Part One

Plane Ideal Aerodynamics

1

Preliminary Notions

1.1 Aerodynamic Force and Moment

An aircraft in flight is subject to several forces: gravity causes the weight force; the propulsion provides a thrust, and the air the *aerodynamic force* $\boldsymbol{A}$.[1]

The central problem of aerodynamics is the prediction of the aerodynamic force; as important is its line of action, or equivalently its moment.

The motion of the aircraft through the air forces the air to move, setting up aerodynamic stresses. In turn, by Newton's Third Law of Motion, the stress in the air is transmitted back across the surface of the aircraft. The stresses include pressure stresses and viscous stresses. The aggregates of the stresses on the surface are the aerodynamic force and moment.

1.1.1 Motion of the Frame of Reference

Newton's equations of motion are unchanged if the frame of reference is replaced with one moving at a constant relative velocity; that is, the aerodynamic force can be computed or measured equally well by an observer in the aircraft in steady flight as by an observer on the ground. In the aeroplane's frame of reference, it's stationary and the air moves at a velocity $-\boldsymbol{V}$.

The equivalence is very useful in aerodynamics; e.g. instead of mounting models of wings on force-measuring apparatus atop an express train as John Stringfellow did in the first half of the nineteenth century, the aerodynamic force can be

- measured in a wind tunnel where a model of the aircraft is held fixed in an air-stream; or
- computed by a numerical solution of the governing equations on a grid fixed to the aircraft surface.

Of course, other factors are involved in the interpretation of wind tunnel data (e.g. the effect of the walls), or computational fluid dynamics (e.g. grid dependence).

[1] Herein **bold** denotes a vector, *italic* a variable, and so ***bold italic*** a variable vector.

Theory of Lift: Introductory Computational Aerodynamics in MATLAB®/Octave, First Edition. G. D. McBain.

1.1.2 Orientation of the System of Coordinates

Various choices of coordinate system in the frame of reference are possible; e.g.

- a geocentric system with coordinates latitude, longitude, and altitude;
- a Cartesian system defined by the instantaneous velocity, curvature, and torsion of the flight; or
- a Cartesian system fixed to the aircraft, moving and rotating with it.

Although all physical results obtained must be independent of the choice of orientation, in aerodynamics we almost always use the last of these, after carefully and explicitly defining it in each case.

Most aircraft have an approximate plane of symmetry naturally dividing the craft into left and right halves. (This symmetry is deliberately broken during independent deployment of the left and right control surfaces.) The reference line is always chosen within this plane, and generally approximately coincides with the long direction of the craft and the usual direction of travel.

Our system of coordinates then consists of:

- the aircraft reference line (x, *longitudinal* or *axial*, positive 'downstream' or 'behind' for the usual direction of flight);
- one axis at right angles to the reference line but still in the plane of symmetry (y, positive in the direction considered up by seated pilots and passengers); and
- a third perpendicular to the plane of symmetry (z, *spanwise*, positive left).

These definitions of 'backwards', 'upwards', and 'leftwards' for increasing x, y, and z give a *right-handed* coordinate system.

1.1.3 Components of the Aerodynamic Force

In simple cases, symmetry considerations imply that the aerodynamic force acts parallel to the plane of symmetry. In such cases, the aerodynamic force $\boldsymbol{A}$ can be resolved into two perpendicular component forces in two different ways.

First, $\boldsymbol{A} = \boldsymbol{L} + \boldsymbol{D}$. The *drag* $\boldsymbol{D}$ is opposed to the direction of motion $\boldsymbol{V}$ of the aircraft (or parallel to the direction of the airstream relative to the aircraft, $-\boldsymbol{V}$). The *lift* $\boldsymbol{L}$ is directed at right-angles to the direction of motion.

Second, we have the simple Cartesian components: $\boldsymbol{A} = A_x\mathbf{i} + A_y\mathbf{j}$; here A_x and A_y are the longitudinal and normal components. The two decompositions are illustrated in Figure 1.1. Remember that in both diagrams, the horizontal x-coordinate corresponds to the (backward) aircraft reference line and is not necessarily the same as horizontal with respect to the ground.

1.1.4 Formulation of the Aerodynamic Problem

The basic task of aerodynamics is to predict the aerodynamic force $\boldsymbol{A}$, or the lift and drag forces.

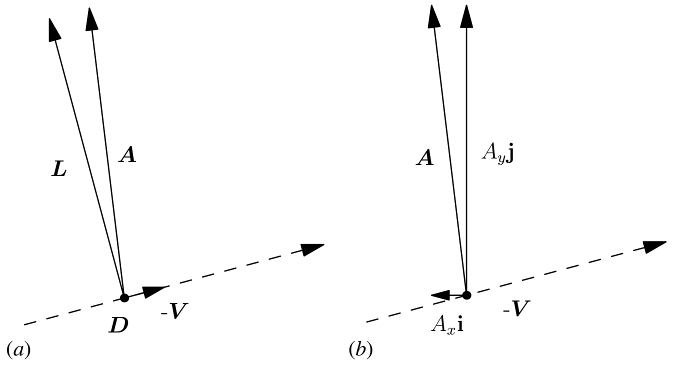

Figure 1.1 Resolving the aerodynamic force into (*a*) lift and drag and (*b*) Cartesian components

Let's see what we can say about this problem without knowing the details of the air flow around the aircraft.

We assume:

- steady flight in a straight line;
- that the air extends to infinity in all directions around the aircraft, and that the air far from the aircraft is otherwise undisturbed.

What then constitutes a specification of the problem?

- We need to know the geometry of the aircraft.
- We need to know the direction and magnitude of the aircraft velocity (or airstream velocity relative to the aircraft).
- We need to know the properties of the air.

These three items are considered in Sections 1.2–1.4. In Section 1.5, we see how *dimensional analysis* can be used to reduce the size of the problem, still without recourse to any detailed fluid mechanics. Finally, in Section 1.6, we look at a real aerodynamical study and see how it incorporates the preliminary notions presented in this chapter.

1.2 Aircraft Geometry

Aircraft have complicated shapes and require a lot of numbers for their complete specification. A number of common terms are in use, e.g. the *span* and *chord* of the wings, but these don't always have standard definitions. For example, Milne-Thomson (1973) lists several definitions of the chord of a two-dimensional wing section:

> As a general definition the *chord* of any profile is an arbitrary fixed line drawn in the plane of the profile.

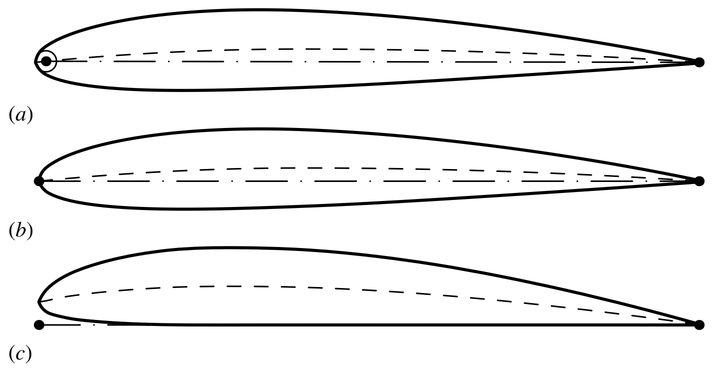

Figure 1.2 Definitions of wing section chord: the line joining the leading edge centre of curvature to the trailing edge (*a*); the line joining the ends of the camber line (*b*); and the double-tangent to the lower surface (*c*). The NACA 2412 aerofoil (*a, b*) has no double-tangent to the lower surface, so (*c*) is illustrated for a Clark Y profile. Dashed curve is the camber line and dot-dashed the chord

> The chord has direction, position, and length. The main requisite is that in each case the chord should be precisely defined, since the chord enters into the constants which describe the aerodynamic properties of the profile.
>
> The official definition is the line which joins the centres of the circles of curvature of minimum radius at the nose and tail.
>
> Another definition is the longest line which can be drawn to join two points of the profile.
>
> A third definition which is sometimes convenient is the projection of the profile on the double tangent to its lower surface (i.e. the tangent which touches the profile at two distinct points).
>
> This definition fails if there is no such point.

A fourth definition was used by NACA: NACA defined wing sections relative to a given *camber line* (actually a curve) and then defined the chord as the straight line joining its ends. For many wing sections, this will coincide at least approximately with the second of Milne-Thomson's definitions.

The definitions are illustrated in Figure 1.2. Notice that, apart from the double-tangent, the other definitions give similar results; this is true for most typical aerofoils.

All of these are or have been in use, and we can do no better than warn of this and stress that while how the chord is defined is relatively unimportant, it is essential that in each case it be defined clearly and precisely. A similar state of affairs persists with regard to the other dimensions of wings. Here we provide a list of common terms with rough definitions; see also the Glossary and Section 1.8, Further Reading.

1.2.1 Wing Section Geometry

Of particular importance in the study of aerodynamics, as we shall see in Chapters 2–8, are the two-dimensional sections of the wing parallel to the aircraft's plane of symmetry; these are called *wing sections* or *aerofoils*. Wing sections are parameterized by:

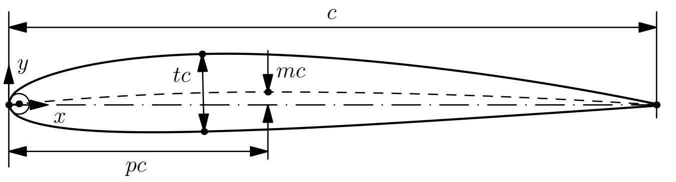

Figure 1.3 NACA 2412 wing section, showing: the camber line (dashed); chord defined as its secant (dot-dashed); leading edge circle of curvature (solid); chord length c; maximum thickness ratio t; maximum camber ratio m; and chordwise fractional position of maximum camber p

chord As above (Section 1.2); see Figure 1.2.
maximum camber The *camber line* is the curve lying halfway between the upper and lower surfaces. The maximum camber is its greatest distance from the chord. The 'halfway' can be measured perpendicular to either the chord or the camber line.
chordwise location of maximum camber
maximum thickness measured perpendicular to either the chord or camber line
location of maximum thickness
leading edge radius of curvature
trailing edge radius of curvature or angle if the radius is zero

These concepts are illustrated in Figures 1.2 and 1.3.

1.2.2 Wing Geometry

The wings are parameterized by:

span b, the length of the line joining the tips of the wings, being the points on the wings furthest from the plane of symmetry
planform area the projection of the wings on the spanwise–longitudinal (z–x) plane, including or excluding the area where the wings and fuselage coincide or meet
root chord c_r, the chord where the wings meet the fuselage or $c(0)$, the chord of the wing section in the plane of symmetry (projected, if required)
tip chord $c_t = c(\pm b/2)$ the chord of the wing sections furthest from the plane of symmetry; zero for a delta-wing
average chord $\bar{c}$, the ratio of the planform area to the span
aspect ratio $\mathcal{R}$, the ratio of the span to the average chord
taper ratio the ratio of the tip and root chords; zero for a delta-wing, and typically ≤ 1
sweep angle of the wing backward from the spanwise (z) axis, which is perpendicular to the root chord, can be measured at leading edge, trailing edge, or one-quarter of the way along the chord

which are all properties of the wing's planform; i.e. its projection on the z–x plane. See Figure 1.4.

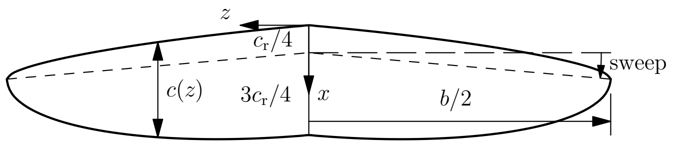

Figure 1.4 Planform of a wing, showing a chord $c(z)$, the root chord c_r, the span b, and the angle of sweep. The plan projection of all chords are parallel. The dashed line joins the quarter-chord points. In this wing, the tip chord is zero

Some additional parameters describe three-dimensional properties:

dihedral angle the upward tilt of the wings relative to the longitudinal–spanwise plane (called *anhedral* if negative)

twist variation of the angle of the chord to the root chord along the span, called *wash in* or *out* as the angle increases or decreases towards the wing tip.

1.3 Velocity

As noted in Section 1.1.2, a reference line for the aircraft must be defined to specify the coordinate system. The usual choice for a whole wing or aircraft is that of the root chord; for a two-dimensional study of a wing section, the section's chord may be used.

For two-dimensional motion (parallel to the plane of symmetry), the angle α between the velocity and the aircraft reference direction is called the *geometric angle of incidence* or *geometric angle of attack*. Note that it depends on the definition of the reference direction. The sign of the angle is that of the y-component of the velocity of the air relative to the aircraft.

In terms of the incidence, and referring to Figure 1.1, the axial and normal components of the aerodynamic force are related to the lift and drag by

$$L = A_y \cos\alpha - A_x \sin\alpha \tag{1.1a}$$

$$D = A_x \cos\alpha + A_y \sin\alpha. \tag{1.1b}$$

1.4 Properties of Air

After density, the two most important properties of air in aerodynamics are its compressibility and its viscosity.

1.4.1 Equation of State: Compressibility and the Speed of Sound

The *equation of state* of a material relates its pressure and density. A material resists changes in density by changing its pressure. The relevant material property is the *compressibility*. Using a

Taylor series for the pressure as a function of density for small changes in density about some reference level ρ_0,

$$p(\rho) \approx p(\rho_0) + (\rho - \rho_0)p'(\rho_0)$$
$$\equiv p(\rho_0) + \frac{\rho - \rho_0}{\rho_0}\frac{1}{\kappa} \quad (1.2)$$

where $\kappa \equiv 1/\rho_0 p'(\rho_0)$ is the compressibility. (The Taylor series approximation to a function matches at a given point first the value, then the derative, then the second derivative, and so on.) Sound waves travel through the material with a speed $a = 1/\sqrt{\rho_0\kappa}$. It is often more convenient to use the speed of sound a as the parameter rather than κ, and we will do so here. In terms of a, the first-order Taylor series approximation to pressure–density Equation (1.2) is

$$p(\rho) \approx p(\rho_0) + (\rho - \rho_0)a^2.$$

The Adiabatic Speed of Sound in an Ideal Gas

The compressibility of a gas depends on how quickly the compression occurs; specifically, how the temperature varies, or is allowed to vary, during the compression. If the compression is very fast, there may be insufficient time for heat transfer between the parcel of gas of interest and its surroundings; this *adiabatic* compression applies to the passage of sound waves.

The work done during expansion by a unit mass is $p\delta(1/\rho)$, and the increase in internal energy is $c_v\delta T$. Without heat transfer, the conservation of energy implies that these sum to zero and so $\mathrm{d}T = p\ \mathrm{d}\rho/c_v\rho^2$.

This can be applied to the differential of the thermal equation of state of an ideal gas

$$p = \rho RT, \quad (1.3)$$

which besides the thermodynamic properties of pressure p, temperature T, and density ρ, involves the gas constant R which for air is about 287.0 J/kg K.

$$\begin{aligned}
\mathrm{d}p &= R(T\ \mathrm{d}\rho + \rho\ \mathrm{d}T) \\
&= R(T\ \mathrm{d}\rho + p\ \mathrm{d}\rho/c_v\rho) \\
&= R(T + p/c_v\rho)\ \mathrm{d}\rho \\
\rho\ \mathrm{d}p &= pR(1/R + 1/c_v)\ \mathrm{d}\rho \\
&= p(1 + R/c_v)\ \mathrm{d}\rho \\
&= p\gamma\ \mathrm{d}\rho \\
\kappa &= p\gamma
\end{aligned}$$

where we have defined the dimensionless number $\gamma \equiv 1 + R/c_v$, a property of the gas, which is independent of temperature and pressure insofar as the gas constant R and specific heat c_v are. When we return to this topic in more detail in Section 18.3.5 and Equation (18.9) we will see that γ is the ratio of the isobaric and isochoric specific heat coefficients.

Thus,

$$a = \frac{1}{\sqrt{\rho\kappa}} = \sqrt{\frac{\gamma p}{\rho}} = \sqrt{\gamma RT}, \quad (1.4)$$

Listing 1.1 `speed_of_sound`: compute the speed of sound in air in m/s at a temperature (or array of temperatures) in Kelvin.

```
function a = speed_of_sound (T), a = sqrt (1.4 * T * 287.0);
```

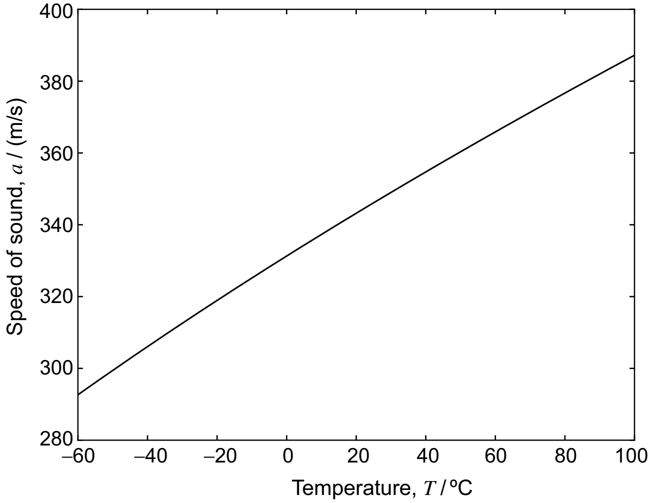

Figure 1.5 Variation of the speed of sound in air with temperature at atmospheric pressures, according to Equation (1.4)

The theoretical value of γ for diatomic gases is $\frac{7}{5}$, and this is a good approximation for air, which, as far as its mechanical properties are concerned, largely consists of diatomic nitrogen and oxygen. Thus the speed of sound depends only on temperature, and at $T = 15°\text{C} = 288.15$ K is $a \doteq 340$ m/s. Values at other (absolute) temperatures are conveniently calculated with a one-line Octave function (Listing 1.1), which was used to generate Figure 1.5.

1.4.2 Rheology: Viscosity

The *constitutive law* of a material relates the stress and stain. A material resists deformation by changing its stress. A fluid has no preferred shape, but resists the rate of change of shape.

Rheology in general is very complicated, but a good model for air is the *linearly viscous fluid*, for which the stress is the sum of an *isotropic* part (the pressure) and a part proportional to the rate of strain. Even for the linearly viscous fluid the constitutive law is complicated, as we shall see in Section 15.2.2. The relevant point here is that the constitutive law introduces another parameter: the coefficient of dynamic viscosity, μ, which has SI units of Pa s (or kg m^{-1} s^{-1}).

Listing 1.2 `viscosity.m.`

```
function m = viscosity (T)
  m = 1.495e-6 * sqrt (T) ./ (1 + 120 ./ T);
```

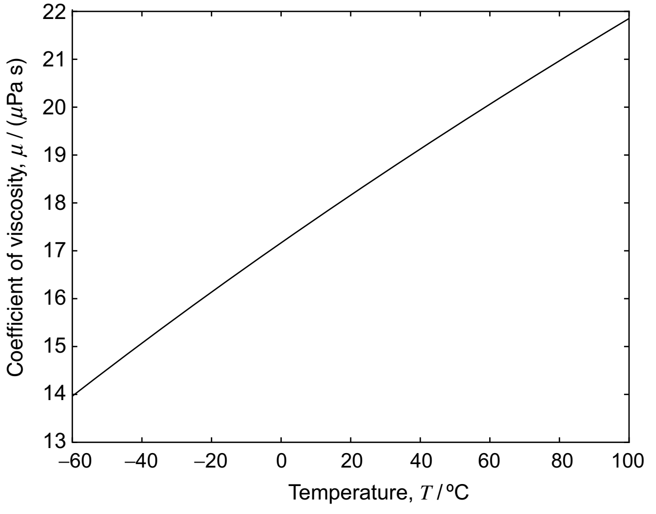

Figure 1.6 Variation of the coefficient of dynamic viscosity of air with temperature at atmospheric pressures, according to Equation (1.5)

Correlating the Viscosity of Air with Sutherland's Law

For sub-hypersonic aerodynamics within the lower parts of the atmosphere, it is usually adequate to take the viscosity of air as depending only on the temperature. A simple correlation is the *Sutherland law*

$$\mu = \frac{S\sqrt{T}}{1 + \frac{C}{T}} \tag{1.5}$$

where T is the absolute temperature and S and C are coefficients determined from correlation with experiments. Reasonable values are $S = 1.495\ \mu\text{Pa s/K}^{-1/2}$ and $C = 120$ K. This is coded in Octave as shown in Listing 1.2 and used to plot Figure 1.6.

Kinematic Viscosity

It is often more convenient to work with the coefficient of kinematic viscosity defined by

$$\nu = \frac{\mu}{\rho} \tag{1.6}$$

which evidently has SI units m^2/s.

1.4.3 The International Standard Atmosphere

For convenience in comparing data gathered at different times and places, an *International Standard Atmosphere* has been defined. This specifies, among other properties, the pressure p, density ρ, speed of sound a, and coefficients of dynamic μ and kinematic ν viscosity as functions of altitude. These values are widely tabulated, but also easily computed.

The lowest layer of the International Standard Atmosphere is the *troposphere*. It begins at sea level at 15°C and 101.325 kPa and extends upwards to the base of the *tropopause* at 11 km with the temperature decreasing by 6.5 K/km. The next lowest layer, up to 20 km, is the lower part of the *stratosphere*, which is isothermal.

The variation of pressure and density with height is then computed from:

- the ideal gas Equation (1.3), relating pressure p, temperature T, and density ρ; and
- hydrostatic equilibrium, relating pressure, density, and altitude y.

1.4.4 Computing Air Properties

The speed of sound and viscosity can be computed directly from Equations (1.4) and (1.5), respectively, and the temperature, so they're easy. Computing the pressure and density requires the solution of the differential equation of hydrostatic equilibrium – which is essentially what remains of the Euler Equations (2.9) in the absence of velocity:

$$\mathrm{d}p = -\rho g \, \mathrm{d}y. \tag{1.7}$$

Troposphere

The temperature in the standard troposphere is defined to be

$$T(y) = T(0) - \Gamma y, \tag{1.8}$$

where $T(0) = 15°\mathrm{C} = 288.15$ K is the standard temperature at sea-level, and $\Gamma = 6.5$ K/km is the standard constant *lapse rate*.

Expressing the density in terms of the ideal gas Equation (1.3) in the hydrostatic balance with $\mathrm{d}y = -\mathrm{d}T/\Gamma$

$$\begin{aligned}
\mathrm{d}p &= \frac{pg}{RT\Gamma} \, \mathrm{d}T \\
\mathrm{d} \ln p &= \frac{g}{\Gamma R} \, \mathrm{d} \ln T \\
\frac{p}{p(0)} &= \left\{ \frac{T}{T(0)} \right\}^{g/\Gamma R},
\end{aligned}$$

where $p(0) = 101325$ Pa $= 1$ atm is the standard atmospheric pressure at sea level.

Listing 1.3 `atmosphere.m`.

```
function [p, T, rho, a, mu] = atmosphere (y)

    g = 9.80665; R = 287.0; cpcv = 7/5;
    T0 = 15 + 273.15; p0 = 101325e0;
    L = 6.5e-3; yt = 11e3; top = 20e3;

    troposphere = y <= yt;
    strat = ~troposphere & (y <= top);
    T = NaN (size (y)); p = T;

    T(troposphere) = T0 - L * y(troposphere);
    Ts = T0 - L * yt;
    T(strat) = Ts;

    p(troposphere) = p0 * (T(troposphere) / T0) .^ (g/L/R);
    pt = p0 * (Ts / T0) ^ (g/L/R);
    p(strat) = pt * exp (g/R/Ts * (yt - y(strat)));

    rho = p ./ (R * T);
    a = speed_of_sound (T);
    mu = viscosity (T);
```

Stratosphere

The lower part of the standard stratosphere is isothermal, so the hydrostatic balance reduces to

$$\begin{aligned} \mathrm{d}p &= -\rho g \, \mathrm{d}y \\ &= \frac{-p}{RT} \, g \, \mathrm{d}y \\ \mathrm{d} \ln p &= \frac{-g}{RT} \, \mathrm{d}y \\ \frac{p}{p_\mathrm{t}} &= \exp \left\{ \frac{g}{RT} (y_\mathrm{t} - y) \right\}. \end{aligned}$$

These properties for these two lowest layers of the atmosphere can be computed with an Octave function (`atmosphere.m`, in Listing 1.3), as shown in Figure 1.7.

1.5 Dimensional Theory

Following the considerations in Section 1.1.4, we expect that the aerodynamic force $\boldsymbol{A}$ for steady symmetrical flight in a straight line is a function of the airspeed and incidence, the aircraft geometry, and the air properties. This involves a lot of parameters.

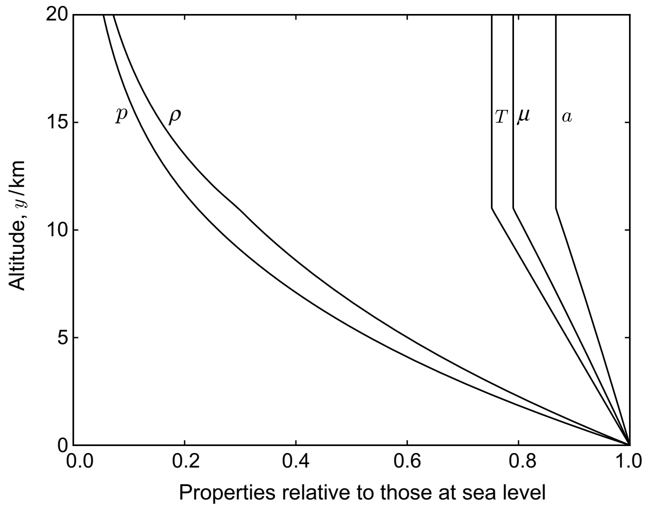

Figure 1.7 Relative variation of the lower International Standard Atmosphere with altitude

First, consider just the two-dimensional flow over a wing section of given shape but variable size parameterized, say, by its chord length c. Then the lift (per unit span) is

$$\ell = f(V, \alpha, c, \rho, \mu, a).$$

This is still a lot of parameters to have to vary in an experimental program. Some further reduction can by achieved using *dimensional theory*.

An equation like 'two metres plus three seconds equals five kilograms' doesn't make any sense. We can only add or compare for equality quantities that have like units. However, we can multiply different physical quantities; e.g.

$$1 \text{ kg} \times 9.8 \text{ m/s} = 9.8 \text{ N}$$

does make sense, since $1 \text{ N} \equiv 1 \text{ kg m/s}^2$.

Say we attempt to correlate the aerodynamic force by a functional form like

$$\ell = \sum k_{pqrstu} V^p \alpha^q c^r \rho^s \mu^t a^u$$

where $p, q, \ldots, u$ are undetermined powers and k_{pqrstu} is a dimensionless coefficient. Then every term in the summation must have the same units as ℓ:

$$\frac{\text{N}}{\text{m}} = (\text{m/s})^p(-)^q(\text{m})^r(\text{kg/m}^3)^s(\text{Pa s})^t(\text{m/s})^u.$$

Reduce all derived units—those except the kilogram, metre, and second—to the fundamental ones using $\text{N} = \text{kg m/s}^2$ and $\text{Pa} = \text{kg/m s}^2$, and equate powers of kg, m, and s. This leads to

$$\begin{aligned}\frac{\text{kg}}{\text{s}^2} &= (\text{m/s})^p(-)^q(\text{m})^r(\text{kg/m}^3)^s(\text{kg/m.s})^t(\text{m/s})^u \\ &= \frac{\text{m}^p}{\text{s}^p}(-)^q\text{m}^r\frac{\text{kg}^s}{\text{m}^{3s}}\frac{\text{kg}^t}{\text{m}^t\text{s}^t}\frac{\text{m}^u}{\text{s}^u} \\ &= \text{m}^{p+r-3s-t+u}\text{kg}^{s+t}\text{s}^{-p-t-u}\end{aligned}$$

so

$$\text{for kg:} \quad s + t = 1 \tag{1.9a}$$

$$\text{for m:} \quad p + r - 3s - t + u = 0 \tag{1.9b}$$

$$\text{for s:} \quad -u - p - t = -2. \tag{1.9c}$$

This is three equations in five unknowns, so we have two free parameters; say t and u. In terms of them, the solution for the other three is $s = 1 - t$, $p = 2 - t - u$, and $r = 1 - t$. Then the formula for the force is

$$\begin{aligned}\ell &= \sum k_{qtu} V^{2-t-u}\alpha^q c^{1-t}\rho^t a^u \\ &= \sum k_{qtu} V^2\alpha^q c\rho\left(\frac{\mu}{\rho Vc}\right)^t\left(\frac{a}{V}\right)^u\end{aligned}$$

where the indices q, t, and u remain arbitrary. Divide through by $\rho V^2 c$ to get

$$\frac{\ell}{\rho V^2 c} = \sum k_{qtu}\alpha^q\left(\frac{\mu}{\rho Vc}\right)^t\left(\frac{a}{V}\right)^u.$$

Define

$$C_\ell \equiv \frac{\ell}{\frac{1}{2}\rho V^2 c} \quad \text{(coefficient of lift)} \tag{1.10}$$

$$\text{Re} \equiv \frac{\rho Vc}{\mu} \equiv \frac{Vc}{\nu} \quad \text{(Reynolds number)} \tag{1.11}$$

$$\text{Ma} \equiv \frac{V}{a} \quad \text{(Mach number).} \tag{1.12}$$

These three quantities have no units: i.e. are pure numbers or *dimensionless* quantities. Then

$$C_\ell = 2\sum k_{qtu}\alpha^q\text{Re}^{-t}\text{Ma}^{-u},$$

so that the lift coefficient must depend only on the incidence and Reynolds and Mach numbers

$$C_\ell = C_\ell(\alpha, \text{Re}, \text{Ma}).$$

Similarly, for the *drag coefficient*

$$C_d = \frac{d}{\frac{1}{2}\rho V^2 c} = C_d(\alpha, \text{Re}, \text{Ma}),$$

where d is the drag per unit span.

In general, *Buckingham's* Π *theorem* states that a function of n parameters involving d fundamental dimensions (e.g. mass, length, time) can be reduced to a dimensionless function of $n - d$ dimensionless parameters. Here we reduced the lift correlation formula from $n = 6$ parameters to $n - d = 6 - 3 = 3$.

We see that the Reynolds number represents the influence of viscosity, and the Mach number the influence of compressibility.

The same result can also be obtained by *nondimensionalizing* the full equations governing the fluid motion: the *Navier–Stokes equations*.

To consider more general problems (e.g. other wing section shapes) we need more parameters. These should also be dimensionless, for example, quantities with dimensions of length like

- maximum camber
- position of maximum camber
- maximum thickness
- position of maximum thickness

are usually introduced using their ratio to the chord length, as in Figure 1.3.

Another advantage of working with dimensionless quantities is that they have the same value, regardless of whether the underlying dimensional quantities are measured in Imperial, SI, or whatever other system of units.

1.5.1 Alternative methods

The Π theorem can be exploited to derive simpler procedures for finding the dimensionless parameters. A popular one is the Hunsaker–Rightmire method; here we present another, taken from Bradshaw (1964).

For each of the relevant dimensions, e.g. mass, length, and time, choose a quantity involving that dimension; e.g. ρ for mass, V for time, and c for length. Then take each of the remaining quantities and multiply or divide by an appropriate power of each of these in turn to eliminate the dimensions. Thus, writing [=] to indicate dimensional equivalence,

$$\frac{\ell}{\rho}[=]\mathrm{m}^3.\mathrm{s}^{-2}$$

$$\frac{\ell}{\rho V^2}[=]\mathrm{m}$$

$$\frac{\ell}{\rho V^2 c}[=]1,$$

which leads us to the lift coefficient. Operating similarly on μ and a would lead us to the Reynolds and Mach numbers, respectively.

1.5.2 Example: Using Octave to Solve a Linear System

Although the system of three linear Equations (1.9a) in five variables was easy enough to solve by hand, this also makes it ideal for demonstrating the use of Octave for these problems. Much larger systems will be encountered later in the lumped vortex (Chapter 7), panel (Chapter 8), and vortex lattice (Chapter 14) methods and they will really require automatic computation.

First, let us rewrite the three Equations (1.9a) as a single matrix equation.

$$\begin{bmatrix} 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & -3 & -1 & 1 \\ -1 & 0 & 0 & -1 & -1 \end{bmatrix} \begin{Bmatrix} p \\ r \\ s \\ t \\ u \end{Bmatrix} = \begin{Bmatrix} 1 \\ 0 \\ -2 \end{Bmatrix} \tag{1.13}$$

Then *partition* the five variables into those remaining free (t and u) and those that will be expressed in terms of them (p, r, and s).

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & -3 \\ -1 & 0 & 0 \end{bmatrix} \begin{Bmatrix} p \\ r \\ s \end{Bmatrix} + \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ -1 & -1 \end{bmatrix} \begin{Bmatrix} t \\ u \end{Bmatrix} = \begin{Bmatrix} 1 \\ 0 \\ -2 \end{Bmatrix}$$

Now, hoping that the square matrix at the left is nonsingular, left-multiply through by its inverse. (If it were singular this step would fail, which would indicate that the partitioning should be reconsidered; this doesn't apply in this example.)

$$\begin{Bmatrix} p \\ r \\ s \end{Bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & -3 \\ -1 & 0 & 0 \end{bmatrix}^{-1} \left(\begin{Bmatrix} 1 \\ 0 \\ -2 \end{Bmatrix} - \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ -1 & -1 \end{bmatrix} \begin{Bmatrix} t \\ u \end{Bmatrix} \right) \tag{1.14}$$

Carrying out the matrix solutions (omitting the details here since it will shortly be redone automatically in Octave using its backslash operator)

$$\begin{Bmatrix} p \\ r \\ s \end{Bmatrix} = \begin{Bmatrix} 2 \\ 1 \\ 1 \end{Bmatrix} - \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 0 \end{bmatrix} \begin{Bmatrix} t \\ u \end{Bmatrix} = \begin{Bmatrix} 2 - t - u \\ 1 - t \\ 1 - t \end{Bmatrix}, \tag{1.15}$$

which is exactly the answer obtained previously.

The matrix from Equation (1.13) can be set up and then Equation (1.14) executed in Octave quite directly, as shown in Listing 1.4, with output in Listing 1.5.

Listing 1.4 Octave code for solving the 5×3 linear system, implementing Equation (1.14) for reducing the number of governing parameters from 5 to 3.

```
A = [ 0,  0,   1,   1,   0;
      1,  1,  -3,  -1,   1;
     -1,  0,   0,  -1,  -1];
bound = 1:size (A, 2) <= size (A, 1);
A(:,bound) \ [1; 0; -2]
A(:,bound) \ A(:,~bound)
```

Listing 1.5 Octave output for Listing 1.4, being the 3 constants and 3 × 2 coefficients in Equation (1.15).

```
ans =
   2
   1
   1

ans =
   1   1
   1   0
   1   0
```

1.6 Example: NACA Report No. 502

Here we look at an example of an actual wind-tunnel study of the aerodynamic force on a wing.

> Silverstein A 1935 Scale effect on Clark Y airfoil characteristics from NACA full-scale wind-tunnel tests. Report 502, NACA.

It exhibits many of the points noted above.

Noting discrepancies in previously published wind-tunnel data, Silverstein (1935) carried out a new set of force measurements on a wing in a wind-tunnel, using different airstream speeds V, wing sizes (varying chord length c and span b), and geometric angle of incidence α.

Silverstein (1935, table I) defines the geometry of the Clark Y wing section using the dimensionless ordinates of the upper y_U and lower y_L surfaces as functions of the dimensionless distance along the chord. Using this data, reproduced here in Table 1.1, we can reconstruct the profile and the camber line, as in Figure 1.8. We see that:

- The maximum camber is about 6% of the chord length, and occurs about 30% along the chord from leading to trailing edge.
- The third of Milne-Thomson's definitions of chord has been used: a line tangent to the lower surface at two points. Note that if, instead, the chord had been defined as the straight line joining the ends of the camber line, the chord would have a different angle, and so the geometric incidence would be modified for the same airstream.
- The coordinates are all normalized by the chord.

Although the airspeed, chord length, and wing span were all varied, Silverstein enables easy and meaningful comparison between the results of different runs by quoting and presenting them in terms of the dimensionless Reynolds number and coefficient of lift (see e.g. his figure 5). Also, although Silverstein measured airspeed in miles per hour and chord length in feet, the dimensionless numbers are exactly the same as if he had used SI units.

Table 1.1 Upper and lower surface ordinates for the Clark Y aerofoil.

$100x/c$	$100y_U/c$	$100y_L/c$
0	3.50	3.50
1.25	5.45	1.93
2.5	6.50	1.47
5	7.90	0.93
7.5	8.85	0.63
10	9.60	0.42
15	10.68	0.15
20	11.36	0.03
30	11.70	0.00
40	11.40	0.00
50	10.52	0.00
60	9.15	0.00
70	7.35	0.00
80	5.22	0.00
90	2.80	0.00
95	1.49	0.00
100	0.12	0.00

Source: After Silverstein (1935, table I).

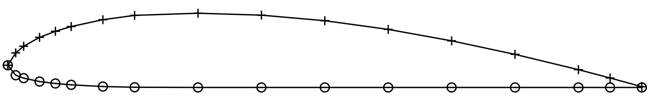

Figure 1.8 The Clark Y profile, from Silverstein (1935, table I); the upper nodes are marked with plus signs

1.7 Exercises

1. A wing has span b and root chord c_r, compute the planform area, mean chord, and aspect ratio if the planform is
   (a) elliptic;
   (b) triangular, with zero tip chord;
   (c) trapezoidal, with taper ratio t.
2. Invert Equations (1.1a) and (1.1b) to obtain the Cartesian components.
   Ans.:

$$A_x = D\cos\alpha - L\sin\alpha$$
$$A_y = D\sin\alpha + L\cos\alpha.$$

3. In the Introduction to NACA Report No. 463, Stack (1933) wrote

> The advantages of model testing as an aid to the solution of full-scale problems are often neutralized by the inaccurate reproduction of the full-scale flow in the model test. The conditions which must be fulfilled in the model test so that the results may be directly applicable to the full-scale problem are twofold. First, the model must be geometrically similar to the full-scale object – a condition usually obtained – and second, the model flow pattern must be similar to the full-scale flow pattern – a condition generally not fulfilled. The principal factors that determine flow similarity and the Reynolds nuber $\rho Vl/\mu$ and the compressibility factor V/V_c where V_c is the velocity of sound in the gas.

   To what quantity in Section 1.5 does the ratio V/V_c correspond?

4. Consider a wing at 12° incidence. The lift and drag coefficients are 1.2 and 0.1092, respectively. Calculate the axial and normal force coefficients. (These figures are from Silverstein 1935, table II.)

5. (a) If $\mathrm{Re} = 1.12 \times 10^6$ for a wing section of chord $c = 4$ ft, what must the air-speed be, assuming sea-level air? (Again, these figures are from Silverstein 1935, table II.)
   (b) What would the free-stream Mach number have been?
   (c) If the chord and span were $c = 4$ ft and $b = 24$ ft, the wing-planform was rectangular, and the lift and drag coefficients were as given in Exercise 4, what must the lift and drag forces have been?
   (d) What would the Reynolds number be in water at the same speed?
   (e) What speed in water would give the same Reynolds number?

6. Nondimensionalize the two-dimensional incompressible Navier–Stokes equations for constant density and viscosity:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

$$\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) = -\frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right).$$

   If the velocity is normalized by q_∞ and the coordinates by c, how should the pressure be nondimensionalized if the pressure term is to remain finite as the Reynolds number $\mathrm{Re} \equiv \rho q_\infty c/\mu$ tends to: (*a*) 0; (*b*) ∞.

7. Golf balls (which have a standard minimum diameter of 42.7 mm) can exceed 300 km/h when driven, although recreational players generally achieve less than half this. What are the Reynolds and Mach numbers of the ball under these conditions, near sea level in the standard atmosphere?

8. The normal altitude of Qantas's A380-800 passenger aircraft is 10 700 – 13 100 m. How much do the properties of air vary over this range, and how different are they from those at sea level.

What is the Mach number at the cruising speed of 920 km/h? What is the cruising Reynolds number, based on the mean wing chord length of 10.6 m?

9. How does the coefficient of kinematic viscosity vary with altitude in the International Standard Atmosphere? Graph and tabulate it at each kilometre from sea level up to the top of the isothermal part of the stratosphere, both in absolute terms and as a ratio to the value at sea level.

10. Between 20 and 32 km, the stratosphere has a lapse rate of $\Gamma = -1$ K/km. Extend Listing 1.3 and Figure 1.7 to this height.

11. Reaction Engines Ltd's A2 hypersonic transport is envisaged as cruising at Mach five at an altitude of 100 000 ft. The total length of the craft is 132 m. Using the results of Exercise 10, what would be the A2's air-speed? What would be its Reynolds number (based on the given overall length)?

12. For each of the properties defined by the International Standard Atmosphere, how high can one go above sea-level before it changes by 1%? By 10%? Given a steady cruising altitude of 5 km, what variations in altitude correspond to these same variations in atmospheric properties?

13. Does humidity affect the properties of air? Under atmospheric conditions, water vapour and dry air combine together ideally; i.e. the water vapour and dry air sharing a given volume of space have a common temperature and their ('partial') densities and pressures are what they would be if they were alone, with each behaving as an ideal gas. The total density and pressure is the sum of that for each component. The gas constant for water vapour is about 461.5 J/kg K.

    The 'saturation pressure' of water is a function of temperature, roughly

$$\log_{10} \frac{p}{\text{mmHg}} = 8.07131 - \frac{1730.63}{T/\text{K} - 39.724},$$

    where 760 mmHg = 101 325 Pa. The partial pressure of water at a given temperature and relative humidity is the saturation pressure times the relative humidity. The partial pressure of dry air, then, is the difference between the total pressure and the partial pressure of water vapour.

    Consider humid air at 101.325 kPa, 15°C, and 50% relative humidity. What is the total density? How different is this from the density of dry air at the same temperature and (total) pressure?

14. If the partial pressure of water vapour in humid air exceeds the saturation pressure, the moisture will begin to condense out as droplets.

    Say a given volume of humid air at the conditions considered in the last question is raised in altitude through the standard atmosphere, so that the temperature and total pressure fall, while the composition (absolute humidity) is conserved. How does the relative humidity of the parcel vary with height? At what height is saturation achieved?

15. What about the droplets of liquid water in a cloud, do they affect atmospheric properties? Consider a typical cloud at 10 km above sea level consisting of 50 000 000 droplets per cubic metre, each of diameter 10 μm. What mass do these droplets add to each unit volume of air?

1.8 Further Reading

The pioneering aeronautical experiments of Henson and Stringfellow were described by Davy (1931). For an introduction to the use of wind tunnels, see Glauert (1926, chapter 14), Prandtl and Tietjens (1957), Bradshaw (1964, chapter 2), and Liepmann and Roshko (1957).

For frames of reference and how to define the geometric angle of incidence, see Milne-Thomson (1973) or Anderson (2007).

For alternative quantitative descriptions of wings and their sections, see Dommasch *et al.* (1967), Kuethe and Chow (1998), Bertin (2002), Houghton and Carpenter (2003), or Anderson (2007). Details on the NACA wing sections can be found in Abbott *et al.* (1945) and Abbott and von Doenhoff (1959).

The Sutherland law is one of the most common forms of equations used to describe the dependence viscosity on temperature (Glasstone 1946; Montgomery 1947).

The International Standard Atmosphere (ISO 1975), its predecessors, and their importance in aerodynamics are discussed by Glauert (1926), Batchelor (1967), Milne-Thomson (1973), Hoerner and Borst (1985), and Kuethe and Chow (1998); tables may be found in Bertin (2002) and Anderson (2007).

The concept of dimensional analysis is fundamental and therefore difficult to reduce to simpler terms. Once grasped it is obvious but until then it can appear abstract. Such fundamental concepts are best acquired by reading different explanations until the darkness suddenly clears; for dimensional analysis, try Glauert (1926), Hunsaker and Rightmire (1947), Karamcheti (1966), Batchelor (1967), Milne-Thomson (1973), Streeter and Wylie (1983), Kuethe and Chow (1998), or Anderson (2007). All textbooks on aerodynamics or fluid mechanics will contain some explanation of dimensional analysis, not least because of the utility of dynamical similarity. Bertin (2002) demonstrates the approach mentioned of nondimensionalizing the Navier–Stokes equations. For the method of Hunsaker and Rightmire (1947), see Streeter and Wylie (1983) or Anderson (2007); the alternative method of Section 1.5.1 is taken from Bradshaw (1964).

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