1 Introduction and Reading Guide

PROLOGUE

We wrote this book with the aim of giving practitioners in computational finance a sound overview of relevant numerical methods. Some of the methods presented in this book are widely used today, while others should, in our opinion, gain more importance in the future. By, "computational finance" we loosely refer to all tasks related to the valuation of financial instruments, risk analysis and some aspects of risk management. Together with our colleagues at MathConsult GmbH, we have been working on a wide range of computational finance projects since 1997. During that time, we have observed that the numerical quality of software used in financial institutions widely varies.

Particular attention is thus given to working out the strengths and weaknesses of the different methods, and to reveal possible traps in their application. We have used real-world examples of valuation, risk analysis and calibration of specific financial instruments and models to introduce each method. A strong emphasis is laid on stable and robust schemes for the numerical treatment.

We have named the book "A Workout in Computational Finance" because due to our experience in training finance professionals, it is our strong belief that computational methods are best studied in a practical, hands-on approach, requiring the student to write at least part of the program code herself. To facilitate this style of learning, the book comes with accompanying software distilled from the UnRisk software package.[1]

The reader is assumed to have a basic knowledge of mathematical finance and financial derivatives, and a strong interest in quantitative methods. The typical reader of the book is a "junior quant" at a financial institution who wants to gain deeper insight into numerical methods, or, if she has a background in economy, wants to take first steps towards a more quantitative approach. Alternatively, university students at the graduate level may find the topics in this book useful when deciding on a possible future career in finance.

WHAT YOU CAN EXPECT FROM THE DIFFERENT CHAPTERS

In the following, we give a short overview of the contents of the different chapters. Together with the reading guide this should allow the reader to select her topics of interest.

Chapter 2: Binomial Trees

Binomial trees a conceptionally elegant method for valuating derivatives: They are explicit (i.e., no system of equations needs to be solved), and they intrinsically include no-arbitrage

[1] The UnRisk ENGINE and the UnRisk FACTORY are software packages for valuation and risk management of financial instruments and portfolios thereof. UnRisk has been developed by MathConsult since 1999 and now contains more than 1 million lines of multi language code. UnRisk is a registered trademark of MathConsult. Details: www.unrisk.com

conditions. From a numerical point of view, their lack of adaptivity as well as stability problems limit their range of applicability.

Chapter 3: Finite Differences and the Black-Scholes PDE

In this chapter the derivation of the Black-Scholes partial differential equation is explained in detail. The differential operators are discretized using finite difference formulae, and various methods for the time discretization are introduced. Stability issues resulting from the chosen spatial and time discretizations are discussed. The application of the finite difference method to the prototype model of the heat equation concludes the chapter.

Chapter 4: Mean Reversion and Trinomial Trees

When models exhibit mean reverting behavior, such as the Vasicek model for interest rates, binomial trees do not recombine anymore. Trinomial trees have been introduced to cure this problem. To retain their stability, up- and down-branching is used, cutting off the calculation domain and therefore implicitly changing the boundary conditions.

Chapter 5: Upwinding Techniques

Particular finite difference formulae need to be applied to cure the instabilities occurring if partial differential equations arising from mean reverting models are discretized. These formulae are derived in Chapter 5 and their ability to cope with the instabilities is examined. The chapter concludes with a detailed example of the application of upwinding techniques to a putable fixed rate bond under a one factor short rate model.

Chapter 6: Boundary, Terminal and Interface Conditions

To valuate a specific financial instrument, its term sheet must be translated into boundary-, terminal- and possibly also interface conditions (for coupons or callabilities) for the differential equation to be solved. These conditions are formulated for a range of instruments. It turns out that for heavily path-dependent instruments, Monte Carlo techniques may be the more appropriate choice. For the case of mean reverting interest rate models, the influence of artificial boundary conditions is studied.

Chapter 7: Finite Element Methods

The basic concepts of the finite element method are described, and for a number of different elements, the element matrices are derived. Particular emphasis is laid on the assembling process of the global matrices and the incorporation of boundary conditions. Similarly to the finite difference technique, stabilization terms need to be added if the finite element method is applied to convection-diffusion-reaction problems. An example comparing the numerical results obtained with the finite element method to results obtained with tree techniques concludes the chapter.

Chapter 8: Solving Systems of Linear Equations

In Chapters 3, 5, 7 and 13 different discretization techniques for partial (integro) differential equations are discussed, all of them leading to systems of linear equations. This chapter provides an overview of different techniques to solve them. Dependent on the type of problem and the problem size, direct methods or iterative solvers methods may be preferable. A number of basic algorithms for both types of methods are discussed and explained based on small examples.

Chapter 9: Monte Carlo Simulation

The principles of Monte Carlo integration techniques and their application for the pricing of derivatives are explained. Furthermore, different discretization techniques for the stochastic differential equations used to model the propagation of the underlying risk factors are discussed. The Libor market model is examined as an example of a high-dimensional model where Monte Carlo methods are typically applied to valuate derivatives. The final part of the chapter emphasizes random number generation, which is one of the major building blocks of every Monte Carlo-based pricing routine.

Chapter 10: Advanced Monte Carlo Techniques

Different techniques exist to reduce the variance of Monte Carlo estimators. Some of the more general algorithms are explained based on examples such as the pricing of exotic options. In contrast to the Monte Carlo method, Quasi Monte Carlo methods do not use sequences of pseudo-random numbers, but use sequences of low-discrepancy numbers instead. The most important sequences are introduced and applied to the pricing of a structured interest rate product under the Libor market model. As a technique to speed up Monte Carlo as well as Quasi Monte Carlo simulations the Brownian bridge method is outlined.

Chapter 11: Least Squares Monte Carlo

A well-known problem when using Monte Carlo methods for pricing is the inclusion of American or Bermudan call- or putability. We present a detailed outline of a Least Squares Monte Carlo algorithm in this chapter and apply this algorithm to a number of structured interest and foreign exchange rate instruments.

Chapter 12: Characteristic Function Methods for Option Pricing

For many distributions the density functions are not known or are not analytically tractable, but their corresponding Fourier transforms, the characteristic functions, are. This circumstance provides the basis for the methods presented in two fast and reliable methods for the valuation of European vanilla options based on the Fast Fourier Transformation and cosine series expansion are discussed. An overview of equity models beyond Black-Scholes is given in this chapter, as the calibration of these models is one of the major areas where the characteristic function methods can be applied.

Chapter 13: Numerical Methods for the Solution of PIDEs

The extension of the methods discussed in Chapters 3, 5 and 7 to cope with partial integro differential equations is discussed in this chapter.

Chapter 14: Copulas and the Pitfalls of Correlation

In the first two parts of this chapter, a number of common measures for dependence and basic concepts of copulas are presented. Subsequently, the most important copulas applied in finance are discussed and some estimation and sampling methods for them are outlined. An example showing the impact of different copula functions on the probability of default closes the chapter.

Chapter 15: Parameter Calibration and Inverse Problems

Market data are changing more or less continuously. To reflect this fact in the valuation of financial instruments, the parameters of the mathematical models describing the movement of underlyings have to be (re)calibrated on a regular basis. This is a classical inverse problem, and therefore instabilities are to be expected. Several examples for equity and for interest rate models are discussed in detail.

Chapter 16: Optimization Techniques

To solve the calibration problems outlined in Chapter 15, optimization techniques need to be applied. We differentiate gradient-based and heuristically motivated methods, discuss algorithms for both types and show that hybrids of the two worlds can successfully be applied to estimate parameters, using the calibration of a Heston model as an example. For constrained optimization problems we introduce the interior point method and show the capability of this method in the field of portfolio optimization.

Chapter 17: Risk Management

Many of the methods discussed up to this point in the book can be used to value single instruments or portfolios. Frequently, such algorithms are used as building blocks in a risk management system where these valuations must be performed over thousands of different scenarios. In this chapter we will discuss the different possibilities to generate such scenarios and how to assess the risk measures from the simulation results. A short outline of extreme value theory and its application to the calculation of Value at Risk and Expected Shortfall concludes the chapter.

Chapter 18: Quantitative Finance on Parallel Architectures

In many fields of quantitative finance it is necessary to perform thousands or even millions of valuations of often highly structured instruments. Parallel computation techniques are used to significantly speed up these valuations. Multicore CPUs and recent GPUs, have fueled this trend by providing affordable and readily available parallel hardware. Based on examples we examine how different parallelization techniques can successfully be applied.

Chapter 19: Building Large Software Systems for the Financial Industry

This is a short chapter on the authors' experiences in building a large software system for risk management of financial instruments.

ACCOMPANYING SOFTWARE

The buyer of the book is entitled to download the accompanying software from www.unrisk.com/Workout after registration. A tutorial on how to use the software together with the book is available on the webpage.

READING GUIDE

The diagram below shows the interrelations between different chapters and the suggested order of reading. Vertical arrows indicate suggested order of reading within a group of associated chapters, all other arrows indicate relations between different groups. For example, the Chapters on P(I)DE methods should be read in the order 3, 5–7, 13, and Chapter 8 (Solving Systems of Linear Equations) depends on material covered the P(I)DE chapters.