1

Combinatorics

Combinatorics deals with the cardinality of classes of objects. The first example that jumps to our minds is the computation of how many triplets can be drawn from 90 different balls. In this chapter and the next we are going to compute the cardinality of several classes of objects.

1.1 Binomial coefficients

1.1.1 Pascal triangle

Binomial coeffcients are defined as

$$\binom{n}{k} := \begin{cases} 1 & \text{if } k = 0, \\ 0 & k > n, \\ \dfrac{n(n-1)\dots(n-k+1)}{k!} & \text{if } n \geq 1 \text{ and } 1 \leq k \leq n. \end{cases}$$

Binomial coefficients are usually grouped in an infinite matrix

$$\mathbf{C} := (\mathbf{C}^n_k), \ \ n, k \geq 0, \qquad \mathbf{C}^n_k := \binom{n}{k}$$

called a *Pascal triangle* given the triangular arrangement of the nonzero entries, see Figure 1.1. Here and throughout the book we denote the entries of a matrix (finite or infinite) $\mathbf{A} = (a^i_j)$ where the superscript i and the subscript j mean the ith row and the jth column, respectively. Notice that the entries of each row of $\mathbf{C}$ are zero if the column index is large enough, $\mathbf{C}^i_j = 0 \ \forall i, j$ with $j > i \geq 0$. We also recall the *Newton binomial formula*,

$$(1+z)^n = \binom{n}{0} + \binom{n}{1}z + \cdots + \binom{n}{n}z^n = \sum_{k=0}^{n} \binom{n}{k} z^k = \sum_{k=0}^{\infty} \binom{n}{k} z^k.$$

A First Course in Probability and Markov Chains, First Edition. Giuseppe Modica and Laura Poggiolini.

$$\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\
1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\
1 & 3 & 3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\
1 & 4 & 6 & 4 & 1 & 0 & 0 & 0 & 0 & 0 & \dots \\
1 & 5 & 10 & 10 & 5 & 1 & 0 & 0 & 0 & 0 & \dots \\
1 & 6 & 15 & 20 & 15 & 6 & 1 & 0 & 0 & 0 & \dots \\
1 & 7 & 21 & 35 & 35 & 21 & 7 & 1 & 0 & 0 & \dots \\
1 & 8 & 28 & 56 & 70 & 56 & 28 & 8 & 1 & 0 & \dots \\
1 & 9 & 36 & 84 & 126 & 126 & 84 & 36 & 9 & 1 & \dots \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots
\end{pmatrix}$$

Figure 1.1 *Pascal matrix of binomial coefficients* $(\mathbf{C}^n_k)$, $k, n \geq 0$.

Thus formula can be proven with an induction argument on n or by means of Taylor formula.

1.1.2 Some properties of binomial coefficients

Many formulas are known on binomial coefficients. In the following proposition we collect some of the simplest and most useful ones.

Proposition 1.1 *The following hold.*

(i) $\displaystyle \binom{n}{k} = \frac{n!}{k!(n-k)!} \quad \forall k, n,\ 0 \leq k \leq n.$

(ii) $\displaystyle \binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1} \qquad \forall k, n,\ 1 \leq k \leq n.$

(iii) Stifel formula $\displaystyle \binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1} \qquad \forall k, n,\ 1 \leq k \leq n.$

(iv) $\displaystyle \binom{n}{j}\binom{j}{k} = \binom{n}{k}\binom{n-k}{j-k} \qquad \forall k, j, n,\ 0 \leq k \leq j \leq n.$

(v) $\displaystyle \binom{n}{k} = \binom{n}{n-k} \qquad \forall k,\ 0 \leq k \leq n.$

(vi) the map $k \mapsto \binom{n}{k}$ *achieves its maximum at* $k = \lfloor \frac{n}{2} \rfloor$.

(vii) $\displaystyle \sum_{k=0}^{n} \binom{n}{k} = 2^n \ \forall n \geq 0.$

(viii) $\displaystyle \sum_{k=0}^{n} (-1)^k \binom{n}{k} = \delta_{0,n} = \begin{cases} 1 & \textit{if } n = 0, \\ 0 & \textit{if } n \neq 0. \end{cases}$

(ix) $\displaystyle \binom{n}{k} \leq 2^n \ \forall k, n, 0 \leq k \leq n.$

Proof. Formulas (i), (ii), (iii), (iv) and (v) directly follow from the definition. (vi) is a direct consequence of (v). (vii) and (viii) follow from the Newton binomial formula $\sum_{k=0}^{n}\binom{n}{k}z^k = (1+z)^n$ choosing $z = 1$ and $z = -1$. Finally, (ix) is a direct consequence of (vii).

Estimate (ix) in Proposition 1.1 can be made more precise. For instance, from the Stirling asymptotical estimate of the factorial,

$$\frac{n!}{n^n e^{-n}\sqrt{2\pi n}} \to 1 \qquad \text{as } n \to \infty$$

one gets

$$(2n)! = 4^n n^{2n} e^{-2n} \sqrt{4\pi n}\,(1 + o(1)),$$

$$(n!)^2 = n^{2n} e^{-2n} 2\pi n\,(1 + o(1)),$$

so that

$$\binom{2n}{n} = \frac{4^n}{\sqrt{\pi n}}\,\frac{1+o(1)}{1+o(1)}$$

or, equivalently,

$$\frac{\binom{2n}{n}}{\frac{4^n}{\sqrt{\pi n}}} \to 1 \qquad \text{as } n \to \infty. \tag{1.1}$$

Estimate (1.1) is ‘accurate’ also for small values of n. For instance, for $n = 4$, one has $\binom{8}{4} = 70$ and $4^4 \frac{1}{\sqrt{\pi 4}} \simeq 72.2$.

1.1.3 Generalized binomial coefficients and binomial series

For $\alpha \in \mathbb{R}$ we define the sequence $\binom{\alpha}{n}$ of *generalized binomial coefficients* as

$$\binom{\alpha}{n} := \begin{cases} 1 & \text{if } n = 0, \\ \dfrac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-n+1)}{n!} & \text{if } n \geq 1. \end{cases}$$

Notice that $\binom{\alpha}{k} \neq 0\ \forall k$ if $\alpha \notin \mathbb{N}$ and $\binom{\alpha}{k} = 0\ \forall k \geq \alpha + 1$ if $\alpha \in \mathbb{N}$. The power series

$$\sum_{n=0}^{\infty} \binom{\alpha}{n} z^n \tag{1.2}$$

is called the *binomial series*.

Proposition 1.2 (Binomial series) *The binomial series converges if* $|z| < 1$ *and*

$$\sum_{n=0}^{\infty} \binom{\alpha}{n} z^n = (1+z)^\alpha \qquad \textit{if } |z| < 1.$$

Proof. Since

$$\frac{\left|\binom{\alpha}{n+1}\right|}{\left|\binom{\alpha}{n}\right|} = \frac{|\alpha - n|}{|n+1|} \to 1 \qquad \text{as } n \to \infty,$$

it is well known that $\sqrt[n]{|a_n|} \to 1$ as well; thus, the radius of the power series in (1.2) is 1.

Differentiating n times the map $z \mapsto (1+z)^\alpha$, one gets $D^n(1+z)^\alpha = \alpha(\alpha - 1)\cdots(\alpha - n + 1)(1+z)^{\alpha-n}$, so that the series on the left-hand side of (1.2) is the McLaurin expansion of $(1+z)^\alpha$.

Another proof is the following. Let $S(z) := \sum_{n=0}^{\infty}\binom{\alpha}{n}z^n$, $|z| < 1$, be the sum of the binomial series. Differentiating one gets

$$(1+z)S'(z) = \alpha S(z), \qquad |z| < 1,$$

hence

$$\left(\frac{S(z)}{(1+z)^\alpha}\right)' = \frac{(1+z)S'(z) - \alpha S(z)}{(1+z)^{\alpha+1}} = 0.$$

Thus there exists $c \in \mathbb{R}$ such that $S(z) = c\,(1+z)^\alpha$ if $|z| < 1$. Finally, $c = 1$ since $S(0) = 1$.

Proposition 1.3 *Let $\alpha \in \mathbb{R}$. The following hold.*

(i) $\displaystyle\binom{\alpha}{k} = \frac{\alpha}{k}\binom{\alpha-1}{k-1} \qquad \forall k \geq 1.$

(ii) $\displaystyle\binom{\alpha}{k} = \binom{\alpha-1}{k} + \binom{\alpha-1}{k-1} \qquad \forall k \geq 1.$

(iii) $\displaystyle\binom{-\alpha}{k} = (-1)^k\binom{\alpha+k-1}{k} \qquad \forall k \geq 0.$

Proof. The proofs of (i) and (ii) are left to the reader. Proving (iii) is a matter of computation:

$$\binom{-\alpha}{k} = \frac{-\alpha(-\alpha-1)\cdots(-\alpha-k+1)}{k!} = (-1)^k\frac{\alpha(\alpha+1)\cdots(\alpha+k-1)}{k!}$$
$$= (-1)^k\binom{\alpha+k-1}{k}.$$

A few negative binomial coefficients are quoted in Figure 1.2.

1.1.4 Inversion formulas

For any N, the matrix $\mathbf{C}_N := (\mathbf{C}_k^n)$, $n, k = 0, \ldots, N$, is lower triangular and all its diagonal entries are equal to 1. Hence 1 is the only eigenvalue of $\mathbf{C}_N$

$$
\left(\begin{array}{ccccccccccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\
1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & \dots \\
1 & -2 & 3 & -4 & 5 & -6 & 7 & -8 & 9 & -10 & \dots \\
1 & -3 & 6 & -10 & 15 & -21 & 28 & -36 & 45 & -55 & \dots \\
1 & -4 & 10 & -20 & 35 & -56 & 84 & -120 & 165 & -220 & \dots \\
1 & -5 & 15 & -35 & 70 & -126 & 210 & -330 & 495 & -715 & \dots \\
1 & -6 & 21 & -56 & 126 & -252 & 462 & -792 & 1\,287 & -2\,002 & \dots \\
1 & -7 & 28 & -84 & 210 & -462 & 924 & -1\,716 & 3\,003 & -5\,005 & \dots \\
1 & -8 & 36 & -120 & 330 & -792 & 1\,716 & -3\,432 & 6\,435 & -11\,440 & \dots \\
1 & -9 & 45 & -165 & 495 & -1\,287 & 3\,003 & -6\,435 & 12\,870 & -24\,310 & \dots \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots
\end{array}\right.
$$

Figure 1.2 The coefficients $\binom{-n}{k}$.

with algebraic multiplicity N. In particular $\mathbf{C}_N$ is invertible, its inverse is lower triangular, all its entries are integers and its diagonal entries are equal to 1.

Theorem 1.4 *For any* $n, k = 0, \dots, N$, $(\mathbf{C}_N^{-1})^n_k = (-1)^{n+k}\binom{n}{k}$.

Proof. Let $\mathbf{B} := (\mathbf{B}^k_n)$, $\mathbf{B}^k_n := (-1)^{n+k}\binom{n}{k}$ so that both $\mathbf{B}$ and $\mathbf{C}_N\mathbf{B}$ are lower triangular, i.e. $(\mathbf{C}_N\mathbf{B})^n_k = 0$ if $0 \le n < k$. Moreover, (iv) and (viii) of Proposition 1.1 yield for any $n \ge k$

$$
\begin{aligned}
(\mathbf{C}_N\mathbf{B})^n_k &= \sum_{j=1}^{N}\binom{n}{j}(-1)^{j+k}\binom{j}{k} = \sum_{j=k}^{n}(-1)^{j+k}\binom{n}{j}\binom{j}{k} \\
&= \binom{n}{k}\sum_{j=k}^{n}(-1)^{j+k}\binom{n-k}{j-k} = \binom{n}{k}\sum_{i=0}^{n-k}(-1)^{i}\binom{n-k}{i} \\
&= \binom{n}{k}\delta_{0,n-k} = \delta_{n,k}.
\end{aligned}
$$

A few entries of the inverse of the matrix of binomial coefficients are shown in Figure 1.3. As a consequence of Theorem 1.4 the following *inversion formulas* hold.

Corollary 1.5 *Two sequences* $\{x_n\}$, $\{y_n\}$ *satisfy*

$$
y_n = \sum_{k=0}^{n}\binom{n}{k}x_k, \qquad \forall n \ge 0
$$

if and only if

$$
x_n = \sum_{k=0}^{n}(-1)^{n+k}\binom{n}{k}y_k, \qquad \forall n \ge 0.
$$

$$\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\
-1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\
1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\
-1 & 3 & -3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\
1 & -4 & 6 & -4 & 1 & 0 & 0 & 0 & 0 & 0 & \dots \\
-1 & 5 & -10 & 10 & -5 & 1 & 0 & 0 & 0 & 0 & \dots \\
1 & -6 & 15 & -20 & 15 & -6 & 1 & 0 & 0 & 0 & \dots \\
-1 & 7 & -21 & 35 & -35 & 21 & -7 & 1 & 0 & 0 & \dots \\
1 & -8 & 28 & -56 & 70 & -56 & 28 & -8 & 1 & 0 & \dots \\
-1 & 9 & -36 & 84 & -126 & 126 & -84 & 36 & -9 & 1 & \dots \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots
\end{pmatrix}$$

Figure 1.3 The inverse of the matrix of binomial coefficients $\mathbf{C}_N^{-1}$.

Similarly,

Corollary 1.6 *Two N-tuples or real numbers* $\{x_n\}$ *and* $\{y_n\}$ *satisfy*

$$y_n = \sum_{k=n}^{N} \binom{k}{n} x_k, \qquad \forall n,\ 0 \le n \le N,$$

if and only if

$$x_n = \sum_{k=n}^{N} (-1)^{n+k} \binom{k}{n} y_k, \qquad \forall n,\ 0 \le n \le N.$$

1.1.5 Exercises

Exercise 1.7 *Prove Newton binomial formula:*

(i) directly, with an induction argument on n;

(ii) applying Taylor expansion formula;

(iii) starting from the formula $D((1+z)^n) = n(1+z)^{n-1}$.

Exercise 1.8 *Differentiating the power series, see Appendix A, prove the formulas in Figure 1.4.*

Solution. Differentiating the identity $\sum_{k=0}^{\infty} z^k = \frac{1}{1-z}$, $|z| < 1$, we get for $|z|<1$

$$\sum_{k=0}^{\infty} kz^k = z \sum_{k=0}^{\infty} D(z^k) = zD\Big(\sum_{k=0}^{\infty} z^k\Big) = zD\Big(\frac{1}{1-z}\Big) = \frac{z}{(1-z)^2};$$

Let $z \in \mathbb{C}$, $|z| < 1$, and $n \in \mathbb{Z}$. We have the followings.

(i) $$\sum_{k=0}^{\infty} kz^k = \frac{z}{(1-z)^2},$$

(ii) $$\sum_{k=0}^{\infty} k^2 z^{k-1} = D\Big(\frac{z}{(1-z)^2}\Big) = \frac{1+z}{(1-z)^3},$$

(iii) $$\sum_{k=0}^{\infty} \binom{n}{k} z^k = (1+z)^n,$$

(iv) $$\sum_{k=0}^{\infty} k\binom{n}{k} z^k = nz(1+z)^{n-1},$$

(v) $$\sum_{k=0}^{\infty} k^2\binom{n}{k} z^k = nz(1+nz)(1+z)^{n-2}.$$

Figure 1.4 The sum of a few series related to the geometric series.

$$\sum_{k=0}^{\infty} k^2 z^{k-1} = \sum_{k=0}^{\infty} D(kz^k)$$
$$= D\Big(\sum_{k=0}^{\infty} kz^k\Big) = D\Big(\frac{z}{(1-z)^2}\Big) = \frac{1+z}{(1-z)^3}.$$

Moreover, for any non-negative integer n, differentiating the identities

$$\sum_{k=0}^{\infty} \binom{n}{k} z^k = (1+z)^n \qquad \text{and} \qquad \sum_{k=0}^{\infty} \binom{-n}{k} z^k = (1+z)^{-n}$$

for any $|z| < 1$, we get

$$\sum_{k=0}^{\infty} k\binom{n}{k} z^k = z\sum_{k=0}^{n} D\Big(\binom{n}{k} z^k\Big)$$
$$= zD\Big(\sum_{k=0}^{n} \binom{n}{k} z^k\Big) = zD((1+z)^n) = nz(1+z)^{n-1};$$

$$\sum_{k=0}^{\infty} k^2\binom{n}{k} z^k = z\sum_{k=0}^{n} D\Big(k\binom{n}{k} z^k\Big) = zD\Big(\sum_{k=0}^{n} \binom{n}{k} z^k\Big)$$
$$= zD(nz(1+z)^{n-1}) = nz(1+nz)(1+z)^{n-2}.$$

1.2 Sets, permutations and functions

1.2.1 Sets

We recall that a *finite set* A is an *unordered* collection of *pairwise different* objects. For example, the collection A hose objects are 1,2,3 is a finite set which we denote as $A = \{1, 2, 3\}$; the collection 1,2,2,3 is not a finite set, and $\{1, 2, 3\}$ and $\{2, 1, 3\}$ are the same set.

If A is a finite set with n objects (or elements), we may enumerate the elements of A so that $A = \{x_1, \ldots, x_n\}$. Therefore, for counting purposes, we can assume without loss of generality that $A = \{1, \ldots, n\}$. The number n is the *cardinality* of A, and we write $|A| = n$.

Proposition 1.9 *Let A be a finite set with n elements,* $n \geq 1$. *There are* $\mathbf{C}_k^n = \binom{n}{k}$ *subsets of A with k elements.*

Proof. Different proofs can be done. We propose one of them. Let $d_{n,k}$ be the number of subsets of A with k elements. Obviously, $d_{n,1} = n$ and $d_{n,n} = 1$. For $2 \leq k \leq n - 1$, assume we have n football players and we want to select a team of k of them, including the captain of the team. We may proceed in the following way: first we choose the team of k-players: $d_{n,k}$ different teams can be selected. Then, among the team, we select the captain: k different choices are possible: so there are $kd_{n,k}$ ways to select the team and the captain. However, we can proceed in another way: first we choose the captain among the n players: there are n different possible choices. Then we choose $k - 1$ players among the remaining $n - 1$ players: there are $d_{n-1,k-1}$ possible choices. Thus

$$d_{n,k} = \frac{n}{k} d_{n-1,k-1}$$

which by induction, gives

$$\begin{aligned} d_{n,k} &= \frac{n}{k}\frac{n-1}{k-1}\cdots\frac{n-k+2}{2}d_{n-k+1,1} \\ &= \frac{n}{k}\frac{n-1}{k-1}\cdots\frac{n-k+2}{2}\frac{n-k+1}{1} = \binom{n}{k}. \end{aligned}$$

1.2.2 Permutations

Let N be a finite set and let n be its cardinality. Without loss of generality, we can assume $N = \{1, 2, \ldots, n\}$. A *permutation* of N is an injective (and thus one-to-one) mapping $\pi : N \to N$. Since composing bijective maps yields another bijective map, the family of permutations of a set N is a group with respect to the composition of maps; the unit element is the identity map; this group is called the *group of permutations of* N. It is denoted as S_n or $\mathcal{P}_n$. Notice that $\mathcal{P}_n$ is a not a commutative group if $n \geq 3$.

Each permutation is characterized by its *image-word* or *image-list*, i.e. by the n-tuple $(\pi(1), \ldots, \pi(n))$. For instance, the permutation $\pi \in \mathcal{P}_6$ defined by

$\pi(1) = 2$, $\pi(2) = 3$, $\pi(3) = 1$, $\pi(4) = 4$, $\pi(5) = 6$ and $\pi(6) = 5$ is denoted as

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 1 & 4 & 6 & 5 \end{pmatrix}.$$

or, in brief, with its image-word 231465.

The set of permutations of $N = \{1 \dots, n\}$ has $n!$ elements,

$$|\mathcal{P}_n| = n!$$

In fact, the image $\pi(1)$ of 1 can be chosen among n possible values, then the image $\pi(2)$ of 2 can be chosen among $n - 1$ possible values and so on. Hence

$$|\mathcal{P}_n| = n(n-1)(n-2)\cdots 3 \cdot 2 \cdots 1 = n!$$

1.2.2.1 Derangements

Let $\pi \in \mathcal{P}_n$ be a permutation of $N = \{1, \dots, n\}$. A point $x \in N$ is a *fixed point* of π if $\pi(x) = x$.

We now compute the cardinality d_n of the set $\mathcal{D}_n$ of *permutations without fixed points*, also called *derangements*.

$$\mathcal{D}_n := \left\{\pi \in \mathcal{P}_n \,\middle|\, \pi(i) \neq i \;\forall i \in N\right\}.$$

Proposition 1.10 *The cardinality of $\mathcal{D}_n$ is*

$$d_n = n! \sum_{j=0}^{n} (-1)^j \frac{1}{j!} \qquad \forall n \geq 1.$$

Proof. If a permutation of N has j fixed points, $0 \leq j \leq n$, then it is a derangement of the other $n - j$ points of N. Thus, a permutation with j fixed points splits as a couple: the set of its fixed points and a derangement of $n - j$ points. There are $\binom{n}{j}$ different choices for the j fixed points and d_{n-j} derangements of the remaining $n - j$ points, so that, the possible permutations of N with exactly j fixed points are $\binom{n}{j} d_{n-j}$ (where $d_0 = 1$). Thus $|\mathcal{P}_n| = \sum_{j=0}^{n} \binom{n}{j} d_{n-j}$ $\forall n \geq 1$, i.e.

$$n! = \sum_{j=0}^{n} \binom{n}{j} d_{n-j} \qquad \forall n \geq 0. \tag{1.3}$$

The inversion formula of binomial coefficients, see Corollary 1.5, reads

$$d_n = \sum_{j=0}^{n} (-1)^{(n+j)} \binom{n}{j} j! = n! \sum_{j=0}^{n} \frac{(-1)^j}{j!} \qquad \forall n \geq 0.$$

0, 0, 1, 2, 9, 44, 265, 1 854, 14 833, 133 496, 1 334 961, 14 684 570, ...

Figure 1.5 *From the left, the numbers* $d_0, d_1, d_2, d_3, \ldots$ *of derangements of* $0, 1, 2, 3, \ldots$ *points.*

Corollary 1.11 *The number* d_n *of derangements of n points is the nearest integer to* $n!/e$.

Proof. The elementary estimate between the exponential and its McLaurin expansion gives

$$\Big|e^x - \sum_{j=0}^{n} \frac{x^j}{j!}\Big| \le \frac{|x|^{n+1}}{(n+1)!}, \qquad \forall x \le 0;$$

hence for $x = -1$ we get

$$\Big|\frac{1}{e} - \sum_{j=0}^{n} \frac{(-1)^j}{j!}\Big| \le \frac{1}{(n+1)!},$$

so that, from Proposition 1.10 one gets

$$\Big|d_n - \frac{n!}{e}\Big| = n!\Big|\sum_{j=0}^{n} \frac{(-1)^j}{j!} - \frac{1}{e}\Big| \le \frac{n!}{(n+1)!} = \frac{1}{n+1} \le \frac{1}{3}$$

for each $n \ge 2$.

Figure 1.5 contains the first elements of the sequence $\{d_n\}$.

1.2.3 Multisets

Another interesting structure is an unordered list of elements taken from a given set A. This structure is called a *multiset* on A. More formally, a *multiset* on a set A is a couple (A, a) where A is a given set and $a : A \to \mathbb{N} \cup \{+\infty\}$ is the *multiplicity function* which counts 'how many times' an element $x \in A$ appears in the multiset. Clearly, each set is a multiset where each object has multiplicity 1. We denote as $\{a^2, b^2, c^5\}$ or $a^2b^2c^5$ the multiset on $A := \{a, b, c\}$ where a and b have multiplicity 2 and c has multiplicity 5. The cardinality of a multiset (A, a) is $\sum_{x\in A} a(x)$ and is denoted by $|(A, a)|$ or $\#(A, a)$. For instance, the cardinality of $a^2b^2c^5$ is 9.

If B is a subset of A, then B is also the multiset (A, a) on A where

$$a(x) = \begin{cases} 1 & \text{if } x \in B, \\ 0 & \text{if } x \notin B. \end{cases}$$

Given two multisets (B, b) and (A, a), we say that (B, b) is *included* in (A, a) if $B \subset A$ and $b(x) \leq a(x)$ $\forall x \in B$. In this case, $(B, b) = (A, \widehat{b})$ where

$$\widehat{b}(x) = \begin{cases} b(x) & \text{if } x \in B, \\ 0 & \text{if } x \notin B. \end{cases}$$

Proposition 1.12 *Let A be a finite set, $|A| = n$. Let (A, a) be a multiset on A and let k be a non-negative integer such that $k \leq a(x)$ $\forall x \in A$. The multisets included in (A, a) with k elements are*

$$\binom{n+k-1}{k}.$$

Proof. Let $A = \{1, \ldots, n\}$. A multiset S of cardinality k included in (A, a) contains the element 1 x_1 times, the element 2 x_2 times, and so on, with $x_1 + x_2 + \cdots + x_n = k$. Moreover, the n-tuple $(x_1, \ldots, x_n)$ characterizes S. We can associate to a n-tuple $(x_1, \ldots, x_n)$ the binary sequence

$$\underbrace{00\ldots0}_{x_1}1\underbrace{00\ldots0}_{x_2}1\ldots1\underbrace{00\ldots0}_{x_{n-1}}1\underbrace{00\ldots0}_{x_n} \tag{1.4}$$

where the symbol 1 denotes the fact that we are changing the element of A. This is a binary word of length $n + k - 1$ with k zeroes.

The correspondence described above is a one-to-one correspondence between the set of multisets of cardinality k included in (A, a) and the set of binary words of length $n + k - 1$ with k zeroes. There are exactly

$$\binom{n+k-1}{k}$$

different words of this kind, so that the claim is proven.

1.2.4 Lists and functions

Given a set A, a *list* of k objects from the set A or a *k-word* with symbols in A is an ordered k-tuple of objects. For instance, if $A = \{1, 2, 3\}$, then the 6-tuples (1,2,3,3,2,1) and (3,2,1,3,2,1) are two different 6-words of objects in A. In these lists, or words, repetitions are allowed and the order of the objects is taken into account. Since each element of the list can be chosen independently of the others, there are n possible choices for each object in the list. Hence, the following holds.

Proposition 1.13 *The number of k-lists of objects from a set A of cardinality n is n^k.*

A *function* $f : X \to A$ is defined by the value it assumes on each element of X: if $f : \{1, \ldots, k\} \to A$, then f is defined by the k-list $(f(1), f(2), \ldots, f(k))$, which we refer to as the *image-list* or *image-word* of f. Conversely, each k-list $(a_1, a_2, \dots, a_k)$ with symbols from A defines the function $f : \{1, \dots, k\} \to A$ given by $f(i) := a_i \ \forall i$. If $|A|$ is finite, $|A| = n$, we have a one-to-one correspondence between the set $\mathcal{F}_n^k$ of maps $f : \{1, \dots, k\} \to A$, and the set of the k-lists with symbols in A. Therefore, we have the following.

Proposition 1.14 *The number of functions in* $\mathcal{F}_n^k$ *is* $F_n^k := |\mathcal{F}_n^k| = n^k$.

1.2.5 Injective functions

We use the symbol $\mathcal{I}_n^k$ to denote the set of injective functions $f : \{1, \dots, k\} \to A$, $|A| = n$, $k \le n$. Let $I_n^k = |\mathcal{I}_n^k|$. Obviously, $I_n^k = 0$ if $k > n$. The image-list of an injective function $f \in \mathcal{I}_n^k$ is a k-word of pairwise different symbols taken from A To form any such image-list, one can choose the first entry among n elements, the second entry can be chosen among $n - 1$ elements, …, the kth entry can be chosen among the remaining $n - k + 1$ elements of A, so that we have the following.

Proposition 1.15 *The cardinality* I_n^k *of* $\mathcal{I}_n^k$ *is*

$$I_n^k = |\mathcal{I}_n^k| = n(n-1) \cdot \dots \cdot (n-k+1) = k!\binom{n}{k} = \frac{n!}{(n-k)!}.$$

Some of the I_n^k*'s are in Figure 1.6.*

1.2.6 Monotone increasing functions

Let $\mathcal{C}_n^k$, $k \le n$, be the set of strictly monotone increasing functions $f : \{1, \dots, k\} \to \{1, \dots, n\}$. The image-list of any such function is an ordered k-tuple of strictly increasing–hence pairwise disjoint–elements of $\{1, \dots, n\}$. The k-tuple is thus identified by the subset of the elements of $\{1, \dots, n\}$ appearing in it, so that we have the following.

$$\left(\begin{array}{rrrrrrrrrrl}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \dots \\
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & \dots \\
0 & 0 & 2 & 6 & 12 & 20 & 30 & 42 & 56 & 72 & \dots \\
0 & 0 & 0 & 6 & 24 & 60 & 120 & 210 & 336 & 504 & \dots \\
0 & 0 & 0 & 0 & 24 & 120 & 360 & 840 & 1\,680 & 3\,024 & \dots \\
0 & 0 & 0 & 0 & 0 & 120 & 720 & 2\,520 & 6\,720 & 15\,120 & \dots \\
0 & 0 & 0 & 0 & 0 & 0 & 720 & 5\,040 & 20\,160 & 60\,480 & \dots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 5\,040 & 40\,320 & 181\,440 & \dots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 40\,320 & 362\,880 & \dots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 362\,880 & \dots \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots
\end{array}\right.$$

Figure 1.6 The cardinality I_n^k *of the set of injective maps* $\mathcal{I}_n^k$ *for* $n, k \ge 0$.

Proposition 1.16 *The cardinality C_n^k of $\mathcal{C}_n^k$ is*

$$C_n^k := |\mathcal{C}_n^k| = \binom{n}{k} = \mathbf{C}_k^n = (\mathbf{C}^T)_n^k.$$

1.2.7 Monotone nondecreasing functions

Let $\mathcal{D}_n^k$ be the class of monotone nondecreasing functions $f : \{1, \dots, k\} \to \{1, \dots, n\}$. The image-list of any such function is a nondecreasing ordered k-tuple of elements of $\{1, \dots, n\}$, so that elements can be repeated. The functions in $\mathcal{D}_n^k$ are as many as the multisets with cardinality k included in a multiset (A, a), where $A = \{1, \dots, n\}$ and $a(x) \geq k \ \forall x \in A$. Thus, see Proposition 1.12, we have the following.

Proposition 1.17 *The cardinality D_n^k of $\mathcal{D}_n^k$ is*

$$D_n^k := |\mathcal{D}_n^k| = |\mathcal{C}_{n+k-1}^k| = \binom{n+k-1}{k}.$$

Another proof of Proposition 1.17. Consider the map $\phi : \mathcal{D}_n^k \to \mathcal{F}_{n+k-1}^k$ defined by $\phi(f)(i) := f(i) + i - 1 \ \forall i \in \{1, \dots, k\}$, $\forall f \in \mathcal{D}_n^k$. Obviously, if $f \in \mathcal{D}_n^k$, then $\phi(f)$ is strictly monotone increasing, $\phi(f) \in \mathcal{C}_{n+k-1}^k$. Moreover, the correspondence $\phi : \mathcal{D}_n^k \to \mathcal{C}_{n+k-1}^k$ is one-to-one, thus

$$D_n^k = |\mathcal{D}_n^k| = |\mathcal{C}_{n+k-1}^k| = \binom{n+k-1}{k}.$$

Yet another proof of Proposition 1.17. We are now going to define a one-to-one correspondence between a family of multisets and $\mathcal{D}_n^k$. Let (A, a) be a multiset on $A = \{1, \dots, n\}$ with $a(x) \geq k \ \forall k$. For any multiset (S, n_S) of cardinality k included in (A, a), let $f_S : A \to \{0, \dots, k\}$ be the function defined by

$$f_S(x) := \sum_{y \leq x} n_S(y),$$

i.e. for each $x \in A$, $f_S(x)$ is the sum of the multiplicities $n_S(y)$ of all elements $y \in A$, $y \leq x$. f_S is obviously a nondecreasing function and $f_S(n) = k$. Moreover, it is easy to show that the map

$$S \mapsto f_S$$

is a one-to-one correspondence between the family of the multisets included in (A, a) of cardinality k and the family of monotone nondecreasing functions from $\{1, \dots, n\}$ to $\{0, \dots, k\}$ such that $f(k) = 1$. In turn, there is an obvious one-to-one correspondence between this class of functions and the class of monotone nondecreasing functions from $\{1, \ldots, n-1\}$ to $\{0, \ldots, k\}$. Thus, applying Proposition 1.17 we get

$$|\mathcal{D}_{k+1}^{n-1}| = \binom{k+1+(n-1)-1}{n-1} = \binom{n+k-1}{k}.$$

1.2.8 Surjective functions

The computation of the number of surjective functions is more delicate. Let $\mathcal{S}_n^k$ denote the family of surjective functions from $\{1, \ldots, k\}$ onto $\{1, \ldots, n\}$ and let

$$S_n^k = \begin{cases} 1 & \text{if } n = k = 0, \\ 0 & \text{if } n = 0,\ k > 0 \\ |\mathcal{S}_n^k| & \text{if } n \geq 1. \end{cases}$$

Obviously, $S_n^k = |\mathcal{S}_n^k| = 0$ if $k < n$. Moreover, if $k = n \geq 1$, then a function $f : \{1, \ldots, n\} \to \{1, \ldots, n\}$ is surjective if and only if f is injective, so that $S_n^n = |\mathcal{S}_n^k| = I_n^n = n!$

If $k > n \geq 1$, then $\mathcal{S}_n^k \neq \emptyset$. Observe that any function is onto on its range. Thus, for each $j = 1, \ldots, n$, consider the set A_j of functions $f : \{1, \ldots, k\} \to \{1, \ldots, n\}$ whose range has cardinality j. We must have

$$n^k = |\mathcal{F}_n^k| = \sum_{j=1}^{n} |A_j|.$$

There are exactly $\binom{n}{j}$ subsets of $\{1, \ldots, n\}$ with cardinality j and there are S_j^k different surjective functions onto each of these sets. Thus, $|A_j| = \binom{n}{j} S_j^k$ and

$$n^k = \sum_{j=1}^{n} \binom{n}{j} S_j^k \qquad \forall n \geq 1.$$

Since we defined $S_0^k = 0$, we get

$$n^k = \sum_{j=0}^{n} \binom{n}{j} S_j^k \qquad \forall n \geq 0. \tag{1.5}$$

Therefore, applying the inversion formula in Corollary 1.5 we conclude the following.

Proposition 1.18 *The cardinality S_n^k of the set $\mathcal{S}_n^k$ of surjective functions from $\{1, \ldots, k\}$ onto $\{1, \ldots, n\}$ is*

$$S_n^k = \sum_{j=0}^{n} (-1)^{n+j} \binom{n}{j} j^k = \sum_{j=0}^{n} (-1)^j \binom{n}{j} (n-j)^k \qquad \forall n, k \geq 1.$$

We point out that the equality holds also if $k \leq n$ so that

$$\frac{1}{n!}\sum_{j=0}^{n}(-1)^j\binom{n}{j}(n-j)^k = \frac{1}{n!}S_n^k = \begin{cases} 1 & \text{if } k = n, \\ 0 & \text{if } k < n. \end{cases}$$

Another useful formula for S_n^k is an inductive one, obtained starting from $S_n^n = n!\ \forall n \geq 0$ and $S_n^k = 0$ for any k and n with $k < n$.

Proposition 1.19 *We have*

$$\begin{cases} S_n^k = n(S_n^{k-1} + S_{n-1}^{k-1}) & \textit{if } k \geq 1, n \geq 0, \\ S_n^n = n!, \\ S_0^k = 0 & \textit{if } k \geq 1. \end{cases} \tag{1.6}$$

Proof. Assume $n \geq 1$ and $k \geq 1$ and let $f : \{1, \dots, k\} \to \{1, \dots, n\}$ be a surjective function. Let $A \subset \mathcal{S}_n^k$ be the class of functions such that the restriction $f : \{1, \dots, k-1\} \to \{1, \dots, n\}$ of f is surjective and let $B := \mathcal{S}_n^k \setminus A$. The cardinality of A is nS_n^{k-1} because there are S_n^{k-1} surjective maps from $\{1, \dots, k-1\}$ onto $\{1, \dots, n\}$ and there are n possible choices for $f(k)$. Since the maps on B have a range of $(n-1)$ elements, we infer that there are nS_{n-1}^{k-1} maps of this kind. In fact, there are $\binom{n}{n-1} = n$ subsets E of $\{1, \dots, n\}$ of cardinality $n-1$ and there are S_{n-1}^{k-1} surjective functions from $\{1, \dots, k-1\}$ onto E. Therefore,

$$S_n^k = |A| + |B| = nS_n^{k-1} + nS_{n-1}^{k-1}.$$

i.e. (1.6).

Some of the S_n^k's are in Figure 1.7.

$$\left(\begin{array}{cccccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\
0 & 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\
0 & 1 & 6 & 6 & 0 & 0 & 0 & 0 & 0 & \dots \\
0 & 1 & 14 & 36 & 24 & 0 & 0 & 0 & 0 & \dots \\
0 & 1 & 30 & 150 & 240 & 120 & 0 & 0 & 0 & \dots \\
0 & 1 & 62 & 540 & 1\,560 & 1\,800 & 720 & 0 & 0 & \dots \\
0 & 1 & 126 & 1\,806 & 8\,400 & 16\,800 & 15\,120 & 5\,040 & 0 & \dots \\
0 & 1 & 254 & 5\,796 & 40\,824 & 126\,000 & 191\,520 & 141\,120 & 40\,320 & \dots \\
0 & 1 & 510 & 18\,150 & 186\,480 & 834\,120 & 1\,905\,120 & 2\,328\,480 & 1\,451\,520 & \dots \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots
\end{array}\right.$$

Figure 1.7 The cardinality S_n^k of the set of surjective maps $\mathcal{S}_n^k$ for $n, k \geq 0$.

1.2.9 Exercises

Exercise 1.20 *How many diagonals are there in a polygon having n edges?*

1.3 Drawings

A *drawing* or *selection* of k objects from a population of n is the choice of k elements among the n available ones. We want to compute how many of such selections are possible. In order to make this computation, it is necessary to be more precise, both on the composition of the population and on the rules of the selection as, for instance, if the order of selection is relevant or not. We consider a few cases:

- The population is made by pairwise different elements, as in a lottery: in other words, the population is a set.
- The population is a multiset (A, a). In this case, we say that we are dealing with a *drawing from A with repetitions*.
- The selected objects may be given an order. In this case we say that we consider an *ordered selection*. Unordered selections are also called *simple selections*.

Some drawing policies simply boil down to the previous cases:

- In the lottery game, numbers are drawn one after another, but the order of drawings is not taken into account: it is a simple selection of objects from a set.
- In ordered selections the k-elements are selected one after another and the order is taken into account.
- A drawing with replacement, i.e. a drawing from a set where each selected object is put back into the population before the next drawing is equivalent to a drawing with repetitions, i.e. to drawing from a multiset where each element has multiplicity larger than the total number of selected objects.

1.3.1 Ordered drawings

Ordered drawings of k objects from a multiset (A, a) are k-words with symbols taken from A.

1.3.1.1 Ordered drawings from a set

Each ordered drawing of k objects from a set A is a k-list with symbols in A that are pairwise different. Thus the number of possible ordered drawings of k elements from A is the number of k-lists with pairwise different symbols in A.

If $|A| = n$, there are n possible choices for the first symbol, $n-1$ for the second and so on, so that there are

$$n(n-1)\dots(n-k+1)$$

different k-words with pairwise different symbols.

1.3.1.2 Ordered drawings from a multiset

Let (A, a) be a multiset where $|A| = n$ and let $k \in \mathbb{N}$ be less than or equal to $\min\{a(x) \mid x \in A\}$. Each ordered drawing of k elements from (A, a) is a k-list with symbols in A, where the same symbol may appear more than once. We have already proven that there are n^k possible k-lists of this kind, so that the following holds.

Proposition 1.21 *The number of ordered drawings of k elements from a multiset (A, a) where $k \leq \min\{a(x) \mid x \in A\}$ is n^k.*

In particular, the number of ordered drawings with replacement of k elements from A is n^k.

1.3.2 Simple drawings

1.3.2.1 Drawings from a set

The population from which we make the selection is a set A. To draw k objects from A is equivalent to selecting a subset of k elements of A: we do not distinguish selections that contain the same objects with a different ordering.

Proposition 1.22 *The number of possible drawings of k elements from a set of cardinality n is $\binom{n}{k}$.*

1.3.2.2 Drawings from a multiset

Let (A, a) be a multiset, $|A| = n$, and let $k \leq \min\{a(x) \mid x \in A\}$. Each sequence S drawn from (A, a) is a sequence of symbols in A where repetitions may occur and the order of the symbols is not taken into accout, e.g.

$$FABADABDF = FBFDDAABA$$

i.e. S is a multiset of k elements included in (A, a) (cf. Proposition 1.17).

Proposition 1.23 *The number of simple drawings of k elements from a multiset (A, a), is $\binom{n+k-1}{k}$ provided $k \leq \min\{a(x) \mid x \in A\}$.*

1.3.3 Multiplicative property of drawings

The previous results on drawings can also be obtained from the following *combinatorics properties* of drawings.

Theorem 1.24 *For each non-negative integer k let a_k and b_k be the numbers of drawings of k objects from the multisets (A, a) and (B, b) made according to policies P_1 and P_2, respectively. If A and B are disjoint, then the number of drawings of k elements from the population obtained by the union of (A, a) and (B, b) made according to policies P_1 and P_2 for the drawings from (A, a) and (B, b), respectively, is*

$$c_k = \sum_{j=0}^{k} a_j b_{k-j}.$$

Proof. A drawing of k objects from the union of the two populations contains, say, j elements from (A, a) and $k - j$ elements from (B, b), where j is an integer, $0 \leq j \leq k$. The j elements drawn from (A, a) can be chosen in a_j different ways, while the $n - j$ elements drawn from (B, b) can be chosen in b_{k-j} different ways and the two choices are independent. Thus,

$$c_k = \sum_{j=0}^{k} a_j b_{k-j}.$$

A similar result holds for ordered drawings

Theorem 1.25 *For each non-negative integer k let a_k and b_k be the number of ordered drawings from the multisets (A, a) and (B, b) made according to policies P_1 and P_2, respectively. If A and B are disjoint, then the number of ordered drawings from the population union of (A, a) and (B, b) made according to policy P_1 for the elements of (A, a) and according to policy P_2 for elements of (B, b) are*

$$c_k = \sum_{j=0}^{k} \binom{k}{j} a_j b_{k-j}.$$

Proof. A drawing of k elements from the union of the two populations contains j elements from (A, a) and $n - j$ elements from (B, b) for some integer $j \in \{0, \dots, k\}$. The j elements from (A, a) can be chosen in a_j different ways, the $k - j$ elements drawn from (B, b) can be chosen in b_{k-j} different ways and the two chosen groups are independent. Finally, there are $\binom{k}{j}$ ways to order such selections. Thus,

$$c_k = \sum_{j=0}^{k} \binom{k}{j} a_j b_{k-j}.$$

1.3.4 Exercises

Exercise 1.26 *A committee of 7 people has to be chosen among 11 women and 8 men. In each of the following cases compute how many different committees can be chosen:*

- *No constraint is imposed.*
- *At least two women and at least one man must be present.*
- *There must be more women than men.*
- *At least two women and no more than three men must be present.*

1.4 Grouping

Many classical counting problems amount to a *collocation* or *grouping problem*: how many different arrangements of k objects in n boxes are there? Putting it another way, how many different ways of grouping k objects into n groups are there? Also in this case a definite answer cannot be given: we must be more precise both on the population to be arranged, on the rules (or *policy*) of the procedure, and on the way the groups are evaluated. For example, one must say whether the objects to be arranged are pairwise different or not, whether the order of the objects in each box must be taken into account or not, whether the boxes are pairwise distinct or not, and if further constraints are imposed. Here we deal with a few cases, all referring to *collocation* or *grouping* in *pairwise different boxes*. We consider the formed groups as a list instead of as a set: for instance, if we start with the objects $\{1, 2, 3\}$ then the two arrangements in two boxes $(\{1\}, \{2, 3\})$ and $(\{2, 3\}, \{1\})$ are considered to be different.

1.4.1 Collocations of pairwise different objects

Arranging k distinct objects in n pairwise different boxes is the act of deciding the box in which each object is going to be located. Since both the objects and the boxes are pairwise distinct, we may identify the objects and the boxes with the sets $\{1, \dots, k\}$ and $\{1, \dots, n\}$, respectively. Each arrangement corresponds to a *grouping map* $f : \{1, \dots, k\} \to \{1, \dots, n\}$ that puts the object j into the box $f(j)$.

1.4.1.1 No further constraint

In this case the set of possible locations is in a one-to-one correspondence with the set $\mathcal{F}_n^k$ of all maps $f : \{1, \dots, k\} \to \{1, \dots, n\}$. Therefore, there are n^k different ways to locate k-different objects in n boxes.

A different way to do the computation is the following. Assume $i_1, \dots, i_n$ objects are placed in the boxes $1, \dots, n$, respectively, so that $i_1 + \dots + i_n = k$. There are $\binom{k}{i_1}$ different choices for the elements located in the first box, $\binom{k-i_1}{i_2}$ different choices for the elements in the second box, and so on, so that there are

$$\binom{k - i_1 - \cdots - i_{n-1}}{i_n}$$

different choices for the elements in the nth box. Thus the different possible arrangements are

$$\binom{k}{i_1}\binom{k-i_1}{i_2}\cdots\binom{k-i_1-\cdots-i_{n-1}}{i_n}=$$
$$=\frac{k!}{i_1!(k-i_1)!}\frac{(k-i_1)!}{i_2!(k-i_1-i_2)!}\cdots=\frac{k!}{i_1!\,i_2!\cdots i_n!}; \tag{1.7}$$

the ratio in (1.7) is called the *multinomial coefficient* and is denoted as

$$\binom{k}{i_1\ i_2\ \cdots\ i_n}.$$

From (1.7) we infer that the possible collocations of k pairwise different objects in n pairwise different boxes are

$$\sum_{i_1+\cdots+i_n=k}\binom{k}{i_1\ i_2\ \cdots\ i_n}$$

where the sum is performed over all the n-tuples $i_1,\ldots,i_n$ of non-negative integers such that $i_1+\cdots+i_n=k$. Thus, from the two different ways of computing collocations, we get the equality

$$n^k=\sum_{i_1+\cdots+i_n=k}\binom{k}{i_1\ i_2\ \cdots\ i_n}.$$

1.4.1.2 At least one object in each box

We now want to compute the number of different arrangements with at least one object in each box. Assuming we have k objects and n boxes, collocations of this type are in a one-to-one correspondence with the class of *surjective maps* $\mathcal{S}_n^k$ from $\{1,\ldots,k\}$ onto $\{1,\ldots,n\}$, thus there are

$$S_n^k=\sum_{j=0}^{n}(-1)^j\binom{n}{j}(n-j)^k$$

collocations of k pairwise different into n pairwise different boxes that place at least one object in each box.

Another way to compute the previous number is the following. Assume $i_1,\ldots,i_n$ objects are located in the boxes $1,\ldots,n$, respectively, with at least one object in each box, i.e. $i_1+\cdots+i_n=k$ and $i_1,\ldots,i_n\geq 1$. As in (1.7), there are

$$\binom{k}{i_1\ i_2\ \cdots\ i_n} \tag{1.8}$$

ways to arrange k different objects in n boxes with i_j objects in the box j. Thus the number of arrangements with no empty box is

$$\sum_{\substack{i_1+\cdots+i_n=k\\ i_1,\ldots,i_n\geq 1}} \binom{k}{i_1\ i_2\ \cdots\ i_n};$$

here, the sum is performed over all the n-tuples $i_1, \ldots, i_n$ with *positive* components with $i_1 + \cdots + i_n = k$. The above two ways of computing the number of such collocations yield the identity

$$S_n^k = \sum_{\substack{i_1+\cdots+i_n=k\\ i_1,\ldots,i_n\geq 1}} \binom{k}{i_1\ i_2\ \cdots\ i_n}. \tag{1.9}$$

1.4.1.3 At most one object in each box

We now impose a different constraint: each box may contain at most one object. Assuming we have k objects and n boxes, collocations of this type are in a one-to-one correspondence with the class of *injective grouping maps* $\mathcal{I}_n^k$ from $\{1, \ldots, k\}$ onto $\{1, \ldots, n\}$, thus there are

$$I_n^k = k!\binom{n}{k}$$

collocations of this type.

1.4.1.4 Grouping into lists

Here, we want to compute the number of ways of grouping k pairwise different objects in n pairwise different boxes and pretend that the order of the objects in each box matters. In other words we want to compute how many different ways exist to group k objects in a list of n lists of objects. We proceed as follows.

The first object can be collocated in one of the n boxes, that is in n different ways. The second object can be collocated in $n+1$ different ways: in fact, it can be either collocated in each of the $n-1$ empty boxes, or it can be collocated in the same box as the first object. In the latter case it can be collocated either as the first or as the second object in that box. So the possible arrangements of the second object are $(n-1)+2 = n+1$. The third object can be collocated in $n+2$ ways. In fact, if the first two objects are collocated in two different boxes, then the third object can either be collocated in one of the $n-2$ empty boxes or in two different ways in each of the two nonempty boxes. Thus, there are $(n-2)+2+2 = n+2$ possible arragements. If the first two objects are in the same box, then the third object can either be collocated in one of the $n-1$ empty boxes or in the nonempty one. In the latter case, it can be collocated in three different ways: either as the first, or between the two objects already present, or as the last one. Again, the third object can be collocated in $(n-1)+3=n+2$ ways. By an induction argument, we infer that there are $n+k-1$ different arrangements for the kth object. Thus, *the number of different ordered locations of k objects in n boxes is*

$$n(n+1)(n+2)\dots(n+k-1)=k!\binom{n+k-1}{k}.$$

1.4.2 Collocations of identical objects

We want to compute the number of ways to arrange k identical objects in n pairwise different boxes. In this case each arrangement is characterized by the number of elements in each box, that is by the map $x:\{1,\dots,n\}\to\{0,\dots,k\}$ which counts how many objects are in each box. Obviously, $\sum_{s=1}^{n}x(s)=k$. If the k objects are copies of the number '0', then each arrangement is identified by the binary sequence

$$\underbrace{00\dots0}_{x(1)}1\underbrace{00\dots0}_{x(2)}1\dots1\underbrace{00\dots0}_{x(n-1)}1\underbrace{00\dots0}_{x(n)} \tag{1.10}$$

where the number '0' denotes the fact that we are changing box.

1.4.2.1 No further constraint

Let us compute the number of such arrangements with no further constraint. There is a one-to-one correspondence between such arrangements and the set of all binary sequences of the type (1.10). Therefore, see Proposition 1.12, *the different collocations of k identical objects in n pairwise different boxes is*

$$\binom{n+k-1}{k}. \tag{1.11}$$

1.4.2.2 At least one in each box

We add now the constraint that each box must contain at least one object. If $k<n$ no such arrangement is possible. If $k\geq n$, we then place one object in each box so that the constraint is satisfied. The remaining $k-n$ objects can be now collocated without constraints. Therefore, cf. (1.11), there are

$$\binom{n+(k-n)-1}{k-n}=\binom{k-1}{k-n}=\binom{k-1}{n-1}$$

ways to arrange k identical objects in n boxes, so that no box remains empty.

1.4.2.3 At most one in each box

We consider arrangements of k identical objects in n pairwise different boxes that place at most one object into each box. In this case, each arrangement is completely characterized by the subset of filled boxes. Since we can choose them in $\binom{n}{k}$ different ways, we conclude that *the collocations of k identical objects in n pairwise different boxes with at most one object per box* is

$$\binom{n}{k}.$$

1.4.3 Multiplicative property

Combinatorial properties hold for collocations as well as for drawings.

Theorem 1.27 *For each non-negative integer k, let a_k and b_k be the number of collocations of k pairwise different objects in two sets S_1 and S_2 of pairwise different boxes with policies P_1 and P_2, respectively. If $S_1 \cap S_2 = \emptyset$, then the different collocations of the k objects in $S_1 \cup S_2$ following policy P_1 for collocations in boxes of S_1 and policy P_2 for collocations in boxes of S_2 is*

$$c_k = \sum_{j=0}^{k} \binom{k}{j} a_j b_{k-j}.$$

Proof. Let j objects, $0 \leq j \leq k$ be collocated in the boxes of the set S_1 and let the other $k - j$ objects be collocated in the boxes of S_2. There are a_j different ways of placing j objects in the boxes of S_1 and b_{k-j} different ways of placing $(k - j)$ objects in the boxes of S_2. Moreover, there are $\binom{k}{j}$ different ways to choose which objects are collocated in the boxes of S_1. Hence,

$$c_k = \sum_{j=0}^{k} \binom{k}{j} a_j b_{k-j} \qquad \forall k \geq 0.$$

A similar result holds for the collocations of identical objects.

Theorem 1.28 *For each non-negative integer k, let a_k and b_k be the number of collocations of k identical objects in two sets S_1 and S_2 of pairwise different boxes with policies P_1 and P_2, respectively. If $S_1 \cap S_2 = \emptyset$, then the collocations of the k objects in the boxes of $S_1 \cup S_2$ made according to policy P_1 for the collocations in the boxes of S_1 and according to policy P_2 for the collocations in the boxes of S_2 is*

$$c_k = \sum_{j=0}^{k} a_j b_{k-j}.$$

Proof. Let j objects, $0 \leq j \leq k$ be collocated in the boxes of the set S_1 and let the other $k - j$ objects be collocated in the boxes of S_2. There are a_j ways of placing j objects in the boxes of S_1 and b_{k-j} different ways of placing $(k - j)$

objects in the boxes of S_2. Since the objects are identical, there is no way to select which the j objects to be placed in the boxes of S_1 are. Then the possible different collocations are

$$c_k = \sum_{j=0}^{k} a_j b_{k-j} \qquad \forall k \geq 0.$$

1.4.4 Collocations in statistical physics

In statistical physics, each 'particle' is allowed to be in a certain 'state'; an 'energy level' is associated with each state. The total energy of a system of particles depends on how many particles are in each of the possible states; the mean value of the energy depends on the probabilities that particles stay in a certain state. Thus, the number of possible collocations of the particles in the available states must be evaluated.

1.4.4.1 Maxwell-Boltzmann statistics

This is the case of classical statistical physics: the particles are distinct and no constraint is imposed on their distribution in different states. The number of possible collocations of k particles in n states is thus the number of collocations of k pairwise different objects in n pairwise different boxes, i.e. n^k.

1.4.4.2 Bose–Einstein statistics

The particles are indistinguishable and no constraint is imposed on their distribution in different states. Particles with this behaviour are called *bosons*. The number of collocations of k particles in n states is then the number of collocations of k identical objects in n pairwise different boxes, i.e.

$$\binom{n+k-1}{k} = \binom{n+k-1}{n-1}.$$

1.4.4.3 Fermi–Dirac statistics

The particles are indistinguishable and each state can be occupied by at most one particle (Pauli exclusion principle). Particles following this behaviour are called *fermions*. Then the collocations of k particles in n states is the number of possible choices for the states to be occupied, i.e. $\binom{n}{k}$. Obviously, the Pauli exclusion principle implies $n \geq k$.

1.4.5 Exercises

Exercise 1.29 *A group of eight people sits around a table with eight seats. How many different ways of sitting are there?*

Exercise 1.30 *Compute the number* $g_{n,k}$ *of subsets of* $\{1\ldots,n\}$ *having cardinality* k *and that do not contain two consecutive integers.*

Solution. There is a one-to-one correspondence between the family of the subsets of cardinality k and the set of binary n-words given by mapping a subset $A \subset \{1,\ldots,n\}$ to its characteristic function $\mathbb{1}_A$. Namely, to the subset $A \subset \{1,\ldots,n\}$ we associate the binary n-word $(a_1, a_2, \ldots, a_n)$ where $a_i = 1$ if $i \in A$ and $a_i = 0$ otherwise. Consequently, the family we are considering is in a one-to-one correspondence with the binary n-words in which there cannot be two consecutive 1's, in

$$0001000101000101$$

Considering the 0's as the sides of a box that contains at most one 1, we have k 1's and $n-k+1$ boxes with at most one 1 per box. Thus, each collocation is uniquely detected by the choice of the k nonempty boxes. Thus, see Section 1.4.2, $g_{n,k} = \binom{n-k+1}{k}$.

Exercise 1.31 *A physical system is made by identical particles. The total energy of the system is* $4E_0$ *where* E_0 *is a given positive constant. The possible energy levels of each particle are* $k\,E_0$, $k = 0, 1, 2, 3, 4$. *How many different configurations are possible if;*

(i) the kth energy level is made of $k^2 + 1$ *states;*

(ii) the kth energy level is made of $2(k^2 + 1)$ *states;*

(iii) the kth energy level is made of $k^2 + 1$ *states and two particles cannot occupy the same state.*

Exercise 1.32 *For any non-negative integer* $n \geq 0$*, define*

$$x^{\overline{n}} := x(x+1)(x+2)\cdots(x+n-1),$$

$$x^{\underline{n}} := x(x-1)(x-2)\cdots(x-n+1).$$

Prove that

$$x^{\overline{n}} = (-1)^n(-x)^{\underline{n}}. \tag{1.12}$$

[*Hint.* Take advantage of (iii) of Proposition 1.3.]

Exercise 1.33 *n players participate in a single-elimination tennis tournament. How many matches shall be played?*

Exercise 1.34 *You are going to place* 4 *mathematics books,* 3 *chemistry books and* 2 *physics books on a shelf. Compute how many different arrangements are possible. What if you want to keep books of the same subject together?*

Exercise 1.35 *A comittee of* 4 *is to be chosen from a group of 20 people. How many different choices are there for the roles of president, chairman, secretary, and treasurer?*

Exercise 1.36 *How many bytes of N digits exist with more zeroes than ones?*

Exercise 1.37 *There are* 40 *cabinet ministers sitting around a circular table. One seat is reserved for the Prime Minister. In how many ways can the other ministers seat themselves?*

Exercise 1.38 *Thirty people meet after a long time and they all shake hands. How many handshakes are there?*

Exercise 1.39 *Find the number of solutions to equation $x + y + z + w = 15$:*

(i) in the set of non-negative integers;

(ii) in the set of positive integers;

(iii) in the set of integers such that $x > 2$, $y > -2$, $z > 0$, $w > 3$.

Exercise 1.40 *Find the number of non-negative integer solutions (x, y, z, t) of $x + y + z + t \leq 6$.*