1

Background

In this chapter, background material that will be referred to in the subsequent chapters is reviewed. In Section 1.1, the statistical characterization of persistent (finite-power) nonstationary stochastic processes is presented. Second-order statistics in both time, frequency, and time-frequency domains are considered. In Section 1.2, definitions of almost-periodic functions and their generalizations (Besicovitch 1932) and related results are reviewed. Almost-cyclostationary (ACS) processes (Gardner 1985, 1987d) are treated in Section 1.3. Finally, in Section 1.4, some results on cumulants are reviewed.

1.1 Second-Order Characterization of Stochastic Processes

1.1.1 Time-Domain Characterization

In the classical stochastic-process framework, statistical functions are defined in terms of ensemble averages of functions of the process and its time-shifted versions. Nonstationary processes have these statistical functions that depend on time.

Let us consider a continuous-time real-valued process $\{x(t, \omega),\ t \in \mathbb{R},\ \omega \in \Omega\}$, with abbreviate notation $x(t)$ when it does not create ambiguity, where Ω is a sample space equipped with a σ-field $\mathcal{F}$ and a probability measure P defined on the elements of $\mathcal{F}$. The cumulative distribution function of $x(t)$ is defined as (Doob 1953)

$$F_x(\xi; t) \triangleq P[x(t, \omega) \leqslant \xi] = \int_\Omega \mathbf{1}_{\{\omega\,:\,x(t,\omega)\leqslant\xi\}}\,\mathrm{d}P(\omega) \triangleq \mathrm{E}\left\{\mathbf{1}_{\{\omega\,:\,x(t,\omega)\leqslant\xi\}}\right\} \tag{1.1}$$

where

$$\mathbf{1}_{\{\omega\,:\,x(t,\omega)\leqslant\xi\}} \triangleq \begin{cases} 1, & \omega\ :\ x(t, \omega) \leqslant \xi, \\ 0, & \omega\ :\ x(t, \omega) > \xi \end{cases} \tag{1.2}$$

Generalizations of Cyclostationary Signal Processing: Spectral Analysis and Applications, First Edition.

is the indicator of the set $\{\omega \in \Omega : x(t,\omega) \leqslant \xi\}$ and $\mathrm{E}\{\cdot\}$ denotes statistical expectation (ensemble average). The expected value corresponding to the distribution $F_x(\xi;t)$ is the statistical mean

$$\int_{\mathbb{R}} \xi \, \mathrm{d}F_x(\xi;t) = \int_{\Omega} x(t,\omega) \, \mathrm{d}P(\omega) = \mathrm{E}\{x(t,\omega)\}. \tag{1.3}$$

Analogously, at second-order, the process is characterized by the second-order joint distribution function (Doob 1953)

$$\begin{aligned} F_x(\xi_1,\xi_2;t,\tau) &\triangleq P[x(t+\tau,\omega) \leqslant \xi_1, x(t,\omega) \leqslant \xi_2] \\ &= \mathrm{E}\left\{\mathbf{1}_{\{\omega : x(t+\tau,\omega)\leqslant\xi_1\}} \, \mathbf{1}_{\{\omega : x(t,\omega)\leqslant\xi_2\}}\right\} \end{aligned} \tag{1.4}$$

and the autocorrelation function

$$\mathrm{E}\{x(t+\tau,\omega)\,x(t,\omega)\} = \int_{\mathbb{R}^2} \xi_1\xi_2 \, \mathrm{d}F_x(\xi_1,\xi_2;t,\tau). \tag{1.5}$$

If $F_x(\xi;t)$ and $F_x(\xi_1,\xi_2;t,\tau)$ depend on t, the process is said to be nonstationary in the strict sense. If $F_x(\xi;t)$ $[F_x(\xi_1,\xi_2;t,\tau)]$ does not depend on t, the process $x(t)$ is said to be 1st-order [2nd-order] stationary in the strict sense. If both mean and autocorrelation function do not depend on t, the process is said to be wide-sense stationary (WSS) (Doob 1953).

In the following, we will focus on the second-order statistics of complex-valued nonstationarity processes.

The complex-valued stochastic process $x(t)$ is said to be a *second-order process* if the second-order moments

$$\mathcal{R}_{\boldsymbol{x}}(t,\tau) \triangleq \mathrm{E}\left\{x(t+\tau)\,x^{(*)}(t)\right\} \tag{1.6}$$

exist $\forall t$ and $\forall \tau$. In Equation (1.6), superscript $(*)$ denotes optional complex conjugation, and subscript $\boldsymbol{x} \triangleq [x, x^{(*)}]$. That is, $\mathcal{R}_{\boldsymbol{x}}(t,\tau)$ denotes one of two different functions depending if the complex conjugation is considered or not in subscript $\boldsymbol{x}$. If conjugation is present, then (1.6) is the *autocorrelation function*. If the conjugation is absent, then (1.6) is the *conjugate autocorrelation function* also referred to as *relation function* (Picinbono and Bondon 1997) or *complementary correlation* (Schreier and Scharf 2003a). Note that, in the complex case the order of the distribution functions turns out to be doubled with respect to the real case. For example, the joint distribution function of $x(t)$ and $x(t+\tau)$ is a fourth-order joint distribution of the real and imaginary parts of $x(t)$ and $x(t+\tau)$.

The *(conjugate) autocovariance* is the (conjugate) autocorrelation of the process reduced to be zero mean by subtracting its mean value

$$\mathcal{C}_{\boldsymbol{x}}(t,\tau) \triangleq \mathrm{E}\Big\{\big[x(t+\tau) - \mathrm{E}\{x(t+\tau)\}\big]\,\big[x(t) - \mathrm{E}\{x(t)\}\big]^{(*)}\Big\}. \tag{1.7}$$

Even if $\mathcal{C}_{\boldsymbol{x}}(t,\tau) = \mathcal{R}_{\boldsymbol{x}}(t,\tau)$ only for zero-mean processes, in some cases the terms autocorrelation, autocovariance, and covariance are used interchangeably. When the terms autocovariance or covariance are adopted, from the context it is understood if the mean value is subtracted or not. In statistics, the definition of autocorrelation includes in (1.6) also a normalization by the standard deviations of $x(t)$ and $x(t+\tau)$.

1.1.2 Spectral-Domain Characterization

The characterization of stochastic processes in the spectral domain can be made by resorting to the concept of harmonizability (Loève 1963). A second-order stochastic process $x(t)$ is said to be *harmonizable* if its (conjugate) autocorrelation function can be expressed by the Fourier-Stieltjes integral

$$\mathrm{E}\left\{x(t_1)\,x^{(*)}(t_2)\right\} = \int_{\mathbb{R}^2} e^{j2\pi[f_1t_1+(-)f_2t_2]}\,\mathrm{d}\gamma_{\boldsymbol{x}}(f_1, f_2) \tag{1.8}$$

where $\gamma_{\boldsymbol{x}}(f_1, f_2)$ is a spectral correlation function of bounded variation (Loève 1963):

$$\int_{\mathbb{R}^2} |\,\mathrm{d}\gamma_{\boldsymbol{x}}(f_1, f_2)| < \infty. \tag{1.9}$$

In (1.8), $(-)$ is an optional minus sign that is linked to $(*)$. $\gamma_{\boldsymbol{x}}(f_1, f_2)$ denotes one of two different functions depending if the complex conjugation is considered or not in subscript $\boldsymbol{x}$.

Under the harmonizability condition, $x(t)$ is said to be *(strongly) harmonizable* and can be expressed by the Cramér representation (Cramér 1940)

$$x(t) = \int_{\mathbb{R}} e^{j2\pi ft}\,\mathrm{d}\chi(f) \tag{1.10}$$

where $\chi(f)$ is the *integrated spectrum* of $x(t)$.

In (Loève 1963), it is shown that a necessary condition for a stochastic process to be harmonizable is that it is second-order continuous (or mean-square continuous) (Definition 2.2.11, Theorem 2.2.12). Moreover, it is shown that a stochastic process is harmonizable if and only if its covariance function is harmonizable. In fact, convergence of integrals in (1.8) and (1.10) is in the mean-square sense. In (Hurd 1973), the harmonizability of processes obtained by some processing of other harmonizable processes is studied.

If the absolutely continuous and the discrete component of $\chi(f)$ are (possibly) nonzero and the singular component of $\chi(f)$ is zero with probability 1 (w.p.1) (Cramér 1940), we can formally write $\mathrm{d}\chi(f) = X(f)\,\mathrm{d}f$ (w.p.1) (Gardner 1985, Chapter 10.1.2), where

$$X(f) = \int_{\mathbb{R}} x(t)\,e^{-j2\pi ft}\,\mathrm{d}t \tag{1.11}$$

is the Fourier transform of $x(t)$ which possibly contains Dirac deltas in correspondence of the jumps of the discrete component of $\chi(f)$. For *finite-power processes*, that is such that the time-averaged power

$$P_x \triangleq \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} \mathrm{E}\left\{|x(t)|^2\right\}\,\mathrm{d}t \tag{1.12}$$

exists and is finite, relation (1.11) is intended in the sense of distributions (Gelfand and Vilenkin 1964, Chapter 3), (Henniger 1970).

Let $x(t)$ be an harmonizable stochastic process. Its *bifrequency spectral correlation function* or *Loève bifrequency spectrum* (Loève 1963; Thomson 1982), also called *generalized spectrum* in (Gerr and Allen 1994), *cointensity spectrum* in (Middleton 1967), or *dual frequency spectral correlation* in (Hanssen and Scharf 2003), is defined as

$$\mathcal{S}_{\boldsymbol{x}}(f_1, f_2) \triangleq \mathrm{E}\left\{X(f_1)\, X^{(*)}(f_2)\right\} \tag{1.13}$$

and if $\chi(f)$ and $\gamma_{\boldsymbol{x}}(f_1, f_2)$ do not contain singular components w.p.1, in the sense of distributions the result is that

$$\mathrm{d}\gamma_{\boldsymbol{x}}(f_1, f_2) = \mathrm{E}\left\{\mathrm{d}\chi(f_1)\ \mathrm{d}\chi^{(*)}(f_2)\right\} \tag{1.14a}$$

$$= \mathrm{E}\left\{X(f_1)\, X^{(*)}(f_2)\right\}\ \mathrm{d}f_1\, \mathrm{d}f_2 \tag{1.14b}$$

and, accordingly with (1.8), we can formally write

$$\mathrm{E}\left\{x(t_1)\, x^{(*)}(t_2)\right\} = \int_{\mathbb{R}^2} \mathrm{E}\left\{X(f_1)\, X^{(*)}(f_2)\right\}\ e^{j2\pi[f_1t_1+(-)f_2t_2]}\,\mathrm{d}f_1\,\mathrm{d}f_2 \tag{1.15}$$

$$\mathrm{E}\left\{X(f_1)\, X^{(*)}(f_2)\right\} = \int_{\mathbb{R}^2} \mathrm{E}\left\{x(t_1)\, x^{(*)}(t_2)\right\} e^{-j2\pi[f_1t_1+(-)f_2t_2]}\,\mathrm{d}t_1\,\mathrm{d}t_2 \tag{1.16}$$

A spectral characterization for nonstationary processes that resembles that for WSS processes (Section 1.1.4) can be obtained starting from the *time-averaged (conjugate) autocorrelation function*

$$\begin{aligned} R_{\boldsymbol{x}}(\tau) &\triangleq \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} \mathrm{E}\left\{x(t+\tau)\, x^{(*)}(t)\right\}\ \mathrm{d}t \\ &\equiv \left\langle \mathrm{E}\left\{x(t+\tau)\, x^{(*)}(t)\right\}\right\rangle_t \end{aligned} \tag{1.17}$$

when the limit exists. Its Fourier transform is called the *power spectrum*, is denoted by $S_{\boldsymbol{x}}(f)$, and represents the spectral density of the time-averaged power $R_{\boldsymbol{x}}(0)$ of the process. The time-averaged autocorrelation function and the power spectrum defined here for nonstationary processes exhibit the same properties of the autocorrelation function and power spectrum defined for wide-sense stationary processes (Wu and Lev-Ari 1997).

1.1.3 Time-Frequency Characterization

The Loève bifrequency spectrum (1.13) provides a description of the nonstationary behavior of the process $x(t)$ in the frequency domain. A description in terms of functions of time and frequency can be obtained by resorting to the time-variant spectrum, the Rihaczek distribution, and the Wigner-Ville spectrum.

The Fourier transform of the second-order moment (1.6) with respect to (w.r.t.) the lag parameter τ is the *time-variant spectrum*

$$\mathcal{S}_{\boldsymbol{x}}(t, f) \triangleq \int_{\mathbb{R}} \mathcal{R}_{\boldsymbol{x}}(t, \tau)\, e^{-j2\pi f\tau}\,\mathrm{d}\tau. \tag{1.18}$$

By substituting (1.6) into (1.18), interchanging the order of the expectation and Fourier-transform operators, and accounting for the formal relation $\mathrm{d}\chi(f) = X(f)\,\mathrm{d}f$, one obtains

$$\mathcal{S}_{\boldsymbol{x}}(t, f)\,\mathrm{d}f = \mathrm{E}\left\{ \mathrm{d}\chi(f)\, x^{(*)}(t) \right\}\, e^{j2\pi f t} \tag{1.19}$$

where the right-hand-side is referred to as the *(conjugate) Rihaczek distribution* of $x(t)$ (Scharf *et al.* 2005).

By the variable change $t' = t + \tau/2$ in (1.6) and Fourier transforming w.r.t. τ, we obtain a time-frequency representation in terms of *Wigner-Ville spectrum* for stochastic processes (Martin and Flandrin 1985)

$$\mathcal{W}_{\boldsymbol{x}}(t', f) \triangleq \int_{\mathbb{R}} \mathrm{E}\left\{ x(t' + \tau/2)\, x^{(*)}(t' - \tau/2) \right\}\, e^{-j2\pi f \tau}\,\mathrm{d}\tau \tag{1.20a}$$

$$= \int_{\mathbb{R}} \mathrm{E}\left\{ X(f + \nu/2)\, X^{(*)}(f - \nu/2) \right\}\, e^{j2\pi \nu t'}\,\mathrm{d}\nu \tag{1.20b}$$

where the second equality follows using (1.11).

Extensive treatments on time-frequency characterizations of nonstationary signals are given in (Amin 1992), (Boashash *et al.* 1995), (Cohen 1989, 1995), (Flandrin 1999), (Hlawatsch and Bourdeaux-Bartels 1992). Most of these references refer to finite-energy signals.

1.1.4 Wide-Sense Stationary Processes

Second-order nonstationary processes have (conjugate) autocorrelation function depending on both time t and lag parameter τ and the function defined in (1.6) is also called the time-lag (conjugate) autocorrelation function. Equivalently, their time-variant spectrum depends on both time t and frequency f. In contrast, second-order WSS processes are characterized by a (conjugate) autocorrelation and time-variant spectrum not depending on t. That is

$$\mathcal{R}_{\boldsymbol{x}}(t, \tau) = R_{\boldsymbol{x}}(\tau) \tag{1.21a}$$

$$\mathcal{S}_{\boldsymbol{x}}(t, f) = S_{\boldsymbol{x}}(f). \tag{1.21b}$$

In such a case, for $(*)$ present, the Fourier-transform (1.18) specializes into the Wiener-Khinchin relation that links the autocorrelation function and the *power spectrum* $S_{\boldsymbol{x}}(f)$ (Gardner 1985)

$$S_{\boldsymbol{x}}(f) = \int_{\mathbb{R}} R_{\boldsymbol{x}}(\tau)\, e^{-j2\pi f \tau}\,\mathrm{d}\tau. \tag{1.22}$$

Condition (1.21a) is equivalent to the fact that the time–time (conjugate) autocorrelation function (1.8) depends only on the time difference $t_1 - t_2$. This time dependence in the spectral domain corresponds to the property that the Loève bifrequency spectrum (1.13) is nonzero only on the diagonal $f_2 = -_{(-)} f_1$. That is,

$$\mathcal{S}_{\boldsymbol{x}}(f_1, f_2) = S_{\boldsymbol{x}}(f_1)\,\delta(f_2 +_{(-)} f_1) \tag{1.23}$$

where $\delta(\cdot)$ denotes Dirac delta. When $(*)$ is present, $S_{\boldsymbol{x}}(f_1)$ is the power spectrum of the process $x(t)$. From (1.23), it follows that for WSS processes distinct spectral component are uncorrelated. In contrast, the presence of spectral correlation outside the diagonal is evidence of nonstationarity in the process $x(t)$ (Loève 1963). Finally, for WSS processes the Wigner-Ville spectrum is independent of t' and is coincident with the power spectrum. That is, $\mathcal{W}_{\boldsymbol{x}}(t', f) = S_{\boldsymbol{x}}(f)$.

Extensive treatments on WSS processes are given in (Brillinger 1981), (Cramér 1940), (Doob 1953), (Grenander and Rosenblatt 1957), (Papoulis 1991), (Prohorov and Rozanov 1989), (Rosenblatt 1974, 1985).

1.1.5 Evolutionary Spectral Analysis

In (Priestley 1965), the class of zero-mean processes for which the autocovariance function admits the representation

$$\mathrm{E}\left\{x(t_1)\,x^*(t_2)\right\} = \int_{\mathbb{R}} \phi_{t_1}(\omega)\,\phi^*_{t_2}(\omega)\,\mathrm{d}\mu(\omega) \tag{1.24}$$

is considered, where $\{\phi_t(\omega)\}$ is a family of functions defined on the real line ($\omega \in \mathbb{R}$) indexed by the suffix t and $\mathrm{d}\mu(\omega)$ is a measure on the real line. In (Grenander and Rosenblatt 1957, paragraph 1.4), it is shown that if the autocovariance has the representation (1.24), then the process $x(t)$ admits the representation

$$x(t) = \int_{\mathbb{R}} \phi_t(\omega)\,\mathrm{d}Z(\omega) \tag{1.25}$$

where $Z(\omega)$ is an orthogonal process with

$$\mathrm{E}\left\{\,\mathrm{d}Z(\omega_1)\,\mathrm{d}Z^*(\omega_2)\right\} = \delta(\omega_1-\omega_2)\,\mathrm{d}\mu(\omega_1). \tag{1.26}$$

In fact, we formally have

$$\begin{aligned}\mathrm{E}\left\{x(t_1)\,x^*(t_2)\right\} &= \mathrm{E}\Big\{\int_{\mathbb{R}} \phi_{t_1}(\omega_1)\,\mathrm{d}Z(\omega_1)\int_{\mathbb{R}} \phi^*_{t_2}(\omega_2)\,\mathrm{d}Z^*(\omega_2)\Big\}\\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} \phi_{t_1}(\omega_1)\,\phi^*_{t_2}(\omega_2)\mathrm{E}\left\{\,\mathrm{d}Z(\omega_1)\,\mathrm{d}Z^*(\omega_2)\right\}\\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} \phi_{t_1}(\omega_1)\,\phi^*_{t_2}(\omega_2)\,\delta(\omega_1-\omega_2)\,\mathrm{d}\mu(\omega_1)\\ &= \int_{\mathbb{R}} \phi_{t_1}(\omega_1)\,\phi^*_{t_2}(\omega_1)\,\mathrm{d}\mu(\omega_1)\end{aligned} \tag{1.27}$$

where, in the last equality, the sampling property of the Dirac delta (Zemanian 1987, Section 1.7) is used.

When the process is second-order WSS, a valid choice for the family $\{\phi_t(\omega)\}$ is $\phi_t(\omega) = e^{j\omega t}$. The autocovariance is

$$\mathrm{E}\left\{x(t_1)\,x^*(t_2)\right\} = \int_{\mathbb{R}} e^{j\omega(t_1-t_2)}\,\mathrm{d}\mu(\omega) \tag{1.28}$$

which is function of $t_1 - t_2$. The function $\mu(\omega)$ is the *integrated power spectrum*. If $\mu(\omega)$ is absolutely continuous or contains jumps and has zero singular component (Cramér 1940), then in the sense of distributions $\mathrm{d}\mu(\omega) = S(\omega)\,\mathrm{d}(\omega)$, where $S(\omega)$ is the power spectrum (with $\omega = 2\pi f$) which contains Dirac deltas in correspondence of the jumps in $\mu(\omega)$.

The function of t, $\phi_t(\omega)$ is said to be an *oscillatory function* if, for some real-valued function $\theta(\omega)$, it results in

$$\phi_t(\omega) = A_t(\omega)\, e^{j\theta(\omega)t} \tag{1.29}$$

where the modulating function $A_t(\omega)$, as a function of t, has a (generalized) Fourier transform with an absolute maximum in the origin (that is, as a function of t, it is a low-pass function) and can be seen as the "envelope" of $x(t)$. In addition, if the function $\theta(\cdot)$ is invertible with inverse $\theta^{-1}(\cdot)$, then by substituting (1.29) into (1.24) and making the variable change $\lambda = \theta(\omega)$ we have

$$\begin{aligned} \mathrm{E}\left\{x(t_1)\, x^*(t_2)\right\} &= \int_{\mathbb{R}} A_{t_1}(\omega)\, A^*_{t_2}(\omega)\, e^{j\theta(\omega)(t_1-t_2)}\, \mathrm{d}\mu(\omega) \\ &= \int_{\mathbb{R}} A_{t_1}(\theta^{-1}(\lambda))\, A^*_{t_2}(\theta^{-1}(\lambda))\, e^{j\lambda(t_1-t_2)}\, \mathrm{d}\mu(\theta^{-1}(\lambda)) \\ &= \int_{\mathbb{R}} \bar{A}_{t_1}(\lambda)\, \bar{A}^*_{t_2}(\lambda)\, e^{j\lambda(t_1-t_2)}\, \mathrm{d}\bar{\mu}(\lambda) \end{aligned} \tag{1.30}$$

where $\bar{A}_t(\cdot) \triangleq A_t(\theta^{-1}(\cdot))$ and $\mathrm{d}\bar{\mu}(\cdot) \triangleq \mathrm{d}\mu(\theta^{-1}(\cdot))$. The process is said to be an *oscillatory process* and admits the representation

$$x(t) = \int_{\mathbb{R}} \bar{A}_t(\lambda)\, e^{j\lambda t}\, \mathrm{d}\bar{Z}(\lambda) \tag{1.31}$$

with respect to the family of oscillatory functions

$$\{\bar{\phi}_t(\lambda)\} \equiv \{\bar{A}_t(\lambda)\, e^{j\lambda t}\} \tag{1.32}$$

where $\bar{Z}(\lambda)$ is an orthogonal process with

$$\mathrm{E}\left\{\mathrm{d}\bar{Z}(\lambda_1)\, \mathrm{d}\bar{Z}^*(\lambda_2)\right\} = \delta(\lambda_1 - \lambda_2)\, \mathrm{d}\bar{\mu}(\lambda_1). \tag{1.33}$$

In fact, by using (1.31) and (1.33) into the autocovariance definition leads to the rhs of (1.30).

Motivated by the fact that, according to (1.30),

$$\mathrm{E}\left\{|x(t)|^2\right\} = \int_{\mathbb{R}} |\bar{A}_t(\lambda)|^2\, \mathrm{d}\bar{\mu}(\lambda) \tag{1.34}$$

the *evolutionary spectrum* at time t with respect to the family (1.32) is defined as

$$\mathrm{d}F_t(\lambda) = |\bar{A}_t(\lambda)|^2\, \mathrm{d}\bar{\mu}(\lambda). \tag{1.35}$$

This definition is consistent with the interpretation of (1.31) as an expression of the process $x(t)$ as the superposition of complex sinewaves with orthogonal time-varying random amplitudes $\bar{A}_t(\lambda)\, \mathrm{d}\bar{Z}(\lambda)$.

The WSS processes are obtained as a special case of oscillatory processes if $A_t(\omega) = 1$ for all t and ω and $\theta(\omega) = \omega$, or, equivalently, $\bar{A}_t(\lambda) = 1$ for all t and λ. In such a case, the evolutionary spectrum is coincident with $\mathrm{d}\bar{\mu}(\lambda)$ and WSS processes are expressed as the superposition of complex sinewaves with orthogonal time-invariant random amplitudes (Cramér 1940).

For generalizations and applications, see (Matz *et al.* 1997), (Hopgood and Rayner 2003).

1.1.6 Discrete-Time Processes

The characterization of discrete-time nonstationary stochastic processes can be made similarly to that of continuous-time processes with the obvious modifications. The harmonizability condition for the discrete-time process $x_{\mathrm{d}}(n)$ is

$$\mathrm{E}\left\{x_{\mathrm{d}}(n_1)\, x_{\mathrm{d}}^{(*)}(n_2)\right\} = \int_{[-1/2,1/2]^2} e^{j2\pi[\nu_1 n_1 + (-)\nu_2 n_2]}\, \mathrm{d}\widetilde{\gamma}_{\boldsymbol{x}_{\mathrm{d}}}(\nu_1, \nu_2) \tag{1.36}$$

with $\widetilde{\gamma}_{\boldsymbol{x}_{\mathrm{d}}}(\nu_1, \nu_2)$ spectral correlation function of bounded variation when $(\nu_1, \nu_2) \in [-1/2, 1/2]^2$. Under the harmonizability condition, $x_{\mathrm{d}}(n)$ can be expressed as

$$x_{\mathrm{d}}(n) = \int_{[-1/2,1/2]} e^{j2\pi\nu n}\, \mathrm{d}\chi_{\mathrm{d}}(\nu) \tag{1.37}$$

where we can formally write $\mathrm{d}\chi_{\mathrm{d}}(\nu) = X_{\mathrm{d}}(\nu)\,\mathrm{d}\nu$ (Gardner 1985, Chapter 10.1.2) with

$$X_{\mathrm{d}}(\nu) = \sum_{n\in\mathbb{Z}} x_{\mathrm{d}}(n)\, e^{-j2\pi\nu n} \tag{1.38}$$

Fourier transform of $x_{\mathrm{d}}(n)$ to be intended in the sense of distributions (Gelfand and Vilenkin 1964, Chapter 3; Henniger 1970).

The possible presence of Dirac deltas on the edges of the integration domain in (1.36) or for $\nu = \pm 1/2$ in (1.38) must be managed, accounting for the periodicity with period 1 w.r.t. variables ν_1 and ν_2 in (1.36) and w.r.t. variable ν in (1.38). If one delta term is considered, then its replica must be neglected.

The Loève bifrequency spectrum of $x_{\mathrm{d}}(n)$ is defined as (Loève 1963)

$$\widetilde{\mathcal{S}}_{\boldsymbol{x}_{\mathrm{d}}}(\nu_1, \nu_2) \triangleq \mathrm{E}\left\{X_{\mathrm{d}}(\nu_1)\, X_{\mathrm{d}}^{(*)}(\nu_2)\right\}. \tag{1.39}$$

Let $x_{\mathrm{d}}(n) \triangleq x(t)|_{t=nT_s}$ be the discrete-time process obtained by uniformly sampling with period $T_s = 1/f_s$ the continuous-time process $x(t)$. The (conjugate) autocorrelation function of $x_{\mathrm{d}}(n)$ turns out to be the sampled version of that of $x(t)$ at sampling instants $t_1 = n_1 T_s$ and $t_2 = n_2 T_s$. The Loève bifrequency spectrum of $x_{\mathrm{d}}(n)$ can be expressed as

$$\widetilde{\mathcal{S}}_{\boldsymbol{x}_{\mathrm{d}}}(\nu_1, \nu_2) = f_s^2 \sum_{p_1\in\mathbb{Z}} \sum_{p_2\in\mathbb{Z}} \mathcal{S}_{\boldsymbol{x}}((\nu_1 - p_1) f_s, (\nu_2 - p_2) f_s). \tag{1.40}$$

Uniform sampling is a linear periodically time-variant transformation of a continuous-time process into a discrete-time process. Since the transformation is time-variant, in general the nonstationary behavior of the discrete-time process can be different from that of the continuous-time one.

1.1.7 Linear Time-Variant Transformations

In this section, linear time-variant (LTV) transformations of stochastic processes are considered. Input/output relations are derived in both time and frequency domains with reference to processes and their second-order moments.

The input/output relationship of a LTV system is given by

$$y(t) = \int_{\mathbb{R}} h(t, u)x(u)\,\mathrm{d}u \tag{1.41}$$

where $h(t, u)$ is the system *impulse-response function*. That is,

$$x(u) = \delta(u - u_0) \quad \Rightarrow \quad y(t) = h(t, u_0) \tag{1.42}$$

where $\delta(\cdot)$ denotes Dirac delta. By Fourier transforming both sides of (1.41), one obtains the input/output relationship in the frequency domain

$$\begin{aligned} Y(f) &\triangleq \int_{\mathbb{R}} y(t)e^{-j2\pi ft}\,\mathrm{d}t \\ &= \int_{\mathbb{R}} H(f, \lambda)X(\lambda)\,\mathrm{d}\lambda \end{aligned} \tag{1.43}$$

where the *transmission function* $H(f, \lambda)$ (Claasen and Mecklenbräuker 1982) is the double Fourier transform of the impulse-response function:

$$H(f, \lambda) \triangleq \int_{\mathbb{R}^2} h(t, u)e^{-j2\pi(ft-\lambda u)}\,\mathrm{d}t\ \mathrm{d}u. \tag{1.44}$$

In (1.43), (1.44), and the following, Fourier transforms are assumed to exist at least in the sense of distributions (generalized functions) (Zemanian 1987).

1.1.7.1 Input/Output Relations in the Time Domain

Let us consider two LTV systems with impulse-response functions $h_1(t, u)$ and $h_2(t, u)$, excited by $x_1(t)$ and $x_2(t)$, respectively (Figure 1.1). The output signals are

$$y_i(t) = \int_{\mathbb{R}} h_i(t, u)\, x_i(u)\,\mathrm{d}u \qquad i = 1, 2. \tag{1.45}$$

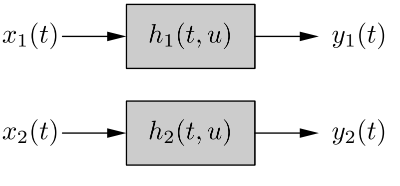

Figure 1.1 LTV systems – time domain

The (conjugate) cross-correlation of the outputs $y_1(t)$ and $y_2(t)$ can be expressed in terms of that of the input signals $x_1(t)$ and $x_2(t)$:

$$\begin{aligned}
&\mathrm{E}\left\{y_1(t+\tau_1)\, y_2^{(*)}(t+\tau_2)\right\} \\
&= \mathrm{E}\left\{\int_{\mathbb{R}} h_1(t+\tau_1, u_1)\, x_1(u_1)\, \mathrm{d}u_1 \int_{\mathbb{R}} h_2^{(*)}(t+\tau_2, u_2)\, x_2^{(*)}(u_2)\, \mathrm{d}u_2\right\} \\
&= \mathrm{E}\left\{\int_{\mathbb{R}} h_1(t+\tau_1, t+s_1)\, x_1(t+s_1)\, \mathrm{d}s_1 \int_{\mathbb{R}} h_2^{(*)}(t+\tau_2, t+s_2)\, x_2^{(*)}(t+s_2)\, \mathrm{d}s_2\right\} \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} h_1(t+\tau_1, t+s_1)\, h_2^{(*)}(t+\tau_2, t+s_2) \\
&\qquad \mathrm{E}\left\{x_1(t+s_1)\ x_2^{(*)}(t+s_2)\right\}\, \mathrm{d}s_1\, \mathrm{d}s_2 \qquad (1.46)
\end{aligned}$$

where, in the second equality, the variable changes $u_i = t + s_i$, $i = 1, 2$ are made in order to allow, in the third equality, the interchange of integral and expectation operators also when the proof is made in the functional approach (see Chapter 6).

A sufficient condition to allow the interchange of integral and expectation operators is given by the Fubini and Tonelli theorem (Champeney 1990, Chapter 3)

$$\begin{aligned}
&\int_{\mathbb{R}}\int_{\mathbb{R}} \left|h_1(t+\tau_1, t+s_1)\, h_2^{(*)}(t+\tau_2, t+s_2)\right| \\
&\qquad \mathrm{E}\left\{\left|x_1(t+s_1)\ x_2^{(*)}(t+s_2)\right|\right\}\, \mathrm{d}s_1\, \mathrm{d}s_2 < \infty. \qquad (1.47)
\end{aligned}$$

1.1.7.2 Input/Output Relations in the Frequency Domain

The frequency-domain counterpart of the input/output relation (1.45) is (Figure 1.2)

$$Y_i(f) = \int_{\mathbb{R}} H_i(f, \lambda)\, X_i(\lambda)\, \mathrm{d}\lambda \qquad i = 1, 2. \qquad (1.48)$$

where $H_i(f, \lambda)$, $i = 1, 2$, are the transmission functions defined according to (1.44).

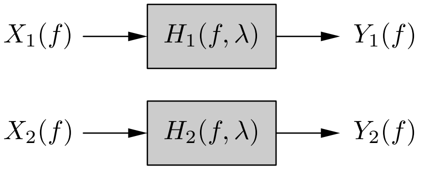

Figure 1.2 LTV systems – frequency domain

The Loève bifrequency cross-spectrum of the outputs $y_1(t)$ and $y_2(t)$ can be expressed in terms of that of the input signals $x_1(t)$ and $x_2(t)$ as follows

$$\begin{aligned}
&\mathrm{E}\left\{Y_1(f_1)\, Y_2^{(*)}(f_2)\right\} \\
&\quad = \mathrm{E}\left\{\int_{\mathbb{R}} H_1(f_1,\lambda_1)\, X_1(\lambda_1)\,\mathrm{d}\lambda_1 \int_{\mathbb{R}} H_2^{(*)}(f_2,\lambda_2)\, X_2^{(*)}(\lambda_2)\,\mathrm{d}\lambda_2\right\} \\
&\quad = \int_{\mathbb{R}}\int_{\mathbb{R}} H_1(f_1,\lambda_1)\, H_2^{(*)}(f_2,\lambda_2)\,\mathrm{E}\left\{X_1(\lambda_1)\, X_2^{(*)}(\lambda_2)\right\}\,\mathrm{d}\lambda_1\,\mathrm{d}\lambda_2
\end{aligned} \tag{1.49}$$

provided that (Fubini and Tonelli theorem (Champeney 1990, Chapter 3))

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \left|H_1(f_1,\lambda_1)\, H_2^{(*)}(f_2,\lambda_2)\right|\,\mathrm{E}\left\{\left|X_1(\lambda_1)\, X_2^{(*)}(\lambda_2)\right|\right\}\,\mathrm{d}\lambda_1\,\mathrm{d}\lambda_2 < \infty. \tag{1.50}$$

1.2 Almost-Periodic Functions

In this section, definitions and main results on almost-periodic (AP) functions and their generalizations are presented for both continuous- and discrete-time cases. For extensive treatments on almost-periodic functions, see (Besicovitch 1932), (Bohr 1933), and (Corduneanu 1989) for continuous-time, and (Corduneanu 1989, Chapter VII), (Jessen and Tornehave 1945), and (von Neumann 1934) for discrete-time.

1.2.1 Uniformly Almost-Periodic Functions

Definition 1.2.1 (Besicovitch 1932, Chapter 1). *A function $z(t)$, $t \in \mathbb{R}$, is said to be* uniformly almost-periodic *if $\forall \epsilon > 0\ \exists\, \ell_\epsilon > 0$ such that for any interval $I_\epsilon = (t_0, t_0 + \ell_\epsilon)\ \exists\, \tau_\epsilon \in I_\epsilon$ such that*

$$\sup_{t\in\mathbb{R}} |z(t+\tau_\epsilon) - z(t)| < \epsilon. \tag{1.51}$$

The quantity τ_ϵ is said translation number of $z(t)$ corresponding to ϵ. □

A set $D \subseteq \mathbb{R}$ is said to be *relatively dense* in $\mathbb{R}$ if $\exists\, \ell > 0$ such that $\forall\, I = (t_0, t_0 + \ell)$ the result is that $D \cap I \neq \emptyset$.

Thus, defined the set of the translation numbers of $z(t)$ corresponding to ϵ

$$E(z,\epsilon) \triangleq \left\{\tau \in \mathbb{R} \,:\, \sup_{t\in\mathbb{R}} |z(t+\tau) - z(t)| < \epsilon\right\} \tag{1.52}$$

according to Definition 1.2.1, the function $z(t)$ is uniformly almost periodic if and only if $\forall \epsilon > 0$ the set $E(z,\epsilon)$ is relatively dense in $\mathbb{R}$. That is, there are many translation numbers of $z(t)$ corresponding to ϵ.

Theorem 1.2.2 (Besicovitch 1932, Chapter 1). *Any uniformly AP function is the limit of a uniformly convergent sequence of trigonometric polynomials in t (generalized Fourier series):*

$$z(t) = \sum_{\alpha \in A} z_\alpha \, e^{j2\pi\alpha t} \tag{1.53}$$

where the frequencies $\alpha \in A$, with A countable set of possibly incommensurate reals and possibly containing cluster points, and

$$z_\alpha \triangleq \lim_{T\to\infty} \frac{1}{T} \int_{t_0-T/2}^{t_0+T/2} z(t)\, e^{-j2\pi\alpha t}\, \mathrm{d}t \tag{1.54}$$

with the limit independent of t_0. Thus, $z(t)$ is bounded and uniformly continuous. □

Theorem 1.2.3 (Besicovitch 1932, Chapter 1). *For any uniformly AP function the following Parseval's equality holds*

$$\lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |z(t)|^2\, \mathrm{d}t = \sum_{\alpha\in A} |z_\alpha|^2. \tag{1.55}$$

□

A function $z(t)$ is *periodic* with period $T_0 > 0$ if T_0 is the smallest nonzero value such that

$$z(t) = z(t + T_0) \qquad \forall t \in \mathbb{R}. \tag{1.56}$$

Thus, periodic functions are obtained as special case of almost-periodic functions with $\tau_\epsilon = kT_0$ independent of ϵ, $k \in \mathbb{Z}$. In such a case, the frequencies of the set A are all multiple of a fundamental frequency $1/T_0$, that is, $A = \{k/T_0\}_{k\in\mathbb{Z}}$ and (1.53) is the ordinary Fourier series expansion of the periodic function $z(t)$.

An example of uniformly AP function which is not periodic is

$$z(t) = \cos(2\pi t/T_0) + \cos(2\pi t/(\sqrt{2}T_0)). \tag{1.57}$$

Both cosines are periodic functions with periods T_0 and $\sqrt{2}T_0$, respectively. However, their sum is not periodic since the ratio of the two periods T_0 and $\sqrt{2}T_0$ is the irrational number $1/\sqrt{2}$.

The functions defined in Definition 1.2.1 and characterized in Theorem 1.2.2 are called almost-periodic in the sense of Bohr (Bohr 1933, paragraphs 84–92) or, equivalently, *uniformly almost periodic* in the sense of Besicovitch (Besicovitch 1932, Chapter 1), or, equivalently, *almost-periodic with respect to the sup norm*. More general classes of almost-periodic functions, including possibly discontinuous functions, are treated in (Besicovitch 1932, Chapter 2) and the following sections.

1.2.2 AP Functions in the Sense of Stepanov, Weyl, and Besicovitch

The almost-periodicity property can be defined with respect to the following norms or seminorms, with $p \geqslant 1$, (Besicovitch 1932, Chapter 2):

1. *Stepanov S_T^p-norm:*

$$\|z\|_{S_T^p} \triangleq \left[\sup_{a\in\mathbb{R}} \frac{1}{T}\int_{a-T/2}^{a+T/2} |z(t)|^p \,\mathrm{d}t\right]^{1/p} \tag{1.58}$$

2. *Weyl W^p-norm:*

$$\|z\|_{W^p} \triangleq \lim_{T\to\infty} \|z\|_{S_T^p} = \left[\lim_{T\to\infty} \sup_{a\in\mathbb{R}} \frac{1}{T}\int_{a-T/2}^{a+T/2} |z(t)|^p \,\mathrm{d}t\right]^{1/p} \tag{1.59}$$

3. *Besicovitch B^p-seminorm:*

$$\|z\|_{B^p} \triangleq \left[\limsup_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} |z(t)|^p \,\mathrm{d}t\right]^{1/p}. \tag{1.60}$$

Functions belonging to $L^p_{\text{loc}}(\mathbb{R})$ with finite Besicovitch B^p-seminorm form a seminormed space called Marcinkiewicz space $\mathcal{M}^p$.

Let $\|\cdot\|_{G^p}$ denote any of the above defined (semi)norms, that is, $\|\cdot\|_{S_T^p}$, $\|\cdot\|_{W^p}$, or $\|\cdot\|_{B^p}$. For each (semi)norm, a definition of almost-periodicity can be given.

Definition 1.2.4 S_T^p, W^p, and B^p Almost Periodicity (Besicovitch 1932, Chapter 2). *A function $z(t)$ is said to be G^p* almost-periodic (G^p-AP)*, $p \geqslant 1$, if $\forall\epsilon > 0\ \exists\ \ell_\epsilon > 0$ such that for any interval $I_\epsilon = (t_0, t_0 + \ell_\epsilon)\ \exists\ \tau_\epsilon \in I_\epsilon$ such that*

$$\|z(t+\tau_\epsilon) - z(t)\|_{G^p} < \epsilon. \tag{1.61}$$

Specifically, if $G^p = S_T^p$, then $z(t)$ is said S_T^p-AP; If $G^p = W^p$, then $z(t)$ is said W^p-AP; If $G^p = B^p$, then $z(t)$ is said B^p-AP. □

Theorem 1.2.5 (Besicovitch 1932, Chapter 2). *Any G^p-AP function is G^p-bounded ($\|z\|_{G^p} < \infty$) and is the G^p-limit of a sequence of trigonometric polynomials in t:*

$$\lim_n \left\| z(t) - \sum_{\alpha\in A_n} z_\alpha\, e^{j2\pi\alpha t} \right\|_{G^p} = 0 \tag{1.62}$$

where A_n is an increasing sequence of countable sets such that $m < n \Rightarrow A_m \subseteq A_n$ and $\lim_n A_n \triangleq \cup_{n\in\mathbb{N}} A_n = A$. The coefficients z_α of the generalized Fourier series are given by (1.54). □

In (Besicovitch 1932, p. 74) it is shown that: If $\|z_1(t) - z_2(t)\|_{S_T^p} = 0$, then $z_1(t) = z_2(t)$ a.e.; If $\|z_1(t) - z_2(t)\|_{W^p} = 0$ or $\|z_1(t) - z_2(t)\|_{B^p} = 0$, then $z_1(t)$ and $z_2(t)$ may differ at a set of points of finite and even of infinite measure. In addition, for $p \geqslant 1$ the result is that (Besicovitch 1932, p. 73) $\sup_{t\in\mathbb{R}} |z(t)| \geqslant \|z(t)\|_{S_T^p} \geqslant \|z(t)\|_{W^p} \geqslant \|z(t)\|_{B^p}$.

Theorem 1.2.6 (Besicovitch 1932, Chapter 2). *For any G^2-AP function the following Parseval's equality holds*

$$\|z(t)\|_{G^2}^2 = \sum_{\alpha \in A} |z_\alpha|^2. \tag{1.63}$$

□

Further generalizations of almost-periodic functions can be found in (Besicovitch 1932, Chapter 2), (Bohr 1933, paragraphs 94–102), and (Corduneanu 1989, Chapter VI).

1.2.3 Weakly AP Functions in the Sense of Eberlein

Definition 1.2.7 Weakly Almost-Periodic Functions (Eberlein 1949, 1956). *A continuous and bounded function $z(t)$ is said to be* weakly almost-periodic (w.a.p.) *(in the sense of Eberlein) if the set of translates $z(t+\tau)$, $\tau \in \mathbb{R}$, is (conditionally) weakly compact in the set of continuous and bounded functions $C^0(\mathbb{R}) \cap L^\infty(\mathbb{R})$.* □

Examples of w.a.p. functions are the uniformly almost periodic functions (in the sense of Definition 1.2.1), the positive definite functions (hence Fourier-Stieltjes transforms), and functions vanishing at infinity (Eberlein 1949, Theorems 11.1 and 11.2). A w.a.p. function is uniformly continuous (Eberlein 1949, Theorem 13.1).

Theorem 1.2.8 (Eberlein 1956). *Every w.a.p. function $z(t)$ admits a unique decomposition*

$$z(t) = z_{\text{uap}}(t) + z_0(t) \tag{1.64}$$

where $z_{\text{uap}}(t)$ is a uniformly almost-periodic function in the sense of Definition 1.2.1 and $z_0(t)$ is a zero-power function

$$\lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |z_0(t)|^2 \, \mathrm{d}t = 0. \tag{1.65}$$

Moreover, it results that

$$z_\alpha = \lim_{T\to\infty} \frac{1}{T} \int_{t_0-T/2}^{t_0+T/2} z_{\text{uap}}(t)\, e^{-j2\pi\alpha t} \, \mathrm{d}t \tag{1.66a}$$

$$= \lim_{T\to\infty} \frac{1}{T} \int_{t_0-T/2}^{t_0+T/2} z(t)\, e^{-j2\pi\alpha t} \, \mathrm{d}t \tag{1.66b}$$

and, accordingly with (1.65) and using the notation of Theorem 1.2.5, the result is that

$$\lim_{n} \lim_{T\to\infty} \frac{1}{T} \int_{t_0-T/2}^{t_0+T/2} \left| z(t) - \sum_{\alpha\in A_n} z_\alpha\, e^{j2\pi\alpha t} \right|^2 \mathrm{d}t = 0. \tag{1.67}$$

□

Theorem 1.2.9 (Eberlein 1956). *For any w.a.p. function the following Parseval's equality holds*

$$\lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |z(t)|^2 \,\mathrm{d}t = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |z_{\mathrm{uap}}(t)|^2 \,\mathrm{d}t = \sum_{\alpha\in A} |z_\alpha|^2. \tag{1.68}$$

□

It is worthwhile emphasizing that the set of w.a.p. functions, unlike other classes of generalized AP functions, is closed under multiplication (Eberlein 1949). That is, the product of two w.a.p. functions is in turn a w.a.p. function.

Other definitions of w.a.p. functions different from Definition 1.2.7 are given in (Amerio and Prouse 1971, Chapter 3), (Corduneanu 1989, Section VI.5), (Zhang and Liu 2010).

Theorem 1.2.10 (Eberlein 1949, Theorem 15.1). *Let $z_1(t)$ and $z_2(t)$ be w.a.p. functions. Then*

$$z(t) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} z_1(t+s)\, z_2(s) \,\mathrm{d}s \tag{1.69}$$

exists and is a uniformly almost periodic function of t. A similar result also holds with different definitions of w.a.p. functions (Zhang and Liu 2010). □

1.2.4 Pseudo AP Functions

Definition 1.2.11 (Ait Dads and Arino 1996). *The function $z(t)$ is said to be* pseudo almost-periodic in the sense of Ait Dads and Arino*, shortly $z(t) \in \widetilde{\mathcal{PAP}}(\mathbb{R})$, if it admits the (unique) decomposition*

$$z(t) = z_{\mathrm{uap}}(t) + z_0(t) \tag{1.70}$$

where $z_{\mathrm{uap}}(t)$ is a uniformly almost-periodic function in the sense of Definition 1.2.1 and $z_0(t)$, referred to as the ergodic perturbation, is a Lebesgue measurable function such that

$$\lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |z_0(t)| \,\mathrm{d}t = 0 \tag{1.71}$$

shortly $z_0(t) \in \widetilde{\mathcal{PAP}}_0(\mathbb{R})$. □

The classes $\widetilde{\mathcal{PAP}}(\mathbb{R})$ and $\widetilde{\mathcal{PAP}}_0(\mathbb{R})$ are slight generalizations of the classes $\mathcal{PAP}(\mathbb{R})$ and $\mathcal{PAP}_0(\mathbb{R})$, respectively, of the pseudo almost-periodic functions in the sense of Zhang (Zhang 1994, 1995), where $z(t)$ and $z_0(t)$ are assumed to be continuous and bounded.

Theorem 1.2.12 (Ait Dads and Arino 1996). *Let be* $z(t) \in \widetilde{\mathcal{PAP}}(\mathbb{R})$. *It results that*

$$z_\alpha = \lim_{T\to\infty} \frac{1}{T} \int_{t_0-T/2}^{t_0+T/2} z_{\text{uap}}(t)\, e^{-j2\pi\alpha t}\, \mathrm{d}t \tag{1.72a}$$

$$= \lim_{T\to\infty} \frac{1}{T} \int_{t_0-T/2}^{t_0+T/2} z(t)\, e^{-j2\pi\alpha t}\, \mathrm{d}t \tag{1.72b}$$

□

Proposition 1.2.13 (Ait Dads and Arino 1996). *If* $\lim_{|t|\to\infty} z_0(t)$ *exists, then* $\lim_{|t|\to\infty} z_0(t) = 0$ *and* $z(t)$ *belong to the class of the* asymptotically almost-periodic functions *in the sense of Frechet.*

Properties of asymptotically almost-periodic functions are given in (Leśkow and Napolitano 2006, Section 6.2).

1.2.5 AP Functions in the Sense of Hartman and Ryll-Nardzewski

Definition 1.2.14 (Kahane 1962), (Andreas *et al.* 2006, Definition 7.1). *The function* $z(t) \in L^1_{\text{loc}}(\mathbb{R})$ *is said to be* almost periodic in the sense of Hartman*, shortly* $z(t) \in H^1_{\text{ap}}$*, if,* $\forall\alpha \in \mathbb{R}$

$$z_\alpha \triangleq \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} z(t)\, e^{-j2\pi\alpha t}\, \mathrm{d}t \tag{1.73}$$

exists and is finite. □

Definition 1.2.15 (Kahane 1962), (Andreas *et al.* 2006, Definition 7.2). *The function* $z(t) \in L^1_{\text{loc}}(\mathbb{R})$ *is said to be* almost periodic in the sense of Ryll-Nardzewski*, shortly* $z(t) \in R^1_{\text{ap}}$*, if,* $\forall\alpha \in \mathbb{R}$

$$z'_\alpha \triangleq \lim_{T\to\infty} \frac{1}{T} \int_{t_0}^{t_0+T} z(t)\, e^{-j2\pi\alpha t}\, \mathrm{d}t \tag{1.74}$$

exists uniformly with respect to $t_0 \in \mathbb{R}$ *and is finite.* □

Obviously, if $z(t) \in R^1_{\text{ap}}$, then $z(t) \in H^1_{\text{ap}}$ and $z'_\alpha = z_\alpha$ $\forall\alpha \in \mathbb{R}$, but the converse is not true. That is, $R^1_{\text{ap}} \subset H^1_{\text{ap}}$.

If $z(t) \in S^p_T(\mathbb{R})$ or $z(t) \in W^p(\mathbb{R})$, then $z(t) \in R^1_{\text{ap}}$ and the Fourier coefficients of the (generalized) Fourier series in (1.62) are coincident with those in (1.74). If $z(t) \in B^p(\mathbb{R})$, then $z(t) \in H^1_{\text{ap}}$ (but not necessarily $z(t) \in R^1_{\text{ap}}$) and the Fourier coefficients of the (generalized) Fourier series in (1.62) are (obviously) those in (1.73).

Theorem 1.2.16 (Kahane 1961), (Andreas *et al.* 2006, Theorem 7.5). *The* spectrum *of* $z(t) \in H^1_{\text{ap}}$*, that is the set* $\{\alpha \in \mathbb{R} : z_\alpha \neq 0\}$*, is at most countable. Consequently, also the spectrum of* $z(t) \in R^1_{\text{ap}}$ *is at most countable.* □

Theorem 1.2.17 (Urbanik 1962), (Kahane 1961). *Let $z(t) \in H^1_{\mathrm{ap}} \cap \mathcal{M}^p$, $p > 1$. Then, the following unique decomposition holds*

$$z(t) = z_{\mathrm{Bap}}(t) + z_0(t) \tag{1.75}$$

where $z_{\mathrm{Bap}}(t)$ is a B^p-AP function and $z_0(t) \in H^1_{\mathrm{ap}}$ with empty spectrum, that is $\forall \alpha \in \mathbb{R}$

$$\lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} z_0(t)\, e^{-j2\pi\alpha t}\, \mathrm{d}t = 0. \tag{1.76}$$

□

In particular, since uniformly AP functions in the sense of Definition 1.2.1 are special cases of B^p-AP functions, the function $z_{\mathrm{Bap}}(t)$ in decomposition (1.75) can reduce to a uniformly AP function.

Let us define the sets

$$H_0 \triangleq \{z(t) \in H^1_{\mathrm{ap}} \ : \ z(t) \text{ has empty spectrum}\} \tag{1.77}$$

$$R_0 \triangleq \{z(t) \in R^1_{\mathrm{ap}} \ : \ z(t) \text{ has empty spectrum}\}. \tag{1.78}$$

Obviously $R_0 \subset H_0$.

Theorem 1.2.18 (Kahane 1962). *Let be $x(t) \in L^p_{\mathrm{loc}}(\mathbb{R})$. Then, there exists $z(t) \in R_0 \cap L^p_{\mathrm{loc}}(\mathbb{R})$ such that $|z(t)| = |x(t)|$. Let be $x(t) \in C^0(\mathbb{R})$. Then, there exists $z(t) \in R_0 \cap C^0(\mathbb{R})$ such that $|z(t)| = |x(t)|$.* □

Theorem 1.2.19 (Kahane 1962). *Let be $x(t) \in C^0(\mathbb{R})$. Then, there exist $z_1, z_2 \in R_0$ such that $x(t) = z_1(t)\, z_2(t)$.* □

Theorem 1.2.20 (Kahane 1962). *Let $x(t)$ be uniformly continuous and bounded. Then, there exist $z_1, z_2 \in R_0$ uniformly continuous and bounded such that $x(t) = z_1(t)\, z_2(t)$. In particular, $x(t)$ can be uniformly almost periodic.* □

1.2.6 AP Functions Defined on Groups and with Values in Banach and Hilbert Spaces

Almost-periodic functions and their generalizations on groups are treated in (Corduneanu 1989, Chapter VII), (von Neumann 1934), (Casinovi 2009).

Almost-periodic functions with values in Hilbert spaces are treated in (Phong 2007). Further classes of AP functions with values in Banach spaces and a survey of their properties are presented in (Andreas *et al.* 2006), (Chérif 2011a,b).

1.2.7 AP Functions in Probability

Let $\{z(t, \omega), t \in \mathbb{R}, \omega \in \Omega\}$, denoted shortly by $z(t)$, be a random process defined on a probability space $(\Omega, \mathcal{F}, P)$.

Definition 1.2.21 Random Functions Almost-Periodic in Probability (Corduneanu 1989, Sect. II.3). *A random process $z(t)$, $t \in \mathbb{R}$, is called* almost-periodic in probability *if* $\forall \epsilon > 0$, $\eta > 0$, *there exists* $\ell_{\epsilon,\eta} > 0$ *such that for every set of length* $\ell_{\epsilon,\eta}$, *say* $I_{\epsilon,\eta} = (t_0, t_0 + \ell_{\epsilon,\eta})$, *there exists at least one number* $\tau_{\epsilon,\eta} \in I_{\epsilon,\eta}$ *such that*

$$\sup_{t \in \mathbb{R}} P\left\{ \left| z(t + \tau_{\epsilon,\eta}, \omega) - z(t, \omega) \right| \geqslant \eta \right\} < \epsilon. \tag{1.79}$$

The real number $\tau_{\epsilon,\eta}$ is said (ϵ, η)-almost period in probability. □

Theorem 1.2.22 (Corduneanu 1989, Sect. II.3). *Any random process AP in probability is bounded in probability and is the limit in probability of a sequence of random trigonometric polynomials in t. That is,* $\forall \eta > 0$,

$$\lim_{n} \sup_{t \in \mathbb{R}} P\left\{ \left| z(t, \omega) - \sum_{\alpha \in A_n} z_\alpha(\omega)\, e^{j2\pi\alpha t} \right| \geqslant \eta \right\} = 0 \tag{1.80}$$

where A_n is an increasing sequence of countable sets of real numbers α and $z_\alpha(\omega)$ are random variables. □

1.2.8 AP Sequences

Definition 1.2.23 Discrete-Time Almost-Periodic Functions (Corduneanu 1989, Chapter VII), (Jessen and Tornehave 1945), (von Neumann 1934). *A sequence $z(n)$, $n \in \mathbb{Z}$, is said to be* almost-periodic *if* $\forall \epsilon > 0\ \exists\ \ell_\epsilon \in \mathbb{N}$ *such that for any set* $I_\epsilon \triangleq \{n_0, n_0 + 1, \ldots, n_0 + \ell_\epsilon\}$ $\exists\ m_\epsilon \in I_\epsilon$ *such that*

$$\sup_{n \in \mathbb{Z}} |z(n + m_\epsilon) - z(n)| < \epsilon \tag{1.81}$$

The integer m_ϵ is said to be the translation number of $z(n)$ corresponding to ϵ. □

Theorem 1.2.24 (Corduneanu 1989, Chapter VII), (von Neumann 1934). *Every AP sequence $\{z(n)\}_{n \in \mathbb{Z}}$ is the limit of a sequence of trigonometric polynomials in n:*

$$z(n) = \sum_{\widetilde{\alpha} \in \widetilde{A}} z_{\widetilde{\alpha}}\, e^{j2\pi\widetilde{\alpha} n} \tag{1.82}$$

where the frequencies $\widetilde{\alpha} \in \widetilde{A}$, with $\widetilde{A}$ countable set with possibly incommensurate elements in $[-1/2, 1/2)$ *and possibly containing cluster points,*

$$z_{\widetilde{\alpha}} \triangleq \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=n_0-N}^{n_0+N} z(n)\, e^{-j2\pi\widetilde{\alpha} n} \tag{1.83}$$

with the limit independent of n_0 and

$$\sum_{\widetilde{\alpha} \in \widetilde{A}} |z_{\widetilde{\alpha}}| < \infty. \tag{1.84}$$

Thus, $z(n)$ is bounded. □

A sequence $z(n)$ is *periodic* with period N_0 if N_0 is the smallest non-zero integer such that

$$z(n) = z(n + N_0) \qquad \forall n \in \mathbb{Z}. \tag{1.85}$$

Thus, periodic sequences are obtained as a special case of almost-periodic sequences with $m_\epsilon = kN_0$ independent of ϵ, $k \in \mathbb{Z}$. In such a case, $\widetilde{A}$ is the finite set $\{0, 1/N_0, \ldots, (N_0 - 1)/N_0\}$ or any equivalent set $k_0 + A \triangleq \{k_0, k_0 + 1/N_0, \ldots, k_0 + (N_0 - 1)/N_0\}$ with k_0 integer, and (1.82) is the discrete Fourier series (DFS) of $z(n)$.

In continuous-time, the complex sinewave $z(t) = e^{j2\pi f_0 t}$ is periodic with period $T_0 = 1/f_0$ and the polynomial phase signal $z(t) = e^{j2\pi f_0 t^\gamma}$, $\gamma > 1$, is not almost periodic. Complex discrete-time sinewaves and polynomial phase sequences require more attention. The sequence $z(n) = e^{j2\pi\nu_0 n}$ with $\nu_0 = p/q \in \mathbb{Q}$, p, q relative prime integers or co-prime (that is, they have no common positive divisor other than 1 or, equivalently, their greatest common divisor is 1 or, equivalently, p/q is an irreducible fraction), is periodic with period q. In contrast, the sequence $z(n) = e^{j2\pi\nu_0 n}$ with $\nu_0 \notin \mathbb{Q}$, is almost periodic (not periodic). For every positive integer $L \geqslant 2$, the sequence $z(n) = e^{j2\pi\nu_0 n^L}$ with $\nu_0 = p/q \in \mathbb{Q}$, p, q relative prime integers, is periodic with period q whereas it is not almost periodic for $\nu_0 \notin \mathbb{Q}$.

1.2.9 AP Sequences in Probability

Definition 1.2.25 Almost-Periodic Random Sequences in Probability (Han and Hong 2007). *A random sequence $z(n)$, $n \in \mathbb{Z}$, is called* almost-periodic in probability *if $\forall \epsilon > 0$, $\eta > 0$, there exists $\ell_{\epsilon,\eta} \in \mathbb{N}$ such that for every set of length $\ell_{\epsilon,\eta}$, say $I_{\epsilon,\eta} \triangleq \{n_0, n_0 + 1, \ldots, n_0 + \ell_{\epsilon,\eta}\}$, there exists at least one number $m_{\epsilon,\eta} \in I_{\epsilon,\eta}$ such that*

$$\sup_{n \in \mathbb{Z}} P\left\{ \left| z(n + m_{\epsilon,\eta}) - z(n) \right| \geqslant \eta \right\} < \epsilon. \tag{1.86}$$

The integer $m_{\epsilon,\eta}$ is said to be (ϵ, η)-almost period in probability. □

1.3 Almost-Cyclostationary Processes

Many processes encountered in telecommunications, radar, mechanics, radio astronomy, biology, atmospheric science, and econometrics, are generated by underlying periodic phenomena. These processes, even if not periodic, give rise to random data whose statistical functions vary periodically with time and are called cyclostationary processes. In this section, the properties of cyclostationary, or more generally, of almost-cyclostationary processes are briefly reviewed. For extensive treatments, see (Gardner 1985, Chapter 12, 1987d), (Gardner *et al.* 2006), (Giannakis 1998), (Hurd and Miamee 2007).

1.3.1 Second-Order Wide-Sense Statistical Characterization

The (finite-power) process $x(t)$ is said to be second-order *cyclostationary* in the wide sense or *periodically correlated* with period T_0 if its first- and second-order moments are periodic functions of time with period T_0. More generally, first- and second-order moments can be almost-periodic functions of time (in one of the senses considered in Section 1.2) and the process is said to be second-order *almost-cyclostationary* (ACS) in the wide sense or *almost-periodically correlated* (Gardner 1985, Chapter 12, 1987d). In such a case, its (conjugate) autocorrelation function (1.6) under mild regularity conditions can be expressed by the (generalized) Fourier series expansion

$$\mathcal{R}_{\boldsymbol{x}}(t,\tau) = \sum_{\alpha \in A} R_{\boldsymbol{x}}^{\alpha}(\tau)\, e^{j2\pi\alpha t} \tag{1.87}$$

where subscript $\boldsymbol{x} \triangleq [x x^{(*)}]$, A is the countable set (depending on $(*)$) of possibly incommensurate *cycle frequencies* α, and the Fourier coefficients

$$R_{\boldsymbol{x}}^{\alpha}(\tau) \triangleq \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} \mathcal{R}_{\boldsymbol{x}}(t,\tau)\, e^{-j2\pi\alpha t}\, \mathrm{d}t$$

referred to as the *(conjugate) cyclic autocorrelation functions*, are complex-valued functions whose magnitude and phase represent the amplitude and phase of the finite-strength additive sinewave component at frequency α contained in $\mathcal{R}_{\boldsymbol{x}}(t,\tau)$. Thus, $R_{\boldsymbol{x}}^{\alpha}(\tau) \not\equiv 0$ if $\alpha \in A$ and $R_{\boldsymbol{x}}^{\alpha}(\tau) \equiv 0$ if $\alpha \notin A$. Cyclostationary processes are obtained as a special case when $A = \{k/T_0\}_{k\in\mathbb{Z}}$. WSS processes are a further specialization when A contains the only element $\alpha = 0$.

The function (1.87) is the *time-lag* (conjugate) autocorrelation function of ACS processes. By substituting $t = t_2$ and $\tau = t_1 - t_2$ into (1.87), one obtains the *time-time* (conjugate) autocorrelation function of ACS processes

$$\mathrm{E}\{x(t_1)\, x^{(*)}(t_2)\} = \sum_{\alpha \in A} R_{\boldsymbol{x}}^{\alpha}(t_1 - t_2)\, e^{j2\pi\alpha t_2}. \tag{1.88}$$

For a zero-mean process $x(t)$ with finite or practically finite memory the result is that $|R_{\boldsymbol{x}}^{\alpha}(\tau)| \to 0$ as $|\tau| \to \infty$. In contrast, if the process has non-zero almost-periodic expectation $\mathrm{E}\{x(t)\}$, then some $R_{\boldsymbol{x}}^{\alpha}(\tau)$ contain additive sinusoidal functions of τ which arise from products of finite-strength sinusoidal terms contained in $\mathrm{E}\{x(t)\}$ (Gardner and Spooner 1994). ACS signals in communications are zero mean unless pilot tones are present for synchronization purposes. In mechanical applications, ACS signals generally have an almost-periodic expected value (Antoni 2009). For the following, let us assume that

$$\varrho(\tau) \triangleq \sum_{\alpha \in A} \left| R_{\boldsymbol{x}}^{\alpha}(\tau) \right| \in L^1(\mathbb{R}) \cap L^{\infty}(\mathbb{R}). \tag{1.89}$$

Such an assumption is verified by finite-memory or practically finite-memory signals. From (1.89) it follows that $\varrho(\tau) \in L^p(\mathbb{R})$ for $p \geqslant 1$ and $R_{\boldsymbol{x}}^{\alpha}(\tau) \in L^p(\mathbb{R})$ for $p \geqslant 1$.

By Fourier transforming with respect to τ both sides of (1.87) we obtain the almost-periodically time-variant spectrum

$$\mathcal{S}_{\boldsymbol{x}}(t, f) = \sum_{\alpha \in A} S_{\boldsymbol{x}}^{\alpha}(f)\, e^{j2\pi\alpha t} \tag{1.90}$$

where

$$S_{\boldsymbol{x}}^{\alpha}(f) = \int_{\mathbb{R}} R_{\boldsymbol{x}}^{\alpha}(\tau)\, e^{-j2\pi f\tau}\, \mathrm{d}\tau \tag{1.91}$$

is the *(conjugate) cyclic spectrum* which can be expressed as

$$S_{\boldsymbol{x}}^{\alpha}(f) = \lim_{\Delta f \to 0} \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} \Delta f\, \mathrm{E}\left\{ X_{1/\Delta f}(t, f)\, X_{1/\Delta f}^{(*)}(t, {}_{(-)}(\alpha - f)) \right\} \mathrm{d}t \tag{1.92}$$

where the order of the two limits cannot be reversed and

$$X_Z(t, f) \triangleq \int_{t-Z/2}^{t+Z/2} x(s)\, e^{-j2\pi f s}\, \mathrm{d}s \tag{1.93}$$

is the short-time Fourier transform (STFT) of $x(t)$. Therefore, the (conjugate) cyclic spectrum is also called the (conjugate) *spectral correlation function* and (1.91) is the *Gardner Relation* (Gardner 1985) (also called *Cyclic Wiener-Khinchin Relation*). Under assumption (1.89) the Fourier transform $S_{\boldsymbol{x}}^{\alpha}(f)$ exists in the ordinary sense. Accordingly with (1.90), the Wigner-Ville spectrum (1.20a) of an ACS signal is an almost-periodic function of t' with frequencies $\alpha \in A$ and Fourier-series coefficients $S_{\boldsymbol{x}}^{\alpha}(f + \alpha/2)$.

From (1.8), (1.14b), and (1.87), in the case of ACS processes, one obtains the following Loève bifrequency spectrum

$$\mathcal{S}_{\boldsymbol{x}}(f_1, f_2) = \sum_{\alpha \in A} S_{\boldsymbol{x}}^{\alpha}(f_1)\, \delta(f_2 + {}_{(-)}(f_1 - \alpha)). \tag{1.94}$$

That is, ACS processes have Loève bifrequency spectrum with spectral masses concentrated on a countable set of lines with slope ± 1. Equivalently, ACS processes have distinct spectral components that are correlated only if the spectral separation belongs to a countable set which is the set A of the cycle frequencies.

In Figures 1.3 and 1.4, the magnitude of the cyclic autocorrelation as a function of α and τ and the magnitude of the bifrequency spectral correlation density $\bar{\mathcal{S}}_{\boldsymbol{x}}(f_1, f_2)$ (obtained by replacing in (1.23) and (1.94) Dirac deltas with Kronecker ones) as a function of f_1 and f_2 are reported for a WSS and an ACS signal. The ACS signal is a cyclostationary PAM with a rectangular pulse $q(t)$. The considered WSS signal has the same autocorrelation function and power spectrum of the PAM signal. The WSS signal has a nonzero cyclic autocorrelation function only for $\alpha = 0$ (Figure 1.3 (top)). Equivalently, the bifrequency spectral correlation density in nonzero is only on the main diagonal of the bifrequency plane (Figure 1.4 (top)). In contrast, the ACS signal has a nonzero cyclic autocorrelation function in correspondence of the cycle frequencies which are multiples of a fundamental one (Figure 1.3 (bottom)). Equivalently, the bifrequency spectral correlation density is nonzero on lines which are parallel to the main diagonal so that spectral components which are correlated are separated by quantities equal to the cycle frequencies (Figure 1.4 (bottom)).

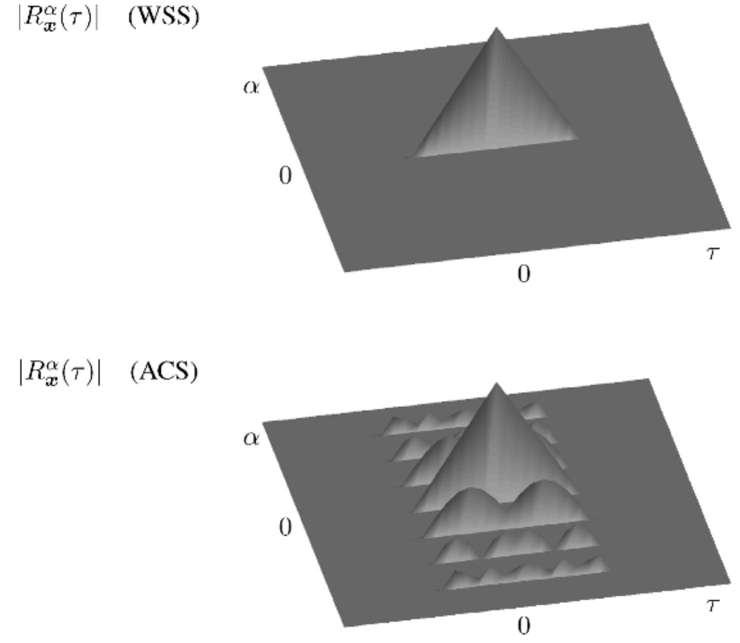

Figure 1.3 Magnitude of the cyclic autocorrelation function of a signal $x(t)$ as functions of α and τ. Top: WSS signal. Bottom: ACS signal

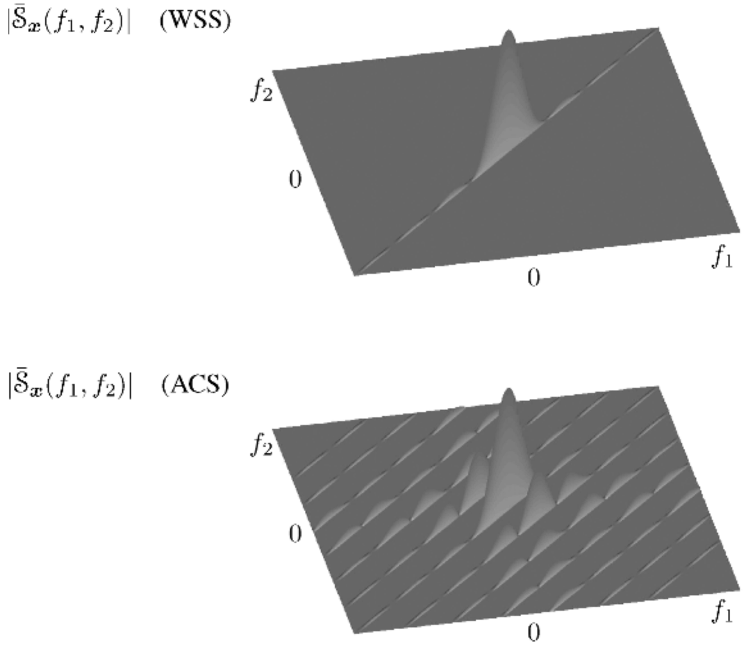

Figure 1.4 Magnitude of the bifrequency spectral correlation density of a signal $x(t)$ as functions of f_1 and f_2. Top: WSS signal. Bottom: ACS signal

1.3.2 Jointly ACS Signals

Let $x_i(t)$, $i = 1, 2$, $t \in \mathbb{R}$, be two complex-valued almost-cyclostationary (ACS) continuous-time processes with second-order (conjugate) cross-correlation function

$$R_{x_1 x_2^{(*)}}(t, \tau) \triangleq \mathrm{E}\left\{x_1(t+\tau)\, x_2^{(*)}(t)\right\} = \sum_{\alpha \in A_{12}} R^{\alpha}_{x_1 x_2^{(*)}}(\tau)\, e^{j2\pi\alpha t} \tag{1.95}$$

where

$$R^{\alpha}_{x_1 x_2^{(*)}}(\tau) \triangleq \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} \mathrm{E}\left\{x_1(t+\tau)\, x_2^{(*)}(t)\right\} e^{-j2\pi\alpha t}\, \mathrm{d}t \tag{1.96}$$

is the *(conjugate) cyclic cross-correlation function* of x_1 and x_2 at (conjugate) cycle frequency α and

$$A_{12} \triangleq \{\alpha \in \mathbb{R} \ : \ R^{\alpha}_{x_1 x_2^{(*)}}(\tau) \not\equiv 0\} \tag{1.97}$$

is a countable set. If the set A_{12} contains at least one nonzero element, then the time series $x_1(t)$ and $x_2(t)$ are said to be *jointly ACS*. Note that, in general, the set A_{12} depends on whether $(*)$ is a conjugation or not and can be different from the sets A_{11} and A_{22} (both defined according to (1.97)).

The *(conjugate) cyclic cross-spectrum* of x_1 and x_2 at (conjugate) cycle frequency α is defined as

$$S^{\alpha}_{x_1 x_2^{(*)}}(f) \triangleq \lim_{\Delta f\to 0} \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} \Delta f\, \mathrm{E}\left\{X_{1,1/\Delta f}(t, f)\, X^{(*)}_{2,1/\Delta f}(t, {}_{(-)}(\alpha - f))\right\} \mathrm{d}t \tag{1.98}$$

and is related to the (conjugate) cyclic cross-correlation function by the *Gardner relation*

$$S^{\alpha}_{x_1 x_2^{(*)}}(f) = \int_{\mathbb{R}} R^{\alpha}_{x_1 x_2^{(*)}}(\tau)\, e^{-j2\pi f\tau}\, \mathrm{d}\tau. \tag{1.99}$$

In (1.98), $X_{i,1/\Delta f}(t, f)$ is the STFT of $x_i(t)$ defined according to (1.93).

In the case of distinct processes (or processes denoted as distinct), the use of optional conjugation can be avoided without a lack of generality. In fact, if $x_i(t) = x_0^*(t)$, then $X_i(f) = X_0^*(-f)$. However, the following results are useful for future reference. Let $\boldsymbol{x} \triangleq [x_1^{(*)_1}\ x_2^{(*)_2}]$, where superscript $(*)_i$ denotes ith optional complex conjugation.

If

$$\mathrm{E}\left\{x_1^{(*)_1}(t+\tau)\, x_2^{(*)_2}(t)\right\} = \sum_{\alpha \in A_{\boldsymbol{x}}} R^{\alpha}_{\boldsymbol{x}}(\tau)\, e^{j2\pi\alpha t} \tag{1.100}$$

then

$$\mathrm{E}\left\{X_1^{(*)_1}(f_1)\, X_2^{(*)_2}(f_2)\right\} = \sum_{\alpha \in A_{\boldsymbol{x}}} S^{\alpha}_{\boldsymbol{x}}({}_{(-)_1} f_1)\, \delta(f_2 - {}_{(-)_2}(\alpha - {}_{(-)_1} f_1)) \tag{1.101}$$

where ${}_{(-)_i}$ is an optional minus sign linked to $(*)_i$.

In fact, by making the variable change $t_1 = t + \tau$, $t_2 = t$ into (1.100) the result is that

$$\mathrm{E}\left\{x_1^{(*)_1}(t_1)\, x_2^{(*)_2}(t_2)\right\} = \sum_{\alpha \in A_{\boldsymbol{x}}} R_{\boldsymbol{x}}^{\alpha}(t_1 - t_2)\, e^{j2\pi\alpha t_2} \tag{1.102}$$

and in the sense of distributions we formally have

$$\begin{aligned}
&\mathrm{E}\left\{X_1^{(*)_1}(f_1)\, X_2^{(*)_2}(f_2)\right\} \\
&\quad = \mathrm{E}\left\{\left[\int_{\mathbb{R}} x_1(t_1)\, e^{-j2\pi f_1 t_1}\, \mathrm{d}t_1\right]^{(*)_1} \left[\int_{\mathbb{R}} x_2(t_2)\, e^{-j2\pi f_2 t_2}\, \mathrm{d}t_2\right]^{(*)_2}\right\} \\
&\quad = \int_{\mathbb{R}} \int_{\mathbb{R}} \mathrm{E}\left\{x_1^{(*)_1}(t_1)\, x_2^{(*)_2}(t_2)\right\}\, e^{-j2\pi[(-)_1 f_1 t_1 + (-)_2 f_2 t_2]}\, \mathrm{d}t_1\, \mathrm{d}t_2 \\
&\quad = \int_{\mathbb{R}} \int_{\mathbb{R}} \sum_{\alpha \in A_{\boldsymbol{x}}} R_{\boldsymbol{x}}^{\alpha}(t_1 - t_2)\, e^{j2\pi\alpha t_2}\, e^{-j2\pi[(-)_1 f_1 t_1 + (-)_2 f_2 t_2]}\, \mathrm{d}t_1\, \mathrm{d}t_2 \\
&\quad = \sum_{\alpha \in A_{\boldsymbol{x}}} \int_{\mathbb{R}} R_{\boldsymbol{x}}^{\alpha}(\tau)\, e^{-(-)_1 j2\pi f_1 \tau}\, \mathrm{d}\tau \int_{\mathbb{R}} e^{j2\pi[\alpha - ((-)_1 f_1 + (-)_2 f_2)]t}\, \mathrm{d}t
\end{aligned} \tag{1.103}$$

from which (1.101) follows since

$$\begin{aligned}
\int_{\mathbb{R}} e^{j2\pi[\alpha - ((-)_1 f_1 + (-)_2 f_2)]t}\, \mathrm{d}t &= \delta(\alpha - ((-)_1 f_1 + (-)_2 f_2)) \\
&= \delta(f_2 - (-)_2(\alpha - (-)_1 f_1)).
\end{aligned} \tag{1.104}$$

By specializing (1.100) and (1.101) we have

$$\mathrm{E}\left\{y(t + \tau)\, x^{(*)}(t)\right\} = \sum_{\alpha \in A_{yx^{(*)}}} R_{yx^{(*)}}^{\alpha}(\tau)\, e^{j2\pi\alpha t} \tag{1.105}$$

$$\mathrm{E}\left\{Y(f_1)\, X^{(*)}(f_2)\right\} = \sum_{\alpha \in A_{yx^{(*)}}} S_{yx^{(*)}}^{\alpha}(f_1)\, \delta(f_2 - (-)(\alpha - f_1)). \tag{1.106}$$

1.3.3 LAPTV Systems

Linear almost-periodically time-varying (LAPTV) systems have an impulse-response function that can be expressed by the (generalized) Fourier series expansion

$$h(t, u) = \sum_{\sigma \in J} h_{\sigma}(t - u) e^{j2\pi\sigma u} \tag{1.107}$$

or, equivalently,

$$h(t + \tau, t) = \sum_{\sigma \in J} h_{\sigma}(\tau) e^{j2\pi\sigma t}. \tag{1.108}$$

where J is a countable set of frequency shifts.

By substituting (1.107) into (1.41), the output $y(t)$ can be expressed in the two equivalent forms (Franks 1994), (Gardner 1987d)

$$y(t) = \sum_{\sigma \in J} h_\sigma(t) \otimes [x(t)\, e^{j2\pi\sigma t}] \tag{1.109a}$$

$$= \sum_{\sigma \in J} [g_\sigma(t) \otimes x(t)]\, e^{j2\pi\sigma t} \tag{1.109b}$$

where

$$g_\sigma(t) \triangleq h_\sigma(t)\, e^{-j2\pi\sigma t}. \tag{1.110}$$

Equations (1.109a) and (1.109b) can be re-expressed in the frequency domain:

$$Y(f) = \sum_{\sigma \in J} H_\sigma(f)\, X(f-\sigma) \tag{1.111a}$$

$$= \sum_{\sigma \in J} G_\sigma(f-\sigma)\, X(f-\sigma) \tag{1.111b}$$

where $H_\sigma(f)$ and $G_\sigma(f)$ are the Fourier transforms of $h_\sigma(t)$ and $g_\sigma(t)$, respectively.

From (1.109a) it follows that the LAPTV systems combine linear time-invariant filtered versions of frequency-shifted versions of the input signal. For this reason, LAPTV filtering is also referred to as *frequency-shift* (FRESH) filtering (Gardner 1993). Equivalently, from (1.109b) the result is that LAPTV systems can be realized by combining frequency shifted versions of linear time-invariant filtered versions of the input (Figure 1.5).

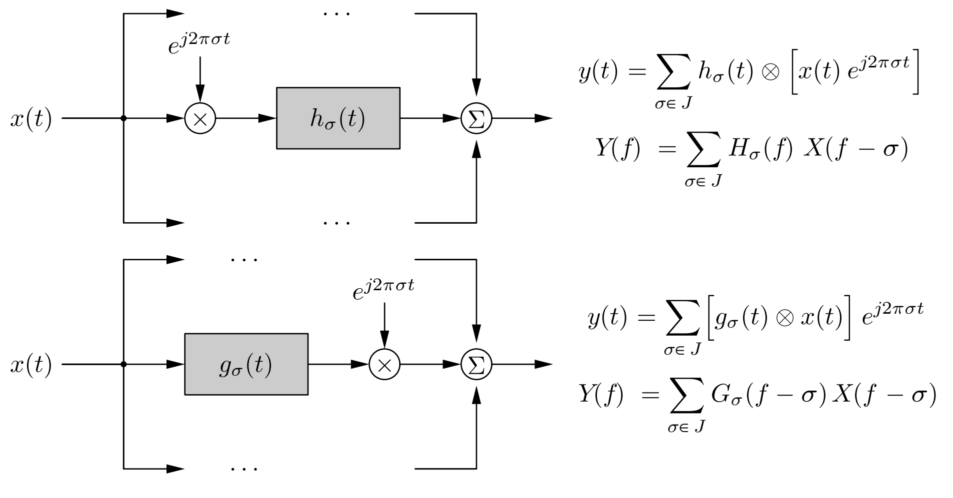

Figure 1.5 LAPTV systems: two equivalent realizations and corresponding input/output relations in time and frequency domains (see (1.109a)–(1.111b))

In the special case for which $J \equiv \{k/T_0\}_{k\in\mathbb{Z}}$ for some period T_0, the system is linear periodically time-variant (LPTV). If J contains only the element $\sigma = 0$, then the system is LTI and $h(t+\tau, t) = h_0(\tau)$.

1.3.3.1 Input/Output Relations

Let us consider two LAPTV systems with impulse-response functions

$$h_i(t,u) = \sum_{\sigma_i \in J_i} h_{\sigma_i}(t-u)\, e^{j2\pi\sigma_i u} \qquad i = 1, 2. \tag{1.112}$$

The (conjugate) cross-correlation of the outputs

$$y_i(t) = \int_{\mathbb{R}} h_i(t,u)\, x_i(u)\, \mathrm{d}u \qquad i = 1, 2 \tag{1.113}$$

is given by (see (Gardner *et al.* 2006, eq. (3.78)))

$$\begin{aligned} R_{y_1 y_2^{(*)}}(t,\tau) &\triangleq \mathrm{E}\left\{ y_1(t+\tau)\, y_2^{(*)}(t) \right\} \\ &= \sum_{\alpha\in A_{12}} \sum_{\sigma_1\in J_1} \sum_{\sigma_2 \in J_2} \left[R^{\alpha}_{x_1 x_2^{(*)}}(\tau)\, e^{j2\pi\sigma_1\tau} \right] \\ &\qquad \underset{\tau}{\otimes} r^{\alpha+\sigma_1+(-)\sigma_2}_{\sigma_1\sigma_2(*)}(\tau)\, e^{j2\pi(\alpha+\sigma_1+(-)\sigma_2)t} \end{aligned} \tag{1.114}$$

where $\underset{\tau}{\otimes}$ denotes convolution with respect to τ, $(-)$ is an optional minus sign that is linked to $(*)$, and

$$\begin{aligned} r^{\gamma}_{\sigma_1\sigma_2(*)}(\tau) &\triangleq \int_{\mathbb{R}} h_{\sigma_1}(s+\tau)\, h^{(*)}_{\sigma_2}(s)\, e^{-j2\pi\gamma s}\, \mathrm{d}s \\ &= h_{\sigma_1}(\tau) \otimes \left[h^{(*)}_{\sigma_2}(-\tau)\, e^{j2\pi\gamma\tau} \right] \end{aligned} \tag{1.115}$$

Thus (see (Napolitano 1995, eqs. (32) and (34)) and (Gardner *et al.* 2006, eqs. (3.80) and (3.81))), the (conjugate) cyclic cross-correlation function and the (conjugate) cyclic cross-spectrum of the outputs $y_1(t)$ and $y_2(t)$ are given by

$$\begin{aligned} R^{\beta}_{y_1 y_2^{(*)}}(\tau) &\triangleq \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} \mathrm{E}\left\{ y_1(t+\tau)\, y_2^{(*)}(t) \right\}\, e^{-j2\pi\beta t}\, \mathrm{d}t \\ &= \sum_{\sigma_1\in J_1} \sum_{\sigma_2\in J_2} \left[R^{\beta-\sigma_1-(-)\sigma_2}_{x_1 x_2^{(*)}}(\tau)\, e^{j2\pi\sigma_1\tau} \right] \underset{\tau}{\otimes} r^{\beta}_{\sigma_1\sigma_2(*)}(\tau) \end{aligned} \tag{1.116}$$

$$\begin{aligned} S^{\beta}_{y_1 y_2^{(*)}}(f) &\triangleq \int_{\mathbb{R}} R^{\beta}_{y_1 y_2^{(*)}}(\tau)\, e^{-j2\pi f\tau}\, \mathrm{d}\tau \\ &= \sum_{\sigma_1\in J_1} \sum_{\sigma_2\in J_2} S^{\beta-\sigma_1-(-)\sigma_2}_{x_1 x_2^{(*)}}(f-\sigma_1)\, H_{\sigma_1}(f)\, H^{(*)}_{\sigma_2}((-)(\beta-f)) \end{aligned} \tag{1.117}$$

where

$$H_{\sigma_i}(f) \triangleq \int_{\mathbb{R}} h_{\sigma_i}(\tau)\, e^{-j2\pi f\tau}\, \mathrm{d}\tau \tag{1.118}$$

and, in the sums in (1.116) and (1.117), only those $\sigma_1 \in J_1$ and $\sigma_2 \in J_2$ such that $\beta - \sigma_1 - {}_{(-)}\sigma_2 \in A_{12}$ give nonzero contribution. In the derivation of (1.117), the Fourier transform pair

$$r^{\gamma}_{\sigma_1\sigma_2(*)}(\tau) = h_{\sigma_1}(\tau) \otimes \left[h^{(*)}_{\sigma_2}(-\tau)\, e^{j2\pi\gamma\tau}\right] \stackrel{\mathcal{F}}{\longleftrightarrow} H_{\sigma_1}(f)\, H^{(*)}_{\sigma_2}({}_{(-)}(\gamma - f)) \tag{1.119}$$

is used.

Equations (1.116) and (1.117) can be specialized to several cases of interest. For example, if $x_1 = x_2 = x$, $h_1 = h_2 = h$, $y_1 = y_2 = y$, and $(*)$ is conjugation, then we obtain the input-output relations for LAPTV systems in terms of cyclic autocorrelation functions and cyclic spectra:

$$R^{\beta}_{yy^*}(\tau) = \sum_{\sigma_1 \in J} \sum_{\sigma_2 \in J} \left[R^{\beta-\sigma_1+\sigma_2}_{xx^*}(\tau)\, e^{j2\pi\sigma_1\tau}\right] \underset{\tau}{\otimes} r^{\beta}_{12}(\tau) \tag{1.120}$$

$$S^{\beta}_{yy^*}(f) = \sum_{\sigma_1 \in J} \sum_{\sigma_2 \in J} S^{\beta-\sigma_1+\sigma_2}_{xx^*}(f - \sigma_1)\, H_{\sigma_1}(f)\, H^*_{\sigma_2}(f - \beta) \tag{1.121}$$

where

$$r^{\beta}_{12}(\tau) \triangleq \int_{\mathbb{R}} h_{\sigma_1}(\tau + s)\, h^*_{\sigma_2}(s)\, e^{-j2\pi\beta s}\, \mathrm{d}s. \tag{1.122}$$

By specializing (1.117) to the case $x_1 = x_2 = x$, $h_1 = h_2 = h$ LTI system, and $y_1 = y_2 = y$ we obtain

$$S^{\alpha}_{yy^{(*)}}(f) = S^{\alpha}_{xx^{(*)}}(f)\, H(f)\, H^{(*)}({}_{(-)}(\alpha - f)). \tag{1.123}$$

Analogously, by specializing (1.116), we obtain the inverse Fourier transform of both sides of (1.123)

$$R^{\alpha}_{yy^{(*)}}(\tau) = R^{\alpha}_{xx^{(*)}}(\tau) \otimes r^{\alpha}_{hh^{(*)}}(\tau) \tag{1.124}$$

where, accounting for the Fourier pairs

$$h(-\tau) \stackrel{\mathcal{F}}{\longleftrightarrow} H(-f) \qquad h^{(*)}(-\tau) \stackrel{\mathcal{F}}{\longleftrightarrow} H^{(*)}({}_{(-)}(-f))$$

$$h^{(*)}(-\tau)\, e^{j2\pi\alpha\tau} \stackrel{\mathcal{F}}{\longleftrightarrow} H^{(*)}({}_{(-)}(-(f - \alpha))) = H^{(*)}({}_{(-)}(\alpha - f))$$

the *(conjugate) ambiguity function* $r^{\alpha}_{hh^{(*)}}(\tau)$ can be written in one of the following equivalent forms

$$\begin{aligned} r^{\alpha}_{hh^{(*)}}(\tau) &\triangleq \int_{\mathbb{R}} h(\tau+s)\, h^{(*)}(s)\, e^{-j2\pi\alpha s}\, \mathrm{d}s = \int_{\mathbb{R}} h(\tau-s')\, h^{(*)}(-s')\, e^{j2\pi\alpha s'}\, \mathrm{d}s' \\ &= h(\tau) \otimes \left[h^{(*)}(-\tau)\, e^{j2\pi\alpha\tau} \right] \end{aligned} \tag{1.125a}$$

$$\begin{aligned} r^{\alpha}_{hh^{(*)}}(\tau) &\triangleq \int_{\mathbb{R}} h(\tau+s)\, h^{(*)}(s)\, e^{-j2\pi\alpha s}\, \mathrm{d}s = \underset{s\to\alpha}{\mathcal{F}} \left[h(\tau+s)\, h^{(*)}(s) \right] \\ &= \left[H(\alpha)\, e^{j2\pi\alpha\tau} \right] \underset{\alpha}{\otimes} H^{(*)}((-)\alpha) = \int_{\mathbb{R}} H(\lambda)\, H^{(*)}((-)(\alpha-\lambda))\, e^{j2\pi\lambda\tau}\, \mathrm{d}\lambda \end{aligned} \tag{1.125b}$$

By specializing (1.116) and (1.117) to the case $x_1 = x_2 = x$, $h_2(t+\tau, t) = \delta(\tau)$, $y_1 = y$, $y_2 = x$, so that $J_2 = \{0\}$ and

$$r^{\beta}_{\sigma_1\sigma_2(*)}(\tau) = \int_{\mathbb{R}} h_{\sigma_1}(\tau+s)\, \delta(s)\, e^{-j2\pi\beta s}\, \mathrm{d}s = h_{\sigma_1}(\tau) \tag{1.126}$$

we obtain the cyclic cross-statistics of the output and input signals

$$R^{\beta}_{yx^{(*)}}(\tau) = \sum_{\sigma_1\in J_1} \left[R^{\beta-\sigma_1}_{xx^{(*)}}(\tau)\, e^{j2\pi\sigma_1\tau} \right] \underset{\tau}{\otimes} h_{\sigma_1}(\tau) \tag{1.127}$$

$$S^{\beta}_{yx^{(*)}}(f) = \sum_{\sigma_1\in J_1} S^{\beta-\sigma_1}_{xx^{(*)}}(f-\sigma_1)\, H_{\sigma_1}(f). \tag{1.128}$$

By specializing (1.117) to the case h_1 and h_2 LTI and $(*)$ absent, for $\alpha = 0$ we obtain

$$S_{y_1y_2}(f) = S_{x_1x_2}(f)\, H_1(f)\, H_2(-f) \tag{1.129}$$

$$\overset{\mathcal{F}}{\longleftrightarrow} \quad R_{y_1y_2}(\tau) = R_{x_1x_2}(\tau) \otimes h_1(\tau) \otimes h_2(-\tau) \tag{1.130}$$

1.3.4 Products of ACS Signals

Let $x_1(t)$, $x_2(t)$, $c_1(t)$, and $c_2(t)$ be ACS signals with (conjugate) cross-correlation functions

$$R_{x_1x_2^{(*)}}(t,\tau) = \sum_{\alpha_x\in A_{x_1x_2^{(*)}}} R^{\alpha_x}_{x_1x_2^{(*)}}(\tau)\, e^{j2\pi\alpha_x t} \tag{1.131}$$

$$R_{c_1c_2^{(*)}}(t,\tau) = \sum_{\alpha_c\in A_{c_1c_2^{(*)}}} R^{\alpha_c}_{c_1c_2^{(*)}}(\tau)\, e^{j2\pi\alpha_c t}. \tag{1.132}$$

If $x_1(t)$ and $x_2(t)$ are statistically independent of $c_1(t)$ and $c_2(t)$, then their fourth-order joint probability density function factors into the product of the two second-order joint probability densities of x_1, x_2 and of c_1, c_2 (Gardner 1987d, 1994), (Brown 1987), and the (conjugate) cross-correlation functions of the product waveforms

$$y_i(t) = c_i(t)\, x_i(t) \qquad i = 1, 2 \tag{1.133}$$

also factors,

$$R_{y_1 y_2^{(*)}}(t, \tau) = R_{c_1 c_2^{(*)}}(t, \tau)\, R_{x_1 x_2^{(*)}}(t, \tau). \tag{1.134}$$

Therefore, the (conjugate) cyclic cross-correlation function and the (conjugate) cyclic cross-spectrum of $y_1(t)$ and $y_2(t)$ are

$$R^{\alpha}_{y_1 y_2^{(*)}}(\tau) = \sum_{\alpha_x \in A_{x_1 x_2^{(*)}}} R^{\alpha_x}_{x_1 x_2^{(*)}}(\tau)\, R^{\alpha-\alpha_x}_{c_1 c_2^{(*)}}(\tau) \tag{1.135}$$

$$S^{\alpha}_{y_1 y_2^{(*)}}(f) = \sum_{\alpha_x \in A_{x_1 x_2^{(*)}}} \int_{\mathbb{R}} S^{\alpha_x}_{x_1 x_2^{(*)}}(\lambda)\, S^{\alpha-\alpha_x}_{c_1 c_2^{(*)}}(f - \lambda)\, \mathrm{d}\lambda \tag{1.136}$$

where, in the sums, only those (conjugate) cycle frequencies α_x such that $\alpha - \alpha_x \in A_{c_1 c_2^{(*)}}$ give nonzero contribution. From (1.135) and (1.136), it follows that the set of (conjugate) cycle-frequencies of the (conjugate) cross-correlation of $y_1(t)$ and $y_2(t)$ is

$$\{\alpha\} = \left\{\alpha = \alpha_c + \alpha_x,\ \alpha_c \in A_{c_1 c_2^{(*)}}, \alpha_x \in A_{x_1 x_2^{(*)}}\right\}. \tag{1.137}$$

In the special case where $c_1(t)$ and $c_2(t)$ are almost-periodic functions with (generalized) Fourier series

$$c_i(t) = \sum_{\gamma_i \in G_i} c_{i,\gamma_i}\, e^{j2\pi\gamma_i t} \qquad i = 1, 2 \tag{1.138}$$

then

$$\begin{aligned} R_{c_1 c_2^{(*)}}(t, \tau) &= c_1(t + \tau)\, c_2^{(*)}(t) \\ &= \sum_{\gamma_1 \in G_1} \sum_{\gamma_2 \in G_2} c_{1,\gamma_1}\, c^{(*)}_{2,\gamma_2}\, e^{j2\pi\gamma_1\tau}\, e^{j2\pi(\gamma_1 + {\scriptstyle (-)}\gamma_2)t} \end{aligned} \tag{1.139}$$

and

$$R^{\alpha}_{c_1 c_2^{(*)}}(\tau) = \sum_{\gamma_1 \in G_1} c_{1,\gamma_1}\, c^{(*)}_{2,(-)(\alpha-\gamma_1)}\, e^{j2\pi\gamma_1\tau} \tag{1.140}$$

$$S^{\alpha}_{c_1 c_2^{(*)}}(f) = \sum_{\gamma_1 \in G_1} c_{1,\gamma_1}\, c^{(*)}_{2,(-)(\alpha-\gamma_1)}\, \delta(f - \gamma_1) \tag{1.141}$$

where, in the sum, only those frequencies γ_1 such that ${\scriptstyle (-)}(\alpha - \gamma_1) \in G_2$ give nonzero contribution. From (1.140) and (1.141) it follows that the set of the (conjugate) cycle-frequencies of the (conjugate) cross-correlation of $y_1(t)$ and $y_2(t)$ is

$$\{\alpha\} = \{\alpha = \gamma_1 + {\scriptstyle (-)}\gamma_2,\ \gamma_1 \in G_1, \gamma_2 \in G_2\}. \tag{1.142}$$

In the special case $x_1 \equiv x_2$ and $c_1 \equiv c_2$, (1.139)–(1.141) reduce to (corrected versions of) (3.93)–(3.95) in (Gardner *et al.* 2006)). By substituting (1.140) and (1.141) into (1.135) and (1.136), respectively, and making the variable change $\gamma_2 = {}_{(-)}(\alpha - \alpha_x - \gamma_1)$, we get (see (Napolitano 1995, eqs. (46)–(48)))

$$R^{\alpha}_{y_1 y_2^{(*)}}(\tau) = \sum_{\gamma_1 \in G_1} \sum_{\gamma_2 \in G_2} c_{1,\gamma_1}\, c^{(*)}_{2,\gamma_2}\, R^{\alpha-(\gamma_1 + {}_{(-)}\gamma_2)}_{x_1 x_2^{(*)}}(\tau)\, e^{j2\pi\gamma_1\tau} \tag{1.143}$$

$$S^{\alpha}_{y_1 y_2^{(*)}}(f) = \sum_{\gamma_1 \in G_1} \sum_{\gamma_2 \in G_2} c_{1,\gamma_1}\, c^{(*)}_{2,\gamma_2}\, S^{\alpha-(\gamma_1 + {}_{(-)}\gamma_2)}_{x_1 x_2^{(*)}}(f - \gamma_1). \tag{1.144}$$

where, in the sums, only those values of γ_1 and γ_2 such that $\alpha - (\gamma_1 + {}_{(-)}\gamma_2) \in A_{x_1 x_2^{(*)}}$ give nonzero contribution. From (1.143) and (1.144), it follows that the set of the (conjugate) cycle-frequencies of the (conjugate) cross-correlation of $y_1(t)$ and $y_2(t)$ is

$$\{\alpha\} = \left\{\alpha = \gamma_1 + {}_{(-)}\gamma_2 + \alpha_x,\ \alpha_x \in A_{x_1 x_2^{(*)}}, \gamma_1 \in G_1, \gamma_2 \in G_2\right\}. \tag{1.145}$$

1.3.5 Cyclic Statistics of Communications Signals

Cyclostationarity in man-made communications signals is due to signal processing operations used in the construction and/or subsequent processing of signals, such as modulation, sampling, scanning, multiplexing, and coding. Consequently, cycle frequencies are related to parameters such as sinewave carrier frequency, sampling rate, etc. In the following, the cyclic autocorrelation functions and cyclic spectra of two basic communications signals are reported. Further examples can be found in (Gardner 1985, Chapter 12, 1987d), (Gardner *et al.* 2006) and references therein.

The first signal is the double side-band amplitude-modulated (DSB-AM) signal

$$x(t) \triangleq s(t)\cos(2\pi f_0 t + \phi_0). \tag{1.146}$$

If the modulating process $s(t)$ is WSS, then $x(t)$ is cyclostationary with period $1/(2f_0)$, cyclic autocorrelation functions (Gardner 1985, Chapter 12, 1987d), (Gardner *et al.* 2006)

$$R^{\alpha}_{x}(\tau) = \begin{cases} \frac{1}{2} R^0_s(\tau)\ \cos(2\pi f_0 \tau) & \alpha = 0 \\ \frac{1}{4} R^0_s(\tau)\, e^{\pm j2\pi f_0 \tau}\ e^{\pm j2\phi_0} & \alpha = \pm 2f_0 \\ 0 & \text{otherwise} \end{cases} \tag{1.147}$$

and cyclic spectra

$$S^{\alpha}_{x}(f) = \begin{cases} \frac{1}{4}\left\{S^0_s(f - f_0) + S^0_s(f + f_0)\right\} & \alpha = 0 \\ \frac{1}{4} S^0_s(f \mp f_0)\, e^{\pm j2\phi_0} & \alpha = \pm 2f_0 \\ 0 & \text{otherwise.} \end{cases} \tag{1.148}$$

The second signal is the pulse-amplitude-modulated (PAM) signal

$$x(t) \triangleq \sum_{k \in \mathbb{Z}} a_k\, q(t - kT_0) \tag{1.149}$$

where $\{a_k\}_{k\in\mathbb{Z}}$ is a WSS sequence and $q(t)$ a finite-energy pulse. This PAM signal is cyclostationary with period T_0 and (conjugate) cyclic autocorrelation functions and cyclic spectra (Gardner 1985, Chapter 12, 1987d), (Gardner *et al.* 2006)

$$R_{\boldsymbol{x}}^{\alpha}(\tau) = \frac{\mathrm{E}\{a_k a_k^{(*)}\}}{T_0}\, r_{\boldsymbol{q}}^{\alpha}(\tau) \tag{1.150}$$

$$S_{\boldsymbol{x}}^{\alpha}(f) = \frac{\mathrm{E}\{a_k a_k^{(*)}\}}{T_0}\, Q(f)\, Q^{(*)}((-)(\alpha - f)) \tag{1.151}$$

for $\alpha = m/T_0$, $m \in \mathbb{Z}$, and zero otherwise. In (1.150),

$$r_{\boldsymbol{q}}^{\alpha}(\tau) \triangleq \int_{\mathbb{R}} q(t+\tau)\, q^{(*)}(t)\, e^{-j2\pi\alpha t}\, \mathrm{d}t \tag{1.152}$$

and in (1.151) $Q(f)$ is the Fourier transform of $q(t)$.

1.3.6 Higher-Order Statistics

A more complete characterization of stochastic processes can be obtained by considering higher-order statistics. Processes that have almost-periodically time-variant moment and cumulant functions are called higher-order almost-cyclostationary. They have been characterized in the time and frequency domains in (Dandawaté and Giannakis 1994, 1995), (Gardner and Spooner 1994), (Napolitano 1995), (Spooner and Gardner 1994).

A continuous-time complex-valued process $x(t)$ is said to *exhibit Nth-order wide-sense cyclostationarity* with cycle frequency $\alpha \neq 0$, for a given conjugation configuration, if the Nth-order *cyclic temporal moment function* (CTMF)

$$\begin{aligned}\mathcal{R}_{\boldsymbol{x}}^{\alpha}(\boldsymbol{\tau}) &\triangleq \mathcal{R}_{x^{(*)_1},\ldots,x^{(*)_N}}^{\alpha}(\boldsymbol{\tau}) \\ &\triangleq \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} \mathrm{E}\Big\{\prod_{n=1}^{N} x^{(*)_n}(t+\tau_n)\Big\}\, e^{-j2\pi\alpha t}\, \mathrm{d}t \end{aligned} \tag{1.153}$$

exists and is not zero for some delay vector $\boldsymbol{\tau}$ (Gardner and Spooner 1994). In (1.153), $\boldsymbol{\tau} \triangleq [\tau_1, \ldots, \tau_N]^{\mathrm{T}}$ and $\boldsymbol{x} \triangleq [x^{(*)_1}, \ldots, x^{(*)_N}]^{\mathrm{T}}$ are column vectors, and $(*)_n$ represents the nth optional complex conjugation. The process is said to be *Nth-order wide-sense almost-cyclostationary* if its Nth-order temporal moment function (TMF) is an almost-periodic function of t. Under mild regularity conditions the TMF can be expressed by the (generalized) Fourier series expansion

$$\begin{aligned}\mathcal{R}_{\boldsymbol{x}}(t, \boldsymbol{\tau}) &\triangleq \mathrm{E}\Big\{\prod_{n=1}^{N} x^{(*)_n}(t+\tau_n)\Big\} \\ &= \sum_{\alpha\in A_{\boldsymbol{x}}} \mathcal{R}_{\boldsymbol{x}}^{\alpha}(\boldsymbol{\tau})\, e^{j2\pi\alpha t} \end{aligned} \tag{1.154}$$

where $A_{\boldsymbol{x}}$ is a countable set and the sense of equality in (1.154) is one of those discussed in Section 1.2.

The N-fold Fourier transform of the CTMF

$$\mathcal{S}_{\boldsymbol{x}}^{\alpha}(\boldsymbol{f}) = \int_{\mathbb{R}^N} \mathcal{R}_{\boldsymbol{x}}^{\alpha}(\boldsymbol{\tau})\, e^{-j2\pi \boldsymbol{f}^{\mathrm{T}}\boldsymbol{\tau}}\, \mathrm{d}\boldsymbol{\tau} \tag{1.155}$$

which is called the Nth-order *cyclic spectral moment function* (CSMF), can be written as

$$\mathcal{S}_{\boldsymbol{x}}^{\alpha}(\boldsymbol{f}) = S_{\boldsymbol{x}}^{\alpha}(\boldsymbol{f}')\, \delta(\boldsymbol{f}^{\mathrm{T}}\mathbf{1} - \alpha) \tag{1.156}$$

where $\delta(\cdot)$ is the Dirac delta function, $\mathbf{1} \triangleq [1, ..., 1]^{\mathrm{T}}$, and primes denote the operator that transforms $\boldsymbol{v} = [v_1, \ldots, v_N]^{\mathrm{T}}$ into $\boldsymbol{v}' = [v_1, \ldots, v_{N-1}]^{\mathrm{T}}$. The function $S_{\boldsymbol{x}}^{\alpha}(\boldsymbol{f}')$, referred to as the Nth-order reduced-dimension CSMF (RD-CSMF), can be expressed as the $(N-1)$-fold Fourier transform of the Nth-order reduced-dimension CTMF (RD-CTMF) defined as

$$R_{\boldsymbol{x}}^{\alpha}(\boldsymbol{\tau}') \triangleq \mathcal{R}_{\boldsymbol{x}}^{\alpha}(\boldsymbol{\tau})\big|_{\tau_N=0}\,. \tag{1.157}$$

For $N = 2$ and conjugation configuration $[xx^*]$, the RD-CTMF is coincident with the *cyclic autocorrelation function* $R_{xx^*}^{\alpha}(\tau_1)$, whereas, for conjugation configuration $[xx]$ it is coincident with the *conjugate cyclic autocorrelation function* $R_{xx}^{\alpha}(\tau_1)$ that can be nonzero only if the signal is noncircular (Picinbono and Bondon 1997), (Schreier and Scharf 2003a,b). It is shown in (Gardner and Spooner 1994) that the RD-CSMF $S_{\boldsymbol{x}}^{\beta}(\boldsymbol{f}')$ can contain impulsive terms if the vector $\boldsymbol{f}$ with $f_N = \beta - \sum_{n=1}^{N-1} f_n$ lies on the β-submanifold, i.e., if there exists at least one partition $\{\mu_1, \cdots, \mu_p\}$ of $\{1, \cdots, N\}$ with $p > 1$ such that each sum $\alpha_{\mu_i} = \sum_{n\in\mu_i} f_n$ is an $|\mu_i|$th-order cycle frequency of $x(t)$, where $|\mu_i|$ is the number of elements in μ_i.

Note that for second-order statistics, unlike in (1.155), in the complex exponential of double Fourier transforms, an optional minus sign $(-)$ linked to the optional complex conjugation $(*)$ is accounted for (see (1.15), (1.16), and (1.101)).

A well-behaved frequency-domain function that characterizes a signal's higher-order cyclostationarity can be obtained starting from the Nth-order *cyclic temporal cumulant function* (CTCF), that is, the coefficient

$$\mathcal{C}_{\boldsymbol{x}}^{\beta}(\boldsymbol{\tau}) \triangleq \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} \operatorname{cum}\Big\{x^{(*)_n}(t+\tau_n),\ n = 1, ..., N\Big\}\, e^{-j2\pi\beta t}\, \mathrm{d}t \tag{1.158}$$

of the (generalized) Fourier-series expansion of the Nth-order temporal cumulant function (Gardner and Spooner 1994)

$$\mathcal{C}_{\boldsymbol{x}}(t, \boldsymbol{\tau}) \equiv \operatorname{cum}\Big\{x^{(*)_n}(t+\tau_n),\ n = 1, \ldots, N\Big\}$$

$$\triangleq (-j)^N \frac{\partial^N}{\partial\omega_1 \cdots \partial\omega_N} \log_e \mathrm{E}\Big\{\exp\Big[j \sum_{n=1}^{N} \omega_n x^{(*)_n}(t+\tau_n)\Big]\Big\}\Big|_{\boldsymbol{\omega}=\mathbf{0}} \tag{1.159a}$$

$$= \sum_{\mathrm{P}} \Big[(-1)^{p-1}(p-1)! \prod_{i=1}^{p} \mathcal{R}_{\boldsymbol{x}_{\mu_i}}(t, \boldsymbol{\tau}_{\mu_i})\Big] \tag{1.159b}$$

where, $\boldsymbol{\omega} \triangleq [\omega_1, ..., \omega_N]$; P is the set of distinct partitions of $\{1, ..., N\}$, each constituted by the subsets $\{\mu_i,\ i = 1, ..., p\}$; $\boldsymbol{x}_{\mu_i}$ is the $|\mu_i|$-dimensional vector whose components are those of $\boldsymbol{x}$ having indices in μ_i, with $|\mu_i|$ the number of elements in μ_i. See Section 1.4.2 for a discussion of the definition of (cross-) cumulant of complex random variables and processes.

The N-fold Fourier transform of $\mathcal{C}_{\boldsymbol{x}}^{\beta}(\boldsymbol{\tau})$ is the Nth-order *cyclic spectral cumulant function* $\mathcal{P}_{\boldsymbol{x}}^{\beta}(\boldsymbol{f})$, which can be written as $\mathcal{P}_{\boldsymbol{x}}^{\beta}(\boldsymbol{f}) = P_{\boldsymbol{x}}^{\beta}(\boldsymbol{f}')\,\delta(\boldsymbol{f}^{\mathrm{T}}\mathbf{1} - \beta)$, where the Nth-order *cyclic polyspectrum* (CP) $P_{\boldsymbol{x}}^{\beta}(\boldsymbol{f}')$ is the $(N-1)$-fold Fourier transform of the reduced-dimension CTCF (RD-CTCF) $C_{\boldsymbol{x}}^{\beta}(\boldsymbol{\tau}')$ obtained by setting $\tau_N = 0$ into (1.158). The CP turns out to be a well-behaved function (i.e., it does not contain impulsive terms) under the mild assumption that the time series $x(t)$ and $x(t+\tau)$ are asymptotically ($\tau \to \infty$) independent (Section 1.4.1). Moreover, except on a β-submanifold, it is coincident with the RD-CSMF $S_{\boldsymbol{x}}^{\beta}(\boldsymbol{f}')$.

Higher-order cyclostationarity can be exploited when the second-order cyclostationarity features of the signal of interest are zero or weak. Moreover, since they provide a more complete characterization of signals, higher-order cyclostationarity is suitable to be used in modulation format classification and cognitive radio (Spooner and Nicholls 2009).

1.3.7 Cyclic Statistic Estimators

Consistent estimates of second-order statistical functions of an ACS stochastic process can be obtained provided that the stochastic process has finite or “effectively finite” memory. Such a property is generally expressed in terms of mixing conditions or summability of second- and fourth-order cumulants. Under such mixing conditions, the cyclic correlogram is a consistent estimator of the cyclic autocorrelation function. Moreover, a properly normalized version of the cyclic correlogram is asymptotically zero-mean complex Normal as the observation interval approaches infinity. In the frequency domain, the cyclic periodogram is an asymptotically unbiased but not consistent estimator of the cyclic spectrum. However, under appropriate conditions, the frequency-smoothed cyclic periodogram is a consistent estimator of the cyclic spectrum. Moreover, a properly normalized version of the frequency-smoothed cyclic periodogram is asymptotically zero-mean complex Normal as the observation interval approaches infinity and the width of the frequency-smoothing window approaches zero. In addition, the time-smoothed cyclic periodogram is asymptotically equivalent to the frequency-smoothed cyclic periodogram.

Consistent estimators of (conjugate) cyclic autocorrelation function and cyclic spectrum are proposed and analyzed in (Hurd and Leśkow 1992a,b), (Dandawaté and Giannakis 1994, 1995), (Dehay 1994), (Dehay and Hurd 1994), (Hurd and Miamee 2007), and references therein. These results can be obtained by specializing to ACS processes the results presented in Chapters 2 and 4. Higher-order cyclic statistic estimators are presented in (Dandawaté and Giannakis 1994, 1995), and (Napolitano and Spooner 2000).

1.3.8 Discrete-Time ACS Signals

ACS processes, such as WSS processes, can also be defined in discrete time. The definitions are similar to those in continuous time with the obvious modifications (Gladyshev 1961), (Gardner 1985, 1994), (Dandawaté and Giannakis 1994), (Napolitano 1995), (Giannakis 1998), (Gardner *et al.* 2006).

Let us consider a discrete-time complex-valued stochastic process $\{x_{\mathrm{d}}(n, \omega), n \in \mathbb{Z}, \omega \in \Omega\}$, with abbreviated notation $x_{\mathrm{d}}(n)$ when this does not create ambiguity. The stochastic process

$x_{\mathrm{d}}(n)$ is said to be *second-order almost-cyclostationary in the wide sense* (Gardner 1985) if its mean value $\mathrm{E}\{x_{\mathrm{d}}(n)\}$ and its (conjugate) autocorrelation function

$$\widetilde{R}_{\boldsymbol{x}_{\mathrm{d}}}(n,m) \triangleq \mathrm{E}\{x_{\mathrm{d}}(n+m)\, x_{\mathrm{d}}^{(*)}(n)\} \tag{1.160}$$

with subscript $\boldsymbol{x}_{\mathrm{d}} \triangleq [x_{\mathrm{d}}\; x_{\mathrm{d}}^{(*)}]$, are almost-periodic functions of the discrete-time parameter n. Thus, under mild regularity assumptions the (conjugate) autocorrelation function can be expressed by the (generalized) Fourier series

$$\widetilde{R}_{\boldsymbol{x}_{\mathrm{d}}}(n,m) = \sum_{\widetilde{\alpha}\in\widetilde{A}} \widetilde{R}^{\widetilde{\alpha}}_{\boldsymbol{x}_{\mathrm{d}}}(m)\, e^{j2\pi\widetilde{\alpha}n} \tag{1.161}$$

where

$$\widetilde{R}^{\widetilde{\alpha}}_{\boldsymbol{x}_{\mathrm{d}}}(m) \triangleq \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} \widetilde{R}_{\boldsymbol{x}_{\mathrm{d}}}(n,m)\, e^{-j2\pi\widetilde{\alpha}n} \tag{1.162}$$

is the *(conjugate) cyclic autocorrelation function* at *(conjugate) cycle frequency* $\widetilde{\alpha}$ and

$$\widetilde{A} \triangleq \left\{\widetilde{\alpha} \in [-1/2, 1/2) \;:\; \widetilde{R}^{\widetilde{\alpha}}_{\boldsymbol{x}_{\mathrm{d}}}(m) \not\equiv 0\right\} \tag{1.163}$$

is a countable set. Note that the (conjugate) cyclic autocorrelation function $\widetilde{R}^{\widetilde{\alpha}}_{\boldsymbol{x}_{\mathrm{d}}}(m)$ is periodic in $\widetilde{\alpha}$ with period 1. Thus, the sum in (1.161) can be equivalently extended to the set $\widetilde{A}_1 \triangleq \{\widetilde{\alpha} \in [0,1) \;:\; \widetilde{R}^{\widetilde{\alpha}}_{\boldsymbol{x}_{\mathrm{d}}}(m) \not\equiv 0\}$.

In general, the set $\widetilde{A}$ (or $\widetilde{A_1}$) depends on the optional complex conjugation $(*)$ and possibly contains incommensurate cycle frequencies $\widetilde{\alpha}$ and cluster points. In the special case where $\widetilde{A}_1 \equiv \{0, 1/N_0, \ldots, (N_0-1)/N_0\}$ for some integer N_0, the (conjugate) autocorrelation function $\widetilde{R}_{\boldsymbol{x}_{\mathrm{d}}}(n,m)$ is periodic in n with period N_0. If also $\mathrm{E}\{x_{\mathrm{d}}(n)\}$ is periodic with period N_0, the process $x(n)$ is said to be *cyclostationary in the wide sense*. If $N_0 = 1$ then $x(n)$ is *wide-sense stationary*.

Let

$$\widetilde{X}_{\mathrm{d}}(\nu) \triangleq \sum_{n\in\mathbb{Z}} x_{\mathrm{d}}(n)\, e^{-j2\pi\nu n} \tag{1.164}$$

be the stochastic process obtained from Fourier transformation (in a generalized sense (Gelfand and Vilenkin 1964, Chapter 3)) of the ACS process $x(n)$. By using (1.161), in the sense of distributions, it can be shown that (Hurd 1991)

$$\mathrm{E}\left\{\widetilde{X}_{\mathrm{d}}(\nu_1)\, \widetilde{X}_{\mathrm{d}}^{(*)}(\nu_2)\right\} = \sum_{\widetilde{\alpha}\in\widetilde{A}} \widetilde{S}^{\widetilde{\alpha}}_{\boldsymbol{x}_{\mathrm{d}}}(\nu_1) \sum_{\ell\in\mathbb{Z}} \delta(\nu_2 +_{(-)} (\nu_1 - \widetilde{\alpha}) - \ell) \tag{1.165}$$

where

$$\widetilde{S}^{\widetilde{\alpha}}_{\boldsymbol{x}_{\mathrm{d}}}(\nu) \triangleq \sum_{m\in\mathbb{Z}} \widetilde{R}^{\widetilde{\alpha}}_{\boldsymbol{x}_{\mathrm{d}}}(m)\, e^{-j2\pi\nu m} \tag{1.166}$$

is the *(conjugate) cyclic spectrum* at (conjugate) cycle frequency $\widetilde{\alpha}$. The cyclic spectrum $\widetilde{S}^{\widetilde{\alpha}}_{\boldsymbol{x}_{\rm d}}(\nu)$ is periodic in both ν and $\widetilde{\alpha}$ with period 1.

In (Gladyshev 1961) and (Dehay 1994), it is shown that discrete-time cyclostationary processes are harmonizable.

1.3.9 Sampling of ACS Signals

Let

$$x(n) \triangleq x_a(t)|_{t=nT_s} \qquad n \in \mathbb{Z} \tag{1.167}$$

be the discrete-time process obtained by uniformly sampling with period $T_s = 1/f_s$ the continuous-time ACS signal $x_a(t)$ with Loève bifrequency spectrum (see (1.94))

$$\mathrm{E}\left\{X_a(f_1)\, X_a^{(*)}(f_2)\right\} = \sum_{\alpha \in A_{\boldsymbol{x}_a}} S^{\alpha}_{\boldsymbol{x}_a}(f_1)\, \delta(f_2 - {\scriptstyle (-)}(\alpha - f_1)). \tag{1.168}$$

where subscript $\boldsymbol{x}_a \triangleq [x_a\ x_a^{(*)}]$. Accounting for the relationship between the Fourier transform $X_a(f)$ of the continuous-time signal $x_a(t)$ and the Fourier transform $X(\nu)$ of the discrete-time sampled signal $x(n)$ (Lathi 2002, Section 9.5)

$$X(\nu) = \frac{1}{T_s} \sum_{k=-\infty}^{+\infty} X_a((\nu - k) f_s) \tag{1.169}$$

we have the following result.

Lemma 1.3.1 Aliasing Formula for the Loève Bifrequency Spectrum of a Sampled ACS Signal.

$$\begin{aligned}
\mathrm{E}\left\{X(\nu_1)\, X^{(*)}(\nu_2)\right\} &= \mathrm{E}\left\{\frac{1}{T_s} \sum_{n_1 \in \mathbb{Z}} X_a((\nu_1 - n_1) f_s)\, \frac{1}{T_s} \sum_{n_2 \in \mathbb{Z}} X_a^{(*)}((\nu_2 - n_2) f_s)\right\} \\
&= \frac{1}{T_s^2} \sum_{n_1 \in \mathbb{Z}} \sum_{n_2 \in \mathbb{Z}} \mathrm{E}\left\{X_a((\nu_1 - n_1) f_s)\, X_a^{(*)}((\nu_2 - n_2) f_s)\right\} \\
&= \frac{1}{T_s^2} \sum_{n_1 \in \mathbb{Z}} \sum_{n_2 \in \mathbb{Z}} \sum_{\alpha \in A_{\boldsymbol{x}_a}} S^{\alpha}_{\boldsymbol{x}_a}((\nu_1 - n_1) f_s) \\
&\quad \delta((\nu_2 - n_2) f_s - {\scriptstyle (-)}(\alpha - (\nu_1 - n_1) f_s)) \\
&= \frac{1}{T_s} \sum_{\alpha \in A_{\boldsymbol{x}_a}} \sum_{n_1 \in \mathbb{Z}} S^{\alpha}_{\boldsymbol{x}_a}((\nu_1 - n_1) f_s) \\
&\quad \sum_{n_2 \in \mathbb{Z}} \delta((\nu_2 - n_2) - {\scriptstyle (-)}(\alpha / f_s - (\nu_1 - n_1)))
\end{aligned} \tag{1.170}$$

where, in the last equality, the scaling property of the Dirac delta $f_s \delta(\nu f_s) = \delta(\nu)$ is used (Zemanian 1987, Section 1.7). □

Corollary 1.3.2 *From (1.170), it follows that the discrete-time signal* $x(n) \triangleq x_a(t)|_{t=nT_s}$ *obtained by uniformly sampling a continuous-time ACS signal* $x_a(t)$ *is a discrete-time ACS signal. In fact, its Loève bifrequency spectrum has support concentrated on lines with slope* ± 1 *in the bifrequency plane* (ν_1, ν_2) *and can be written in the form (1.165).*

The countable set $\widetilde{A}$ *in (1.165) is linked to* $A_{\boldsymbol{x}_a}$ *by the relationships*

$$\widetilde{A} = \bigcup_{\alpha \in A_{\boldsymbol{x}_a}} \left\{ \widetilde{\alpha} \in (-1/2, 1/2] \; : \; \widetilde{\alpha} = (\alpha/f_s) \bmod 1 \right\} \tag{1.171}$$

$$A_{\boldsymbol{x}_a} \subseteq \bigcup_{\widetilde{\alpha} \in \widetilde{A}} \bigcup_{p \in \mathbb{Z}} \left\{ \alpha \in \mathbb{R} \; : \; \alpha = \widetilde{\alpha} f_s - p f_s \right\} \tag{1.172}$$

with mod 1 *denoting the modulo 1 operation with values in* $(-1/2, 1/2]$ *and equality can hold in (1.172) only if* $x_a(t)$ *is not strictly bandlimited.* □

The (conjugate) cyclic spectrum and the (conjugate) cyclic autocorrelation function of $x(n)$ are given in the following result.

Lemma 1.3.3 Aliasing Formulas for Cyclic Statistics of ACS Signals (Izzo and Napolitano 1996), (Napolitano 1995).

$$\mathrm{E}\{x(n+m)\, x^{(*)}(n)\} = \mathrm{E}\{x_a(t+\tau)\, x_a(t)\}|_{t=nT_s, \tau=mT_s} \, . \tag{1.173}$$

$$\widetilde{R}_{\boldsymbol{x}}^{\widetilde{\alpha}}(m) = \sum_{p \in \mathbb{Z}} R_{\boldsymbol{x}_a}^{\alpha - p f_s}(\tau) \Big|_{\tau = m T_s, \alpha = \widetilde{\alpha} f_s} \tag{1.174}$$

$$\widetilde{S}_{\boldsymbol{x}}^{\widetilde{\alpha}}(\nu) = \frac{1}{T_s} \sum_{p \in \mathbb{Z}} \sum_{q \in \mathbb{Z}} S_{\boldsymbol{x}_a}^{\alpha - p f_s}(f - q f_s) \Big|_{f = \nu f_s, \alpha = \widetilde{\alpha} f_s} \, . \tag{1.175}$$

where $\boldsymbol{x} \triangleq [x\, x^{(*)}]$. □

Assumption 1.3.4 *The continuous-time process* $x_a(t)$ *is strictly band-limited, i.e.,* $S^0_{x_a x_a^*}(f) = 0 \quad |f| > B$ *(Gardner et al. 2006, Section 3.8).* □

Lemma 1.3.5 Support of the (Conjugate) Cyclic Spectrum. *Under Assumption 1.3.4 ($x_a(t)$ strictly band-limited), it results that (Gardner et al. 2006, eqs. (3.100) and (3.111), Figure 1):*

$$\operatorname{supp}\left\{ S_{\boldsymbol{x}_a}^{\alpha}(f) \right\} \subseteq \{(\alpha, f) \in \mathbb{R} \times \mathbb{R} \; : \; |f| \leqslant B, \; |\alpha - f| \leqslant B\} \tag{1.176a}$$

$$\subseteq \{(\alpha, f) \in \mathbb{R} \times \mathbb{R} \; : \; |f| \leqslant B, \; |\alpha| \leqslant 2B\} \, . \tag{1.176b}$$

Therefore,

$$S_{\boldsymbol{x}_a}^{\alpha}(f) = 0 \quad |f| > B, \; |\alpha| > 2B. \tag{1.177}$$

Moreover, in (1.168) we have

$$\begin{aligned} &\text{supp}\left\{S^{\alpha}_{\boldsymbol{x}_a}(f)\delta(f_2 - {}_{(-)}(\alpha - f_1))\right\} \\ &\quad\subseteq \{(f_1, f_2) \in \mathbb{R}\times\mathbb{R} \,:\, |f_1| \leqslant B,\ |\alpha - f_1| \leqslant B,\ f_2 = {}_{(-)}(\alpha - f_1)\} \qquad (1.178a) \\ &\quad= \{(f_1, f_2) \in \mathbb{R}\times\mathbb{R} \,:\, |f_1| \leqslant B,\ |f_2| \leqslant B,\ f_2 = {}_{(-)}(\alpha - f_1)\}\,. \qquad (1.178b) \end{aligned}$$

□

Consequently, under Assumption 1.3.4 ($x_a(t)$ strictly band-limited), and accounting for (1.176a), the support of the replica with $n_1 = n_2 = 0$ in the aliasing formula (1.170) is given by

$$\begin{aligned} &\text{supp}\left\{S^{\alpha}_{\boldsymbol{x}_a}(\nu_1 f_s)\,\delta(\nu_2 - {}_{(-)}(\alpha/f_s - \nu_1))\right\} \\ &\quad\subseteq \{(\nu_1, \nu_2) \in \mathbb{R}\times\mathbb{R} \,:\, |\nu_1 f_s| \leqslant B,\ |\alpha - \nu_1 f_s| \leqslant B, \\ &\qquad \nu_2 - {}_{(-)}(\alpha/f_s - \nu_1) = 0\} \qquad (1.179a) \\ &\quad= \{(\nu_1, \nu_2) \in \mathbb{R}\times\mathbb{R} \,:\, |\nu_1| \leqslant B/f_s,\ |\nu_2| \leqslant B/f_s,\ \alpha = (\nu_1 + {}_{(-)}\nu_2) f_s\} \qquad (1.179b) \end{aligned}$$

Theorem 1.3.6 ACS Signals: Sampling Theorem for the Discrete-Time Loève Bifrequency Spectrum. *Under Assumption 1.3.4 and for $f_s \geqslant 4B$, it results in*

$$\mathrm{E}\left\{X(\nu_1)\,X^{(*)}(\nu_2)\right\} = \frac{1}{T_s} \sum_{\alpha\in A_{\boldsymbol{x}_a}} S^{\alpha}_{\boldsymbol{x}_a}(\nu_1 f_s)\,\delta\left(\nu_2 - {}_{(-)}(\alpha/f_s - \nu_1)\right) \qquad |\nu_1| \leqslant \frac{1}{2},\ |\nu_2| \leqslant \frac{1}{2}. \qquad (1.180)$$

Moreover,

$$\mathrm{E}\left\{X(\nu_1)\,X^{(*)}(\nu_2)\right\} = 0 \ \text{ for }\ 1/4 \leqslant |\nu_1| \leqslant 1/2 \ \text{ and }\ 1/4 \leqslant |\nu_2| \leqslant 1/2. \qquad (1.181)$$

Proof: Replicas in (1.170) are separated by 1 in both ν_1 and ν_2 variables. Thus, from (1.179b) it follows that $B/f_s \leqslant 1/2$, that is,

$$f_s \geqslant 2B \qquad (1.182)$$

is a *sufficient condition such that replicas do not overlap*. Note that, however, (1.182) does not assure that the mappings $f_1 = \nu_1 f_s$ and $\alpha = \widetilde{\alpha} f_s$ in

$$\widetilde{S}^{\widetilde{\alpha}}_{\boldsymbol{x}}(\nu_1) = \frac{1}{T_s}\, S^{\alpha}_{\boldsymbol{x}_a}(f_1)\Big|_{f_1=\nu_1 f_s, \alpha=\widetilde{\alpha} f_s} \qquad (1.183)$$

holds $\forall\widetilde{\alpha} \in [-1/2, 1/2)$ and $\forall\nu_1 \in [-1/2, 1/2)$. For example, for $(*)$ present and $\alpha = -B$, equality (1.183) holds only for $\nu_1 \in [-1/2, 0]$ and not for $\nu_1 \in [0, 1/2]$. In fact, for $\nu_1 \in [0, 1/2]$ the density of the replica with $n_1 = 0, n_2 = 1$ should be present in the right-hand side of (1.183).

In contrast, condition $B/f_s \leqslant 1/4$, that is,

$$f_s \geqslant 4B \qquad (1.184)$$

assures in (1.179b)

$$|\nu_1| \leqslant 1/4, \quad |\nu_2| \leqslant 1/4, \quad |\alpha/f_s| = |\nu_1 +_{(-)} \nu_2| \leqslant |\nu_1| + |\nu_2| \leqslant 1/2 \tag{1.185}$$

and, consequently, the mappings $f_1 = \nu_1 f_s$ and $\alpha = \widetilde{\alpha} f_s$ in (1.183) hold $\forall \widetilde{\alpha} \in [-1/2, 1/2)$ and $\forall \nu_1 \in [-1/2, 1/2)$. Moreover, (1.180) holds. Finally, note that the effect of sampling at two times the Nyquist rate (see (1.184)) leads also to (1.181). □

Theorem 1.3.7 Sampling Theorem for Cyclic Statistics of ACS Signals (Napolitano 1995), (Izzo and Napolitano 1996). *Under Assumption 1.3.4 and for $f_s \geqslant 4B$, the result is*

$$\widetilde{R}_{\boldsymbol{x}}^{\widetilde{\alpha}}(m) = R_{\boldsymbol{x}_a}^{\alpha}(\tau)\Big|_{\tau = mT_s, \alpha = \widetilde{\alpha} f_s} \qquad |\widetilde{\alpha}| \leqslant \frac{1}{2} \tag{1.186a}$$

$$R_{\boldsymbol{x}_a}^{\alpha}(\tau)\Big|_{\tau = mT_s} = \begin{cases} \widetilde{R}_{\boldsymbol{x}}^{\widetilde{\alpha}}(m)\Big|_{\widetilde{\alpha} = \alpha/f_s} & |\alpha| \leqslant \frac{f_s}{2} \\ 0 & \text{otherwise} \end{cases} \tag{1.186b}$$

$$R_{\boldsymbol{x}_a}^{\alpha}(\tau) = \sum_{m \in \mathbb{Z}} \widetilde{R}_{\boldsymbol{x}}^{\widetilde{\alpha}}(m) \operatorname{sinc}\left(\frac{\tau}{T_s} - m\right) \qquad \forall \tau \in \mathbb{R},\ \alpha = \widetilde{\alpha} f_s \tag{1.187}$$

$$\widetilde{S}_{\boldsymbol{x}}^{\widetilde{\alpha}}(\nu) = \frac{1}{T_s}\, S_{\boldsymbol{x}_a}^{\alpha}(f)\Big|_{f = \nu f_s, \alpha = \widetilde{\alpha} f_s} \qquad |\widetilde{\alpha}| \leqslant \frac{1}{2},\ |\nu| \leqslant \frac{1}{2} \tag{1.188a}$$

$$S_{\boldsymbol{x}_a}^{\alpha}(f) = \begin{cases} T_s\, \widetilde{S}_{\boldsymbol{x}}^{\widetilde{\alpha}}(\nu)\Big|_{\nu = f/f_s, \widetilde{\alpha} = \alpha/f_s} & |\alpha| \leqslant \frac{f_s}{2},\ |f| \leqslant \frac{f_s}{2} \\ 0 & \text{otherwise} \end{cases} \tag{1.188b}$$

where $\boldsymbol{x}_a = [x_a\, x_a^{(*)}]$ and $\boldsymbol{x} = [x\, x^{(*)}]$. By Fourier transforming both sides in (1.187) we get the following expression for the cyclic spectra of a strictly band-limited signal

$$S_{\boldsymbol{x}_a}^{\alpha}(f) = \sum_{m \in \mathbb{Z}} \widetilde{R}_{\boldsymbol{x}}^{\widetilde{\alpha}}(m)\, e^{-j2\pi f m T_s}\, T_s \operatorname{rect}(f T_s), \qquad \alpha = \widetilde{\alpha} f_s \tag{1.189}$$

which is equivalent to (1.188b). □

1.3.10 Multirate Processing of Discrete-Time ACS Signals

Expansion and decimation are linear time-variant transformations that are not almost-periodically time-variant. Consequently, the results of Section 1.3.3 cannot be used. In Section 4.3.1, it is observed that expansion and decimation belong to the class of linear time-variant systems that can be classified as deterministic in the FOT probability sense. These systems transform input ACS signals into output ACS signals with different almost-cyclostationarity characteristics (Sathe and Vaidyanathan 1993), (Izzo and Napolitano 1998b), (Akkarakaran and Vaidyanathan 2000). In contrast, in Section 4.10, it is shown that cross-statistics of signals generated with different rates by the same ACS signal give rise to signals that are jointly SC.

Results on multirate processing of discrete-time ACS processes are obtained as special cases of results on multirate processing of spectrally correlated processes in Section 4.10.

1.3.11 Applications

The existence of consistent estimators for the cyclic statistics is one of the main motivations for many applications of cyclostationarity in several fields such as communications, circuits, systems and control, acoustics, mechanics, econometrics, biology, and astronomy (Gardner *et al.* 2006). Applications in mechanics and vibroacoustic signal analysis are discussed in (Antoni 2009) and references therein.

In communications and radar/sonar applications, cyclostationarity properties can be used to design signal selective detection and estimation algorithms. In fact, in the case of additive noise uncorrelated with the signal of interest (SOI), if the SOI does not share at least one cycle frequency, say α_0, with the disturbance, then the cyclic autocorrelation function of the SOI plus noise at α_0 is coincident with the cyclic autocorrelation function of the SOI alone. Therefore, detection or estimation algorithms operating at α_0 are potentially immune to the effects of noise and interference, provided that a sufficiently long observation interval is adopted to estimate the cyclic statistics (Gardner 1985, 1987d), (Gardner and Chen 1992), (Chen and Gardner 1992), (Gardner and Spooner 1992, 1993), (Flagiello *et al.* 2000). In (Gardner *et al.* 2006) and references therein, applications of cyclostationarity for interference tolerant channel identification and equalization, signal detection and classification, source separation, periodic autoregressive (AR) and autoregressive moving-average (ARMA) modeling and prediction, are described. The spectral line regeneration property of ACS signal can be exploited for synchronization purposes (Gardner 1987d).

Cyclostationarity properties can be suitably exploited in minimum mean-square error (MMSE) linear filtering or Wiener filtering. If the useful signal, the data, and the disturbance signal are singularly and jointly ACS, then it can be shown that the optimum linear filter is linear almost-periodically time variant. By exploiting the spectral correlation property of ACS signals, the optimum filter cancels the spectral bands of the useful signal which are corrupted by noise and then reconstructs these bands exploiting the spectral components of the useful signal that are noise-free and correlated with the canceled bands (Gardner 1993). This MMSE filtering procedure is referred to as *cyclic Wiener filtering* or *frequency shift (FRESH) filtering*. It provides significant performance improvement with respect to the classical Wiener filtering that assumes a stationary model for the involved signals.

Second- and higher-order cyclostationarity properties can be exploited for modulation format classification. The classification is performed by comparing estimated second- and higher-order cyclic features of the received signal with those stored in a catalog (Spooner and Gardner 1994), (Spooner and Nicholls 2009).

1.4 Some Properties of Cumulants

In this section, some properties of cumulants that will be used in the subsequent chapters are reviewed. For comprehensive treatment of higher-order statistics of random variables and stationary and nonstationary signals see (Brillinger 1965, 1981), (Rosenblatt 1974), (Gardner and Spooner 1994), (Spooner and Gardner 1994), (Dandawaté and Giannakis 1994, 1995), (Boashash *et al.* 1995), (Napolitano 1995), (Izzo and Napolitano 1998a), (Napolitano and Tesauro 2011).

1.4.1 Cumulants and Statistical Independence

Theorem 1.4.1 *If two real-valued random-variable vectors* $\boldsymbol{X}_1 \triangleq [X_1, \ldots, X_h]^{\mathrm{T}}$ *and* $\boldsymbol{X}_2 \triangleq [X_{h+1}, \ldots, X_k]^{\mathrm{T}}$ *are statistically independent, then*

$$\operatorname{cum}\{X_1, \ldots, X_k\} = 0. \tag{1.190}$$

Proof: Let $\boldsymbol{X} \triangleq [\boldsymbol{X}_1^{\mathrm{T}}\ \boldsymbol{X}_2^{\mathrm{T}}]^{\mathrm{T}}$, $\boldsymbol{\omega}_1 \triangleq [\omega_1, \ldots, \omega_h]^{\mathrm{T}}$, $\boldsymbol{\omega}_2 \triangleq [\omega_{h+1}, \ldots, \omega_k]^{\mathrm{T}}$, and $\boldsymbol{\omega} \triangleq [\boldsymbol{\omega}_1^{\mathrm{T}}\ \boldsymbol{\omega}_2^{\mathrm{T}}]^{\mathrm{T}}$. The *characteristic function* of the random vector $\boldsymbol{X}$ factorizes into the product of the characteristic functions of $\boldsymbol{X}_1$ and $\boldsymbol{X}_2$. That is,

$$\begin{aligned} \mathrm{E}\left\{e^{j\boldsymbol{\omega}^{\mathrm{T}}\boldsymbol{X}}\right\} &= \mathrm{E}\left\{e^{j\boldsymbol{\omega}_1^{\mathrm{T}}\boldsymbol{X}_1}\, e^{j\boldsymbol{\omega}_2^{\mathrm{T}}\boldsymbol{X}_2}\right\} \\ &= \mathrm{E}\left\{e^{j\boldsymbol{\omega}_1^{\mathrm{T}}\boldsymbol{X}_1}\right\}\, \mathrm{E}\left\{e^{j\boldsymbol{\omega}_2^{\mathrm{T}}\boldsymbol{X}_2}\right\} \end{aligned} \tag{1.191}$$

where the second equality is consequence of the statistical independence of the random vectors $\boldsymbol{X}_1$ and $\boldsymbol{X}_2$. Thus,

$$\begin{aligned} \operatorname{cum}\{X_1, \ldots, X_k\} &\triangleq (-j)^k \frac{\partial^k}{\partial\omega_1 \cdots \partial\omega_k} \log_e \mathrm{E}\left\{e^{j\boldsymbol{\omega}^{\mathrm{T}}\boldsymbol{X}}\right\}\Big|_{\boldsymbol{\omega}=\mathbf{0}} \\ &= (-j)^k \frac{\partial^k}{\partial\omega_1 \cdots \partial\omega_k} \left[\log_e \mathrm{E}\left\{e^{j\boldsymbol{\omega}_1^{\mathrm{T}}\boldsymbol{X}_1}\right\} + \log_e \mathrm{E}\left\{e^{j\boldsymbol{\omega}_2^{\mathrm{T}}\boldsymbol{X}_2}\right\}\right]_{\boldsymbol{\omega}=\mathbf{0}} \\ &= 0 \end{aligned} \tag{1.192}$$

where the kth-order derivative of each of the two terms in the square brackets is zero since the first one depends only on $\boldsymbol{\omega}_1 \triangleq [\omega_1, \ldots, \omega_h]^{\mathrm{T}}$, and the second one depends only on $\boldsymbol{\omega}_2 \triangleq [\omega_{h+1}, \ldots, \omega_k]^{\mathrm{T}}$.

This result can be extended with minor changes to the case of complex-valued random-variable vectors. □

As a consequence of Theorem 1.4.1, we have that if the stochastic process is asymptotically independent, that is, for every t the random variables $x(t)$ and $x(t+\tau)$ are asymptotically ($|\tau| \to \infty$) independent, then

$$\operatorname{cum}\{x(t),\ x(t+\tau_i), i = 1, \ldots, k-1\} \to 0 \quad \text{as } \|\boldsymbol{\tau}\| \to \infty \tag{1.193}$$

where $\|\boldsymbol{\tau}\|^2 \triangleq \tau_1^2 + \cdots + \tau_{k-1}^2$. In fact, if $\|\boldsymbol{\tau}\| \to \infty$, then at least one $\tau_i \to \infty$ and $x(t+\tau_i)$ becomes statistically independent of $x(t)$.

1.4.2 Cumulants of Complex Random Variables and Joint Complex Normality

In this section, the non-obvious result is proved that the Nth-order cumulant for complex random variables defined in (Spooner and Gardner 1994, App. A) (see also (1.209)) is zero for $N \geqslant 3$ when the random variables are jointly complex Normal (Napolitano 2007a, App. E).

Let $\boldsymbol{V} = [V_1, \ldots, V_N, V_{N+1}, \ldots, V_{2N}]^{\mathrm{T}} \triangleq [\boldsymbol{X}^{\mathrm{T}}, \boldsymbol{Y}^{\mathrm{T}}]^{\mathrm{T}}$ be the $2N$-dimensional column vector of real-valued random variables obtained by the N-dimensional column vectors $\boldsymbol{X} \triangleq [X_1, \ldots, X_N]^{\mathrm{T}}$, and $\boldsymbol{Y} \triangleq [Y_1, \ldots, Y_N]^{\mathrm{T}}$ of real-valued random variables. It is characterized by the $2N$th-order joint probability density function (pdf)

$$f_{\boldsymbol{V}}(\boldsymbol{v}) = f_{V_1 \cdots V_{2N}}(v_1, \ldots, v_{2N}) = f_{X_1 \ldots X_N Y_1 \ldots Y_N}(x_1, \ldots, x_N, y_1, \ldots, y_N). \tag{1.194}$$

Its *moment generating function* is the $2N$-dimensional Laplace transform

$$\Phi_V(\boldsymbol{s}) \triangleq \mathrm{E}\{e^{\boldsymbol{s}^{\mathrm{T}}\boldsymbol{V}}\} = \int_{\mathbb{R}^{2N}} f_{\boldsymbol{V}}(\boldsymbol{v})\, e^{\boldsymbol{s}^{\mathrm{T}}\boldsymbol{v}}\, \mathrm{d}\boldsymbol{v} \tag{1.195}$$

with $\boldsymbol{s} \triangleq [\boldsymbol{s}_X^{\mathrm{T}}, \boldsymbol{s}_Y^{\mathrm{T}}]^{\mathrm{T}} \in \mathbb{C}^{2N}$, which is analytic in the region of convergence of the integral. The mean vector of $\boldsymbol{V}$ is

$$\boldsymbol{\mu}_V \triangleq \mathrm{E}\{\boldsymbol{V}\} = [\boldsymbol{\mu}_X^{\mathrm{T}}, \boldsymbol{\mu}_Y^{\mathrm{T}}]^{\mathrm{T}} \tag{1.196}$$

where $\boldsymbol{\mu}_X$ and $\boldsymbol{\mu}_Y$ are the mean vectors of $\boldsymbol{X}$ and $\boldsymbol{Y}$, respectively. The covariance matrix is

$$\boldsymbol{C}_{VV} \triangleq \mathrm{E}\{(\boldsymbol{V} - \mathrm{E}\{\boldsymbol{V}\})(\boldsymbol{V} - \mathrm{E}\{\boldsymbol{V}\})^{\mathrm{T}}\} = \begin{bmatrix} \boldsymbol{C}_{XX} & \boldsymbol{C}_{XY} \\ \boldsymbol{C}_{YX} & \boldsymbol{C}_{YY} \end{bmatrix} \tag{1.197}$$

where $\boldsymbol{C}_{XX}$ and $\boldsymbol{C}_{YY}$ are the covariance matrices of $\boldsymbol{X}$ and $\boldsymbol{Y}$, respectively, and

$$\boldsymbol{C}_{XY} \triangleq \mathrm{E}\{(\boldsymbol{X} - \mathrm{E}\{\boldsymbol{X}\})(\boldsymbol{Y} - \mathrm{E}\{\boldsymbol{Y}\})^{\mathrm{T}}\} = \boldsymbol{C}_{YX}^{\mathrm{T}} \tag{1.198}$$

is their cross-covariance matrix.

The $2N$th-order cumulant of the real random variables $V_1, \ldots, V_{2N}$ is defined as

$$\begin{aligned} \mathrm{cum}\{V_1, \ldots, V_{2N}\} \triangleq (-j)^{2N} & \frac{\partial^{2N}}{\partial \omega_{X1} \cdots \partial \omega_{XN} \partial \omega_{Y1} \cdots \partial \omega_{YN}} \\ & \log_e \mathrm{E}\left\{\exp[j(\boldsymbol{\omega}_X^{\mathrm{T}} \boldsymbol{X} + \boldsymbol{\omega}_Y^{\mathrm{T}} \boldsymbol{Y})]\right\}\big|_{\boldsymbol{\omega}_X = \boldsymbol{\omega}_Y = \boldsymbol{0}} \end{aligned} \tag{1.199}$$

where $\boldsymbol{\omega}_X \triangleq [\omega_{X1}, \ldots, \omega_{XN}]^{\mathrm{T}}$ and $\boldsymbol{\omega}_Y \triangleq [\omega_{Y1}, \ldots, \omega_{YN}]^{\mathrm{T}}$.

Let us consider, now, the N-dimensional complex-valued column vector $\boldsymbol{Z} = [Z_1, \ldots, Z_N]^{\mathrm{T}} \triangleq \boldsymbol{X} + j\boldsymbol{Y}$. It is characterized by the same $2N$th-order joint pdf $f_{\boldsymbol{V}}(\boldsymbol{v})$ of the $2N$-dimensional real-valued vector $\boldsymbol{V}$ defined above.

Instead of considering the vector $\boldsymbol{V}$, the following complex augmented-dimension vector

$$\boldsymbol{\zeta} \triangleq \begin{bmatrix} \boldsymbol{Z} \\ \boldsymbol{Z}^* \end{bmatrix} \tag{1.200}$$

can be considered. Its mean vector and covariance matrix are given by

$$\boldsymbol{\mu}_\zeta \triangleq \begin{bmatrix} \mathrm{E}\{\boldsymbol{Z}\} \\ \mathrm{E}\{\boldsymbol{Z}^*\} \end{bmatrix} \tag{1.201}$$

$$\boldsymbol{\Gamma} \triangleq \mathrm{E}\left\{(\boldsymbol{\zeta}-\boldsymbol{\mu}_\zeta)(\boldsymbol{\zeta}-\boldsymbol{\mu}_\zeta)^{\mathrm{H}}\right\} = \begin{bmatrix} \boldsymbol{C}_{ZZ^*} & \boldsymbol{C}_{ZZ} \\ \boldsymbol{C}_{ZZ}^* & \boldsymbol{C}_{ZZ^*}^* \end{bmatrix} \tag{1.202}$$

where

$$\begin{aligned} \boldsymbol{C}_{ZZ^*} &\triangleq \operatorname{cov}\{\boldsymbol{Z},\boldsymbol{Z}\} = \mathrm{E}\left\{(\boldsymbol{Z}-\mathrm{E}\{\boldsymbol{Z}\})(\boldsymbol{Z}-\mathrm{E}\{\boldsymbol{Z}\})^{\mathrm{H}}\right\} \\ &= \boldsymbol{C}_{XX} + \boldsymbol{C}_{YY} - j\boldsymbol{C}_{XY} + j\boldsymbol{C}_{YX} \end{aligned} \tag{1.203}$$

$$\begin{aligned} \boldsymbol{C}_{ZZ} &\triangleq \operatorname{cov}\left\{\boldsymbol{Z},\boldsymbol{Z}^{\mathrm{H}}\right\} = \mathrm{E}\left\{(\boldsymbol{Z}-\mathrm{E}\{\boldsymbol{Z}\})(\boldsymbol{Z}-\mathrm{E}\{\boldsymbol{Z}\})^{\mathrm{T}}\right\} \\ &= \boldsymbol{C}_{XX} - \boldsymbol{C}_{YY} + j\boldsymbol{C}_{YX} + j\boldsymbol{C}_{XY} \end{aligned} \tag{1.204}$$

$$\boldsymbol{C}_{XX} = \frac{1}{2}\mathrm{Re}\{\boldsymbol{C}_{ZZ^*} + \boldsymbol{C}_{ZZ}\} \tag{1.205}$$

$$\boldsymbol{C}_{YY} = \frac{1}{2}\mathrm{Re}\{\boldsymbol{C}_{ZZ^*} - \boldsymbol{C}_{ZZ}\} \tag{1.206}$$

$$\boldsymbol{C}_{XY} = -\frac{1}{2}\mathrm{Im}\{\boldsymbol{C}_{ZZ^*} - \boldsymbol{C}_{ZZ}\} \tag{1.207}$$

$$\boldsymbol{C}_{YX} = \frac{1}{2}\mathrm{Im}\{\boldsymbol{C}_{ZZ^*} + \boldsymbol{C}_{ZZ}\} \tag{1.208}$$

with $\boldsymbol{C}_{ZZ^*}$ and $\boldsymbol{C}_{ZZ}$ referred to as *covariance matrix* and *conjugate covariance matrix*, respectively.

The Nth-order cumulant of the complex random variables $Z_1, \ldots, Z_N$ can be defined as (Spooner and Gardner 1994, App. A)

$$\begin{aligned} \operatorname{cum}\{Z_1,\ldots,Z_N\} &\triangleq (-j)^N \frac{\partial^N}{\partial\omega_1\cdots\partial\omega_N} \log_e \mathrm{E}\left\{\exp[j\boldsymbol{\omega}^{\mathrm{T}}(\boldsymbol{X}+j\boldsymbol{Y})]\right\}\Big|_{\boldsymbol{\omega}=\mathbf{0}} \\ &= \sum_{\mathrm{P}} (-1)^{p-1}(p-1)! \prod_{i=1}^{p} \mathrm{E}\Big\{\prod_{\ell\in\mu_i} Z_\ell\Big\} \end{aligned} \tag{1.209}$$

where $\boldsymbol{\omega} \triangleq [\omega_1,\ldots,\omega_N]^{\mathrm{T}}$ and P are the set of distinct partitions of $\{1,\ldots,N\}$ each constituted by the subsets $\{\mu_i,\ i=1,\ldots,p\}$. Note that since each complex variable Z_k is arbitrary, it can also be the complex conjugate of another complex variable. Such a definition turns out to be useful when applied to complex-valued stochastic processes or time series as shown in (Spooner and Gardner 1994, App. A), (Izzo and Napolitano 1998a), (Izzo and Napolitano 2002a). In particular, it is useful since it preserves the same relationship between moments and cumulants of real random variables (Leonov and Shiryaev 1959), (Brillinger 1965, 1981), (Brillinger and Rosenblatt 1967), as such a relationship is purely algebric. In addition, the characteristic function $\mathrm{E}\left\{\exp[j\boldsymbol{\omega}^{\mathrm{T}}(\boldsymbol{X}+j\boldsymbol{Y})]\right\}$ in (1.209) is an analytic function of the complex variables $X_k + jY_k$, $k=1,\ldots,N$. In contrast, the characteristic function $\mathrm{E}\left\{\exp[j(\boldsymbol{\omega}_X^{\mathrm{T}}\boldsymbol{X}+\boldsymbol{\omega}_Y^{\mathrm{T}}\boldsymbol{Y})]\right\}$ in (1.199) depends separately on X_k and Y_k and hence, in general, is not an analytic function of the complex variables $X_k + jY_k$. Finally, note that, for deterministic liner time-variant systems, input/output relationships in terms of cumulants defined as in (1.209) have the same form as that in terms of moments (Napolitano 1995), (Izzo and Napolitano 2002a).

Let $\boldsymbol{Z}$ be a N-dimensional column vector of jointly complex Normal random variables. That is, $\boldsymbol{V}$ is a $2N$-dimensional column vector of jointly Normal real-valued random variables:

$$f_{\boldsymbol{V}}(\boldsymbol{v}) = \frac{1}{(2\pi)^N |\det \boldsymbol{C}_{VV}|^{1/2}} \exp\left[-\frac{1}{2}(\boldsymbol{v}-\boldsymbol{\mu}_V)^{\mathrm{T}} \boldsymbol{C}_{VV}^{-1}(\boldsymbol{v}-\boldsymbol{\mu}_V)\right] \tag{1.210}$$

that can also be written in the complex form (Picinbono 1996), (van den Bos 1995)

$$f_{\boldsymbol{Z},\boldsymbol{Z}^*}(\boldsymbol{z},\boldsymbol{z}^*) = \frac{1}{\pi^N |\det \boldsymbol{\Gamma}|^{1/2}} \exp\left[-\frac{1}{2}(\boldsymbol{\zeta_1}-\boldsymbol{\mu}_\zeta)^{\mathrm{H}} \boldsymbol{\Gamma}^{-1}(\boldsymbol{\zeta_1}-\boldsymbol{\mu}_\zeta)\right] \tag{1.211}$$

where $\boldsymbol{\zeta_1} \triangleq [\boldsymbol{z}^{\mathrm{T}}, \boldsymbol{z}^{\mathrm{H}}]^{\mathrm{T}}$, which is denoted as $\mathcal{N}(\boldsymbol{\mu}_\zeta, \boldsymbol{C}_{ZZ^*}, \boldsymbol{C}_{ZZ})$.

The moment-generating function (1.195) of $f_{\boldsymbol{V}}(\boldsymbol{v})$ is the $2N$-dimensional Laplace transform

$$\Phi_V(\boldsymbol{s}) = \exp[\boldsymbol{s}^{\mathrm{T}} \boldsymbol{\mu}_V] \; \exp\left[\frac{1}{2} \boldsymbol{s}^{\mathrm{T}} \boldsymbol{C}_{VV} \boldsymbol{s}\right] \tag{1.212}$$

whose region of convergence is the whole complex space $\mathbb{C}^{2N}$. Therefore, both characteristic functions involved in the cumulant definitions (1.199) and (1.209) can be expressed as slices of the moment-generating function $\Phi_V(\boldsymbol{s})$ given in (1.212). Specifically, in (1.199) we have

$$\begin{aligned} &\mathrm{E}\left\{\exp[j(\boldsymbol{\omega}_X^{\mathrm{T}} \boldsymbol{X} + \boldsymbol{\omega}_Y^{\mathrm{T}} \boldsymbol{Y})]\right\} \\ &\quad = \Phi_V(\boldsymbol{s})|_{s_X = j\omega_X, s_Y = j\omega_Y} \\ &\quad = \exp[j(\boldsymbol{\omega}_X^{\mathrm{T}} \boldsymbol{\mu}_X + \boldsymbol{\omega}_Y^{\mathrm{T}} \boldsymbol{\mu}_Y)] \\ &\qquad \exp\left[-\frac{1}{2}\left(\boldsymbol{\omega}_X^{\mathrm{T}} \boldsymbol{C}_{XX} \boldsymbol{\omega}_X + \boldsymbol{\omega}_X^{\mathrm{T}} \boldsymbol{C}_{XY} \boldsymbol{\omega}_Y + \boldsymbol{\omega}_Y^{\mathrm{T}} \boldsymbol{C}_{YX} \boldsymbol{\omega}_X + \boldsymbol{\omega}_Y^{\mathrm{T}} \boldsymbol{C}_{YY} \boldsymbol{\omega}_Y\right)\right] \end{aligned} \tag{1.213}$$

from which it follows that $\log_e \mathrm{E}\left\{\exp[j(\boldsymbol{\omega}_X^{\mathrm{T}} \boldsymbol{X} + \boldsymbol{\omega}_Y^{\mathrm{T}} \boldsymbol{Y})]\right\}$ is a quadratic homogeneous polynomial in the real variables $\omega_{X1}, \ldots, \omega_{XN}, \omega_{Y1}, \ldots, \omega_{YN}$. As a consequence, we have the well-known result that jointly real-valued Normal random variables have kth-order cumulants of order $k \geqslant 3$ equal to zero. In addition, in (1.209) we have

$$\begin{aligned} &\mathrm{E}\left\{\exp[j\boldsymbol{\omega}^{\mathrm{T}}(\boldsymbol{X} + j\boldsymbol{Y})]\right\} \\ &\quad = \Phi_V(\boldsymbol{s})|_{s_X = j\omega, s_Y = -\omega} \\ &\quad = \exp[j\boldsymbol{\omega}^{\mathrm{T}} \boldsymbol{\mu}_X - \boldsymbol{\omega}^{\mathrm{T}} \boldsymbol{\mu}_Y] \\ &\qquad \exp\left[-\frac{1}{2}\boldsymbol{\omega}^{\mathrm{T}}(\boldsymbol{C}_{XX} - \boldsymbol{C}_{YY})\boldsymbol{\omega} - j\frac{1}{2}\boldsymbol{\omega}^{\mathrm{T}}(\boldsymbol{C}_{XY} + \boldsymbol{C}_{YX})\boldsymbol{\omega}\right]. \end{aligned} \tag{1.214}$$

Hence, $\log_e \mathrm{E}\left\{\exp[j\boldsymbol{\omega}^{\mathrm{T}}(\boldsymbol{X} + j\boldsymbol{Y})]\right\}$ is a quadratic homogeneous polynomial in the real variables ω_i. Therefore, jointly *complex* Normal random variables are characterized by the fact that their cumulants of order $N \geqslant 3$ defined as in (1.209) are equal to zero.