1
Basic Equity Derivatives Theory

In finance, an equity derivative belongs to a class of derivative instruments whose underlying asset is a stock or stock index. Hence, the value of an equity derivative is a function of the value of the stock or index. With a growing interest in the stock markets of the world, and the prevalence of employee stock options as a form of compensation, equity derivatives continue to expand with new product structures continuously being offered. In this chapter, we introduce the concept of equity derivatives with emphasis on forwards, futures, option contracts and also different types of hedging strategies.

1.1 INTRODUCTION

Among the many equity derivatives that are actively traded in the market, options and futures are by far the most commonly traded financial instruments. The following is the basic vocabulary of different types of derivatives contracts:

- **Option** A contract that gives the holder the right but not the obligation to buy or sell an asset for a fixed price (strike/exercise price) at or before a fixed expiry date.
- **Call Option** A contract that gives the holder the right to buy an asset for a fixed price (strike/exercise price) at or before a fixed expiry date.
- **Put Option** A contract that gives the holder the right to sell an asset for a fixed price (strike/exercise price) at or before a fixed expiry date.
- **Payoff** Difference between the market price and the strike price depending on derivative type.
- **Intrinsic Value** The payoff that would be received/paid if the option was exercised when the underlying asset is at its current level.
- **Time Value** Value that the option is above its intrinsic value. The relationship can be written as

$$\text{Option Price} = \text{Intrinsic Value} + \text{Time Value}.$$

- **Forward/Futures** A contract that obligates the buyer and seller to trade an underlying, usually a commodity or stock price index, at some specified time in the future. The difference between a forward and a futures contract is that forwards are over-the-counter (OTC) products which are customised agreements between two counterparties. In contrast, futures are standardised contracts traded on an official exchange and are marked to market on a daily basis. Hence, futures contracts do not carry any credit risk (the risk that a party will not meet its contractual obligations).
- **Swap** An OTC contract in which two counterparties exchange cash flows.
- **Stock Index Option** A contract that gives the holder the right but not the obligation to buy or sell a specific amount of a particular stock index for an agreed fixed price at or before

a fixed expiry date. As it is not feasible to deliver an actual stock index, this contract is usually settled in cash.

- **Stock Index Futures** A contract that obligates the buyer and seller to trade a quantity of a specific stock index on an official exchange at a price agreed between two parties with delivery on a specified future date. Like the stock index option, this contract is usually settled in cash.
- **Strike/Exercise Price** Fixed price at which the owner of an option can buy (for a call option) or sell (for a put option) the underlying asset.
- **Expiry Date/Exercise Date** The last date on which the option contract is still valid. After this date, the option contract becomes worthless.
- **Delivery Date** The last date by which the underlying commodity or stock price index (usually cash payment based on the underlying stock price index) for a forward/futures contract must be delivered to fulfil the requirements of the contract.
- **Discounting** Multiplying an amount by a discount factor to compute its present value (discounted value). It is the opposite of compounding, where interest is added to an amount so that the added interest also earns interest from then on. If we assume the risk-free interest rate r is a constant and continuously compounding, then the present value at time t of a certain payoff M at time T, for $t < T$, is $Me^{-r(T-t)}$.
- **Hedge** An investment position intended to reduce the risk from adverse price movements in an asset. A hedge can be constructed using a combination of stocks and derivative products such as options and forwards.
- **Contingent Claim** A claim that depends on a particular event such as an option payoff, which depends on a stock price at some future date.

Within the context of option contracts we subdivide them into *option style* or *option family*, which denotes the class into which the type of option contract falls, usually defined by the dates on which the option may be exercised. These include:

- **European Option** An option that can only be exercised on the expiry date.
- **American Option** An option that can be exercised any time before the expiry date.
- **Bermudan Option** An option that can only be exercised on predetermined dates. Hence, this option is intermediate between a European option and an American option.

Unless otherwise stated, all the options discussed in this chapter are considered to be European.

Option Trading

In option trading, the transaction involves two parties: a buyer and a seller.

- The buyer of an option is said to take a *long position* in the option, whilst the seller is said to take a *short position* in the option.
- The buyer or owner of a call (put) option has the right to buy (sell) an asset at a specified price by paying a premium to the seller or writer of the option, who will assume the obligation to sell (buy) the asset should the owner of the option choose to exercise (enforce) the contract.

- The payoff of a call option at expiry time T is defined as

$$\Psi(S_T) = \max\{S_T - K, 0\}$$

where S_T is the price of the underlying asset at expiry time T and K is the strike price. If $S_T > K$ at expiry, then the buyer of the call option should exercise the option by paying a lower amount K to obtain an asset worth S_T. However, if $S_T \leq K$ then the buyer of the call option should not exercise the option because it would not make any financial sense to pay a higher amount K to obtain an asset which is of a lower value S_T. Here, the option expires worthless.

In general, the profit earned by the buyer of the call option is

$$\Upsilon(S_T) = \max\{S_T - K, 0\} - C(S_t, t; K, T)$$

where $C(S_t, t; K, T)$ is the premium paid at time $t < T$ (written on the underlying asset S_t) in order to enter into a call option contract.

Neglecting the premium for buying an option, a call option is said to be *in-the-money* (ITM) if the buyer profits when the option is exercised ($S_T > K$). In contrast, a call option is said to be *out-of-the-money* (OTM) if the buyer loses when the option is exercised ($S_T < K$). Finally, a call option is said be to *at-the-money* (ATM) if the buyer neither loses nor profits when the option is exercised ($S_T = K$). Figure 1.1 illustrates the concepts we have discussed.

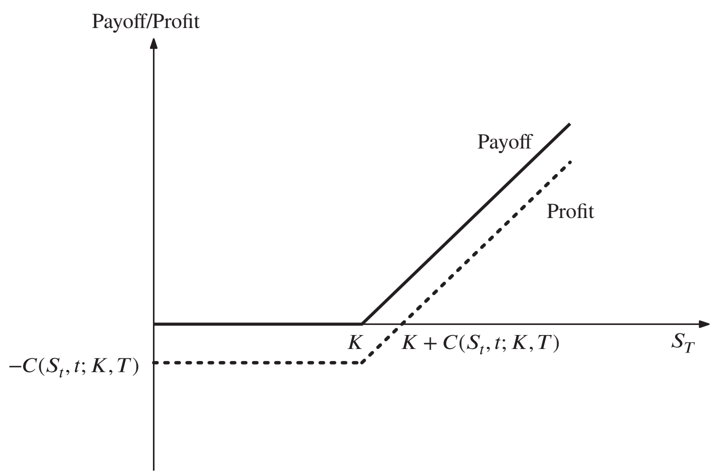

Figure 1.1 Long call option payoff and profit diagram.

- The payoff of a put option at expiry time T is defined as

$$\Psi(S_T) = \max\{K - S_T, 0\}$$

where S_T is the price of the underlying asset at expiry time T and K is the strike price. If $K > S_T$ at expiry, then the buyer of the put option should exercise the option by selling the asset worth S_T for a higher amount K. However, if $K \leq S_T$ then the buyer of the put option should not exercise the option because it would not make any financial sense to sell the asset worth S_T for a lower amount K. Here, the option expires worthless.

In general, the profit earned by the buyer of the put option is

$$\Upsilon(S_T) = \max\{K - S_T, 0\} - P(S_t, t; K, T)$$

where $P(S_t, t; K, T)$ is the premium paid at time $t < T$ (written on the underlying asset S_t) in order to enter into a put option contract.

Neglecting the premium for buying an option, a put option is said to be ITM if the buyer profits when the option is exercised ($K > S_T$). In contrast, a put option is said to be OTM if the buyer loses when the option is exercised ($K < S_T$). Finally, a put option is said to be ATM if the buyer neither loses nor profits when the option is exercised ($S_T = K$). Figure 1.2 illustrates the concepts we have discussed.

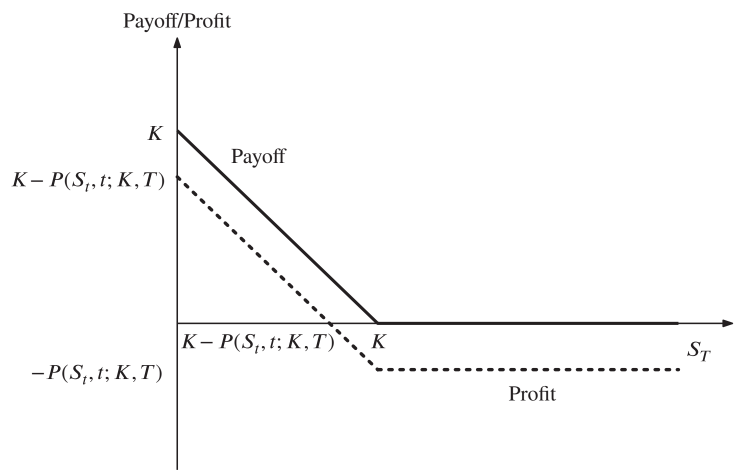

Figure 1.2 Long put option payoff and profit diagram.

Forward Contract

In a forward contract, the transaction is executed between two parties: a buyer and a seller.

- The buyer of the underlying commodity or stock index is referred to as the *long side* whilst the seller is known as the *short side*.
- The contractual obligation to buy the asset at the agreed price on a specified future date is known as the *long position*. A long position profits when the price of an asset rises.
- The contractual obligation to sell the asset at the agreed price on a specified future date is known as the *short position*. A short position profits when the price of an asset falls.
- For a long position, the payoff of a forward contract at the delivery time T is

$$\Pi_T = S_T - F(t, T)$$

where S_T is the spot price (or market price) at the delivery time T and $F(t, T)$ is the forward price initiated at time $t < T$ to be delivered at time T.

- For a short position, the payoff of a forward contract at the delivery time T is

$$\Pi_T = F(t,T) - S_T$$

  where S_T is the spot price (or market price) at the delivery time T and $F(t,T)$ is the forward price initiated at time $t < T$ to be delivered at time T.
- Since there is no upfront payment to enter into a forward contract, the profit at delivery time T is the same as the payoff of a forward contract at time T. Figure 1.3 illustrates the concepts we have discussed.

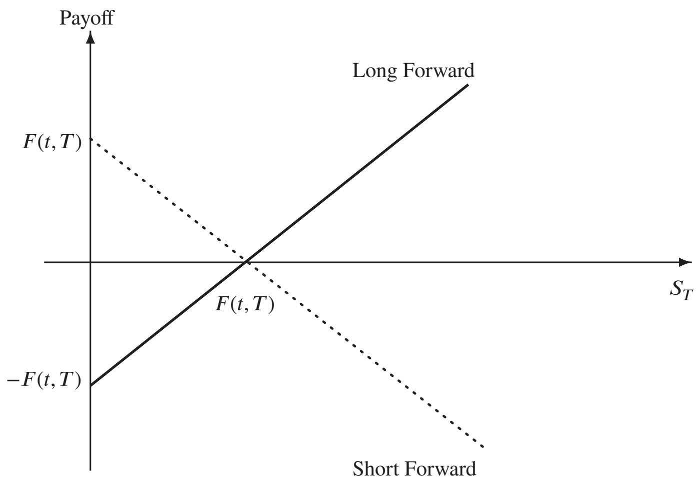

Figure 1.3 Long and short forward payoffs diagram.

Futures Contract

Similar to a forward contract, a futures contract is also an agreement between two parties in which the buyer agrees to buy an underlying asset from the seller. The delivery of the asset occurs at a specified future date, where the price is determined at the time of initiation of the contract. As in the case of a forward contract, it costs nothing to enter into a futures contract. However, the differences between futures and forwards are as follows:

- In a futures contract, the terms and conditions are standardised where trading takes place on a formal exchange with deep liquidity.
- There is no default risk when trading futures contracts, since the exchange acts as a counterparty guaranteeing delivery and payment by use of a clearing house.
- The clearing house protects itself from default by requiring its counterparties to settle profits and losses or mark to market their positions on a daily basis.
- An investor can hedge his/her future position by engaging in an opposite transaction before the delivery date of the contract.

In the futures market, margin is a performance guarantee. It is money deposited with the clearing house by both the buyer and the seller. There is no loan involved and hence, no interest is charged. To safeguard the clearing house, the exchange requires buyers/sellers to post margin (i.e., deposit funds) and settle their accounts on a daily basis. Prior to trading, the trader must post margin with their broker who in return will post margin with the clearing house.

- **Initial Margin** Money that must be deposited in order to initiate a futures position.
- **Maintenance Margin** Minimum margin amount that must be maintained; when the margin falls below this amount it must be brought back up to its initial level. Margin calculations are based on the daily settlement price, the average of the prices for trades during the closing period set by the exchange.
- **Variation Margin** Money that must be deposited to bring it back to the initial margin amount. If the account margin is more than the initial margin, the investor can withdraw the funds for new positions.
- **Settlement Price** Known also as the closing price for a stock. The settlement price is the price at which a derivatives contract settles once a given trading day has ended. The settlement price is used to calculate the margin at the end of each trading day.
- **Marking-to-Market** Process of adding gains to or subtracting losses from the margin account daily, based on the change in the settlement prices from one day to the next.

Termination of a futures position can be achieved by:

- An offsetting trade (known as a back-to-back trade), entering into an opposite position in the same contract.
- Payment of cash at expiration for a cash-settlement contract.
- Delivery of the asset at expiration.
- Exchange of physicals.

Stock Split (Divide) Effect

When a company issues a stock split (e.g., doubling the number of shares), the price is adjusted so as to keep the net value of all the stock the same as before the split.

Stock Dividend Effect

When dividends are paid during the life of an option contract they will inadvertently affect the price of the stock or asset. Here, the direction of the stock price will be determined based on the choice of the company whether it pays dividends to its shareholders or reinvests the money back in the business. Since we may regard dividends as a cash return to the shareholders, the reinvestment of the cash back into the business could create more profit and, depending on market sentiment, lead to an increase in stock price. Conversely, paying dividends to the shareholders will effectively reduce the stock price by the amount of the dividend payment, and as a result will affect the premium prices of options as well as futures and forwards.

Hedging Strategies

In the following we discuss how an investor can use options to design investment strategies with specific views on the stock price behaviour in the future.

- **Protective** This hedging strategy is designed to insure an investor's asset position (long buy or short sell).

An investor who owns an asset and wishes to be protected from falling asset values can insure his asset by buying a put option written on the same asset. This combination of owning an asset and purchasing a put option on that asset is called a protective put.

In contrast, an investor shorting an asset who will experience a loss if the asset price rises in value can insure his position by purchasing a call option written on the same asset. Such a combination of selling an asset and purchasing a call option on that asset is called a protective call.

- **Covered** This hedging strategy involves the investor writing an option whilst holding an opposite position on the asset. The motivation for doing so is to generate additional income by receiving premiums from option buyers, and this strategy is akin to selling insurance. When the writer of an option has no position in the underlying asset, this form of option writing is known as naked writing.

  In a covered call, the investor would hold a long position on an asset and sell a call option written on the same asset.

  In a covered put, the investor would short sell an asset and sell a put option written on the same asset.

- **Collar** This hedging strategy uses a combination of protective strategy and selling options to collar the value of an asset position within a specific range. By using a protective strategy, the investor can insure his asset position (long or short) whilst reducing the cost of insurance by selling an option.

  In a purchased collar, the strategy consists of a protective put and selling a call option whilst in a written collar, the strategy consists of a protective call and selling a put option.

- **Synthetic Forward** A synthetic forward consists of a long call, $C(S_t, t; K, T)$ and a short put, $P(S_t, t; K, T)$ written on the same asset S_t at time t with the same expiration date $T > t$ and strike price K.

  At expiry time T, the payoff is

  $$C(S_T, T; K, T) - P(S_T, T; K, T) = S_T - K$$

  and, assuming a constant risk-free interest rate r and by discounting the payoff back to time t, we have

  $$C(S_t, t; K, T) - P(S_t, t; K, T) = S_t - Ke^{-r(T-t)}.$$

  The above equation is known as the *put–call parity*, tying the relationship between options and forward markets together.

- **Bull Spread** An investor who enters a bull spread expects the stock price to rise and wishes to exploit this.

  For a bull call spread, it is composed of

  $$\text{Bull Call Spread} = C(S_t, t; K_1, T) - C(S_t, t; K_2, T)$$

  which consists of buying a call at time t with strike price K_1 and expiry T and selling a call at time t with strike price K_2, $K_2 > K_1$ and same expiry T.

  For a bull put spread, it is composed of

  $$\text{Bull Put Spread} = P(S_t, t; K_1, T) - P(S_t, t; K_2, T)$$

  which consists of buying a put at time t with strike price K_1 and expiry T and selling a put at time t with strike price K_2, $K_2 > K_1$ and same expiry T.

- **Bear Spread** The strategy behind the bear spread is the opposite of a bull spread. Here, the investor who enters a bear spread expects the stock price to fall.

  For a bear call spread, it is composed of

$$\text{Bear Call Spread} = C(S_t.t; K_2, T) - C(S_t, t; K_1, T)$$

  which consists of selling a call at time t with strike price K_1 and expiry T and buying a call at time t with strike price K_2, $K_2 > K_1$ and same expiry T.

  For a bear put spread, it is composed of

$$\text{Bear Put Spread} = P(S_t, t; K_2, T) - P(S_t, t; K_1, T)$$

  which consists of selling a put at time t with strike price K_1 and expiry T and buying a put at time t with strike price K_2, $K_2 > K_1$ and same expiry T.
- **Butterfly Spread** The investor who enters a butterfly spread expects that the stock price will not change significantly. It is a neutral strategy combining bull and bear spreads.
- **Straddle** This strategy is used if an investor believes that a stock price will move significantly, but is unsure in which direction. Here such a strategy depends on the volatility of the stock price rather than the direction of the stock price changes.

  For a long straddle, it is composed of

$$\text{Long Straddle} = C(S_t, t; K, T) + P(S_t, t; K, T)$$

  which consists of buying a call and a put option at time t with the same strike price K and expiry T.

  For a short straddle, it is composed of

$$\text{Short Straddle} = -C(S_t, t; K, T) - P(S_t, t; K, T)$$

  which consists of selling a call and a put option at time t with the same strike price K and expiry T.
- **Strangle** The strangle hedging strategy is a variation of the straddle with the key difference that the options have different strike prices but expire at the same time.
- **Strip/Strap** The strip and strap strategies are modifications of the straddle, principally used in volatile market conditions. However, unlike a straddle which has an unbiased outlook on the stock price movement, investors who use a strip (strap) strategy would exploit on downward (upward) movement of the stock price.

1.2 PROBLEMS AND SOLUTIONS

1.2.1 Forward and Futures Contracts

1. Consider an investor entering into a forward contract on a stock with spot price \$10 and delivery date 6 months from now. The forward price is \$12.50. Draw the payoff diagrams for both the long and short forward position of the contract.

   Solution: See Figure 1.4.

   □

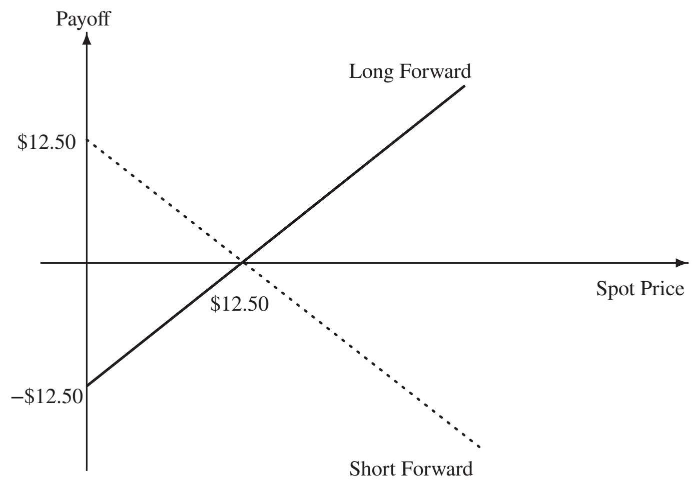

Figure 1.4 Long and short forward payoff diagram.

2. In terms of credit risk, is a forward contract riskier than a futures contract? Explain.

   Solution: Given that forward contracts are traded OTC between two parties and futures contracts are traded on exchanges which require margin accounts, forward contracts are riskier than futures contracts.

   □

3. Suppose ABC company shares are trading at $25 and pay no dividends and that the risk-free interest rate is 5% per annum. The forward price for delivery in 1 year's time is $28. Draw the payoff and profit diagrams for a long position for this contract.

   Solution: As there is no cost involved in entering into a forward contract, the payoff and profit diagrams coincide (see Figure 1.5).

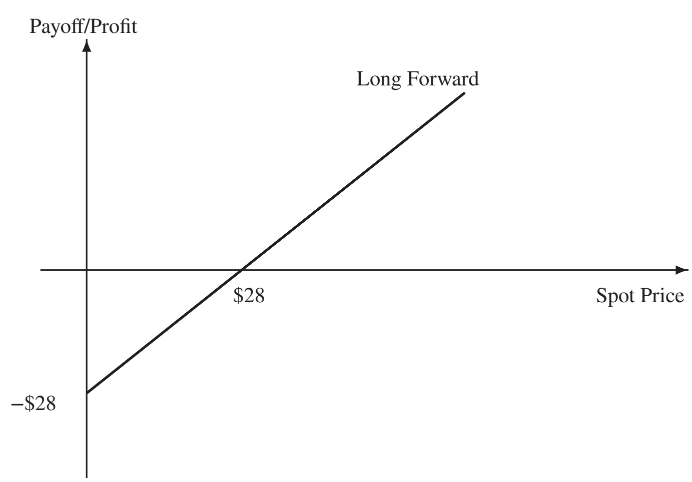

Figure 1.5 Long forward payoff and profit diagram.

□

4. Consider a stock currently worth \$100 per share with the risk-free interest rate 2% per annum. The futures price for a 1-year contract is worth \$104. Show that there exists an arbitrage opportunity by entering into a short position in this futures contract.

Solution: At current time $t = 0$, a speculator can borrow \$100 from the bank, buy the stock and short a futures contract.

At delivery time $T = 1$ year, the outstanding loan is now worth $100e^{0.02\times 1} = \$102.02$. By delivering the stock to the long contract holder and receiving \$104, the speculator can make a riskless profit of $\$104 - \$102.02 = \$1.98$.

□

5. Let the current stock price be \$75 with the risk-free interest rate 2.5% per annum. Assume the futures price for a 1-year contract is worth \$74. Show that there exists an arbitrage opportunity by entering into a long position in this futures contract.

Solution: At current time $t = 0$, a speculator can short sell the stock, invest the proceeds in a bank account at the risk-free rate and then long a futures contract.

At time $T = 1$ year, the amount of money in the bank will grow to $75e^{0.025\times 1} = \$76.89$. After paying for the futures contract which is priced at \$74, the speculator can then return the stock to its owner. Thus, the speculator can make a riskless profit of $\$76.89 - \$74 = \$2.89$.

□

6. An investor holds a long position in a stock index futures contract with a delivery date 3 months from now. The value of the contract is \$250 times the level of the index at the start of the contract, and each index point movement represents a gain or a loss of \$250 per contract. The futures contract at the start of the contract is valued at \$250,000, and the initial margin deposit is \$15,000 with a maintenance margin of \$13,750 per contract.

Table 1.1 shows the stock index movement over a 4-day period.

Table 1.1 Daily closing stock index.

Day	Closing Stock Index
1	1002
2	994
3	998
4	997

Calculate the initial stock index at the start of the contract. By setting up a table, calculate the daily marking-to-market, margin balance and the variation margin over a 4-day period.

Solution: Since the futures contract is valued at \$250,000 at the start of the contract, the initial stock index is $\frac{250,000}{250} = 1000$.

Table 1.2 displays the daily marking-to-market, margin balance and the variation margin in order to maintain the maintenance margin.

On Day 0, the initial balance is the initial margin requirement of \$15,000 while on Day 1, as the change in the stock index is increased by 2 points, the margin balance is increased by $\$250 \times 2 = \$500$. On Day 2, the margin balance is \$13,500 which is below the maintenance margin level of \$13,750. Therefore, a deposit of \$1,500 is needed to

Table 1.2 Daily movements of stock index.

Day	Required Deposit	Closing Stock Index	Daily Change	Marking-to-Market	Margin Balance	Variation Margin
0	\$15,000	1000	0	0	\$15,000	0
1	0	1002	+2	\$500	\$15,500	0
2	0	994	−8	−\$2,000	\$13,500	\$1,500
3	\$1,500	998	+4	\$1,000	\$16,000	0
4	0	997	−1	−\$250	\$15,750	0

bring the margin back to the margin requirement of \$15,000. Hence, the variation margin is \$1,500 occurring on Day 2.

□

7. An investor wishes to enter into 10 stock index futures contracts where the value of a contract is \$250 times the level of the index at the start of the contract and each index point movement represents a gain or a loss of \$250 per contract. The stock index at the start of the contract is 1,000 points and the initial margin deposit is 10% of the total futures contract value.

   Let the continuously compounded interest rate be 5% which can be earned on the margin balance and the maintenance margin be 85% of the initial margin deposit. Suppose the investor position is marked on a weekly basis. What does the maximum stock index need to be in order for the investor to receive a margin call on week 1.

   Solution: At the start of the contract the total futures contract value is $\$250 \times 1{,}000 \times 10 = \$2{,}500{,}000$ and the initial margin deposit is $\$2{,}500{,}000 \times \frac{10}{100} = \$250{,}000$. The maintenance margin is therefore $\$250{,}000 \times \frac{85}{100} = \$187{,}500$.

   To describe the movement of the stock index for week 1, see Table 1.3.

Table 1.3 Movement of stock index on week 1.

Week	Closing Stock Index	Weekly Change	Marking-to-Market	Margin Balance	Variation Margin
0	1000	0	0	\$250,000	0
1	x	$x - 1000$	$\$2{,}500 \times (x-1000)$	$\$250{,}000 + \$2{,}500 \times (x-1000)$	\$187,500

   Thus, in order to invoke a margin call we can set

$$\begin{aligned} 2500(x-1000) + 250{,}000 &= 187{,}500 \\ x &= 975. \end{aligned}$$

   Therefore, if the stock index were to fall to values below 975 points then a margin call will be issued on week 1.

□

8. Let S_t denote the price of a stock with a dividend payment $\delta \geq 0$ at time t. What is the price of the stock immediately after the dividend payment?

Solution: Let S_t^+ denote the price of the stock immediately after the dividend payment. Therefore,

$$S_t^+ = S_t - \delta.$$

□

9. Consider the price of a futures contract $F(t,T)$ with delivery time T on a stock with price S_t at time t ($t < T$). Suppose the stock does not pay any dividends. Show that under the no-arbitrage condition the futures contract price is

$$F(t,T) = S_t e^{r(T-t)}$$

where r is the risk-free interest rate.

Solution: We prove this result via contradiction.

If $F(t,T) > S_t e^{r(T-t)}$ then at time t an investor can short the futures contract worth $F(t,T)$ and then borrow an amount S_t from the bank to buy the asset. By time T the bank loan will amount to $S_t e^{r(T-t)}$. Since $F(t,T) > S_t e^{r(T-t)}$ then using the money received at delivery time T, the investor can pay off the loan, deliver the asset and make a risk-free profit $F(t,T) - S_t e^{r(T-t)} > 0$.

In contrast, if $F(t,T) < S_t e^{r(T-t)}$ then at time t an investor can long the futures contract, short sell the stock valued at S_t and then put the money in the bank. By time T the money in the bank will grow to $S_t e^{r(T-t)}$ and after returning the stock (from the futures contract) the investor will make a risk-free profit $S_t e^{r(T-t)} - F(t,T) > 0$.

Therefore, under the no-arbitrage condition we must have $F(t,T) = S_t e^{r(T-t)}$.

□

10. Consider the price of a futures contract $F(t,T)$ with delivery time T on a stock with price S_t at time t ($t < T$). Throughout the life of the futures contract the stock pays discrete dividends δ_i, $i = 1, 2, \ldots, n$ where $t < t_1 < t_2 < \cdots < t_n < T$. Show that under the no-arbitrage condition the futures contract price is

$$F(t,T) = S_t e^{r(T-t)} - \sum_{i=1}^{n} \delta_i e^{r(T-t_i)}$$

where r is the risk-free interest rate.

Solution: Suppose that over the life of the futures contract the stock pays dividends δ_i at time t_i, $i = 1, 2, \ldots, n$ where $t < t_1 < t_2 < \cdots < t_n < T$. When dividends are paid, the stock price S_t is reduced by the present values of all the dividends paid, that is

$$S_t - \sum_{i=1}^{n} \delta_i e^{-r(t_i - t)}.$$

Hence, using the same steps as discussed in Problem 1.2.1.9 (page 12), the futures price is

$$\begin{aligned}F(t,T) &= \left(S_t - \sum_{i=1}^{n} \delta_i e^{-r(t_i - t)}\right) e^{r(T-t)} \\ &= S_t e^{r(T-t)} - \sum_{i=1}^{n} \delta_i e^{r(T-t_i)}.\end{aligned}$$

□

11. Consider the number of stocks owned by an investor at time t as A_t where each of the stocks pays a continuous dividend yield D. Assume that all the dividend payments are reinvested in the stock. Show that the number of stocks owned by time T $(t < T)$ is

$$A_T = A_t e^{D(T-t)}.$$

Next consider the price of a futures contract $F(t,T)$ with delivery time T on a stock with price S_t at time t $(t < T)$. Suppose the stock pays a continuous dividend yield D. Using the above result, show that under the no-arbitrage condition the futures contract price is

$$F(t,T) = S_t e^{(r-D)(T-t)}$$

where r is the risk-free interest rate.

Solution: We first divide the time interval $[t,T]$ into n sub-intervals such that $t_i = t + \dfrac{i(T-t)}{n}$, $i = 1, 2, \ldots, n$ with $t_0 = t$ and $t_n = T$. By letting the dividend payment at time t_i be

$$\delta_i = \frac{D(T-t)}{n} S_t$$

for $i = 1, 2, \ldots, n$, and because all the dividends are reinvested in the stock, the number of stocks held becomes

$$\begin{aligned}A_{t_1} &= A_{t_0}\left[1 + \frac{D(T-t)}{n}\right] \\ A_{t_2} &= A_{t_1}\left[1 + \frac{D(T-t)}{n}\right] = A_{t_0}\left[1 + \frac{D(T-t)}{n}\right]^2 \\ A_{t_3} &= A_{t_2}\left[1 + \frac{D(T-t)}{n}\right] = A_{t_0}\left[1 + \frac{D(T-t)}{n}\right]^3 \\ &\vdots \\ A_{t_n} &= A_{t_{n-1}}\left[1 + \frac{D(T-t)}{n}\right] = A_{t_0}\left[1 + \frac{D(T-t)}{n}\right]^n.\end{aligned}$$

Because $A_{t_0} = A_t$ and $A_{t_n} = A_T$, therefore

$$A_T = A_t \left[1 + \frac{D(T-t)}{n}\right]^n$$

and taking limits $n \to \infty$ we have

$$\lim_{n\to\infty} A_T = A_t \lim_{n\to\infty} \left[1 + \frac{D(T-t)}{n}\right]^n = A_t e^{D(T-t)}.$$

From the above result we can deduce that investing one stock at time t will lead to a total growth of $e^{D(T-t)}$ by time T. Hence, if we start by buying $e^{-D(T-t)}$ number of stocks S_t at time t it will grow to one stock at time T. The total value of the stock at time t is therefore

$$S_t e^{-D(T-t)}$$

and following the arguments in Problem 1.2.1.9 (page 12) the futures price is

$$\begin{aligned} F(t,T) &= S_t e^{-D(T-t)} e^{r(T-t)} \\ &= S_t e^{(r-D)(T-t)}. \end{aligned}$$

□

12. Suppose an asset is currently worth \$20 and the 6-month futures price of this asset is \$22.50. By assuming the stock does not pay any dividends and the risk-free interest rate is the same for all maturities, calculate the 1-year futures price of this asset.

Solution: By definition the futures price is

$$F(t,T) = S_t e^{r(T-t)}$$

where t is the time of the start of the contract, T is the delivery time, S_t is the spot price at time t and r is the risk-free interest rate.

By setting $t = 0$, $S_0 = \$20$ and $T_1 = 0.5$ years we have

$$F(0,T_1) = S_0 e^{rT_1} = \$22.50.$$

Hence,

$$r = 2\log\left(\frac{22.50}{20}\right) = 2\log 1.125.$$

Therefore, for a 1-year futures price, $T_2 = 1$ year

$$F(0,T_2) = S_0 e^{rT_2} = \$20 e^{2\log 1.125 \times 1} = \$25.31.$$

□

13. Assume an investor buys 100,000 stocks of XYZ company and holds them for 3 years. Each of the stocks held pays a continuous dividend yield of 4% per annum and the investor

reinvests all the dividends when they are paid. Calculate the additional number of shares the investor would have at the end of 3 years.

Solution: Let $A_0 = 100{,}000$, $D = 0.04$ and $T = 3$ years. Therefore, by the end of 3 years, the number of shares owned by the investor is

$$A_T = A_0 e^{DT} = 100{,}000e^{0.04\times 3} = 112{,}749.69.$$

Therefore, the additional number of shares the investor has by the end of year 3 is

$$A_T - A_0 = 112{,}749.69 - 100{,}000 = 12{,}749.69 \simeq 12{,}749.$$

□

14. Let the current stock price be \$30 with two dividend payments in 6 months and 9 months from today of \$1.50 and \$1.80, respectively. The continuously compounded risk-free interest rate is 5% per annum. Find the price of a 1-year futures contract.

    Solution: Let $S_0 = \$30$, $t_1 = \frac{6}{12} = 0.5$ years, $t_2 = \frac{9}{12} = 0.75$ years, $\delta_1 = \$1.50$, $\delta_2 = \$1.80$, $r = 0.05$ and $T = 1$ year.
    Therefore, the price of a 1-year futures contract is

$$\begin{aligned} F(0,T) &= S_0 e^{rT} - \delta_1 e^{r(T-t_1)} - \delta_2 e^{r(T-t_2)} \\ &= 30e^{0.05\times 1} - 1.50e^{0.05\times(1-0.5)} - 1.80e^{0.05\times(1-0.75)} \\ &= \$28.18. \end{aligned}$$

□

15. Let the current price of a stock be \$12.75 that pays a continuous dividend yield D. Suppose the risk-free interest rate is 6% per annum and the price of a 6-month forward contract is \$13.25. Find D.

    Solution: Let $S_0 = \$12.50$, $r = 0.06$, $T = 0.5$ years and $F(0,T) = \$13.25$. Since $F(0,T) = S_0 e^{(r-D)T}$,

$$\begin{aligned} 12.75e^{(0.06-D)\times 0.5} &= 13.25 \\ D &= 0.06 - \log\left(\frac{13.25}{12.75}\right) \times \frac{1}{0.5} \\ &= 0.020395. \end{aligned}$$

    Hence, the dividend yield is $D = 2.0395\%$ per annum.

□

1.2.2 Options Theory

1. Consider a long call option with strike price $K = \$100$. The current stock price is $S_t = \$105$ and the call premium is \$10. What is the intrinsic value of the call option at time t? Find the payoff and profit if the spot price at the option expiration date T is $S_T = \$120$. Draw the payoff and profit diagrams.

Solution: By defining $S_t = \$105$, $S_T = \$120$, $K = \$100$ and the call premium as $C(S_t, t; K, T) = \$10$, the intrinsic value of the call option at time t is

$$\Psi(S_t) = \max\{S_t - K, 0\} = \max\{105 - 100, 0\} = \$5.$$

At expiry time T, the payoff is

$$\Psi(S_T) = \max\{S_T - K, 0\} = \max\{120 - 100, 0\} = \$20$$

and the profit is

$$\Upsilon(S_T) = \Psi(S_T) - C(S_t, t; K, T) = \$20 - \$10 = \$10.$$

Figure 1.6 shows the payoff and profit diagrams for a long call option at the expiry time T. Here the profit diagram is a vertical shift of the call payoff based on the premium paid.

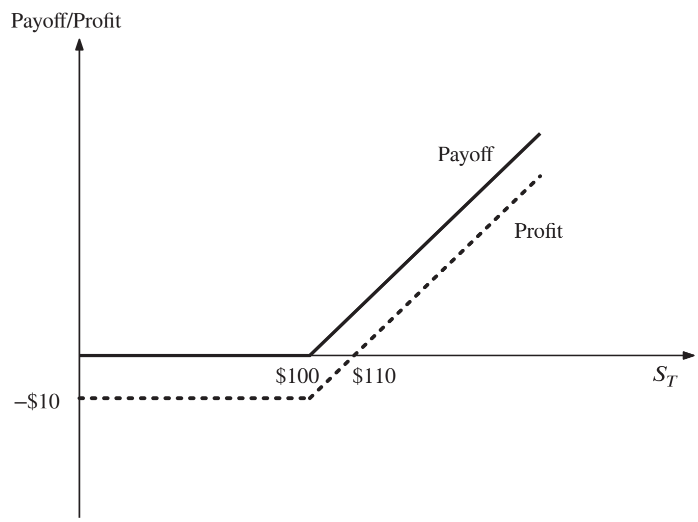

Figure 1.6 Long call option payoff and profit diagrams.

□

2. Consider a long put option with strike price $K = \$100$. The current stock price is $S_t = \$80$ and the put premium is \$5. What is the intrinsic value of the put option at time t? Find the payoff and profit if the spot price at the option expiration date T is $S_T = \$75$. Draw the payoff and profit diagrams.

Solution: By defining $S_t = \$80$, $S_T = \$75$, $K = \$100$ and the put premium as $P(S_t, t; K, T) = \$5$, the intrinsic value of the call option at time t is

$$\Psi(S_t) = \max\{K - S_t, 0\} = \max\{100 - 80, 0\} = \$20.$$

At expiry time T, the payoff is

$$\Psi(S_T) = \max\{K - S_T, 0\} = \max\{100 - 75, 0\} = \$25$$

and the profit is

$$\Upsilon(S_T) = \Psi(S_T) - P(S_t, t; K, T) = \$25 - \$5 = \$20.$$

Figure 1.7 shows the payoff and profit diagrams for a long put option at the expiry time T. Here the profit diagram is a vertical shift of the put payoff based on the premium paid.

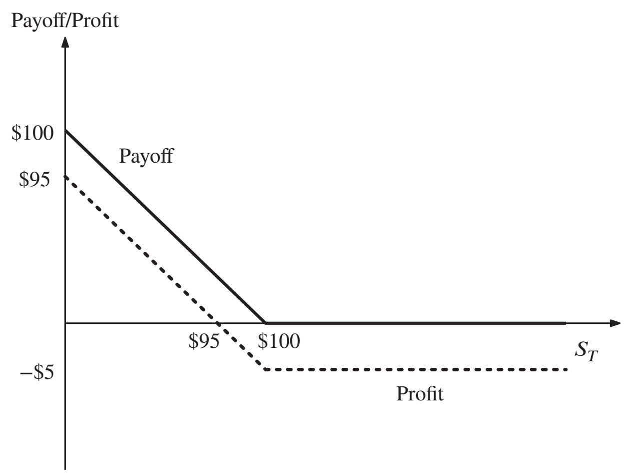

Figure 1.7 Long put option payoff and profit diagrams.

□

3. At time t we consider a long call option with a strike price K and a long forward contract with price K on the same underlying asset S_t. The premium for the call option is $C(S_t, t; K, T)$. Draw the profit diagram for these two financial instruments at the option expiry date T.

   Under what conditions is the call option more profitable than the forward contract, and vice versa?

   Solution: Figure 1.8 shows the profit diagram for a long call and a long forward contract at expiry date T.

   At time T the break even at the profit level is at $S_T = K - C(S_t, t; K, T)$. Therefore, if the stock $S_T \leq K - C(S_t, t; K, T)$ then the call option is more profitable as the loss is fixed with the amount of premium paid. However, if $S_T > K - C(S_t, t; K, T)$ then the forward contract is more profitable since there is no cost in entering a forward contract.

□

4. At time t we consider a long put option with strike price K and a short forward contract with price K on the same underlying asset S_t. The premium for the put option is $P(S_t, t; K, T)$. Draw the profit diagram for these two financial instruments at the option expiry date T.

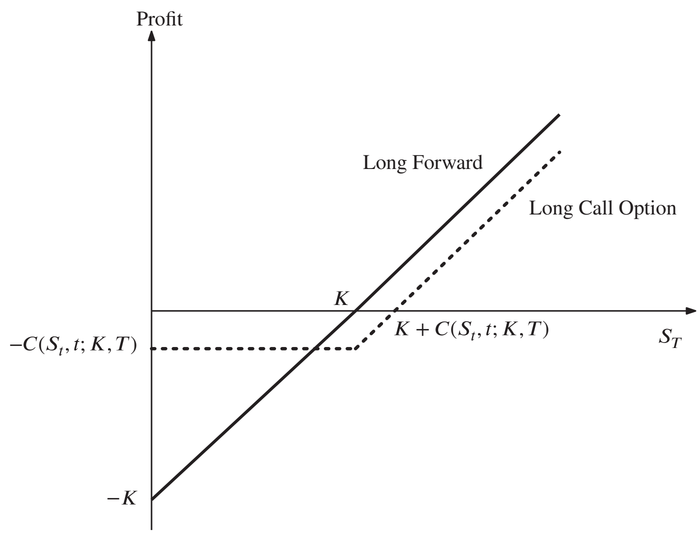

Figure 1.8 Long call option and long forward profit diagrams.

Under what conditions is the put option more profitable than the forward contract, and vice versa?

Solution: Figure 1.9 shows the profit diagram for a long put and a short forward contract at expiry date T.

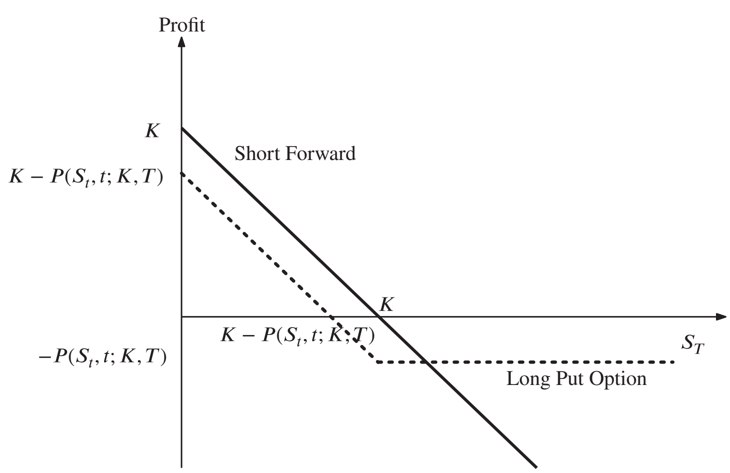

Figure 1.9 Long put option and short forward profit diagrams.

At time T the break-even point is at $S_T = K + P(S_t, t; K, T)$. Therefore, if the stock $S_T \geq K + P(S_t, t; K, T)$ then the put option is more profitable as the loss is fixed with the amount of premium paid. However, if $S_T < K + P(S_t, t; K, T)$ then the forward contract is more profitable since there is no cost in entering a forward contract.

$\square$

5. Consider a short call option with strike price $K = \$100$. The current stock price is $S_t = \$105$ and the call premium is \$10. What is the intrinsic value of the call option at time t? Find the payoff and profit if the spot price at the option expiration date T is $S_T = \$120$. Draw the payoff and profit diagrams.

   Solution: By defining $S_t = \$105$, $S_T = \$120$, $K = \$100$ and the call premium as $C(S_t, t; K, T) = \$10$, the intrinsic value of the short call option at time t is

$$\Psi(S_t) = -\max\{S_t - K, 0\} = \min\{K - S_t, 0\} = \min\{100 - 105, 0\} = -\$5.$$

   At expiry time T, the payoff is

$$\Psi(S_T) = -\max\{S_T - K, 0\} = \min\{K - S_T, 0\} = -\min\{100 - 120, 0\} = -\$20$$

   and the profit is

$$\Upsilon(S_T) = \Psi(S_T) + C(S_t, t; K, T) = -\$20 + \$10 = -\$10.$$

   Figure 1.10 shows the payoff and profit diagram for a short call option at the expiry time T. Here the profit diagram is a vertical shift of the short call payoff based on the premium received.

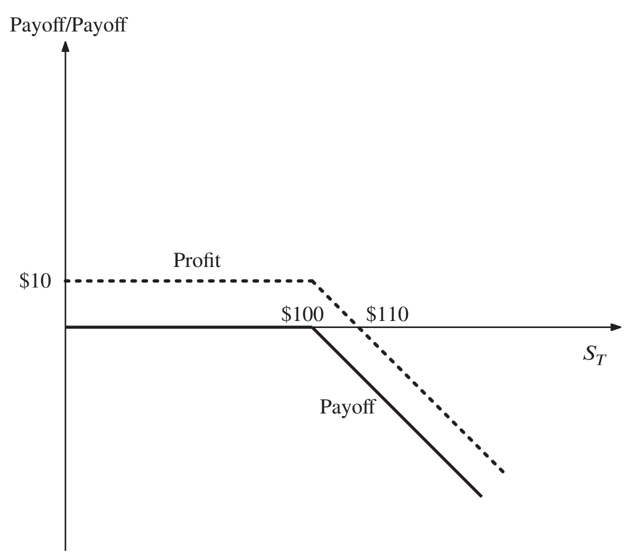

Figure 1.10 Short call option payoff and profit diagrams.

□

6. Consider a short put option with strike price $K = \$100$. The current stock price is $S_t = \$80$ and the put premium is \$5. What is the intrinsic value of the put option at time t? Find the payoff and profit if the spot price at the option expiration date T is $S_T = \$75$. Draw the payoff and profit diagrams.

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Solution: By defining $S_t = \$80$, $S_T = \$75$, $K = \$100$ and the put premium as $P(S_t, t; K, T) = \$5$, the intrinsic value of the short put option at time t is

$$\Psi(S_t) = -\max\{K - S_t, 0\} = \min\{S_t - K, 0\} = \min\{80 - 100, 0\} = -\$20.$$

At expiry time T, the payoff is

$$\Psi(S_T) = -\max\{K - S_T, 0\} = \min\{S_T - K, 0\} = \min\{75 - 100, 0\} = -\$25$$

and the profit is

$$\Upsilon(S_T) = \Psi(S_T) + P(S_t, t; K, T) = -\$25 + \$5 = -\$20.$$

Figure 1.11 shows the payoff and profit diagrams for a short put option at the expiry time T. Here the profit diagram is a vertical shift of the short put payoff based on the premium received.

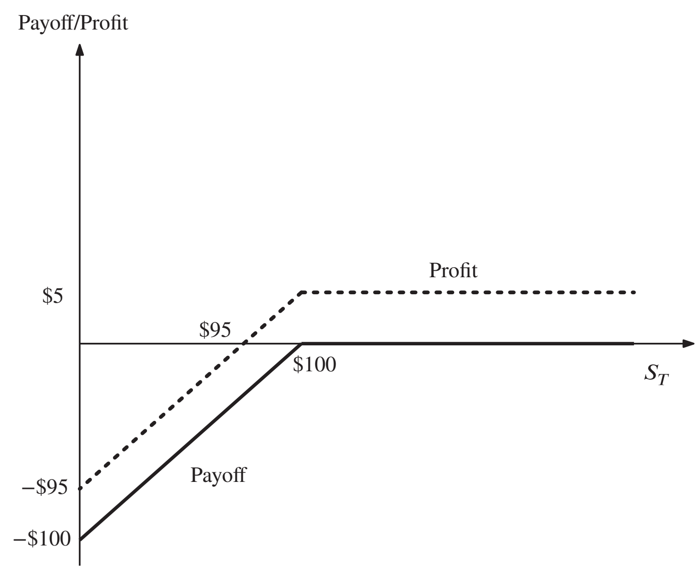

Figure 1.11 Short put option payoff and profit diagrams.

□

7. At time t we consider a short call option with strike price K and a short forward contract with price K on the same underlying asset S_t and expiry time $T > t$. The premium for the call option is $C(S_t, t; K, T)$ at time t. Draw the profit diagram for these two financial instruments at the option expiry time T.

   Under what conditions is the call option more profitable than the forward contract, and vice versa?

Solution: Figure 1.12 shows the profit diagram for a short call and a short forward contract at expiry time T.

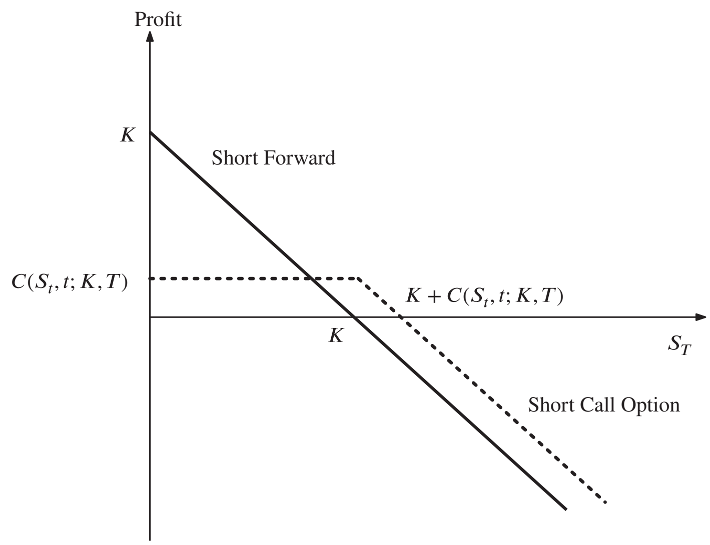

Figure 1.12 Short call option and short forward profit diagrams.

At time T the break-even point is at $S_T = K - C(S_t, t; K, T)$. Therefore, if the stock $S_T \leq K - C(S_t, t; K, T)$ then the forward contract is more profitable as the short call profit is fixed with the amount of premium received. However, if $S_T > K - C(S_t, t; K, T)$ then the forward contract is less profitable since the short call is augmented by the amount of premium paid to it.

□

8. At time t we consider a short put option with strike price K and a long forward contract with price K on the same underlying asset S_t and expiry time $T > t$. The premium for the put option is $P(S_t, t; K, T)$. Draw the profit diagram for these two financial instruments at the option expiry time T.

   Under what conditions is the put option more profitable than the forward contract, and vice versa?

   Solution: Figure 1.13 shows the profit diagram for a short put and a long forward contract at expiry time T.

   At time T the break-even point is at $S_T = K + P(S_t, t; K, T)$. Therefore, if the stock $S_T \geq K + P(S_t, t; K, T)$ then the forward contract is more profitable as the short put profit is fixed by the amount of premium received. However, if $S_T < K + P(S_t, t; K, T)$ then the forward contract is less profitable since the short put is augmented by the amount of premium paid to it.

□

9. *Put–Call Parity I.* At time t we consider a non-dividend-paying stock with spot price S_t and a risk-free interest rate r. Show that by taking a long European call option price at time t, $C(S_t, t; K, T)$ and a short European put option price at time t, $P(S_t, t; K, T)$ on the same

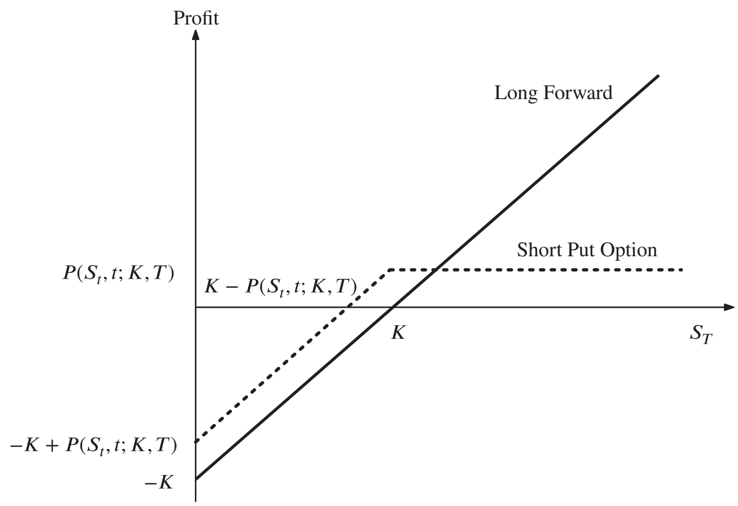

Figure 1.13 Short put option and long forward profit diagrams.

underlying stock S_t, strike price K and expiry time T ($t < T$) we have

$$C(S_t, t; K, T) - P(S_t, t; K, T) = S_t - Ke^{-r(T-t)}.$$

Solution: At time t we define the call option price as $C(S_t, t; K, T)$ and the put option price as $P(S_t, t; K, T)$, and we set the portfolio Π_t as

$$\Pi_t = C(S_t, t; K, T) - P(S_t, t; K, T).$$

At expiry time T

$$\begin{aligned}
\Pi_T &= C(S_T, T; K, T) - P(S_T, T; K, T) \\
&= \max\{S_T - K, 0\} - \max\{K - S_T, 0\} \\
&= \begin{cases} S_T - K & \text{if } S_T \geq K \\ S_T - K & \text{if } S_T < K \end{cases} \\
&= S_T - K.
\end{aligned}$$

In order for the portfolio to generate a guaranteed K at expiry time T, at time t we can discount the final value of the portfolio to

$$C(S_t, t; K, T) - P(S_t, t; K, T) = S_t - Ke^{-r(T-t)}$$

since a share valued at S_t will be worth S_T at expiry time T.

□

10. *Put–Call Parity II.* At time t we consider a discrete dividend-paying stock with spot price S_t where the stock pays dividend $\delta_i \geq 0$ at time t_i, $i = 1, 2, \ldots, n$ for $t < t_1 < t_2 < \cdots <$

$t_n < T$. Show that by taking a long European call option $C(S_t, t; K, T)$ and a short European put option $P(S_t, t; K, T)$ on the same underlying stock S_t, strike price K and expiry time T $(t < T)$ we have

$$C(S_t, t; K, T) - P(S_t, t; K, T) = S_t - \sum_{i=1}^{n} \delta_i e^{-r(t_i - t)} - K e^{-r(T-t)}$$

where r is the risk-free interest rate.

Solution: At time t we set up the portfolio Π_t as

$$\Pi_t = C(S_t, t; K, T) - P(S_t, t; K, T)$$

and at expiry time T

$$\begin{aligned}
\Pi_T &= C(S_T, T; K, T) - P(S_T, T; K, T) \\
&= \max\{S_T - K, 0\} - \max\{K - S_T, 0\} \\
&= \begin{cases} S_T - K & \text{if } S_T \geq K \\ \\ S_T - K & \text{if } S_T < K \end{cases} \\
&= S_T - K.
\end{aligned}$$

In order for the portfolio to generate one stock S_T with guaranteed K at expiry time T, at time t we can discount the final value of the portfolio to

$$C(S_t, t; K, T) - P(S_t, t; K, T) = S_t - \sum_{i=1}^{n} \delta_i e^{-r(t_i - t)} - K e^{-r(T-t)}$$

since when dividends are paid the stock price S_t is reduced by the present value of all the dividends paid.

□

11. *Put–Call Parity III.* At time t we consider a continuous dividend-paying stock with spot price S_t where D is the continuous dividend yield and r is the risk-free interest rate. Show that by taking a long European call option $C(S_t, t; K, T)$ and a short European put option $P(S_t, t; K, T)$ on the same underlying stock S_t, strike price K and expiry time T $(t < T)$ we have

$$C(S_t, t; K, T) - P(S_t, t; K, T) = S_t e^{-D(T-t)} - K e^{-r(T-t)}.$$

Solution: At time t we define the call option price as $C(S_t, t; K, T)$ and the put option price as $P(S_t, t; K, T)$, and we set the portfolio Π_t as

$$\Pi_t = C(S_t, t; K, T) - P(S_t, t; K, T).$$

At expiry time T

$$\begin{aligned}\Pi_T &= C(S_T, T; K, T) - P(S_T, T; K, T)\\ &= \max\{S_T - K, 0\} - \max\{K - S_T, 0\}\\ &= \begin{cases} S_T - K & \text{if } S_T \geq K\\ S_T - K & \text{if } S_T < K\end{cases}\\ &= S_T - K.\end{aligned}$$

In order for the portfolio to generate one stock S_T with guaranteed K at expiry time T, at time t we can discount the final value of the portfolio to

$$C(S_t, t; K, T) - P(S_t, t; K, T) = S_t e^{-D(T-t)} - Ke^{-r(T-t)}$$

since $e^{-D(T-t)}$ number of shares valued at $S_t e^{-D(T-t)}$ will become one share worth S_T at expiry time T.

□

12. At time t we consider a call option with strike price $K = \$100$. Calculate the intrinsic value of this option if the current spot price is $S_t = \$105$, $S_t = \$100$ or $S_t = \$95$ and state whether it is ITM, OTM or ATM.

Solution: At time t the intrinsic call option value is $\Psi(S_t) = \max\{S_t - K, 0\}$. Hence, if $S_t = \$105$

$$\Psi(S_t) = \max\{105 - 100, 0\} = \$5$$

and since $S_t > K$, the intrinsic value of the call option is ITM.

As for $S_t = \$100$

$$\Psi(S_t) = \max\{100 - 100, 0\} = 0$$

and because $\Psi(S_t) = 0$ and $S_t = K$, the intrinsic value of the call option is ATM.

Finally, for $S_t = \$95$

$$\Psi(S_t) = \max\{95 - 100, 0\} = 0$$

and since $S_t < K$, the intrinsic value of the call option is OTM.

□

13. At time t we consider a put option with strike price $K = \$100$. Compute the intrinsic value of this option if the current spot price is $S_t = \$105$, $S_t = \$100$ or $S_t = \$95$ and state whether it is ITM, OTM or ATM.

Solution: At time t the intrinsic put option value is $\Psi(S_t) = \max\{K - S_t, 0\}$. Hence, if $S_t = \$105$

$$\Psi(S_t) = \max\{100 - 105, 0\} = 0$$

and since $S_t > K$, the intrinsic value of the put option is OTM.

As for $S_t = \$100$

$$\Psi(S_t) = \max\{100 - 100, 0\} = 0$$

and because $\Psi(S_t) = 0$ and $S_t = K$, the intrinsic value of the put option is ATM.

Finally, for $S_t = \$95$

$$\Psi(S_t) = \max\{100 - 95, 0\} = \$5$$

and since $S_t < K$, the intrinsic value of the put option is OTM.

□

14. Suppose we have a quote for a 3-month European put option, with a strike price $K = \$60$ of \$1.25. The current stock price $S_0 = \$62$ and the risk-free interest rate $r = 5\%$ per annum. Owing to small trading in call options, there is no listing for the 3-month \$60 call (a call option price with strike \$60 expiring in 3 months). Suppose the stock does not pay any dividends then find the price of the 3-month European call option.

Solution: We first denote $C(S_0, 0; K, T)$ and $P(S_0, 0; K, T)$ as the call and put option prices, respectively at time $t = 0$ with strike price K and option expiry time $T = 3$ months. Given $P(S_0, 0; K, T) = \$1.25$ and $T = \frac{3}{12} = 0.25$ years, by rearranging the put–call parity we can write

$$\begin{aligned} C(S_0, 0; K, T) &= P(S_0, 0; K, T) + S_0 - Ke^{-rT} \\ &= \$1.25 + \$62 - 60e^{-0.05\times 0.25} \\ &= \$4.00. \end{aligned}$$

Hence, if the 3-month \$60 call is available, it should be priced at \$4.00.

□

15. At time t we consider a European call option $C(S_t, t; K, T)$ and a European put option $P(S_t, t; K, T)$ on the same underlying asset S_t, strike price K and expiry time T. Suppose the underlying asset pays a continuous dividend yield D and there is a risk-free interest rate r, then under what condition is a European call option more expensive than a European put option?

Solution: From the put–call parity

$$C(S_t, t; K, T) - P(S_t, t; K, T) = S_t e^{-D(T-t)} - Ke^{-r(T-t)}$$

then $C(S_t, t; K, T) > P(S_t, t; K, T)$ if

$$S_t e^{-D(T-t)} - Ke^{-r(T-t)} > 0$$

or

$$S_t > Ke^{-(r-D)(T-t)}.$$

□

16. Suppose that a 6-month European call option, with a strike price of $K = \$85$, has a premium of \$2.75. The futures price for a 6-month contract is worth \$75 and the risk-free rate $r = 5\%$ per annum. Find the price of a 6-month European put option with the same strike price.

Solution: At initial time $t = 0$ we denote $C(S_0, 0; K, T)$ and $P(S_0, 0; K, T)$ as the call and put option prices, respectively on the underlying asset S_0, strike K and expiry time $T = 6$

months. Let the expiry time $T = \frac{6}{12} = 0.5$ years, $C(S_0, 0; K, T) = \$2.75$ and set the futures price $F(0, T) = \$75$. From the put–call parity we have

$$C(S_0, 0; K, T) - P(S_0, 0; K, T) = S_0 e^{-DT} - Ke^{-rT}$$

where D is the continuous dividend yield. Since

$$F(0, T) = S_0 e^{(r-D)T}$$

we can write

$$P(S_0, 0; K, T) = C(S_0, 0; K, T) - (F(0, T) - K)e^{-rT}$$

and by substituting $C(S_0, 0; K, T) = \$2.75$, $F(0, T) = \$75$, $K = \$85$, $r = 0.05$ and $T =$ 0.5 years, the put option price is

$$P(S_0, 0; K, T) = \$2.75 - (\$75 - \$85)e^{-0.05\times 0.5} = \$12.50.$$

□

17. Suppose a 12-month European call option, with a strike price of $K = \$35$, has a premium of \$2.15. The stock pays a dividend valued at \$1.50 four months from now and another dividend valued at \$1.75 eight months from now. Given that the current stock price is $S_0 = \$32$ and the risk-free rate $r = 5\%$ per annum, find the price of a 12-month European put option with the same strike price.

    Solution: Using the put–call parity for a stock with discrete dividends at time $t = 0$ we have

    $$C(S_0, 0; K, T) - P(S_0, 0; K, T) = S_0 - \delta_1 e^{-rt_1} - \delta_2 e^{-rt_2} - Ke^{-rT}$$

    where $S_0 = \$32$, $\delta_1 = \$1.50$, $t_1 = \frac{4}{12} = \frac{1}{3}$ years, $\delta_2 = \$1.75$, $t_2 = \frac{8}{12} = \frac{2}{3}$ years, $K = \$35$, $r = 0.05$ and $T = 1$ year with European call option price $C(S_0, 0; K, T) = \$2.15$ and unknown European put option price $P(S_0, 0; K, T)$.

    Thus,

    $$\begin{aligned} P(S_0, 0; K, T) &= C(S_0, 0; K, T) - S_0 + \delta_1 e^{-rt_1} + \delta_2 e^{-rt_2} + Ke^{-rT} \\ &= 2.15 - 32 + 1.50e^{-0.05\times\frac{1}{3}} + 1.75e^{-0.05\times\frac{2}{3}} + 35e^{-0.05} \\ &= \$6.61. \end{aligned}$$

    □

18. Consider a European put option priced at \$2.50 with strike price \$22 and a European call option priced at \$4.75 with strike price \$30. What are the maximum losses to the writer of the put and the buyer of the call?

Solution: At time t the maximum loss of a short put is $-K + P(S_t, t; K, T)$ where K is the strike price, $T > t$ is the expiry date and $P(S_t, t; K, T)$ is the put option price. Therefore, the maximum loss of a short put is $-\$22 + \$2.50 = -\$19.50$.

In contrast, the maximum loss to the buyer of the call option is the premium paid, that is $-\$4.75$.

□

19. Let the current stock price be \$35 and the European put option is ITM by \$3.50. Find the corresponding strike price.

Solution: At current time t, the intrinsic value of the put option is defined as

$$\Psi(S_t) = \max\{K - S_t, 0\}$$

where S_t and K are the current stock price and strike price, respectively.

Given $S_t = \$35$, $\Psi(S_t) = \$3.50$ and since $K > S_t$ (intrinsic value is ITM), then

$$K - \$35 = \$3.50 \quad \text{or} \quad K = \$38.50.$$

□

20. Given the current spot price $S_0 = \$55$ we consider a European call option and a European put option with premiums \$1.98 and \$0.79, respectively on a common strike price $K = \$58$ and having the same expiry time T. By setting the risk-free interest rate $r = 3\%$ per annum, find T.

Solution: From the put–call parity at time $t = 0$,

$$C(S_0, 0; K, T) + P(S_0, 0; K, T) = S_0 - Ke^{-rT}$$

where the call option $C(S_0, 0; K, T) = \$1.98$ and the put option $P(S_0, 0; K, T) = \$0.79$. Substituting $S_0 = \$55$, $K = \$58$ and $r = 0.03$, we have

$$\begin{aligned} 1.98 - 0.79 &= 55 - 58e^{-0.03\times T} \\ 1.19 &= 55 - 58e^{-0.03\times T} \end{aligned}$$

and solving the equation, we have

$$e^{-0.03\times T} = 0.9278$$

or

$$T \simeq 2.5 \text{ years.}$$

□

1.2.3 Hedging Strategies

1. *Covered Call.* A covered call is an investment strategy constructed by buying a stock and selling an OTM call option on the same stock.

Explain why a call writer would set up this portfolio trading strategy and show that this strategy is undertaken for $C(S_t, t; K, T) \geq S_t - K$ where $C(S_t, t; K, T)$ is the call option price written on stock S_t with strike price K at time t with option expiry time $T > t$.

Draw the profit diagram of this strategy at expiry time T $(T > t)$.

Solution: In writing a covered call where the writer owns the stock, the writer can cover the obligation of delivering the stock if the holder of the call exercises the option at expiry date. In addition, by writing a covered call, the writer assumes the stock price will not be higher than the strike price and thus enhance his income by receiving the call option's premium. In contrast, if the stock price declines in value then the writer will lose money.

At time T the payoff of this portfolio is

$$\begin{aligned}\Psi(S_T) &= S_T - C(S_T, T; K, T)\\ &= S_T - \max\{S_T - K, 0\}\\ &= \begin{cases} K & \text{if } S_T \geq K \\ \\ S_T & \text{if } S_T < K \end{cases}\end{aligned}$$

whilst the profit is

$$\Upsilon(S_T) = \begin{cases} K - S_t + C(S_t, t; K, T) & \text{if } S_T \geq K \\ S_T - S_t + C(S_t, t; K, T) & \text{if } S_T < K \end{cases}$$

where we need to deduct the cost of acquiring S_t and also to add the call option premium at the start of the contract.

Since the break-even point occurs when $S_T = S_t - C(S_t, t; K, T)$ where $\Upsilon(S_T) = 0$, in order for the strategy to take place we require $S_t - C(S_t, t; K, T) \geq K$ or $C(S_t, t; K, T) \leq S_t - K$.

Figure 1.14 shows the profit diagram of a covered call.

Since $C(S_t, t; K, T) \leq S_t - K$, the maximum gain from this strategy is $K - S_t + C(S_t, t; K, T) \geq 0$ whilst the maximum loss is $-S_t + C(S_t, t; K, T) \leq 0$.

□

2. *Covered Put.* A covered put is a hedging strategy constructed by selling a stock and selling an OTM put option on the same stock.

   Explain why a put writer would set up this portfolio trading strategy and show that this strategy is undertaken for $P(S_t, t; K, T) \geq K - S_t$ where $P(S_t, t; K, T)$ is the put option price written on stock S_t with strike price K at time t and expiry time $T > t$.

   Draw the profit diagram of this strategy at expiry time T $(T > t)$.

Solution: In writing a covered put the writer expects the stock price will decline in value relative to the strike and thus enhance his income by receiving the put option's premium. By selling the stock short, the writer does not need to worry if the stock price drops further. However, if the stock price is much greater than the strike price at expiry then the writer will lose money.

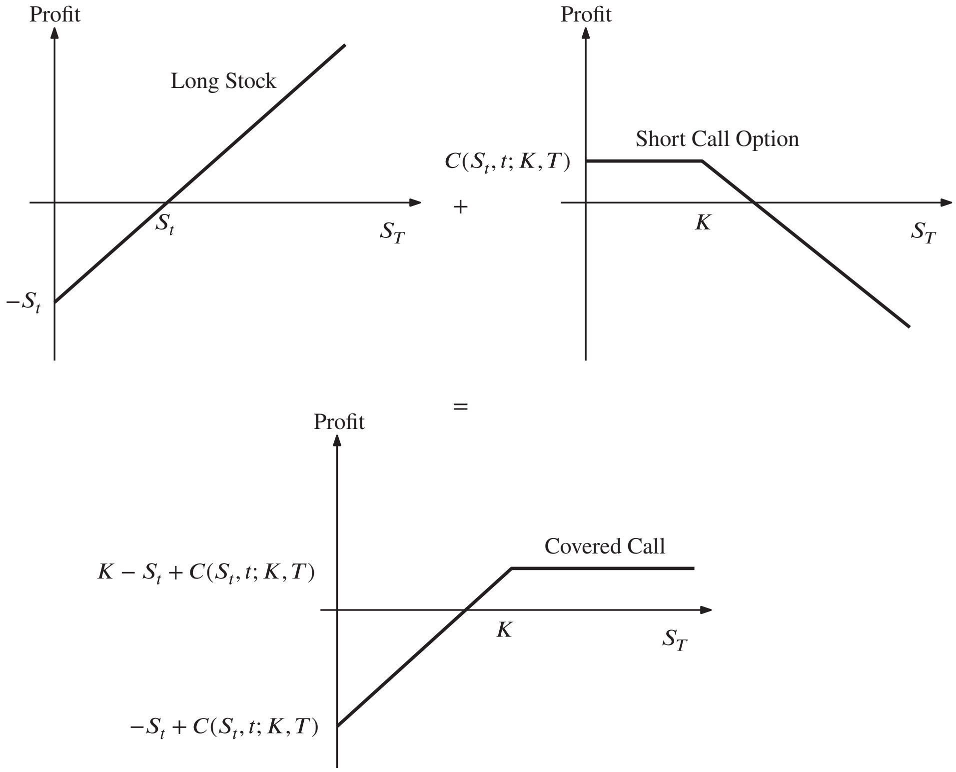

Figure 1.14 Construction of a covered call.

At time T the payoff of this portfolio is

$$\begin{aligned}\Psi(S_T) &= -S_T - P(S_T, T; K, T)\\ &= -S_T - \max\{K - S_T, 0\}\\ &= \begin{cases} -S_T & \text{if } S_T \geq K \\ -K & \text{if } S_T < K \end{cases}\end{aligned}$$

whilst the profit is

$$\begin{aligned}\Upsilon(S_T) &= \Psi(S_T) + S_t + P(S_t, t; K, T)\\ &= \begin{cases} S_t - S_T + P(S_t, t; K, T) & \text{if } S_T \geq K \\ S_t - K + P(S_t, t; K, T) & \text{if } S_T < K \end{cases}\end{aligned}$$

where we need to add the sale of S_t and also the put option premium received at the start of the contract.

Since the break-even point occurs when $S_T = S_t + P(S_t, t; K, T)$ where $\Upsilon(S_T) = 0$, this strategy is undertaken when $S_t + P(S_t, t; K, T) \geq K$ or $P(S_t, t; K, T) \geq S_t - K$.

For a detailed construction of a covered put, see Figure 1.15.

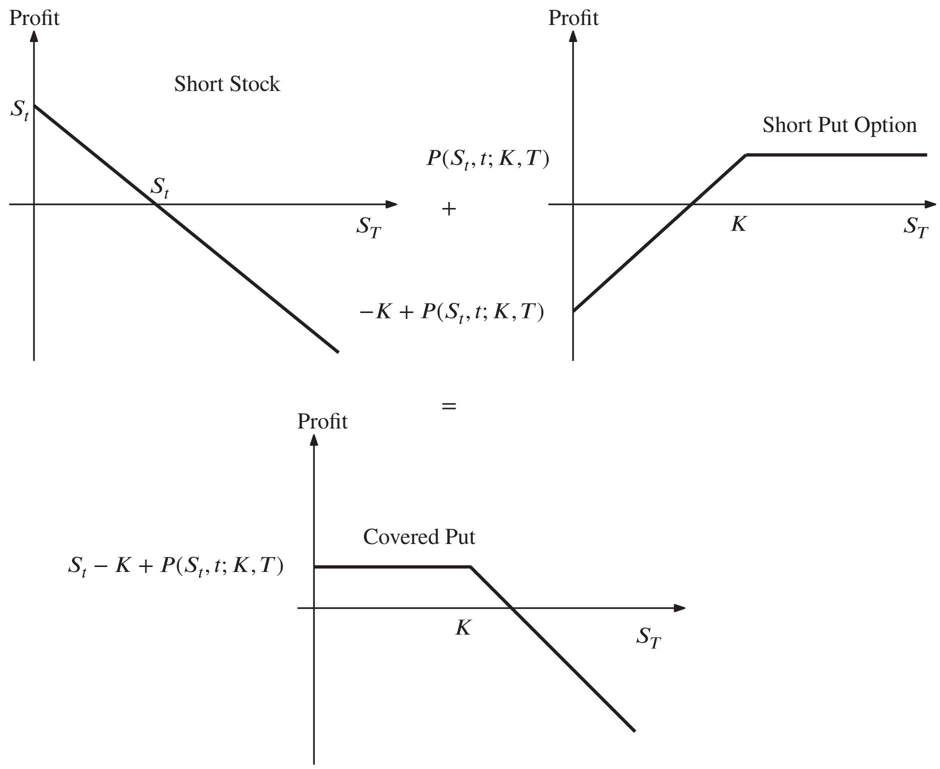

Figure 1.15 Construction of a covered put.

Since $P(S_t, t; K, T) \geq K - S_t$, the maximum gain from this strategy is capped at $S_t - K + P(S_t, t; K, T) \geq 0$ whilst the loss is unlimited.

□

3. *Protective Call.* A protective call is an investment strategy constructed by selling a stock and buying an OTM call option on the same stock.

Explain why a call holder would set up this portfolio trading strategy and show that this strategy is undertaken for $C(S_t, t; K, T) \geq S_t - K$ where $C(S_t, t; K, T)$ is the call option price written on stock S_t with strike price K at time t and expiry time $T > t$.

Draw the profit diagram of this strategy at expiry time T ($T > t$).

Solution: In buying a protective call the investor strategy is to protect profits from the rising stock price with respect to the strike price. By selling the stock the call holder assumes that the stock price will decline further. If the stock price is less than the strike at expiry time then the option will not be exercised and the call buyer will only lose the premium paid.

At time T the payoff of this portfolio is

$$\begin{aligned}\Psi(S_T) &= -S_T + C(S_T, T; K, T)\\ &= -S_T + \max\{S_T - K, 0\}\\ &= \begin{cases} -K & \text{if } S_T \geq K \\ -S_T & \text{if } S_T < K \end{cases}\end{aligned}$$

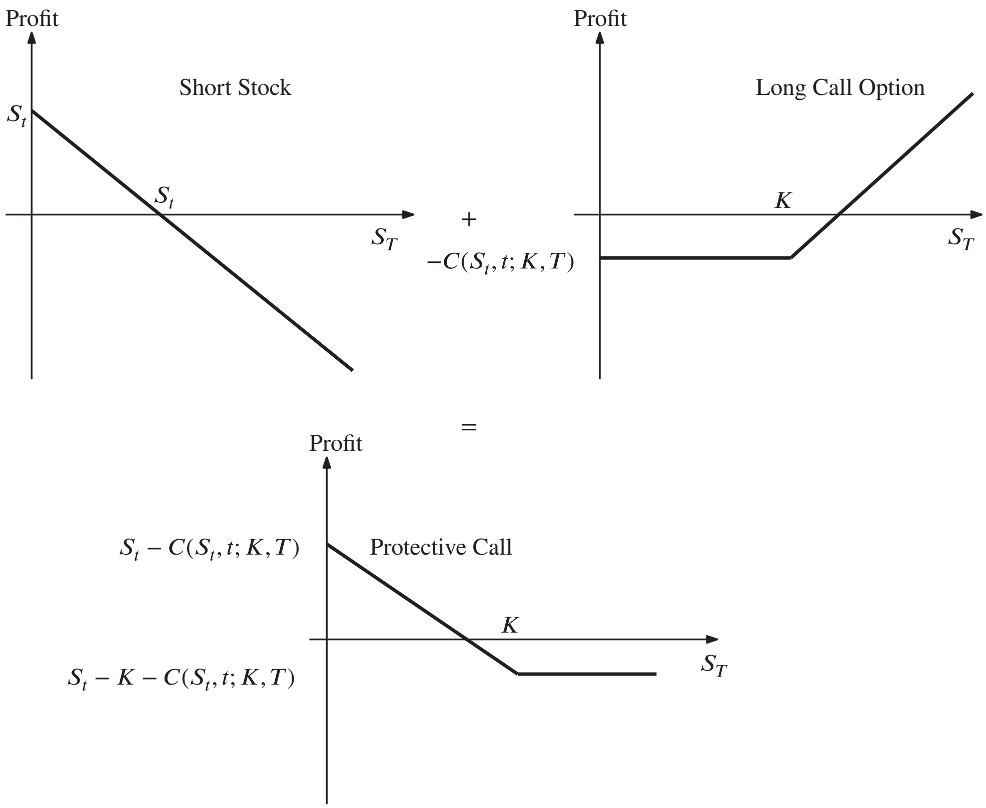

Figure 1.16 Construction of a protective call.

whilst the profit is

$$\begin{aligned}\Upsilon(S_T) &= \Psi(S_T) + S_t - C(S_t, t; K, T)\\ &= \begin{cases} S_t - K - C(S_t, t; K, T) & \text{if } S_T \geq K \\ S_t - S_T - C(S_t, t; K, T) & \text{if } S_T < K \end{cases}\end{aligned}$$

where we need to add the sale of the stock at time t and deduct the call option premium paid at the beginning of the contract.

Since the break-even point occurs when $S_T = S_t - C(S_t, t; K, T)$ so that $\Upsilon(S_T) = 0$, this strategy should be undertaken when $S_t - C(S_t, t; K, T) \leq K$ or $C(S_t, t; K, T) \geq S_t - K$.

Figure 1.16 shows the profit diagram of a protective call.

From the profit formula we can see that the protective call has a maximum upside gain of $S_t - C(S_t, t; K, T)$, which is the difference between the sale of the stock and the call option premium, and a limited loss of $S_t - K - C(S_t, t; K, T) \leq 0$.

□

4. *Protective Put.* A protective put is a hedging strategy constructed by buying a stock and buying an OTM put option on the same stock.

   Explain why a put buyer would set up this portfolio trading strategy and show that this strategy is undertaken for $P(S_t, t) \geq K - S_t$ where $P(S_t, t; K, T)$ is the put option price written on stock S_t with strike price K at time t and expiry time $T > t$.

   Draw the profit diagram of this strategy at expiry time T $(T > t)$.

   Solution: In buying a protective put the investor strategy is to protect profits from the stock declining in value with respect to the strike price. By owning the stock the put holder can cover the obligation of delivering the stock to the put writer if the option is exercised at expiry time. If the stock price is above the strike at expiry time then the option will not be exercised and the put buyer will only lose the premium paid.

   At time T the payoff of this portfolio is

$$\begin{aligned}\Psi(S_T) &= S_T + P(S_T, T; K, T)\\ &= S_T + \max\{K - S_T, 0\}\\ &= \begin{cases} S_T & \text{if } S_T \geq K\\ K & \text{if } S_T < K\end{cases}\end{aligned}$$

   whilst the profit is

$$\Upsilon(S_T) = \begin{cases} S_T - S_t - P(S_t, t; K, T) & \text{if } S_T \geq K\\ K - S_t - P(S_t, t; K, T) & \text{if } S_T < K\end{cases}$$

   where we need to deduct both the cost of acquiring S_t and the put option premium paid at the beginning of the contract.

   Since the break-even point occurs when $S_T = S_t + P(S_t, t; K, T)$ so that $\Upsilon(S_T) = 0$, this strategy is undertaken if $S_t + P(S_t, t; K, T) \geq K$ or $P(S_t, t; K, T) \geq K - S_t$.

   Figure 1.17 shows the profit diagram of a protective put.

   From the profit formula we can see that the protective put has an unlimited upside gain and, because $P(S_t, t; K, T) \geq K - S_t$, it has a limited loss of $K - S_t - P(S_t, t; K, T) \leq 0$.

   □

5. At time t consider a writer of a covered call on a \$35 stock with a strike price of \$40. The premium of the call option is \$1.75. Calculate the writer's maximum gain and loss at expiry time $T > t$.

   Solution: At time t at the start of the contract let $S_t = \$35$, $K = \$40$ and the call option price $C(S_t, t; K, T) = \$1.75$.

   From Problem 1.2.3.1 (page 28) the maximum gain of a covered call at expiry time T $(T > t)$ is

$$\Upsilon_G(S_T) = K - S_t + C(S_t, t; K, T) = 40 - 35 + 1.75 = \$6.75$$

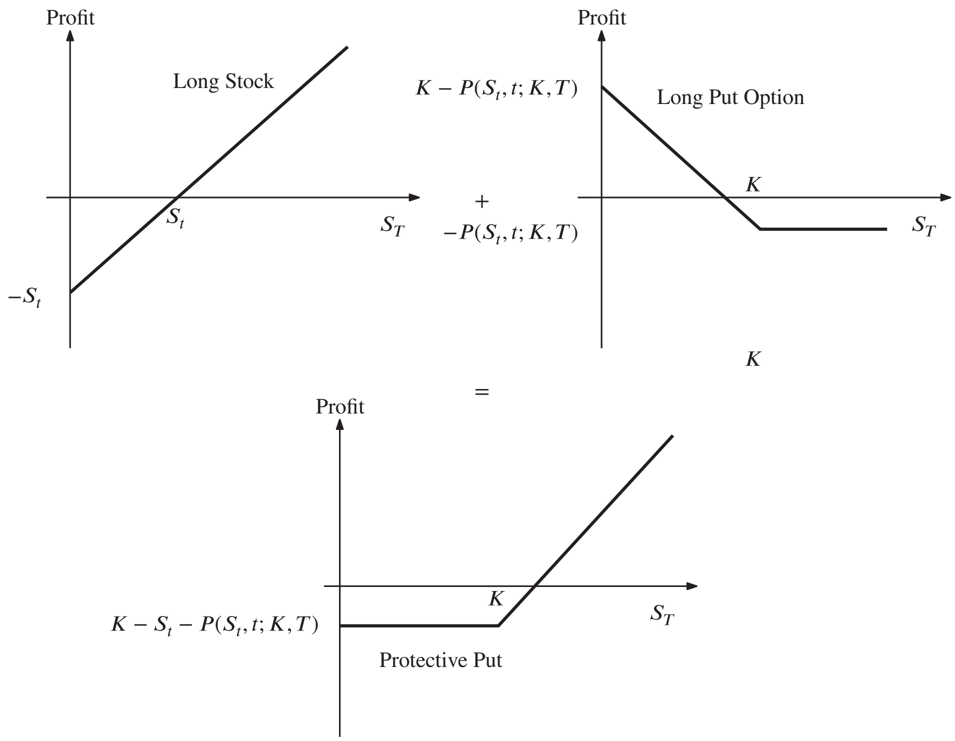

Figure 1.17 Construction of a protective put.

and the maximum loss is

$$\Upsilon_L(S_T) = -S_t + C(S_t, t; K, T) = -35 + 1.75 = -\$33.25.$$

□

6. At time t a writer short sells a stock for \$33 and sells a put option with strike price \$25 for \$1.75. What is the maximum gain and loss for the writer of this protective put at expiry time $T > t$?

   Solution: At current time t we let the spot price of the stock be $S_t = \$33$, strike $K = \$25$ and put option price $P(S_t, t; K, T) = \$1.75$. Therefore, from Problem 1.2.3.4 (page 33)

   $$\text{Maximum Gain of Protective Put} = +\infty$$

   and

   $$\begin{aligned}\text{Maximum Loss of Protective Put} &= K - S_t - P(S_t, t; K, T)\\ &= \$25 - \$33 - \$1.75\\ &= -\$9.75.\end{aligned}$$

   □

7. *Bull Call Spread.* A bull call spread is a hedging position designed to buy a call option $C(S_t, t; K_1, T)$ with strike K_1 and simultaneously sell a call option $C(S_t, t; K_2, T)$ with

strike K_2, $K_1 \leq K_2$ on the same underlying asset S_t and having the same expiry time T ($T > t$).

Show that

$$C(S_t, t; K_1, T) \geq C(S_t, t; K_2, T), \quad K_1 \leq K_2$$

and draw the payoff and profit diagrams of a bull call spread.

Discuss under what conditions an investor should invest in such a hedging strategy.

Solution: We first assume $C(S_t, t; K_1, T) < C(S_t, t; K_2, T)$ and we set up a portfolio

$$\Pi_t = C(S_t, t; K_1, T) - C(S_t, t; K_2, T) < 0.$$

At expiry time T

$$\begin{aligned}
\Pi_T &= C(S_T, T; K_1) - C(S_T, T; K_2) \\
&= \max\{S_T - K_1, 0\} - \max\{S_T - K_2, 0\} \\
&= \begin{cases} 0 & \text{if } S_T \leq K_1 \\ S_T - K_1 & \text{if } K_1 < S_T \leq K_2 \\ K_2 - K_1 & \text{if } S_T > K_2 \end{cases} \\
&\geq 0
\end{aligned}$$

which constitutes an arbitrage opportunity. Therefore, $C(S_t, t; K_1, T) \geq C(S_t, t; K_2, T)$, $K_1 \leq K_2$.

At time T the payoff of this hedging strategy is

$$\begin{aligned}
\Psi(S_T) &= C(S_T, T; K_1, T) - C(S_T, T; K_2, T) \\
&= \max\{S_T - K_1, 0\} - \max\{S_T - K_2, 0\} \\
&= \begin{cases} 0 & \text{if } S_T \leq K_1 \\ S_T - K_1 & \text{if } K_1 < S_T \leq K_2 \\ K_2 - K_1 & \text{if } S_T > K_2 \end{cases}
\end{aligned}$$

and the corresponding profit is

$$\begin{aligned}
\Upsilon(S_T) &= \Psi(S_T) - C(S_t, t; K_1, T) + C(S_t, t; K_2, T) \\
&= \begin{cases} C(S_t, t; K_2, T) - C(S_t, t; K_1, T) & \text{if } S_T \leq K_1 \\ S_T - K_1 + C(S_t, t; K_2, T) - C(S_t, t; K_1, T) & \text{if } K_1 < S_T \leq K_2 \\ K_2 - K_1 + C(S_t, t; K_2, T) - C(S_t, t; K_1, T) & \text{if } S_T > K_2 \end{cases}
\end{aligned}$$

where the break-even point is $S_T = K_1 + C(S_t, t; K_1, T) - C(S_t, t; K_2, T)$ so that $\Upsilon(S_T) = 0$.

Figure 1.18 shows the payoff and profit diagrams of a bull call spread.

Based on the payoff and profit diagrams of a bull call spread, this hedging strategy would appeal to investors who have a bullish sentiment that the stock price

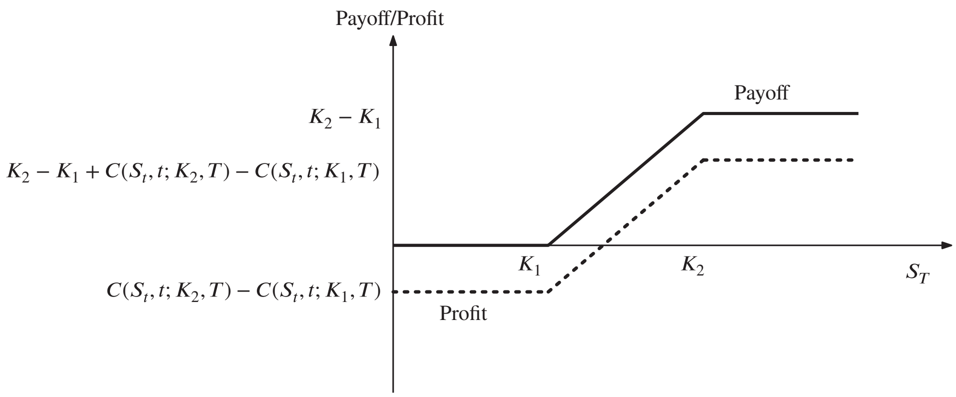

Figure 1.18 Construction of a bull call spread.

will increase in value relative to the strike prices K_1 and K_2. Hence, if $S_T > K_1 + C(S_t, t; K_1, T) - C(S_t, t; K_2, T)$ then the investor would make a profit but capped at a maximum gain of $K_2 - K_1 + C(S_t, t; K_2, T) - C(S_t, t; K_1, T) > 0$. However, if $S_T < K_1 + C(S_t, t; K_1, T) - C(S_t, t; K_2, T)$ then the investor would make a loss but limited to the difference between premiums received and paid.

□

8. *Bull Put Spread.* A bull put spread is an investment strategy constructed by buying a put option $P(S_t, t; K_1, T)$ with strike K_1 and simultaneously selling a put option $P(S_t, t; K_2, T)$ with strike K_2, $K_1 \le K_2$ on the same underlying asset S_t and having the same expiry time T $(T > t)$.

   Show that

$$P(S_t, t; K_2, T) \ge P(S_t, t; K_1, T), \quad K_1 \le K_2$$

and draw the payoff and profit diagrams of a bull put spread.

Discuss under what conditions an investor should invest in such an investment strategy.

Solution: We first assume $P(S_t, t; K_2, T) < P(S_t, t; K_1, T)$ and we set up a portfolio

$$\Pi_t = P(S_t, t; K_2, T) - P(S_t, t; K_1, T) < 0.$$

At expiry time T

$$\begin{aligned}
\Pi_T &= P(S_T, T; K_2, T) - P(S_T, T; K_1, T) \\
&= \max\{K_2 - S_T, 0\} - \max\{K_1 - S_T, 0\} \\
&= \begin{cases} K_2 - K_1 & \text{if } S_T \le K_1 \\ K_2 - S_T & \text{if } K_1 < S_T \le K_2 \\ 0 & \text{if } S_T > K_2 \end{cases} \\
&\ge 0
\end{aligned}$$

which constitutes an arbitrage opportunity. Therefore, $P(S_t, t; K_2, T) \ge P(S_t, t; K_1, T)$, $K_1 \le K_2$.

At time T the payoff of this hedging strategy is

$$\begin{aligned}\Psi(S_T) &= P(S_T, T; K_1, T) - P(S_T, T; K_2, T)\\ &= \max\{K_1 - S_T, 0\} - \max\{K_2 - S_T, 0\}\\ &= \begin{cases} K_1 - K_2 & \text{if } S_T \le K_1 \\ \\ S_T - K_2 & \text{if } K_1 < S_T \le K_2 \\ \\ 0 & \text{if } S_T > K_2 \end{cases}\end{aligned}$$

and the corresponding profit is

$$\begin{aligned}\Upsilon(S_T) &= \Psi(S_T) - P(S_t, t; K_1, T) + P(S_t, t; K_2, T)\\ &= \begin{cases} K_1 - K_2 + P(S_t, t; K_2, T) - P(S_t, t; K_1, T) & \text{if } S_T \le K_1 \\ \\ S_T - K_2 + P(S_t, t; K_2, T) - P(S_t, t; K_1, T) & \text{if } K_1 < S_T \le K_2 \\ \\ P(S_t, t; K_2, T) - P(S_t, t; K_1, T) & \text{if } S_T > K_2 \end{cases}\end{aligned}$$

where the break-even point is $S_T = K_2 + P(S_t, t; K_1, T) - P(S_t, t; K_2, T)$ so that $\Upsilon(S_T) = 0$.

Figure 1.19 shows the payoff and profit diagrams of a bull put spread.

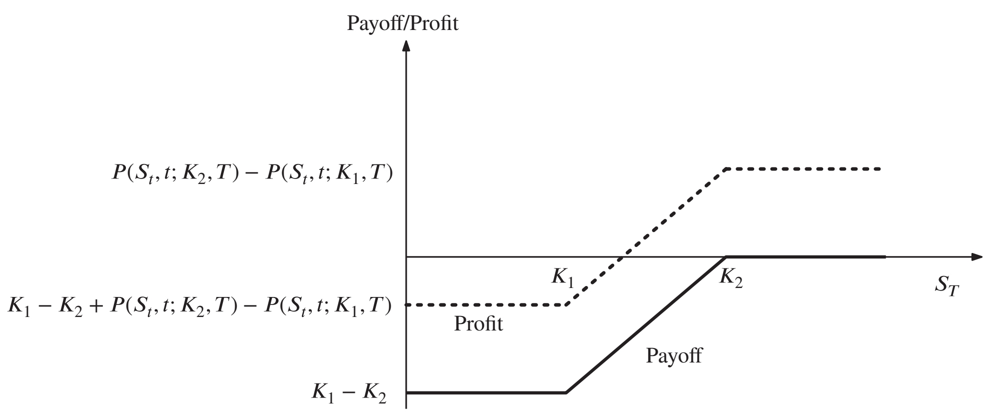

Figure 1.19 Construction of a bull put spread.

Based on the payoff and profit diagrams of a bull put spread, this hedging strategy would appeal to investors who have a bullish sentiment that the stock price will increase in value relative to the strike prices K_1 and K_2. Hence, if $S_T > K_2 + P(S_t, t; K_1, T) - P(S_t, t; K_2, T)$ then the investor would make a profit but capped at a maximum gain based on the difference between the put premiums received and paid. However, if $S_T < K_2 + P(S_t, t; K_1, T) - P(S_t, t; K_2, T)$ then the investor would make a loss but limited to a maximum loss of $K_1 - K_2 + P(S_t, t; K_2, T) - P(S_t, t; K_1, T) < 0$.

□

9. *Bear Call Spread.* A bear call spread is a hedging position designed to sell a call option $C(S_t, t; K_1, T)$ with strike K_1 and simultaneously buy a call option $C(S_t, t; K_2, T)$ with strike K_2, $K_1 \leq K_2$ on the same underlying asset S_t and having the same expiry time T $(T > t)$.

   Show that

$$C(S_t, t; K_1, T) \geq C(S_t, t; K_2, T), \quad K_1 \leq K_2$$

   and draw the payoff and profit diagrams of a bear call spread.

   Discuss under what conditions an investor should invest in such a hedging strategy.

   Solution: To show that $C(S_t, t; K_1, T) \geq C(S_t, t; K_2, T)$ for $K_1 \leq K_2$ see Problem 1.2.3.7 (page 34).

   At time T the payoff of this hedging strategy is

$$\begin{aligned}
\Psi(S_T) &= -C(S_T, T; K_1, T) + C(S_T, T; K_2, T) \\
&= -\max\{S_T - K_1, 0\} + \max\{S_T - K_2, 0\} \\
&= \begin{cases} 0 & \text{if } S_T \leq K_1 \\ \\ K_1 - S_T & \text{if } K_1 < S_T \leq K_2 \\ \\ K_1 - K_2 & \text{if } S_T > K_2 \end{cases}
\end{aligned}$$

   and the corresponding profit is

$$\begin{aligned}
\Upsilon(S_T) &= \Psi(S_T) + C(S_t, t; K_1, T) - C(S_t, t; K_2, T) \\
&= \begin{cases} C(S_t, t; K_1, T) - C(S_t, t; K_2, T) & \text{if } S_T \leq K_1 \\ \\ K_1 - S_T + C(S_t, t; K_1, T) - C(S_t, t; K_2, T) & \text{if } K_1 < S_T \leq K_2 \\ \\ K_1 - K_2 + C(S_t, t; K_1, T) - C(S_t, t; K_2, T) & \text{if } S_T > K_2 \end{cases}
\end{aligned}$$

   where the break-even point is $S_T = K_1 + C(S_t, t; K_1, T) - C(S_t, t; K_2, T)$ so that $\Upsilon(S_T) = 0$.

   Figure 1.20 is the payoff and profit diagrams of a bear call spread.

   Based on the payoff and profit diagrams of a bear call spread, this hedging strategy would appeal to investors who have a bearish attitude that the stock price will decrease in value relative to the strike prices K_1 and K_2. Hence, if $S_T < K_1 + C(S_t, t; K_1, T) - C(S_t, t; K_2, T)$ then the investor would make a profit but capped at a maximum gain based on the difference between the call premiums received and paid. On the other hand, if $S_T > K_1 + C(S_t, t; K_1, T) - C(S_t, t; K_2, T)$ then the investor would make a loss but limited to a maximum loss of $K_1 - K_2 + C(S_t, t; K_1, T) - C(S_t, t; K_2, T) < 0$.

   □

10. *Bear Put Spread.* A bear put spread is an investment strategy constructed by selling a put option $P(S_t, t; K_1, T)$ with strike K_1 and simultaneously buying a put option $P(S_t, t; K_2, T)$ with strike K_2, $K_1 \leq K_2$ on the same underlying asset S_t and having the same expiry time T $(T > t)$.

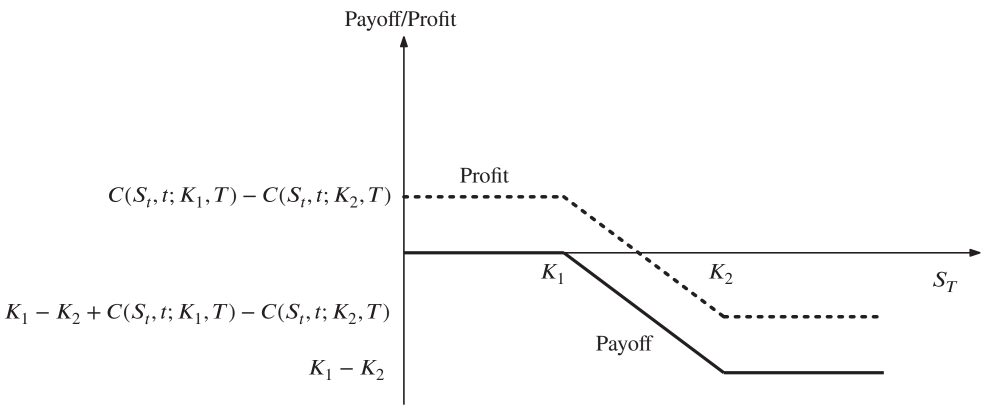

Figure 1.20 Construction of a bear call spread.

Show that

$$P(S_t, t; K_2, T) \geq P(S_t, t; K_1, T), \quad K_1 \leq K_2$$

and draw the payoff and profit diagrams of a bear put spread.

Discuss under what conditions an investor should invest in such an investment strategy.

Solution: To show that $P(S_t, t; K_2, T) \geq P(S_t, t; K_1, T)$ for $K_1 \leq K_2$ see Problem 1.2.3.8 (page 36).

At time T the payoff of this hedging strategy is

$$\begin{aligned}\Psi(S_T) &= -P(S_T, T; K_1, T) + P(S_T, T; K_2, T) \\ &= -\max\{K_1 - S_T, 0\} + \max\{K_2 - S_T, 0\} \\ &= \begin{cases} K_2 - K_1 & \text{if } S_T \leq K_1 \\ K_2 - S_T & \text{if } K_1 < S_T \leq K_2 \\ 0 & \text{if } S_T > K_2 \end{cases}\end{aligned}$$

and the corresponding profit is

$$\begin{aligned}\Upsilon(S_T) &= \Psi(S_T) + P(S_t, t; K_1, T) - P(S_t, t; K_2, T) \\ &= \begin{cases} K_2 - K_1 + P(S_t, t; K_1, T) - P(S_t, t; K_2, T) & \text{if } S_T \leq K_1 \\ K_2 - S_T + P(S_t, t; K_1, T) - P(S_t, t; K_2, T) & \text{if } K_1 < S_T \leq K_2 \\ P(S_t, t; K_1, T) - P(S_t, t; K_2, T) & \text{if } S_T > K_2 \end{cases}\end{aligned}$$

where the break-even point is $S_T = K_2 + P(S_t, t; K_1, T) - P(S_t, t; K_2, T)$ so that $\Upsilon(S_T) = 0$.

Figure 1.21 shows the payoff and profit diagrams of a bear put spread.

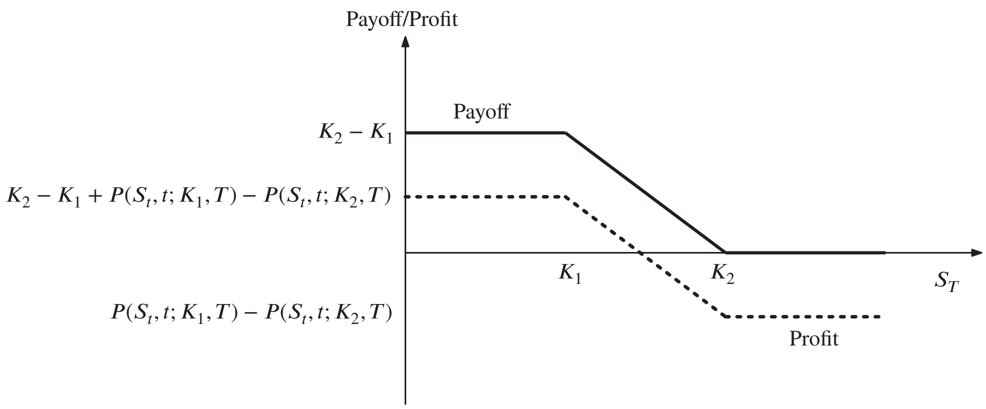

Figure 1.21 Construction of a bear put spread.

From the payoff and profit diagrams of a bear put spread we can see that this hedging strategy would appeal to investors who have a bearish attitude that the stock price will decrease in value relative to the strike prices K_1 and K_2. Hence, if $S_T < K_2 + P(S_t, t; K_1, T) - P(S_t, t; K_2, T)$ then the investor would make a profit but capped at a maximum gain of $K_2 - K_1 + P(S_t, t; K_1, T) - P(S_t, t; K_2, T)$. However, if $S_T > K_2 + P(S_t, t; K_1, T) - P(S_t, t; K_2, T)$ then the investor would make a loss but limited to a maximum loss of $P(S_t, t; K_1, T) - P(S_t, t; K_2, T) < 0$.

□

11. *Box Spread.* A long box spread is an investment strategy constructed by buying a bull call spread $C(S_t, t; K_1, T) - C(S_t, t; K_2, T)$ and buying a bear put spread $P(S_t, t; K_2, T) - P(S_t, t; K_1, T)$ with strikes K_1 and K_2, $K_1 \leq K_2$ on the same underlying asset S_t and having the same expiry time T $(T > t)$. Let the risk-free interest rate be r and the stock pays a continuous dividend D.

    Show that the premium paid to enter into this portfolio strategy is the same as the present value of the payoff.

    Show also that at time T the profit based on this strategy is independent of the terminal stock price S_T.

    Give a financial interpretation of this investment strategy.

Solution: At initial time t the portfolio is worth

$$\Pi_t = C(S_t, t; K_1, T) - C(S_t, t; K_2, T) + P(S_t, t; K_2, T) - P(S_t, t; K_1, T)$$

and from the put–call parity

$$C(S_t, t; K_1, T) - P(S_t, t; K_1, T) = S_t e^{-D(T-t)} - K_1 e^{-r(T-t)}$$

and

$$C(S_t, t; K_2, T) - P(S_t, t; K_2, T) = S_t e^{-D(T-t)} - K_2 e^{-r(T-t)}.$$

Therefore,

$$\Pi_t = (K_2 - K_1)e^{-r(T-t)}.$$

At expiry time T the portfolio is

$$\begin{aligned}
\Pi_T &= C(S_T, T; K_1, T) - C(S_T, T; K_2, T) \\
&\quad + P(S_T, T; K_2, T) - P(S_T, T; K_1, T) \\
&= \max\{S_T - K_1, 0\} - \max\{K_1 - S_T, 0\} \\
&\quad - \max\{S_T - K_2, 0\} + \max\{K_2 - S_T, 0\} \\
&= \begin{cases} K_2 - K_1 & \text{if } S_T \le K_1 \\ \\ K_2 - K_1 & \text{if } K_1 < S_T \le K_2 \\ \\ K_2 - K_1 & \text{if } S_T > K_2 \end{cases} \\
&= K_2 - K_1.
\end{aligned}$$

Thus, at time t we can discount back the final value of the portfolio to become

$$\Pi_t = (K_2 - K_1)e^{-r(T-t)}$$

which is also the price of the premium paid.

The profit of entering such a hedging strategy is therefore

$$\begin{aligned}
\Upsilon(S_T) &= \Pi_T - \Pi_t \\
&= (K_2 - K_1)(1 - e^{-r(T-t)}) \\
&> 0
\end{aligned}$$

which guarantees a positive cash flow irrespective of the terminal stock price value S_T. Thus, the box spread is clearly an arbitrage opportunity provided the transaction cost is low.

□

12. At current time $t = 0$ a stock is trading at \$20 and for a risk-free interest rate of 4% per annum the prices of 6-month European options are given in Table 1.4.

Table 1.4 European option prices for different strikes.

Strike Price	European Call	European Put
\$20	\$1.98	\$1.58
\$27	\$1.21	\$7.68

Determine the payoff, premium paid and profit at the expiry time for a box spread constructed by buying a 20-strike European call, selling a 27-strike European call, selling a 20-strike European put and buying a 27-strike European put.

Solution: By setting $S_0 = \$20$, $K_1 = \$20$, $K_2 = \$27$, $r = 0.04$ and $T = \frac{6}{12} = 0.5$ years, from Problem 1.2.3.11 (page 40) the payoff of a box spread is

$$\Psi(S_T) = K_2 - K_1 = \$27 - \$20 = \$7.$$

The premium paid is therefore

$$\Psi(S_0) = (K_2 - K_1)e^{-rT} = \$7e^{-0.04\times 0.5} = \$6.86$$

and the profit is

$$\Upsilon(S_T) = \Psi(S_T) - \Psi(S_0) = \$7 - \$6.86 = \$0.14.$$

□

13. *Purchased Collar.* A purchased collar is a hedging strategy whereby at time t an investor buys an asset S_t, buys a put option $P(S_t, t; K_1, T)$ with strike price K_1 and sells a call option $C(S_t, t; K_2, T)$ with strike price K_2, $K_1 < K_2$ on the same underlying S_t and having the same expiry time $T (T > t)$.

    Show that in order to prevent any arbitrage opportunity

$$S_t + P(S_t, t; K_1, T) - C(S_t, t; K_2, T) \geq 0.$$

Draw the profit diagram and give a financial interpretation of this hedging strategy.

Solution: We assume at time t that $S_t + P(S_t, t; K_1, T) - C(S_t, t; K_2, T) < 0$ and we set up a portfolio

$$\Psi(S_t) = S_t + P(S_t, t; K_1, T) - C(S_t, t; K_2, T) < 0.$$

At expiry time T

$$\begin{aligned}\Psi(S_T) &= S_T + P(S_T, T; K_1, T) - C(S_T, T; K_2, T)\\ &= S_T + \max\{K_1 - S_T, 0\} - \max\{S_T - K_2, 0\}\\ &= \begin{cases} K_1 & \text{if } S_T \leq K_1\\ \\ S_T & \text{if } K_1 < S_T \leq K_2\\ \\ K_2 & \text{if } S_T > K_2\end{cases}\\ &\geq 0\end{aligned}$$

which is an arbitrage opportunity. Hence, $S_t + P(S_t, t; K_1, T) - C(S_t, t; K_2, T) \geq 0$.

Given that the investor has paid for both the stock S_t and the put premium $P(S_t, t; K_1, T)$ and simultaneously received $C(S_t, t; K_2, T)$, the profit at expiry time T is

$$\Upsilon(S_T) = \Psi(S_T) + C(S_t, t; K_2, T) - P(S_t, t; K_1, T) - S_t$$

where $\Psi(S_T) = S_T + P(S_T, T; K_1, T) - C(S_T, T; K_2, T)$ is the payoff of the purchased collar contract. Thus,

$$\Upsilon(S_T) = \begin{cases} K_1 + C(S_t, t; K_2, T) - P(S_t, t; K_1, T) - S_t & \text{if } S_T \le K_1 \\ S_T + C(S_t, t; K_2, T) - P(S_t, t; K_1, T) - S_t & \text{if } K_1 < S_T \le K_2 \\ K_2 + C(S_t, t; K_2, T) - P(S_t, t; K_1, T) - S_t & \text{if } S_T > K_2 \end{cases}$$

with break even at $S_T = S_t + P(S_t, t; K_1, T) - C(S_t, t; K_2, T) \ge 0$.

Figure 1.22 shows the payoff and profit diagrams of a purchased collar.

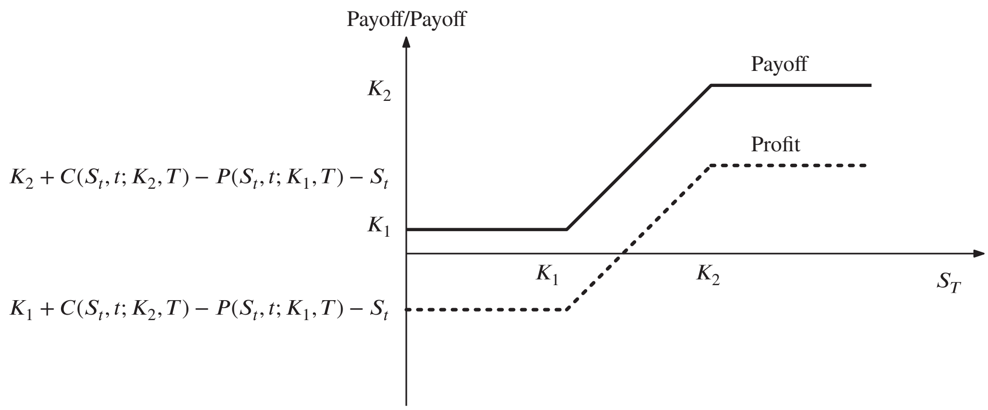

Figure 1.22 Construction of a purchased collar.

From the combination of options and asset we can see that the purchased collar is a hedging strategy consisting of buying a protective put and selling a call option. By buying a protective put the investor is able to insure the asset, whilst selling a call reduces the cost of insurance. Therefore, this position would be beneficial when the asset price $S_T > S_t + P(S_t, t; K_1, T) - C(S_t, t; K_2, T)$ (up to a maximum gain of $K_2 + C(S_t, t; K_2, T) - P(S_t, t; K_1, T) - S_t$) but if $S_T < S_t + P(S_t, t; K_1, T) - C(S_t, t; K_2, T)$ then the investor would lose money (up to a maximum loss of $K_1 + C(S_t, t; K_2, T) - P(S_t, t; K_1, T) - S_t$). $\square$

14. *Written Collar.* A written collar is an investment strategy whereby at time t an investor would sell an asset S_t, sell a put option $P(S_t, t; K_1, T)$ with strike price K_1 and buy a call option $C(S_t, t; K_2, T)$ with strike price K_2, $K_1 < K_2$ on the same underlying S_t and having the same expiry time $T > t$.

    Show that in order to prevent any arbitrage opportunity

$$S_t + P(S_t, t; K_1, T) - C(S_t, t; K_2, T) \ge 0.$$

    Draw the payoff and profit diagrams and give a financial interpretation of this hedging strategy.

Solution: To show that at time t, $S_t + P(S_t, t; K_1, T) - C(S_t, t; K_2, T) \geq 0$ see Problem 1.2.3.13 (page 42).

At expiry time T the payoff is

$$\begin{aligned}\Psi(S_T) &= -S_T - P(S_T, T; K_1, T) + C(S_T, T; K_2, T)\\ &= -S_T - \max\{K_1 - S_T, 0\} + \max\{S_T - K_2, 0\}\\ &= \begin{cases} -K_1 & \text{if } S_T \leq K_1 \\ \\ -S_T & \text{if } K_1 < S_T \leq K_2 \\ \\ -K_2 & \text{if } S_T > K_2 \end{cases}\end{aligned}$$

while the profit at expiry time T is

$$\begin{aligned}\Upsilon(S_T) &= \Psi(S_T) + S_t + P(S_t, t; K_1, T) - C(S_t, t; K_2, T)\\ &= \begin{cases} -K_1 + S_t + P(S_t, t; K_1, T) - C(S_t, t; K_2, T) & \text{if } S_T \leq K_1 \\ \\ -S_T + S_t + P(S_t, t; K_1, T) - C(S_t, t; K_2, T) & \text{if } K_1 < S_T \leq K_2 \\ \\ -K_2 + S_t + P(S_t, t; K_1, T) - C(S_t, t; K_2, T) & \text{if } S_T > K_2 \end{cases}\end{aligned}$$

with break even at $S_T = S_t + P(S_t, t; K_1, T) - C(S_t, t; K_2, T) \geq 0$.

Figure 1.23 shows the payoff and profit diagrams of a written collar.

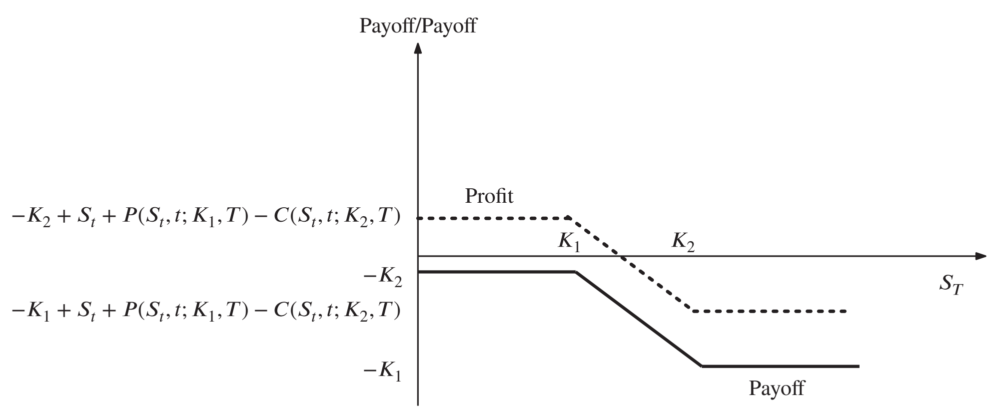

Figure 1.23 Construction of a written collar.

Given the combination of options and asset we can see that the written collar is a hedging strategy consisting of buying a protective call and selling a put option. By buying a protective call the investor is able to insure the short sale of the asset, whilst selling a put reduces the cost of insurance. Hence, this position would be beneficial when the asset price declines $S_T < S_t + P(S_t, t; K_1, T) - C(S_t, t; K_2, T)$ (up to a maximum gain of $-K_2 + S_t + P(S_t, t; K_1, T) - C(S_t, t; K_2))$ but if $S_T > S_t + P(S_t, t; K_1, T) - C(S_t, t; K_2, T)$ then the investor would lose money (up to a maximum loss of $-K_1 + S_t + P(S_t, t; K_1, T) - C(S_t, t; K_2, T)$).

□

15. *Long Straddle.* A long straddle is an investment strategy whereby at time t an investor would buy a call option $C(S_t, t; K, T)$ and buy a put option $P(S_t, t; K, T)$ on the same strike K using the same stock S_t and having the same expiry time T $(t < T)$.

    Draw the payoff and profit diagrams of this investment strategy and give a financial interpretation based on this combination of options portfolio.

Solution: We consider at time t that there is a stock worth S_t and for a strike price K the investor buys a call option $C(S_t, t; K, T)$ and a put option $P(S_t, t; K, T)$ on the same stock S_t with expiry time T.

The payoff at time T is

$$\begin{aligned}\Psi(S_T) &= C(S_T, T; K, T) + P(S_T, T; K, T)\\ &= \max\{S_T - K, 0\} + \max\{K - S_T, 0\}\\ &= \begin{cases} K - S_T & \text{if } S_T \le K \\ S_T - K & \text{if } S_T > K \end{cases}\end{aligned}$$

and the profit is

$$\begin{aligned}\Upsilon(S_T) &= \Psi(S_T) - C(S_t, t; K, T) - P(S_t, t; K, T)\\ &= \begin{cases} K - S_T - C(S_t, t; K, T) - P(S_t, t; K, T) & \text{if } S_T \le K \\ S_T - K - C(S_t, t; K, T) - P(S_t, t; K, T) & \text{if } S_T > K \end{cases}\end{aligned}$$

with break even at the profit level occurring at $S_T = K - C(S_t, t; K, T) - P(S_t, t; K, T)$ (provided $C(S_t, t; K, T) + P(S_t, t; K, T) \le K$) and $S_T = K + C(S_t, t; K, T) + P(S_t, t; K, T)$.

Figure 1.24 shows the payoff and profit diagrams of a long straddle.

From the profit diagram we can see that the investor would make an unlimited profit if $S_T > K + C(S_t, t; K, T) + P(S_t, t; K, T)$ or a limited profit if $S_T < K - C(S_t, t; K, T) - P(S_t, t; K, T)$. In contrast, the maximum loss for the investor is the cost of purchasing the option. Hence, this strategy is dependent on how high the volatility of the stock is rather than the direction of the stock price. In short, the profit at expiry time relies on how much the stock price moves instead of whether it is increasing or decreasing in value.

□

16. *Short Straddle.* A short straddle is an investment strategy where at time t an investor would sell a call option $C(S_t, t; K, T)$ and sell a put option $P(S_t, t; K, T)$ on the same strike K using the same stock S_t and having the same expiry time T $(t < T)$.

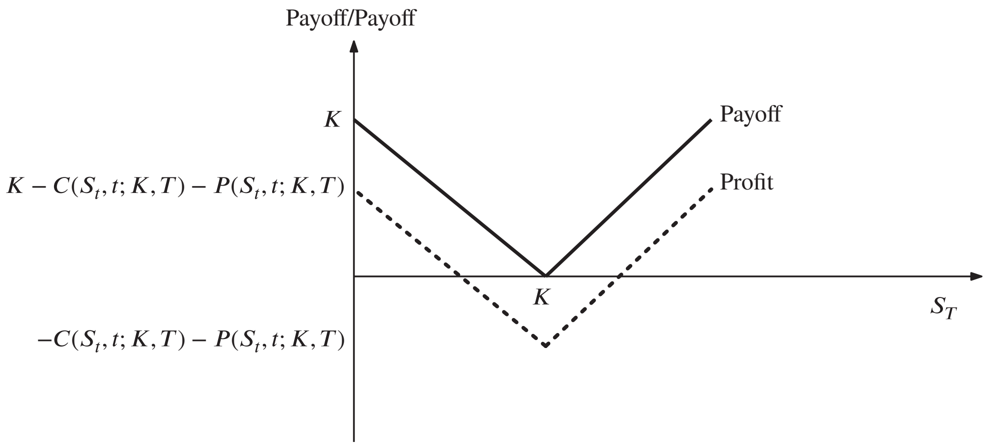

Figure 1.24 Construction of a long straddle.

Draw the payoff and profit diagrams of this investment strategy and give a financial interpretation based on this combination of options portfolio.

Solution: We consider at time t that there is a stock worth S_t and for a strike price K the writer sells a call option $C(S_t, t; K, T)$ and a put option $P(S_t, t; K, T)$ on the same stock S_t with expiry time T.

The payoff at time T is

$$\begin{aligned}\Psi(S_T) &= -C(S_T, T; K, T) - P(S_T, T; K, T) \\ &= -\max\{S_T - K, 0\} - \max\{K - S_T, 0\} \\ &= \begin{cases} S_T - K & \text{if } S_T \leq K \\ K - S_T & \text{if } S_T > K \end{cases}\end{aligned}$$

and the profit is

$$\begin{aligned}\Upsilon(S_T) &= \Psi(S_T) + C(S_t, t; K, T) + P(S_t, t; K, T) \\ &= \begin{cases} S_T - K + C(S_t, t; K, T) + P(S_t, t; K, T) & \text{if } S_T \leq K \\ K - S_T + C(S_t, t; K, T) + P(S_t, t; K, T) & \text{if } S_T > K \end{cases}\end{aligned}$$

with break even at the profit level occurring either at $S_T = K - C(S_t, t; K, T) - P(S_t, t; K, T)$ (provided $C(S_t, t; K, T) + P(S_t, t; K, T) \leq K$) or $S_T = K + C(S_t, t; K, T) + P(S_t, t; K, T)$.

Figure 1.25 illustrates the payoff and profit diagrams of a short straddle.

From the profit diagram we can see that the writer would make an unlimited loss if $S_T > K + C(S_t, t; K, T) + P(S_t, t; K, T)$ or a limited loss if $S_T < K - C(S_t, t; K, T) - P(S_t, t; K, T)$. In contrast, the maximum gain for the writer is only the options premium received. Thus, unlike the long straddle, the short straddle depends very much on low

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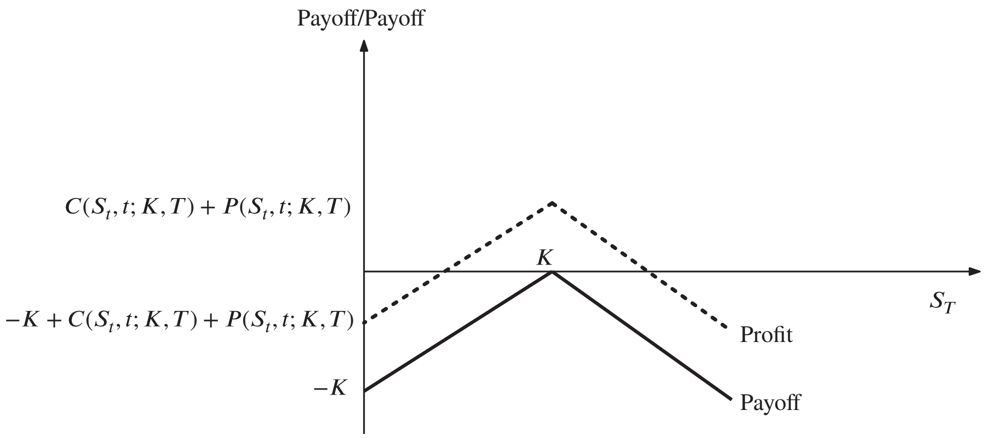

Figure 1.25 Construction of a short straddle.

volatility and is most profitable if $S_T = K$. Hence, this strategy is dependent on the volatility of the stock rather than the direction of the stock price, and the writer who sells this portfolio of options would bet on the low volatility of the stock price.

□

17. Consider an investor buying a 35-strike call option and a 25-strike put option on a stock for prices of \$0.69 and \$0.52, respectively. Both of the options have the same expiry time of 6 months from now. Given that the current price of a stock is \$28 and the risk-free interest rate is $r = 3\%$ per annum, what is the position of this investment strategy? Find the profit and determine the break-even price of this hedging position at expiry time.

Solution: Let $S_0 = \$28$, $K_1 = \$25$, $K_2 = \$35$, $T = \frac{6}{12} = \frac{1}{2}$ years and we can write the portfolio at time $t = 0$ as

$$\Pi_0 = P(S_0, 0; K_1, T) + C(S_0, 0; K_2, T)$$

where $P(S_0, 0; K_1, T) = \$0.52$ is the put option price with strike $K_1 = \$25$ and $C(S_0, 0; K_2, T) = \$0.69$ is the call option price with strike price $K_2 = \$35$.

Since $K_1 < K_2$ and the options have the same expiry time on the same underlying asset price, the position is a long straddle.

At expiry time T the payoff is

$$\begin{aligned}\Psi(S_T) &= P(S_T, T; K_1, T) + C(S_T, T; K_2, T)\\ &= \max\{K_1 - S_T, 0\} + \max\{S_T - K_2, 0\}\\ &= \begin{cases} K_1 - S_T & \text{if } S_T \le K_1\\ K_1 - K_2 & \text{if } K_1 < S_T \le K_2\\ S_T - K_2 & \text{if } S_T > K_2.\end{cases}\end{aligned}$$

Given the total cost of the premium paid is $P(S_0, 0; K_1, T) + C(S_0, 0; K_2, T) = \$0.52 + \$0.69 = \$1.21$, the profit at expiry time T is

$$\begin{aligned}
\Upsilon(S_T) &= \Psi(S_T) - \$1.21 \\
&= \begin{cases} \$25 - S_T - \$1.21 & \text{if } S_T \leq K_1 \\ \$25 - \$35 - \$1.21 & \text{if } K_1 < S_T \leq K_2 \\ S_T - \$35 - \$1.21 & \text{if } S_T > K_2 \end{cases} \\
&= \begin{cases} \$23.79 - S_T & \text{if } S_T \leq K_1 \\ -\$11.21 & \text{if } K_1 < S_T \leq K_2 \\ S_T - \$36.21 & \text{if } S_T > K_2 \end{cases}
\end{aligned}$$

with break-even points occurring at $S_T = \$23.79$ and $S_T = \$36.21$.

□

18. *Long Strangle.* A long strangle is a hedging technique where at time t an investor would buy a put option $P(S_t, t; K_1, T)$ with strike K_1 and buy a call option $C(S_t, t; K_2, T)$ with strike K_2, $K_1 < K_2$ on the same stock S_t and having the same expiry time T $(t < T)$.

    Draw the payoff and profit diagrams of this investment strategy and give a financial interpretation based on this combination of options portfolio.

    Solution: At expiry time T the payoff of a long strangle portfolio is

$$\begin{aligned}
\Psi(S_T) &= P(S_T; T; K_1, T) + C(S_T; T; K_2, T) \\
&= \max\{K_1 - S_T, 0\} + \max\{S_T - K_2, 0\} \\
&= \begin{cases} K_1 - S_T & \text{if } S_T \leq K_1 \\ 0 & \text{if } K_1 < S_T \leq K_2 \\ S_T - K_2 & \text{if } S_T > K_2. \end{cases}
\end{aligned}$$

Given that the investor paid for the call and put premiums at time t, the profit at expiry time T is

$$\begin{aligned}
\Upsilon(S_T) &= \Psi(S_T) - P(S_t, t; K_1, T) - C(S_t, t; K_2, T) \\
&= \begin{cases} K_1 - S_T - P(S_t, t; K_1, T) - C(S_t, t; K_2, T) & \text{if } S_T \leq K_1 \\ -P(S_t, t; K_1, T) - C(S_t, t; K_2, T) & \text{if } K_1 < S_T \leq K_2 \\ S_T - K_2 - P(S_t, t; K_1) - C(S_t, t; K_2) & \text{if } S_T > K_2 \end{cases}
\end{aligned}$$

with break even occurring at $S_T = K_1 - P(S_t, t; K_1, T) - C(S_t, t; K_2, T)$ (provided $P(S_t, t; K_1, T) + C(S_t, t; K_2, T) \leq K_1$) and $S_T = K_2 + C(S_t, t; K_1, T) + P(S_t, t; K_1, T)$.

Figure 1.26 shows the payoff and profit diagrams of a long strangle.

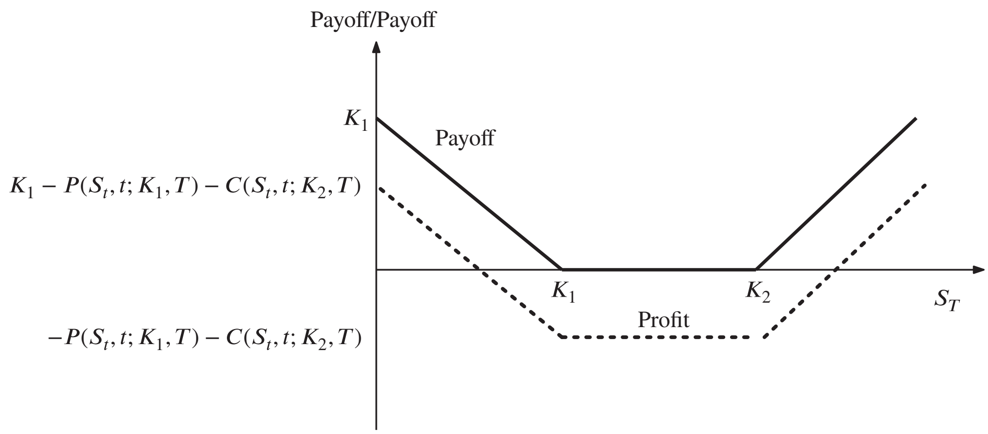

Figure 1.26 Construction of a long strangle.

Like the long straddle, the long strangle also exploits the volatility of the stock price where the profit is based on how much the price of the stock moves instead of its direction. Here the investor makes a positive gain at the expiry time T if $S_T > K_2 + P(S_t; t; K_1, T) + C(S_t, t; K_2, T)$ (unlimited gain) and if $S_T < K_1 - P(S_t, t; K_1, T) - C(S_t, t; K_2, T)$ (gain limited to $K_1 - P(S_t, t; K_1, T) - C(S_t, t; K_2, T)$). As for the downward risk, this strategy has a limited risk which is only the cost of the premiums paid and there is a range between the strikes in which the loss is unaffected by the change in stock price. Hence, this strategy is more suitable for high-volatility stocks.

□

19. *Short Strangle.* A short strangle is a hedging technique where at time t a writer would sell a put option $P(S_t, t; K_1, T)$ with strike K_1 and sell a call option $C(S_t, t; K_2, T)$ with strike K_2, $K_1 < K_2$ on the same stock S_t and having the same expiry time T $(t < T)$.

    Draw the payoff and profit diagrams of this investment strategy and give a financial interpretation based on this combination of options portfolio.

Solution: At expiry time T the payoff of a short strangle portfolio is

$$\begin{aligned}\Psi(S_T) &= -P(S_T; T; K_1, T) - C(S_T; T; K_2, T)\\ &= -\max\{K_1 - S_T, 0\} - \max\{S_T - K_2, 0\}\\ &= \begin{cases} S_T - K_1 & \text{if } S_T \leq K_1 \\ 0 & \text{if } K_1 < S_T \leq K_2 \\ K_2 - S_T & \text{if } S_T > K_2. \end{cases}\end{aligned}$$

Given that the investor received the call and put premiums at time t, the profit at expiry time T is

$$\Upsilon(S_T) = \Psi(S_T) + P(S_t, t; K_1, T) + C(S_t, t; K_2, T)$$
$$= \begin{cases} S_T - K_1 + P(S_t, t; K_1, T) + C(S_t, t; K_2, T) & \text{if } S_T \leq K_1 \\ P(S_t, t; K_1, T) + C(S_t, t; K_2, T) & \text{if } K_1 < S_T \leq K_2 \\ K_2 - S_T + P(S_t, t; K_1, T) + C(S_t, t; K_2, T) & \text{if } S_T > K_2 \end{cases}$$

with break even occurring at $S_T = K_1 - P(S_t, t; K_1, T) - C(S_t, t; K_2, T)$ (provided $P(S_t, t; K_1, T) + C(S_t, t; K_2, T) \leq K_1$) and $S_T = K_2 + P(S_t, t; K_1, T) + C(S_t, t; K_2, T)$.

Figure 1.27 shows the payoff and profit diagrams of a short strangle.

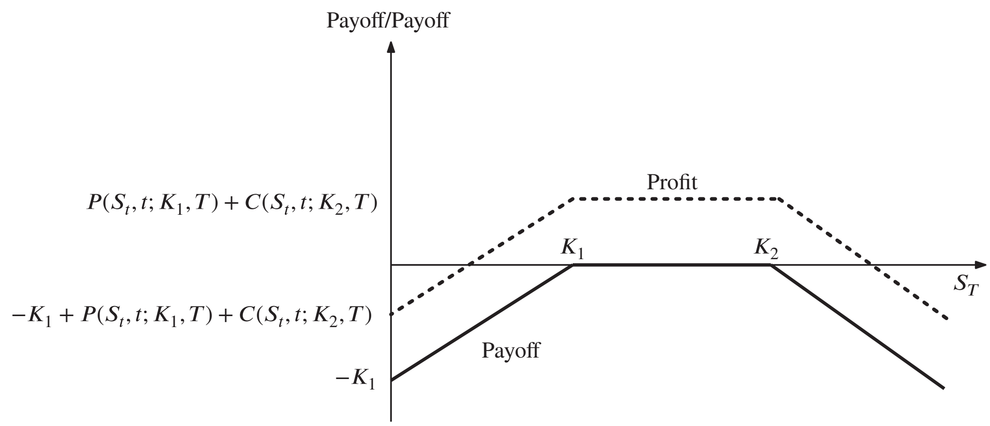

Figure 1.27 Construction of a short strangle.

Like the short straddle, the short strangle also exploits the low volatility of the stock price where the maximum profit is attained from the premiums received. But unlike the short straddle, this contract has a range between the strikes in which the gain is a constant value and unaffected by the change in strike price S_T. In contrast, by taking a short position of a strangle, the writer is exposed to a limited loss if $S_T < K_1 - P(S_t, t; K_1, T) - C(S_t, t; K_2, T)$ and an unlimited loss if $S_T > K_2 + P(S_t, t; K_1, T) + C(S_t, t; K_2, T)$. Hence, this strategy is more suitable for low-volatility stocks.

□

20. *Long Strip.* A long strip is an investment strategy where at time t an investor would buy a call option $C(S_t, t; K, T)$ and buy two put options $P(S_t, t; K, T)$ with strike K on the same stock S_t and having the same expiry time T $(t < T)$.

Draw the payoff and profit diagrams of this investment strategy and give a financial interpretation based on this combination of options portfolio.

Solution: At expiry time T the payoff of this portfolio of options is

$$\begin{aligned}\Psi(S_T) &= C(S_T, T; K, T) + 2P(S_T, T; K, T)\\ &= \max\{S_T - K, 0\} + 2\max\{K - S_T, 0\}\\ &= \begin{cases} 2(K - S_T) & \text{if } S_T \le K\\ S_T - K & \text{if } S_T > K.\end{cases}\end{aligned}$$

Given that the investor paid for the options premiums, the profit at expiry time T is

$$\begin{aligned}\Upsilon(S_T) &= \Psi(S_T) - C(S_t, t; K, T) - 2P(S_t, t; K, T)\\ &= \begin{cases} 2(K - S_T) - C(S_t, t; K, T) - 2P(S_t, t; K, T) & \text{if } S_T \le K\\ S_T - K - C(S_t, t; K, T) - 2P(S_t, t; K, T) & \text{if } S_T > K\end{cases}\end{aligned}$$

with break even at $S_T = K - \frac{1}{2}\left(C(S_t, t; K, T) + 2P(S_t, t; K, T)\right)$ (provided $C(S_t, t; K, T) + 2P(S_t, t; K, T) \le 2K$) and $S_T = K + C(S_t, t; K, T) + 2P(S_t, t; K, T)$.

Figure 1.28 shows the payoff and profit diagrams of a long strip.

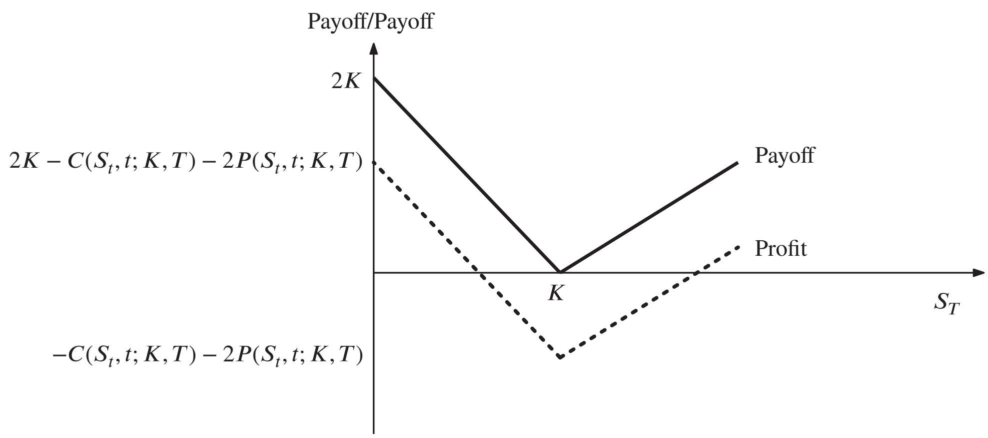

Figure 1.28 Construction of a long strip.

From the profit diagram we can see that the investor would make an unlimited profit if

$$S_T > K + C(S_t, t; K, T) + 2P(S_t, t; K, T)$$

or a limited profit if

$$S_T < K - \frac{1}{2}(C(S_t, t; K, T) + 2P(S_t, t; K, T)).$$

In contrast, the maximum loss for the investor is the cost of purchasing the options. Hence, like the long straddle, this strategy is also dependent on how high the volatility of the stock is rather than the direction of the stock price. However, the only difference between a long straddle and a long strip is that in the latter the strategy exploits more the fall in the stock price as the profit is much higher.

□

21. *Short Strip.* A short strip is an investment strategy where at time t an investor would sell a call option $C(S_t, t; K, T)$ and sell two put options $P(S_t, t; K, T)$ with strike K on the same stock S_t and having the same expiry time T $(t < T)$.

Draw the payoff and profit diagrams of this investment strategy and give a financial interpretation based on this combination of options portfolio.

Solution: At expiry time T the payoff of this portfolio of options is

$$\begin{aligned}\Psi(S_T) &= -C(S_T, T; K, T) - 2P(S_T, T; K, T) \\ &= -\max\{S_T - K, 0\} - 2\max\{K - S_T, 0\} \\ &= \begin{cases} 2(S_T - K) & \text{if } S_T \leq K \\ K - S_T & \text{if } S_T > K. \end{cases}\end{aligned}$$

Given that the writer received the options premium, the profit at expiry time T is

$$\begin{aligned}\Upsilon(S_T) &= \Psi(S_T) + C(S_t, t; K, T) + 2P(S_t, t; K, T) \\ &= \begin{cases} 2(S_T - K) + C(S_t, t; K, T) + 2P(S_t, t; K, T) & \text{if } S_T \leq K \\ K - S_T + C(S_t, t; K, T) + 2P(S_t, t; K, T) & \text{if } S_T > K \end{cases}\end{aligned}$$

with break even at $S_T = K - \frac{1}{2}\big(C(S_t, t; K, T) + 2P(S_t, t; K, T)\big)$ (provided $C(S_t, t; K, T) + 2P(S_t, t; K, T) \leq 2K$) and $S_T = K + C(S_t, t; K, T) + 2P(S_t, t; K, T)$.

Figure 1.29 illustrates the payoff and profit diagrams of a short strip.

From the profit diagram we can see that the writer would make an unlimited loss if

$$S_T > K + C(S_t, t; K, T) + 2P(S_t, t; K, T)$$

or a limited loss if

$$S_T < K - \frac{1}{2}(C(S_t, t; K, T) + 2P(S_t, t; K, T)).$$

In contrast, the maximum gain is the premium received from the options. Hence, like the short straddle, this strategy is also dependent on how high the volatility of the stock is rather than the direction of the stock price. Here the writer bets that the stock price will remain stagnant but incurs a smaller loss if the stock price rises in value.

□

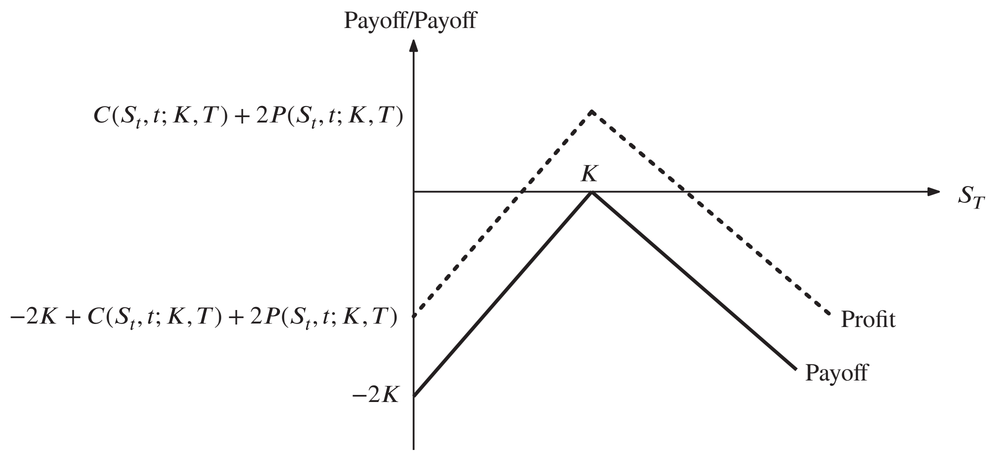

Figure 1.29 Construction of a short strip.

22. *Long Strap.* A long strap is a hedging technique where at time t an investor buys two call options $C(S_t, t; K, T)$ and one put option $P(S_t, t; K, T)$ with strike K on the same stock S_t and having the same expiry time T $(t < T)$.

    Draw the payoff and profit diagrams of this investment strategy and give a financial interpretation based on this combination of options portfolio.

Solution: At expiry time T the payoff of this portfolio of options is

$$\begin{aligned}\Psi(S_T) &= 2C(S_T, T; K, T) + P(S_T, T; K, T)\\ &= 2\max\{S_T - K, 0\} + \max\{K - S_T, 0\}\\ &= \begin{cases} K - S_T & \text{if } S_T \le K\\[2ex] 2(S_T - K) & \text{if } S_T > K.\end{cases}\end{aligned}$$

Given that the investor paid for an options premium, the profit at expiry time T is

$$\begin{aligned}\Upsilon(S_T) &= \Psi(S_T) - 2C(S_t, t; K, T) - P(S_t, t; K, T)\\ &= \begin{cases} K - S_T - 2C(S_t, t; K, T) - P(S_t, t; K, T) & \text{if } S_T \le K\\[2ex] 2(S_T - K) - 2C(S_t, t; K, T) - P(S_t, t; K, T) & \text{if } S_T > K\end{cases}\end{aligned}$$

with break even at $S_T = K - 2C(S_t, t; K, T) - P(S_t, t; K, T)$ (provided $2C(S_t, t; K, T) + P(S_t, t; K, T) \le K$) and $S_T = K + \frac{1}{2}\left(2C(S_t, t; K, T) + P(S_t, t; K, T)\right)$.

Figure 1.30 shows the payoff and profit diagrams of a long strap.

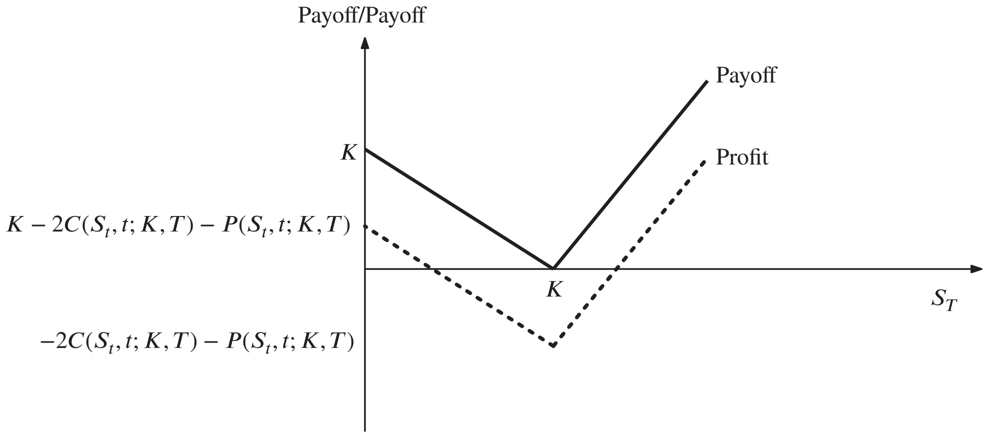

Figure 1.30 Construction of a long strap.

From the profit diagram we can see that the investor would make an unlimited profit if

$$S_T > K + \frac{1}{2}(2C(S_t, t; K, T) + P(S_t, t; K, T))$$

or a limited profit if

$$S_T < K - 2C(S_t, t; K, T) - P(S_t, t; K, T)$$

whilst the maximum loss for the investor is the cost of purchasing the options. Thus, like the long straddle, this strategy is also dependent on how high the volatility of the stock is rather than the direction of the stock price. However, the only difference between a long straddle and a long strap is that in the latter the strategy exploits more the rise in the stock price as the profit is much higher.

□

23. *Short Strap.* A short strap is a hedging technique where at time t a writer sells two call options $C(S_t, t; K, T)$ and one put option $P(S_t, t; K, T)$ with strike K on the same stock S_t and having the same expiry time T $(t < T)$.

    Draw the payoff and profit diagrams of this investment strategy and give a financial interpretation based on this combination of options portfolio.

    Solution: At expiry time T the payoff of this portfolio of options is

$$\begin{aligned}\Psi(S_T) &= -2C(S_T, T; K, T) - P(S_T, T; K, T)\\ &= -2\max\{S_T - K, 0\} - \max\{K - S_T, 0\}\\ &= \begin{cases} S_T - K & \text{if } S_T \le K\\[2ex] 2(K - S_T) & \text{if } S_T > K.\end{cases}\end{aligned}$$

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Given that the writer received the options premium, the profit at expiry time T is

$$\begin{aligned}\Upsilon(S_T) &= \Psi(S_T) + 2C(S_t,t;K,T) + P(S_t,t;K,T)\\ &= \begin{cases} S_T - K + 2C(S_t,t;K,T) + P(S_t,t;K,T) & \text{if } S_T \le K \\ 2(K - S_T) + 2C(S_t,t;K,T) + P(S_t,t;K,T) & \text{if } S_T > K \end{cases}\end{aligned}$$

with break even at $S_T = K - 2C(S_t,t;K,T) - P(S_t,t;K,T)$ (provided $2C(S_t,t;K,T) + P(S_t,t;K,T) \le K$) and $S_T = K + C(S_t,t;K,T) + \frac{1}{2}P(S_t,t;K,T)$.

Figure 1.31 shows the payoff and profit diagrams of a short strap.

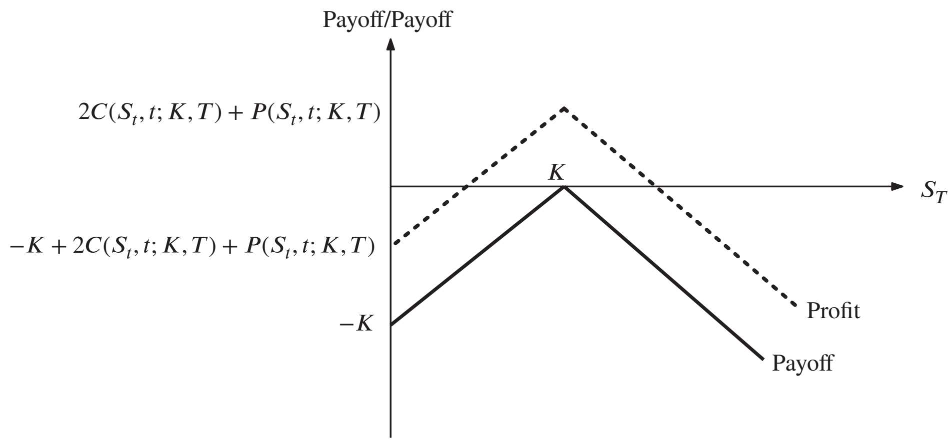

Figure 1.31 Construction of a short strap.

Based on the profit diagram we can see that the writer would make an unlimited loss if

$$S_T > K + C(S_t,t;K,T) + \frac{1}{2}P(S_t,t;K,T)$$

or a limited loss if

$$S_T < K - 2C(S_t,t;K,T) - P(S_t,t;K,T).$$

In contrast, the maximum gain is the premium received from the options. Hence, like the short straddle, this strategy is also dependent on how high the volatility of the stock is rather than the direction of the stock price. Here, the writer bets that the stock is price will remain stagnant but incurs a smaller loss if the stock price falls in value.

□

24. *Butterfly Spread (Using Call Options).* A butterfly spread is a hedging technique that, at time t and for strikes $K_1 < K_2 < K_3$, is constructed by buying one call option $C(S_t,t;K_1,T)$, buying one call option $C(S_t,t;K_3,T)$ and selling two call options $C(S_t,t;K_2,T)$ on the same stock S_t and having the same expiry time T ($t < T$).

Draw the payoff and profit diagrams of this hedging strategy with $K_2 = \frac{1}{2}(K_1 + K_3)$.

Solution: At expiry time T the payoff of a butterfly spread is

$$\begin{aligned}\Psi(S_T) &= C(S_T, T; K_1, T) + C(S_T, T; K_3, T) - 2C(S_T, T; K_2, T)\\ &= \max\{S_T - K_1, 0\} + \max\{S_T - K_3, 0\} - 2\max\{S_T - K_2, 0\}\\ &= \begin{cases} 0 & \text{if } S_T \leq K_1\\ S_T - K_1 & \text{if } K_1 < S_T \leq K_2\\ 2K_2 - K_1 - S_T & \text{if } K_2 < S_T \leq K_3\\ 2K_2 - K_1 - K_3 & \text{if } S_T > K_3\end{cases}\end{aligned}$$

and by setting $\Pi_t = C(S_t, t; K_1, T) + C(S_t, t; K_3, T) - 2C(S_t, t; K_2, T)$ the corresponding profit is

$$\Upsilon(S_T) = \Psi(S_T) - \Pi_t.$$

Hence, provided

$$-K_1 \leq C(S_t, t; K_1, T) + C(S_t, t; K_3, T) - 2C(S_t, t; K_2, T) \leq 2K_2 - 1$$

then break even occurs at

$$S_T = K_1 + C(S_t, t; K_1, T) + C(S_t, t; K_3, T) - 2C(S_t, t; K_2, T)$$

and

$$S_T = 2K_2 - K_1 - C(S_t, t; K_1, T) - C(S_t, t; K_3, T) + 2C(S_t, t; K_2, T).$$

Figure 1.32 shows the payoff and profit diagrams of a butterfly spread using call options with $K_2 = \frac{1}{2}(K_1 + K_3)$.

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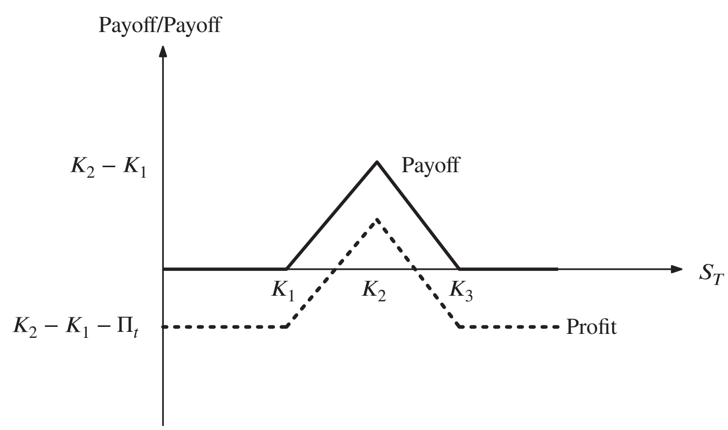

Figure 1.32 Construction of a butterfly spread with $K_2 = \frac{1}{2}(K_1 + K_3)$ and Π_t the net premium paid.

25. *Butterfly Spread (Using Put Options).* A butterfly spread is a hedging technique that, at time t and for strikes $K_1 < K_2 < K_3$, is constructed by buying one put option $P(S_t, t; K_1, T)$, buying one put option $P(S_t, t; K_3, T)$ and selling two put options $P(S_t, t; K_2, T)$ on the same stock S_t and having the same expiry time T $(t < T)$.

Draw the payoff and profit diagrams of this hedging strategy with $K_2 = \frac{1}{2}(K_1 + K_3)$.

Solution: At expiry time T the payoff of a butterfly spread is

$$\begin{aligned}\Psi(S_T) &= P(S_T, T; K_1, T) + P(S_T, T; K_3, T) - 2P(S_T, T; K_2, T)\\ &= \max\{K_1 - S_T, 0\} + \max\{K_3 - S_T, 0\} - 2\max\{K_2 - S_T, 0\}\\ &= \begin{cases} K_1 + K_3 - 2K_2 & \text{if } S_T \le K_1\\ S_T + K_3 - 2K_2 & \text{if } K_1 < S_T \le K_2\\ K_3 - S_T & \text{if } K_2 < S_T \le K_3\\ 0 & \text{if } S_T > K_3\end{cases}\end{aligned}$$

and by setting $\Pi_t = P(S_t, t; K_1, T) + P(S_t, t; K_3, T) - 2P(S_t, t; K_2, T)$ the corresponding profit is

$$\Upsilon(S_T) = \Psi(S_T) - \Pi_t.$$

Hence, provided

$$K_3 - 2K_2 \le P(S_t, t; K_1, T) + P(S_t, t; K_3, T) - 2P(S_t, t; K_2, T) \le K_3$$

then break even occurs at

$$S_T = 2K_2 - K_3 + P(S_t, t; K_1, T) + P(S_t, t; K_3, T) - 2P(S_t, t; K_2, T)$$

and

$$S_T = K_3 - P(S_t, t; K_1, T) - P(S_t, t; K_3, T) + 2P(S_t, t; K_2, T).$$

Figure 1.33 shows the payoff and profit diagrams of a butterfly spread using put options with $K_2 = \frac{1}{2}(K_1 + K_3)$.

□

26. *Butterfly Spread (Using Straddle and Strangle).* A butterfly spread is a hedging technique that, at time t and for strikes $K_1 < K_2 < K_3$, is constructed by selling a straddle:

- sell a call option $C(S_t, t; K_2, T)$ with strike K_2
- sell a put option $P(S_t, t; K_2, T)$ with strike K_2

and buying a strangle:

- buy a call option $C(S_t, t; K_1, T)$ with strike K_1
- buy a put option $P(S_t, t; K_3, T)$ with strike K_3

on the same stock S_t and having the same option expiry time T $(t < T)$.

Draw the payoff and profit diagrams of this investment strategy with $K_2 = \frac{1}{2}(K_1 + K_3)$.

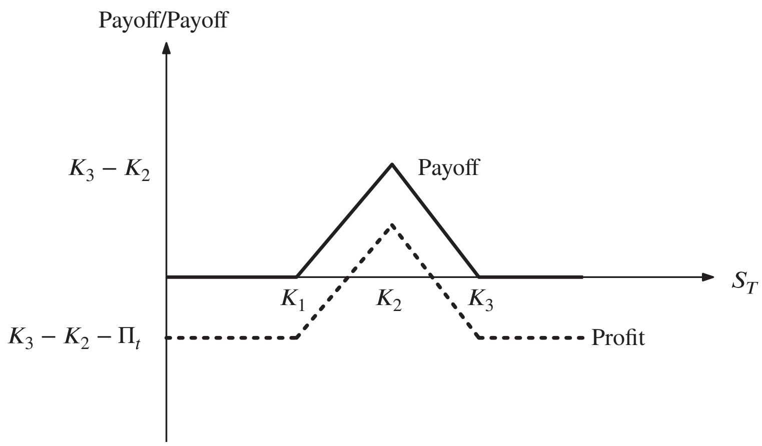

Figure 1.33 Construction of a butterfly spread with $K_2 = \frac{1}{2}(K_1 + K_3)$ and Π_t the net premium paid.

Solution: At expiry time T the payoff of a butterfly spread is

$$\begin{aligned}
\Psi(S_T) &= C(S_T, T; K_1, T) + P(S_T, T; K_3, T) \\
&\quad - C(S_T, T; K_2, T) - P(S_T, T; K_2, T) \\
&= \max\{S_T - K_1, 0\} + \max\{K_3 - S_T, 0\} \\
&\quad - \max\{S_T - K_2, 0\} - \max\{K_2 - S_T, 0\} \\
&= \begin{cases} K_3 - K_1 & \text{if } S_T \leq K_1 \\ S_T + K_3 - K_1 - K_2 & \text{if } K_1 < S_T \leq K_2 \\ -S_T + K_2 + K_3 - K_1 & \text{if } K_2 < S_T \leq K_3 \\ K_2 - K_1 & \text{if } S_T > K_3 \end{cases}
\end{aligned}$$

and by setting $\Pi_t = C(S_t, t; K_1, T) + P(S_t, t; K_3, T) - C(S_t, t; K_2, T) - P(S_t, t; K_2, T)$ the corresponding profit is

$$\Upsilon(S_T) = \Psi(S_T) - \Pi_t.$$

Hence, provided

$$-K_1 - K_2 + K_3 \leq \begin{matrix} C(S_t, t; K_1, T) + P(S_t, t; K_3, T) \\ -C(S_t, t; K_2, T) - P(S_t, t; K_2, T) \end{matrix} \leq -K_1 + K_2 + K_3$$

break even occurs at

$$\begin{aligned}
S_T = {} & K_2 + K_3 - K_1 - C(S_t, t; K_1, T) - P(S_t, t; K_3, T) \\
& + C(S_t, t; K_2, T) + P(S_t, t; K_2, T)
\end{aligned}$$

and

$$S_T = K_1 + K_2 - K_3 + C(S_t, t; K_1, T) + P(S_t, t; K_3, T)$$
$$- C(S_t, t; K_2, T) - P(S_t, t; K_2, T).$$

Figure 1.34 shows the payoff and profit diagrams of a butterfly spread with $K_2 = \frac{1}{2}(K_1 + K_3)$.

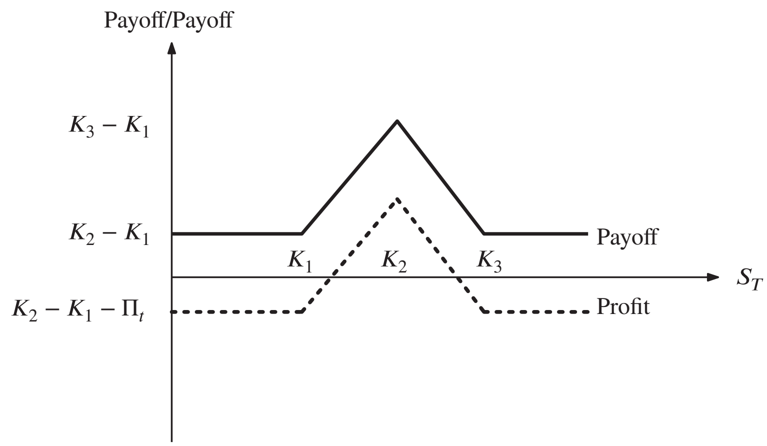

Figure 1.34 Construction of a butterfly spread with $K_2 = \frac{1}{2}(K_1 + K_3)$ and Π_t the net premium paid.

□

27. What is the financial motivation for an investor to enter into a butterfly spread contract?

Solution: Given three strike prices $K_1 < K_2 < K_3$, a butterfly spread can be constructed with a combination of call options, put options or by combining a straddle and a strangle.

At expiry time T for $0 < S_T^{\min} < S_T^{\max}$ such that the profit of a butterfly spread $\Upsilon(S_T^{\min}) = 0$ and $\Upsilon(S_T^{\max}) = 0$ then the investor would make a limited loss if the stock price $S_T > S_T^{\max}$ or $S_T < S_T^{\min}$. In contrast, the maximum gain for the investor occurs when $S_T = K_2$.

Thus, the investor buying a butterfly spread would speculate that the stock price at expiry time T will be between K_1 and K_3 in which the strategy will be most profitable. In essence, by exploiting simultaneously both the low and high volatilities of the stock price based on the combination of options, the investor purchasing a butterfly spread would bet that the stock price will stay close to K_2.

□

28. *Condor Spread (Using Call Options).* A condor spread is a hedging technique that, at time t and for strikes $K_1 < K_2 < K_3 < K_4$, is constructed by buying one call option $C(S_t, t; K_1, T)$, buying one call option $C(S_t, t; K_4, T)$, selling one call option $C(S_t, t; K_2, T)$ and selling one call option $C(S_t, t; K_3, T)$ on the same stock S_t and having the same option expiry time T $(t < T)$.

Draw the payoff and profit diagrams of this hedging strategy with $K_2 - K_1 = K_4 - K_3$.

Solution: Based on the construction of the options, the payoff at expiry time T is

$$\begin{aligned}\Psi(S_T) &= C(S_T, T; K_1, T) - C(S_T, T; K_2, T)\\ &\quad -C(S_T, T; K_3, T) + C(S_T, T; K_4, T)\\ &= \max\{S_T - K_1, 0\} - \max\{S_T - K_2, 0\}\\ &\quad - \max\{S_T - K_3, 0\} + \max\{S_T - K_4, 0\}\\ &= \begin{cases} 0 & \text{if } S_T \le K_1\\ S_T - K_1 & \text{if } K_1 < S_T \le K_2\\ K_2 - K_1 & \text{if } K_2 < S_T \le K_3\\ K_2 - K_1 + K_3 - S_T & \text{if } K_3 < S_T \le K_4\\ K_2 - K_1 + K_3 - K_4 & \text{if } S_T > K_4 \end{cases}\end{aligned}$$

and by setting $\Pi_t = C(S_t, t; K_1, T) - C(S_t, t; K_2, T) - C(S_t, t; K_3, T) + C(S_t, t; K_4, T)$ the corresponding profit is

$$\Upsilon(S_T) = \Psi(S_T) - \Pi_t.$$

Hence, provided

$$-K_1 \le \begin{matrix} C(S_t, t; K_1, T) - C(S_t, t; K_2, T)\\ -C(S_t, t; K_3, T) + C(S_t, t; K_4, T)\end{matrix} \le -K_1 + K_2 + K_3$$

then break even occurs at

$$S_T = K_1 + C(S_t, t; K_1, T) - C(S_t, t; K_2, T) - C(S_t, t; K_3, T) + C(S_t, t; K_4, T)$$

and

$$\begin{aligned}S_T &= K_2 - K_1 + K_3 - C(S_t, t; K_1, T) + C(S_t, t; K_2, T)\\ &\quad +C(S_t, t; K_3, T) - C(S_t, t; K_4, T).\end{aligned}$$

Figure 1.35 shows the payoff and profit diagrams of a condor spread using call options with $K_2 - K_1 = K_4 - K_3$.

□

29. *Condor Spread (Using Put Options).* A condor spread is a hedging technique that, at time t and for strikes $K_1 < K_2 < K_3 < K_4$, is constructed by buying one put option $P(S_t, t; K_1, T)$, buying one put option $P(S_t, t; K_4, T)$, selling one put option $P(S_t, t; K_2, T)$ and selling one put option $P(S_t, t; K_3, T)$ on the same stock S_t and having the same expiry time T $(t < T)$.

    Draw the payoff and profit diagrams of this hedging strategy with $K_2 - K_1 = K_4 - K_3$.

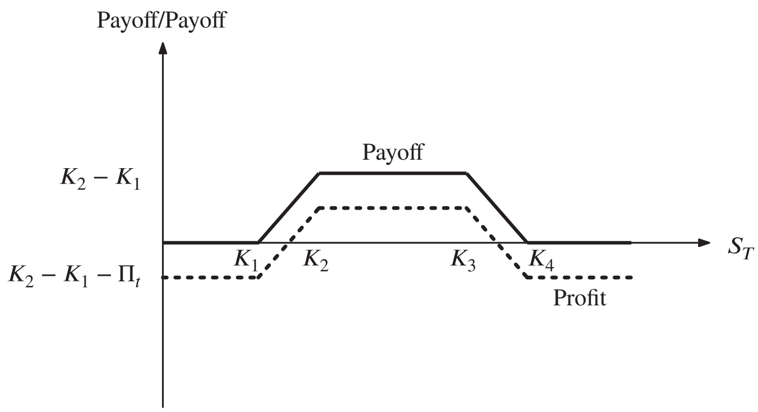

Figure 1.35 Construction of a condor spread with $K_2 - K_1 = K_4 - K_3$ and Π_t the net premium paid.

Solution: Based on the construction of the put options, the payoff at expiry time T is

$$\begin{aligned}
\Psi(S_T) &= P(S_T, T; K_1, T) - P(S_T, T; K_2, T) \\
&\quad - P(S_T, T; K_3, T) + P(S_T, T; K_4, T) \\
&= \max\{K_1 - S_T, 0\} - \max\{K_2 - S_T, 0\} \\
&\quad - \max\{K_3 - S_T, 0\} + \max\{K_4 - S_T, 0\} \\
&= \begin{cases} K_1 - K_2 - K_3 + K_4 & \text{if } S_T \le K_1 \\ S_T - K_2 - K_3 + K_4 & \text{if } K_1 < S_T \le K_2 \\ K_4 - K_3 & \text{if } K_2 < S_T \le K_3 \\ K_4 - S_T & \text{if } K_3 < S_T \le K_4 \\ 0 & \text{if } S_T > K_4 \end{cases}
\end{aligned}$$

and by setting $\Pi_t = P(S_t, t; K_1, T) - P(S_t, t; K_2, T) - P(S_t, t; K_3, T) + P(S_t, t; K_4, T)$ the corresponding profit is

$$\Upsilon(S_T) = \Psi(S_T) - \Pi_t.$$

Hence, provided

$$-K_2 - K_3 + K_4 \le \begin{matrix} P(S_t, t; K_1, T) - P(S_t, t; K_2, T) \\ -P(S_t, t; K_3, T) + P(S_t, t; K_4, T) \end{matrix} \le K_4$$

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then break even occurs at

$$\begin{aligned} S_T = {} & K_2 + K_3 - K_4 + P(S_t, t; K_1, T) - P(S_t, t; K_2, T) \\ & - P(S_t, t; K_3, T) + P(S_t, t; K_4, T) \end{aligned}$$

and

$$S_T = K_4 - P(S_t, t; K_1, T) + P(S_t, t; K_2, T) + P(S_t, t; K_3, T) - P(S_t, t; K_4, T).$$

Figure 1.36 shows the payoff and profit diagrams of a condor spread using put options with $K_2 - K_1 = K_4 - K_3$.

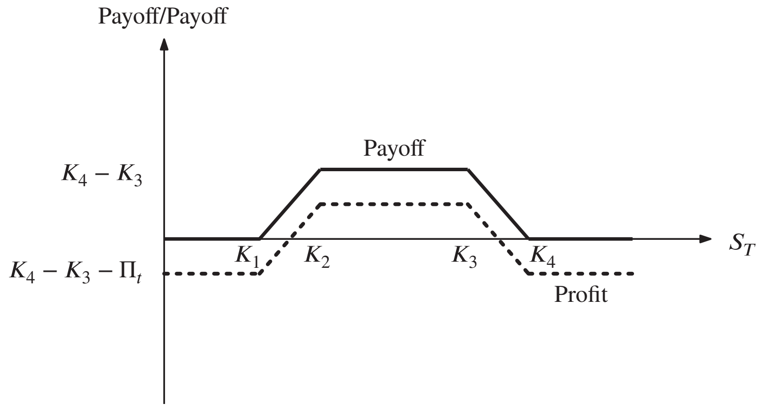

Figure 1.36 Construction of a condor spread with $K_2 - K_1 = K_4 - K_3$ and Π_t the net premium paid.

□

30. Explain the financial motivation for an investor to enter into a condor spread contract.

Solution: Using four different strike prices $K_1 < K_2 < K_3 < K_4$, a condor spread can be constructed with a combination of call options or put options.

At expiry time T for $0 < S_T^{\min} < S_T^{\max}$ such that the profit of a condor spread $\Upsilon(S_T^{\min}) = 0$ and $\Upsilon(S_T^{\max}) = 0$ then the investor would make a limited loss if the stock price $S_T > S_T^{\max}$ or $S_T < S_T^{\min}$.

Thus, the investor who invests in a condor spread speculates that the stock price at expiry time T will be between K_2 and K_3 in which the strategy will be most profitable. But unlike a butterfly spread, a condor spread has a much wider profit range at the expense of a higher premium paid. Thus, this contract is profitable for the investor who has a neutral outlook on the market.

□