the slug and the jet in the laboratory frame in terms of the experimental parameters $(u_0,\, v_0,\, \alpha)$

$$\begin{aligned} v_s &= v_0 \left\{ \frac{\cos[\frac{1}{2}(\beta - \alpha)]}{\sin\beta} - \frac{\cos[\frac{1}{2}(\beta + \alpha)]}{\sin\beta} \right\} &= 2\,\frac{\sin(\beta/2)\,\sin(\alpha/2)}{\sin\beta}\, v_0 \\ v_j &= v_0 \left\{ \frac{\cos[\frac{1}{2}(\beta - \alpha)]}{\sin\beta} + \frac{\cos[\frac{1}{2}(\beta + \alpha)]}{\sin\beta} \right\} &= 2\,\frac{\cos(\beta/2)\,\cos(\alpha/2)}{\sin\beta}\, v_0 \end{aligned} \tag{1.56}$$

respectively, and the masses are

$$m_s = m_0\,\frac{(1 + \cos\beta)}{2} \qquad \text{and} \qquad m_j = m_0\,\frac{(1 - \cos\beta)}{2} \tag{1.57}$$