IN THIS CHAPTER

» Measuring the area of shapes by using classical and analytic geometry

» Using integration to frame the area problem

» Approximating area using Riemann sums

» Applying integration to more complex problems

» Seeing how differential equations are related to integrals

» Looking at sequences and series

Chapter 1
An Aerial View of the Area Problem

Humans have been measuring the area of shapes for thousands of years. One practical use for this skill is measuring the area of a parcel of land. Measuring the area of a square or a rectangle is simple, so land tends to get divided into these shapes.

Discovering the area of a triangle, circle, or polygon is also relatively easy, but as shapes get more unusual, measuring them gets harder. Although the Greeks were familiar with the conic sections — parabolas, ellipses, and hyperbolas — they couldn't reliably measure shapes with edges based on these figures.

René Descartes's invention of analytic geometry — studying lines and curves as equations plotted on the *xy*-graph — brought great insight into the relationships among the conic sections. But even analytic geometry didn't answer the question of how to measure the area inside a shape that includes a curve.

This bit of mathematical history is interesting in its own right, but I tell the story in order to give you, the reader, a sense of what drove those who came up with the concepts that eventually got bundled together as part of a standard Calculus II course. I start out by showing you how *integral calculus* (*integration* for short) was developed from attempts to answer this basic question of measuring the area of weird shapes, called the *area problem.* To do this, you will discover how to approximate the area under a parabola on the *xy*-graph in ways that lead to an ordered system of measuring the exact area under any function.

First, I frame the problem using a tool from calculus called the *definite integral*. I show you how to use the definite integral to define the areas of shapes you already know how to measure, such as circles, squares, and triangles.

With this introduction to the definite integral, you're ready to look at the practicalities of measuring area. The key to approximating an area that you don't know how to measure is to slice it into shapes that you do know how to measure — for example, rectangles. This process of slicing unruly shapes into nice, crisp rectangles — called finding a *Riemann sum* — provides the basis for calculating the exact value of a definite integral.

At the end of this chapter, I give you a glimpse into the more advanced topics in a basic Calculus II course, such as finding volume of unusual solids, looking at some basic differential equations, and understanding infinite series.

Checking Out the Area

Finding the area of certain basic shapes — squares, rectangles, triangles, and circles — is easy using geometric formulas you typically learn in a geometry class. But a reliable method for finding the exact area of shapes containing more esoteric curves eluded mathematicians for centuries. In this section, I give you the basics of how this problem, called the *area problem,* is formulated in terms of a new concept, the definite integral.

The *definite integral* represents the area of a region bounded by the graph of a function, the *x*-axis, and two vertical lines located at the *bounds of integration.* Without getting too deep into the computational methods of integration, I give you the basics of how to state the area problem formally in terms of the definite integral.

Comparing classical and analytic geometry

In *classical geometry*, you discover a variety of simple formulas for finding the area of different shapes. For example, Figure 1-1 shows the formulas for the area of a rectangle, a triangle, and a circle.

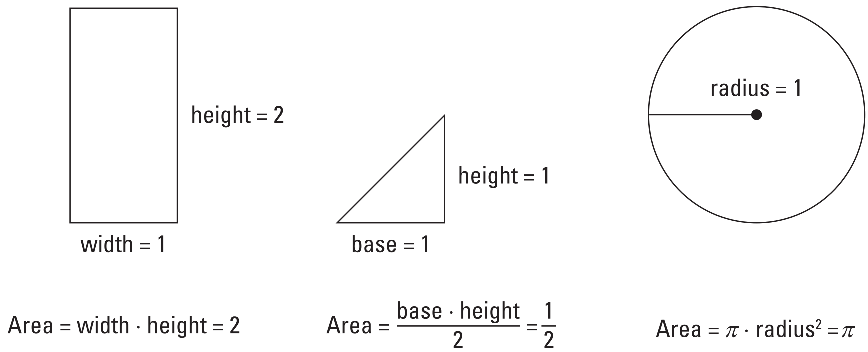

FIGURE 1-1: Formulas for the area of a rectangle, a triangle, and a circle.

On the *xy*-graph, you can generalize the problem of finding area to measure the area under any continuous function of x. To illustrate how this works, the shaded region in Figure 1-2 shows the area under the function $f(x)$ between the vertical lines $x = a$ and $x = b$.

The area problem is all about finding the area under a continuous function between two constant values of x that are called the *bounds of integration*, usually denoted by a and b. This problem is generalized as follows:

$$\text{Area} = \int_a^b f(x)\, dx$$

WISDOM OF THE ANCIENTS

Long before calculus was invented, the ancient Greek mathematician Archimedes used his *method of exhaustion* to calculate the exact area of a segment of a parabola. He was also the first mathematician to come up with an approximation for π (pi) within about a 0.2% margin of error.

Indian mathematicians also developed *quadrature* methods for some difficult shapes before Europeans began their investigations in the 17th century.

These methods anticipated some of the methods of calculus. But before calculus, no single theory could measure the area under arbitrary curves.

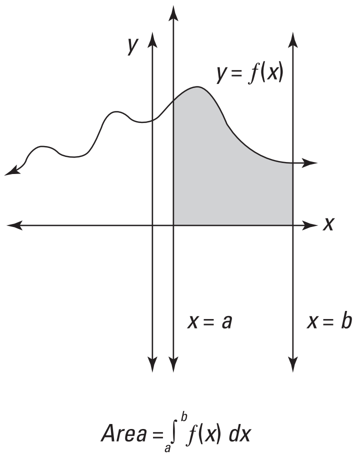

FIGURE 1-2: A typical area problem.

In a sense, this formula for the shaded area isn't much different from the geometric formulas you already know. It's just a formula, which means that if you plug in the right numbers and calculate, you get the right answer.

For example, suppose you want to measure the area under the function x^2 between $x = 1$ and $x = 5$. (You can see what this area looks like by flipping a few pages forward to Figure 1-5.) Here's how you plug these values into the area formula shown previously:

$$\text{Area} = \int_1^5 x^2 \, dx$$

The catch, however, is how exactly to calculate using this new symbol. As you may have figured out, the answer is on the cover of this book: calculus. To be more specific, *integral calculus,* or *integration.*

REMEMBER

Most typical Calculus II courses taught at your friendly neighborhood college or university focus on integration — the study of how to solve the area problem. So, if what you're studying starts to get confusing (and to be honest, you probably will get confused somewhere along the way), try to relate what you're doing to this central question: "How does what I'm working on help me find the area under a function?"

Finding definite answers with the definite integral

You may be surprised to find out that you've known how to integrate some functions for years without even knowing it. (Yes, you can know something without knowing that you know it.)

For example, find the rectangular area under the function $y = 2$ between $x = 1$ and $x = 4$, as shown in Figure 1-3.

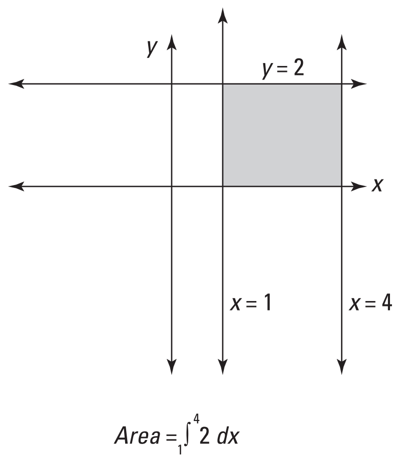

FIGURE 1-3: The rectangular area under the function $f(x) = 2$, between $a = 1$ and $b = 4$ equals 6.

This is just a rectangle with a base of 3 and a height of 2, so its area is 6. But this is also an area problem that can be stated in terms of integration as follows:

$$\text{Area} = \int_1^4 2\,dx = 6$$

As you can see, the function I'm integrating here is $f(x) = 2$. The bounds of integration are 1 and 4 (notice that the greater value goes on top). You already know that the area is 6, so you can solve this calculus problem without resorting to any scary or hairy methods. But you're still *integrating,* so please pat yourself on the back, because I can't quite reach it from here.

The following expression is called a *definite integral:*

$$\int_1^4 2\,dx$$

For now, don't spend too much time worrying about the deeper meaning behind the $\int$ symbol or the *dx* (which you may fondly remember from your days spent differentiating in Calculus I). Just think of $\int$ and *dx* as notation placed around a function — notation that means *area.*

What's so definite about a definite integral? Two things, really:

» **You definitely know the bounds of integration** (in this case, 1 and 4). Their presence distinguishes a definite integral from an indefinite integral, which you find out about in Chapter 5. Definite integrals always include the bounds of integration; indefinite integrals never include them.

- **A definite integral definitely equals a number** (assuming that its limits of integration are also numbers). This number may be simple to find or difficult enough to require a room full of math professors scribbling away with #2 pencils. But, at the end of the day, a number is just a number. And, because a definite integral is a measurement of area, you should expect the answer to be a number.

TECHNICAL STUFF

When the limits of integration *aren't* numbers, a definite integral doesn't necessarily equal a number. For example, expressions such as k and $2k$ might be used as limits of integration to stand in for constants. In such cases, the answer to a definite integral may include the letter k. Similarly, a definite integral whose limits of integration are $\sin\theta$ and $2\sin\theta$ would most likely equal a trig expression that includes θ. To sum up, because a definite integral represents an area, it always equals a number — though you may or may not be able to compute this number.

As another example, find the triangular area under the function $y = x$, between $x = 0$ and $x = 8$, as shown in Figure 1-4.

This time, the shape of the shaded area is a triangle with a base of 8 and a height of 8, so its area is 32 (because the area of a triangle is half the base times the height). But again, this is an area problem that can be stated in terms of integration as follows:

$$\text{Area} = \int_0^8 x\,dx = 32$$

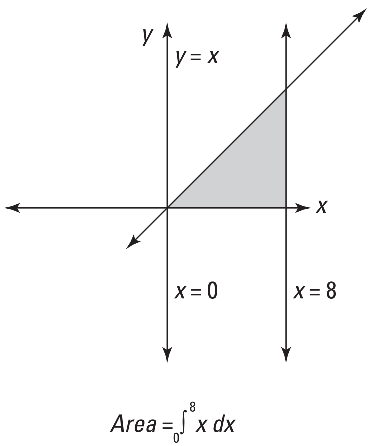

FIGURE 1-4: The triangular area under the function $y = x$, between $x = 0$ and $x = 8$ equals 32.

The function I'm integrating here is $f(x) = x$ and the bounds of integration are 0 and 8. Again, you can evaluate this integral with methods from classical and analytic geometry. And, again, the definite integral evaluates to a number, which is the area below the function and above the x-axis between $x = 0$ and $x = 8$.

Slicing Things Up

One good way of approaching a difficult task — from planning a wedding to climbing Mount Everest — is to break it down into smaller and more manageable pieces.

In this section, I show you the basics of how mathematician Bernhard Riemann used this same type of approach to calculate the definite integral using his self-named *Riemann sums*, which I introduce in the earlier section "Checking Out the Area." Throughout this section I use the example of the area under the function $y = x^2$, between $x = 1$ and $x = 5$. You can find this example in Figure 1-5.

$$\text{Area} = \int_1^5 x^2 dx$$

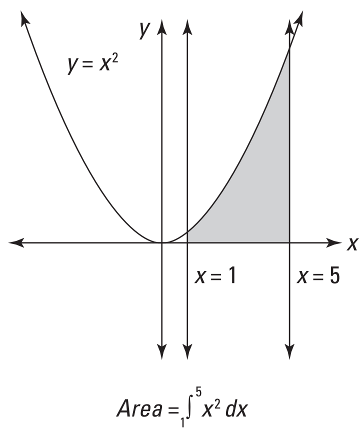

FIGURE 1-5: The area under the function $y = x^2$, between $x = 1$ and $x = 5$.

Untangling a hairy problem using rectangles

The earlier section "Checking Out the Area" tells you how to write the definite integral that represents the area of the shaded region in Figure 1-5:

$$\text{Area} = \int_1^5 x^2 dx$$

Unfortunately, this definite integral — unlike those earlier in this chapter — doesn't respond to the methods of classical and analytic geometry that I use to solve the earlier problems. (If it did, integrating would be much easier and this book would be a lot thinner!)

Even though you can't solve this definite integral directly (yet!), you can approximate it by slicing the shaded region into two pieces, as shown in Figure 1-6.

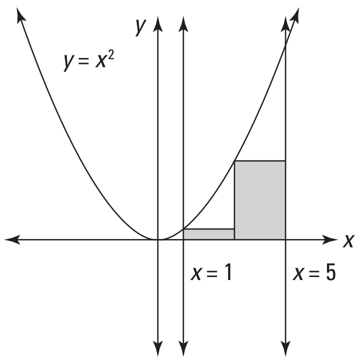

FIGURE 1-6: Area approximated by two rectangles.

Obviously, the region that's now shaded — it looks roughly like two steps going up but leading nowhere — is less than the area that you're trying to find. Fortunately, these steps do lead someplace, because calculating the area under them is fairly easy.

Each rectangle has a width of 2. The tops of the two rectangles cut across where the function x^2 meets $x = 1$ and $x = 3$, so their heights are 1 and 9, respectively. So the total area of the two rectangles is 20, because

$$2\,(1) + 2\,(9) = 2\,(1 + 9) = 2\,(10) = 20$$

With this approximation of the area of the original shaded region, here's the conclusion you can draw:

$$\int_1^5 x^2 dx \approx 20$$

Granted, this is a ballpark approximation with a really big ballpark. But even a lousy approximation is better than none at all. To get a better approximation, try cutting the figure that you're measuring into a few more slices, as shown in Figure 1-7.

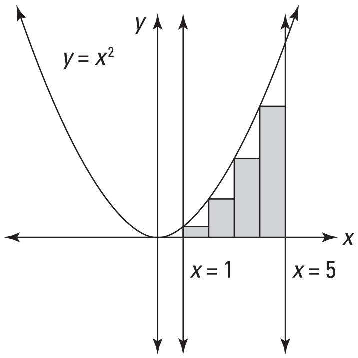

FIGURE 1-7: A closer approximation, where the area is approximated by four rectangles.

Again, this approximation is going to be less than the actual area that you're seeking. This time, each rectangle has a width of 1. And the tops of the four rectangles cut across where the function x^2 meets $x = 1$, $x = 2$, $x = 3$, and $x = 4$, so their heights are 1, 4, 9, and 16, respectively. So the total area of the four rectangles is 30, because

$$1\,(1) + 1\,(4) + 1\,(9) + 1\,(16) = 1\,(1 + 4 + 9 + 16) = 1\,(30) = 30$$

Therefore, here's a second approximation of the shaded area that you're seeking:

$$\int_1^5 x^2 dx \approx 30$$

Your intuition probably tells you that your second approximation is better than your first, because slicing the rectangles more thinly allows them to cut in closer to the function. You can verify this intuition by realizing that both 20 and 30 are *less* than the actual area, so whatever this area turns out to be, 30 must be closer to it.

You might imagine that by slicing the area into more rectangles (say 10, or 100, or 1,000,000), you'd get progressively better estimates. And, again, your intuition would be correct: As the number of slices increases, the result approaches 41.3333 . . . .

In fact, after enough calculation, you may very well decide to write:

$$\int_1^5 x^2 dx = 41.\overline{3}$$

This, in fact, is the correct answer. And it's a good start on solving the area problem. But to verify it, you'll need a more reliable overall method.

Moving left, right, or center

In the previous section, I slice the area I wish to measure into four rectangles in Figure 1-8.

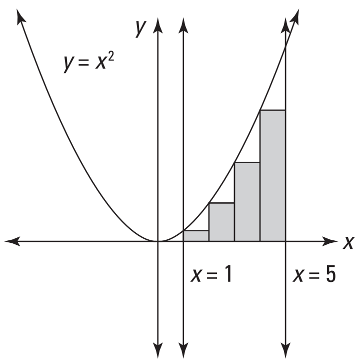

FIGURE 1-8: Approximating area with left rectangles.

As you can see, the heights of the four rectangles are determined by the value of $f(x)$ when x is equal to 1, 2, 3, and 4, respectively — that is, $f(1)$, $f(2)$, $f(3)$, and $f(4)$. Notice that the upper-left corner of each rectangle touches the function and determines the height of each rectangle. This process is called *approximating area with left rectangles*.

However, you can also *approximate area with right rectangles* by drawing the rectangles as shown in Figure 1-9.

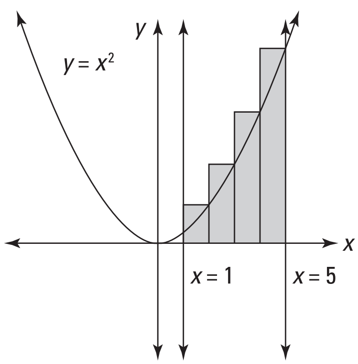

FIGURE 1-9: Approximating area with right rectangles.

In this case, the upper-right corner touches the function, so the heights of the four rectangles are $f(2)$, $f(3)$, $f(4)$, and $f(5)$.

Additionally, you can *approximate area with midpoint rectangles* by drawing the rectangles as shown in Figure 1-10.

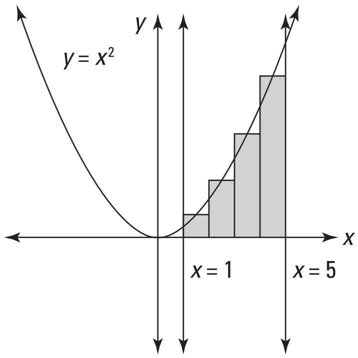

FIGURE 1-10: Approximating area with midpoint rectangles.

This time, the midpoint of the top edge of each rectangle touches the function, so the heights of the rectangles are $f(1.5)$, $f(2.5)$, $f(3.5)$, and $f(4.5)$.

Approximations like these are called *Riemann sums*. In Chapter 4, you work with a variety of methods for calculating Riemann sums.

Defining the Indefinite

Riemann sums allow you to approximate and even calculate areas that you can't measure using classical or analytic geometry. The downside of this method, however, is that it's quite complicated. In fact, at this point most students throw their hands up and say, "There has to be a better way!"

The better way is called the *indefinite integral.* The indefinite integral looks a lot like the definite integral. Compare for yourself:

Definite Integrals	Indefinite Integrals
$\int_1^5 x^2 dx$	$\int x^2 dx$
$\int_0^\pi \sin x \, dx$	$\int \sin x \, dx$
$\int_{-1}^1 e^x dx$	$\int e^x dx$

Like the definite integral, the indefinite integral is a tool for measuring the area under a function. A quick way to tell them apart is by noticing that the definite integral includes bounds of integration and the indefinite integral omits them.

Indefinite integrals provide an easier and faster way of finding the area under a curve. This method relies on finding a general algebraic solution to an indefinite integral, and then plugging in the bounds of integration to this solution to calculate a numerical value for the area. Chapter 5 gives you the details of how definite and indefinite integrals are related.

Indefinite integrals also provide a convenient way to calculate definite integrals. In fact, the indefinite integral is the *inverse* of the derivative, which you know from Calculus I. (Don't worry if you don't remember all about the derivative — Chapter 3 gives you a thorough review.) By inverse, I mean that the indefinite integral of a function is really the *antiderivative* of that function. This connection between integration and differentiation is more than just an odd little fact: It leads to the *Fundamental Theorem of Calculus* (FTC).

For example, you know from Calculus I that the derivative of x^2 is $2x$. So you should expect that the antiderivative — that is, the indefinite integral — of $2x$ is x^2. This is fundamentally correct with one small tweak, as I explain in Chapter 5.

Seeing integration as anti-differentiation allows you to solve tons of integrals without resorting to the Riemann sum formula for integration. But integration by finding the antiderivative can still be sticky depending on the function that you're trying to integrate. Mathematicians have developed a wide variety of algebraic techniques for evaluating integrals. Some of these methods are variable substitution (see Chapter 8), integration by parts (see Chapter 9), trig substitution (see Chapter 10), and integration by partial fractions (see Chapter 11).

Solving Problems with Integration

After you understand how to describe an area problem using the definite integral (Part 2 of this book), and how to calculate integrals (Parts 2, 3, and 4), you're ready to get into action solving a wide range of problems.

Some of these problems know their place and stay in two dimensions. Others rise up and create a revolution in three dimensions. In this section, I give you a taste of these types of problems, with an invitation to check out Part 5 of this book for a deeper look.

Three types of problems that you're almost sure to find on an exam involve finding the area between curves, the arc length of a curve, and the volume of revolution. I focus on these types of problems and many others in Chapters 12 and 13. For now, here's a preview.

We can work it out: Finding the area between curves

When you know how the definite integral represents the area under a curve, finding the area between curves isn't too difficult. Just figure out how to break the problem into several smaller versions of the basic area problem. For example, suppose that you want to find the area between the function $y = \sin x$ and $y = \cos x$, from $x = 0$ to — that is, the shaded area A in Figure 1-11.

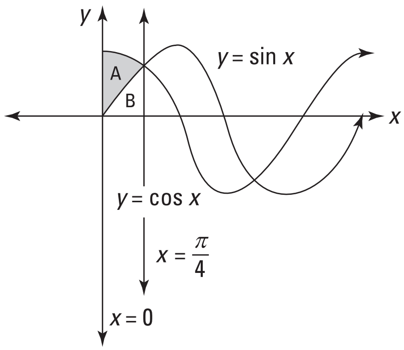

FIGURE 1-11: The area between the function $y = \sin x$ and $y = \cos x$, from $x = 0$ to $x = \frac{\pi}{4}$.

In this case, integrating $y = \cos x$ allows you to find the total area A + B. And integrating $y = \sin x$ gives you the area of B. So you can subtract A + B − B to find the area of A.

For more on how to find an area between curves, flip to Chapter 12.

Walking the long and winding road

Measuring a segment of a straight line or even the arc length of a section of a circle is relatively simple when you're using classical and analytic geometry. But how do you measure arc length along an unusual curve produced by a polynomial, exponential, or trig function?

For example, what's the distance from Point A to Point B along the curve shown in Figure 1-12?

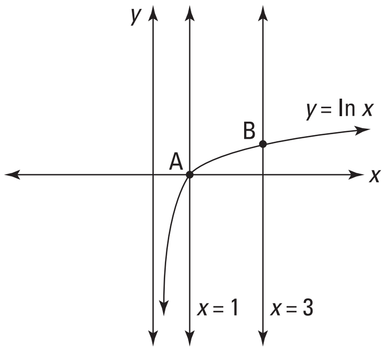

FIGURE 1-12: The distance from Point A to Point B along the function $y = \ln x$.

Once again, integration is your friend. In Chapter 12, I show you how integration provides a formula that allows you to measure arc length.

You say you want a revolution

Calculus also allows you to find the volume of unusual solids. In most cases, calculating volume involves a dimensional leap into *multivariable calculus,* the topic of Calculus III. But in a few situations, setting up an integral just right allows you to calculate volume by integrating over a single variable — that is, by using the methods you discover in Calculus II.

Among the trickiest of these problems involves the *solid of revolution* of a curve. In such problems, you're presented with a region under a curve. You imagine the solid that results when you spin this region around the axis, and then you calculate the volume of this solid as seen in Figure 1-13.

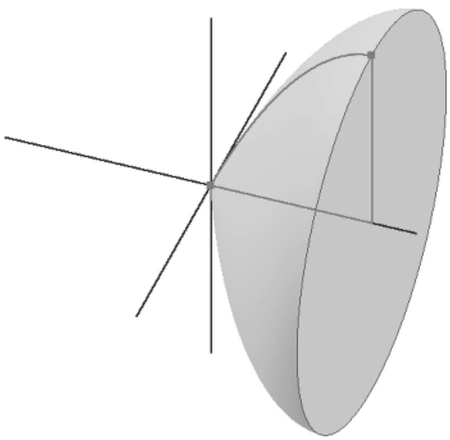

FIGURE 1-13: A solid of revolution produced by spinning the function $y = 2 \sin x$ around the x-axis.

Clearly, you need calculus to find the area of this region. Then you need more calculus and a clear plan of attack to find the volume. I give you all this and more in Chapter 13.

Differential Equations

Many Calculus II courses include a chapter on *differential equations* (*DEs* for short), which are equations that include one or more derivatives. For example:

$$y^2 \frac{dy}{dx} = \cos 2x$$

Here's what this DE is saying in words: When you square *y* (which is a function of *x*) and then multiply it by the derivative of *y* over *x*, the result is the cosine of 2*x*.

Your goal is to solve this DE by finding the original function *y* in terms of *x*. To do this, treat the derivative $\frac{dy}{dx}$ as if it were a fraction. Thus, you can multiply both sides of this equation by *dx*, separating the *x* and *y* terms onto opposite sides of the equation:

$$y^2 \; dy = \cos 2x \; dx$$

Now, you can integrate both sides to solve the problem:

$$\int y^2 \; dy = \int \cos 2x \; dx$$

$$\frac{1}{3} y^3 = \frac{1}{2} \sin 2x + C$$

Don't worry if you don't understand this integration step. Just realize that now, you can use basic algebra to complete the problem:

$$y^3 = \frac{3}{2} \sin 2x + C$$

$$y = \sqrt[3]{\frac{3}{2} \sin 2x + C}$$

This solution provides you with the function *y* in terms of *x* that satisfies the original DE. All this and much more is revealed in Chapter 14.

Understanding Infinite Series

The last third of a typical Calculus II course — roughly five weeks — usually focuses on the topic of infinite series. I cover this topic in detail in Part 6. Here's an overview of some of the ideas you find out about there.

Distinguishing sequences and series

A *sequence* is a string of numbers in a determined order. For example:

$$2, 4, 6, 8, 10, \ldots$$

$$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots$$

$$\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots$$

Sequences can be finite or infinite, but calculus deals well with the infinite, so it should come as no surprise that calculus concerns itself only with *infinite sequences.*

You can turn an infinite sequence into an *infinite series* by changing the commas into plus signs:

$$2 + 4 + 6 + 8 + 10 + \ldots$$

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots$$

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \ldots$$

Sigma notation, which I discuss further in Chapter 2, is useful for expressing infinite series more succinctly. For example, here's how to express these three series using sigma notation:

$$\sum_{n=1}^{\infty} 2n = 2 + 4 + 6 + 8 + 10 + \ldots$$

$$\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots$$

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \ldots$$

Chapter 15 gets you started on sequences and series.

Evaluating series

Evaluating an infinite series is often possible. That is, you can find out what all those numbers add up to. A helpful way to get a handle on some series is to create a related *sequence of partial sums* — that is, a sequence that includes the first term, the sum of the first two terms, the sum of the first three terms, and so forth. For example, here's a sequence of partial sums for the second series shown earlier:

$$1 = 1$$
$$1 + \frac{1}{2} = 1\frac{1}{2}$$
$$1 + \frac{1}{2} + \frac{1}{4} = 1\frac{3}{4}$$
$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = 1\frac{7}{8}$$
$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = 1\frac{15}{16}$$

The resulting sequence of partial sums provides strong evidence of this conclusion:

$$\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots = 2$$

Chapter 15 focuses on understanding related forms of sequences and series.

Identifying convergent and divergent series

When a series evaluates to a number — as does $\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n$ — it's called a *convergent series.* When a series isn't convergent, it's called a *divergent series.*

Identifying whether a series is convergent or divergent isn't always simple. For example, take another look at the third series I introduce earlier in this section:

$$\sum_{n=1}^{\infty}\frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \ldots = ?$$

This is called the *harmonic series,* but can you guess by looking at it whether it converges or diverges? (Before you begin adding fractions, let me warn you that the partial sum of the first 10,000 numbers is approximately 10.)

An ongoing problem as you study infinite series is deciding whether a given series is convergent or divergent. Chapter 16 gives you a slew of tests to help you find out. Then, in Chapter 17, you focus on how *power series* — convergent series that have many of the same properties as polynomials — can be "Taylored" to approximate definite integrals.