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Least Action Principle in Electromagnetics

Laws of nature are governed by following fundamental principles – the action principle, locality, Lorentz invariance, and gauge invariance [1]. Hamilton's principle, or the least action principle, is originally developed for classical mechanics stating that a particle, among all of the trajectories between fixed time instants t_1 and t_2, follows the path which minimizes the *action*. Action is defined as time integral of the difference between the kinetic energy and potential energy, respectively. Thus, Hamilton's principle somehow requires the time averages of the kinetic energy and potential energy to distribute as equally as possible (equipartition) [2]. In classical mechanics, Hamilton's principle and Newton's second law represent equivalent formulations.

An extension of Hamilton's principle from classical mechanics to classical electromagnetics can be undertaken starting with the analysis of the motion of single charged particle [3]. Next step is to construct a Lagrangian for the electromagnetic field by extending the Lagrangian pertaining to classical mechanics. From the corresponding Lagrangians, featuring Noether's theorem and gauge invariance, it is possible to derive equation of continuity for the charge, Lorentz force, and Maxwell's equations, which can be found elsewhere, e.g. [2–5].

Generally, when a functional is extremal, Noether's theorem yields the conservation law. Thus, invariance of the system under a time translation results in the energy conservation. It is also worth noting that space translation invariance corresponds to the conservation of linear momentum, rotation invariance corresponds to the conservation of angular momentum, while gauge invariance yields the charge conservation [1, 2].

These derivations are recently reviewed in [6–8].

This chapter first deals with derivation of continuity equation and Lorentz force, and a derivation of Maxwell's equations from the electromagnetic field

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functional is carried out. Finally, a variational basis of numerical solution methods in electromagnetics is discussed.

1.1 Hamilton Principle

Hamilton variational principle represents not only the basis of modern analytical dynamics but also of universal physical laws, i.e. fundamental laws of classical physics can be understood in terms of action.

This section first deals with Hamilton's variational principle in mechanics, Newton's equation of motion, and Noether's theorem. Then the variational principle in electromagnetics is discussed.

For simplicity, a system with one degree of freedom represented by a generalized coordinate q is considered together with related function of position, velocity, and time $L(q, \dot{q}, t)$ where $\dot{q}$ denotes the time derivative.

The task is to determine how a point particle should move in this one-dimensional space so that the time integral of L is minimized compared with the integral over the conceivable paths between the same starting and end points, as depicted in Fig. 1.1. The solution is given by stating q as a function of time $q = q(\mathrm{t})$.

To compare all paths having the same starting and end points, the variation of function q is zero at both ends [1–4]

$$\delta q = 0 \tag{1.1}$$

i.e. all alternatives start at instant t_1 and arrive together at instant t_2.

The minimum condition is then given by a functional F expressed in terms of the integral [1–4]

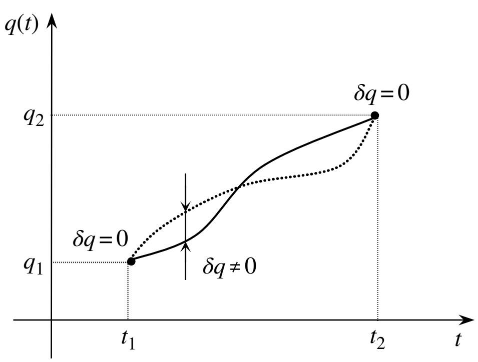

Figure 1.1 The varied function q(t).

$$F = \int_{t_1}^{t_2} L(q, \dot{q}, t)dt = \text{ min} \tag{1.2}$$

In classical mechanics, function L is referred to as Lagrangian and is expressed as

$$L = W_{kin} - W_{pot} \tag{1.3}$$

where W_{kin} and W_{pot} are the kinetic energy and potential energy, respectively.

According to the calculus of variation, the functional approaches minimum value

$$F = \int_{t_1}^{t_2} \left(W_{kin} - W_{pot}\right)dt = \text{ min} \tag{1.4}$$

when its variation vanishes, i.e.:

$$\delta F = 0 \tag{1.5}$$

which can also be written as

$$\delta \int_{t_1}^{t_2} Ldt = 0 \tag{1.6}$$

Therefore, function q(t) minimizes functional (1.2) or (1.4), respectively, and it follows

$$\delta F = \int_{t_1}^{t_2} \delta Ldt \tag{1.7}$$

For the simplest case given by $L(q, \dot{q}, t)$ the variation of function L is given by

$$\delta L = \frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot{q}}\delta \dot{q} \tag{1.8}$$

And by performing some further mathematical manipulation, one readily obtains

$$\delta F = \int_{t_1}^{t_2} \left[\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)\right]\delta q dt + \frac{\partial L}{\partial \dot{q}}\delta q\Big|_{t_1}^{t_2} = 0 \tag{1.9}$$

As $\delta q = 0$ at the ends of the path, the second term at the right-hand side automatically vanishes.

Furthermore, according to the *fundamental lemma* of variational calculus [4] the first integral term at the right-hand side of (1.9) vanishes if the following condition is satisfied

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = 0 \tag{1.10}$$

It is worth noting that the second order differential Eq. (1.10) relates the position q for the time t, and also determines the true path of the system when two end positions and times are given. This equation is known as Lagrange–Euler equation of motion [1, 2].

In the case of a single function of multiple variables, $L(q_k, \dot{q}_k, \xi, t)$ (1.10) becomes

$$\frac{\partial L}{\partial q} - \frac{\partial L}{\partial \xi}\left(\frac{\partial L}{\partial q_\xi}\right) - \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial q_t}\right) = 0 \tag{1.11}$$

Finally, if few functions of multiple independent variables $L(q_k, \dot{q}_k, x, y, z, t)$ are considered, it follows

$$\sum_{k=1}^{N}\left\{\frac{\partial L}{\partial q_k} - \frac{\partial}{\partial x}\left[\frac{\partial L}{\partial (q_k)_x}\right] - \frac{\partial}{\partial y}\left[\frac{\partial L}{\partial (q_k)_y}\right] - \frac{\partial}{\partial z}\left[\frac{\partial L}{\partial (q_k)_z}\right]\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial (q_k)_t}\right)\right\} = 0 \tag{1.12}$$

which, for example, pertains to three-dimensional problems in electromagnetics.

Hamilton variational principle can be considered a general law not only for particle dynamics but also for the dynamics of continuous materials.

An extension to three-dimensional problems in continuous materials, i.e. for physical fields, the variational principle corresponding to Eq. (1.6) is given by

$$\delta \int_{t_1}^{t_2} \int_V \overline{L}_d dV dt = 0 \tag{1.13}$$

where $\overline{L}_d$ is the so-called Lagrange density defined as

$$L = \int_V \overline{L}_d dV \tag{1.14}$$

and has a unit of energy per volume.

It is worth noting that the variational principle is an invariant for coordinate transformations [1].

1.2 Newton's Equation of Motion from Lagrangian

Lagrangian in classical mechanics for a particle with mass m with displacement $\vec{r}$ at time t $L_0\left(\vec{r}, \frac{d\vec{r}}{dt}, t\right)$ is of the form

$$L_0\left(\vec{r}, \frac{d\vec{r}}{dt}, t\right) = \frac{1}{2}m\left(\frac{d\vec{r}}{dt}\right)^2 - W_{pot}\left(\vec{r}\right) \tag{1.15}$$

where $\frac{d\vec{r}}{dt}$ is the particle velocity and $W_{pot}\left(\vec{r}\right)$ stands for potential energy (not due to electromagnetic field).

The corresponding action F_0 related to Lagrangian (1.15) is given by integral

$$F_0 = \int_{t_1}^{t_2} L_0 dt \tag{1.16}$$

Now varying the action (1.16)

$$\delta F_0 = 0 \tag{1.17}$$

It can be written as:

$$\frac{\partial L}{\partial x} = -\frac{\partial W_{pot}}{\partial x}, \frac{\partial L}{\partial y} = -\frac{\partial W_{pot}}{\partial y}, \frac{\partial L}{\partial z} = -\frac{\partial W_{pot}}{\partial z} \tag{1.18}$$

$$\frac{\partial L}{\partial \dot{x}} = m\dot{x}, \quad \frac{\partial L}{\partial \dot{y}} = m\dot{y}, \quad \frac{\partial L}{\partial \dot{z}} = m\dot{z} \tag{1.19}$$

and it follows:

$$\begin{aligned} -\frac{\partial W_{pot}}{\partial x} - \frac{d}{dt}(m\dot{x}) = 0 \\ -\frac{\partial W_{pot}}{\partial y} - \frac{d}{dt}(m\dot{y}) = 0 \\ -\frac{\partial W_{pot}}{\partial z} - \frac{d}{dt}(m\dot{z}) = 0 \end{aligned} \tag{1.20}$$

which simply gives:

$$\begin{aligned} \frac{d^2(mx)}{dt^2} = -\frac{\partial W_{pot}}{\partial x} \\ \frac{d^2(my)}{dt^2} = -\frac{\partial W_{pot}}{\partial y} \\ \frac{d^2(mz)}{dt^2} = -\frac{\partial W_{pot}}{\partial z} \end{aligned} \tag{1.21}$$

If mass m is regarded as constant quantity, right-hand side of (1.21) represents the components of mechanical force:

$$F_{mech,x} = m\frac{d^2x}{dt^2}$$
$$F_{mech,y} = m\frac{d^2y}{dt^2} \tag{1.22}$$
$$F_{mech,z} = m\frac{d^2z}{dt^2}$$

As the mechanical force is defined as a negative gradient of the potential energy

$$\vec{F}_{mech} = -\nabla W_{pot} \tag{1.23}$$

one simply obtains a set of Newton's equations of motion

$$m\frac{d^2\vec{r}}{dt^2} = -\nabla W_{pot}\left(\vec{r}\right) \tag{1.24}$$

where the right-hand side of (1.24) represents a force acting on the particle.

1.3 Noether's Theorem and Conservation Laws

There are physical quantities that do not change throughout the time development of physical systems. These quantities are stated to be conserved under certain conditions which are governed by conservation laws. It can be shown that conservation laws are a consequence of the symmetry properties of a physical system (invariance properties of a system under a group of transformations [2, 9]). The symmetry properties of the system and conservation laws are connected with Noether's theorem, e.g. [2, 9].

Let $u_k(x)$ ($k = 1,2,\ldots n$) be a set of differentiable functions of the independent variable x and let $v_k(x)$ be the first derivatives, i.e. it can be written as

$$v_k(x) = \frac{du_k}{dx} \tag{1.25}$$

Now Lagrangian L is defined as a function of x, and n functions of u_k and n functions of v_k.

$$L = L[x, u_k, v_k] \tag{1.26}$$

If one considers an infinitesimal transformation T

$$x \rightarrow x' = x + \delta x \tag{1.27}$$

Furthermore, under T, one consequently has:

$$u_k(x) \to u'_k(x') = u_k(x) + \delta u_k(x) \tag{1.28}$$

$$v_k(x) \to v'_k(x') = v_k(x) + \delta v_k(x) \tag{1.29}$$

Now assuming the functional

$$S = \int_{\Omega} L dx \tag{1.30}$$

to be invariant under T so that the transformation (1.30) which maps the interval Ω into Ω' does not change, it follows

$$\delta S = \int_{\Omega'} L' dx - \int_{\Omega} L dx = 0 \tag{1.31}$$

Performing some mathematical manipulations, one obtains the following expression

$$\left[\frac{d}{dx}\left(\frac{\partial L}{\partial v_k}\right) - \frac{\partial L}{\partial u_k}\right](\delta u_k - v_k \delta x) = \frac{d}{dx}\left[L\delta x + \frac{\partial L}{\partial v_k}(\delta u_k - v_k \delta x)\right] \tag{1.32}$$

Finally, Noether's theorem states that if functional S is invariant under the infinitesimal one-parameter group of transformations T, then the set of n equations

$$\left[\frac{d}{dx}\left(\frac{\partial L}{\partial v_k}\right) - \frac{\partial L}{\partial u_k}\right] = 0 \tag{1.33}$$

simply gives

$$\frac{d}{dx}\left[L\delta x + \frac{\partial L}{\partial v_k}(\delta u_k - v_k \delta x)\right] = 0 \tag{1.34}$$

i.e. it can be written as

$$L\delta x + \frac{\partial L}{\partial v_k}(\delta u_k - v_k \delta x) = const \tag{1.35}$$

namely, the expression is independent of x variable.

On the other hand, Newton's equation of motion can be expressed in terms of Lagrangian being a function of the generalized coordinates q_k ($k = 1, 2, \ldots, n$) and their derivatives, i.e.

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q} = 0 \tag{1.36}$$

which is equivalent to the request

$$\delta \int_{t_1}^{t_2} L\,dt = 0 \tag{1.37}$$

Furthermore, Hamilton's principle (the least action principle), which states that solutions q_k of the system of mass points for which the integral in (C6) is an extremum contrary to all functions q_k'

$$q_k'(t) = q_k(t) + \delta q_k(t) \tag{1.38}$$

If one compares (1.34) with (1.35) with $\delta x = 0$ and $\delta u_k = 0$, it follows that these statements are identical Namely, it can be written as

$$L\delta t + \frac{\partial L}{\partial \dot{q}_k}(\delta q_k - \dot{q}_k \delta t) = const \tag{1.39}$$

i.e. the left-hand side is a constant of motion.

As any time translation is expressed as $\delta t \neq 0$ and $\delta q_k = 0$, the following expression is obtained

$$H = \frac{\partial L}{\partial \dot{q}_k}\dot{q}_k - L = const \tag{1.40}$$

stating that Hamiltonian H which is equal to the total energy of the system (sum of kinetic energy and potential energy) is constant of the motion.

In other words, time translation symmetry corresponds to the conservation of energy. Therefore, Noether's theorem states that for a system with time translation symmetry, the energy of the system is conserved.

The energy conservation can be demonstrated by using Lagrangian and Hamiltonian in classical mechanics for one-dimensional case.

Thus, Lagrangian is of the form

$$L = \frac{1}{2}m\dot{x}^2 - W_{pot}(x) \tag{1.41}$$

where

$$\dot{x} = \frac{dx}{dt} \tag{1.42}$$

while Hamiltonian of the system, according to (1.40), is then

$$H = \frac{\partial L}{\partial \dot{x}}\dot{x} - L \tag{1.43}$$

As from (1.42), one simply obtains

$$\frac{\partial L}{\partial \dot{x}}\dot{x} = m\dot{x} \tag{1.44}$$

and the Hamiltonian now can be written as

$$H = \frac{1}{2}m\dot{x}^2 + W_{pot}(x) \tag{1.45}$$

For a system with a time translation symmetry

$$t \to t' = t + \delta t \tag{1.46}$$

the Hamiltonian of the system remains invariant.

To check if the Hamiltonian is a conserved quantity under a time translation, it should be differentiated with respect to time, i.e. it follows

$$\frac{dH}{dt} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\dot{x} - L\right) = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\dot{x}\right) - \frac{dL}{dt} \tag{1.47}$$

and, one has

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\dot{x}\right) = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)\dot{x} + \frac{\partial L}{\partial \dot{x}}\frac{d\dot{x}}{dt} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)\dot{x} + \frac{\partial L}{\partial \dot{x}}\ddot{x} \tag{1.48}$$

$$\frac{dL}{dt} = \frac{\partial L}{\partial x}\frac{dx}{dt} + \frac{\partial L}{\partial \dot{x}}\frac{d\dot{x}}{dt} + \frac{\partial L}{\partial t} = \frac{\partial L}{\partial x}\dot{x} + \frac{\partial L}{\partial \dot{x}}\ddot{x} + \frac{\partial L}{\partial t} \tag{1.49}$$

Now, inserting (1.48) and (1.49) into (1.47) yields

$$\frac{dH}{dt} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)\dot{x} + \frac{\partial L}{\partial \dot{x}}\ddot{x} - \left(\frac{\partial L}{\partial x}\dot{x} + \frac{\partial L}{\partial \dot{x}}\ddot{x} + \frac{\partial L}{\partial t}\right) \tag{1.50}$$

and one obtains

$$\frac{dH}{dt} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)\dot{x} - \frac{\partial L}{\partial x}\dot{x} - \frac{\partial L}{\partial t} \tag{1.51}$$

which can be written:

$$\frac{dH}{dt} = \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x}\right)\dot{x} - \frac{\partial L}{\partial t} \tag{1.52}$$

Now, as Euler Lagrange equation vanishes

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0 \tag{1.53}$$

and (1.52) becomes

$$\frac{dH}{dt} = -\frac{\partial L}{\partial t} \tag{1.54}$$

Now, considering Lagrangian (1.41) it is evident that its time derivative is zero, as there is no explicit time dependence. Consequently, the total time derivative of the Hamiltonian is also zero, and it can be concluded that Hamiltonian is conserved under the time translation (1.46).

It is worth noting that space-time invariance corresponds to the conservation of linear momentum, and rotation invariance corresponds to the conservation of angular momentum.

1.4 Equation of Continuity from Lagrangian

A change in system which does not affect the action integral, or the equation of motions, is referred to as invariance or symmetry [1].

Lagrangian L in classical mechanics is defined as difference between kinetic energy (W_{kin}) and potential energy (W_{pot}) of the system

$$L = W_{kin} - W_{pot} \tag{1.55}$$

while the corresponding Euler–Lagrange equation of motion governing the considered trajectory of a particle is given by

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = 0 \tag{1.56}$$

where q pertains to the position of the particle and $\dot{q}$ denotes its time derivative. Furthermore, Lagrangian for particle of mass m with displacement $\vec{r}$ at time t $L\left(\vec{r}, \frac{d\vec{r}}{dt}, t\right)$ is given by [3, 4]

$$L\left(\vec{r}, \frac{d\vec{r}}{dt}, t\right) = \frac{1}{2}m\left|\frac{d\vec{r}}{dt}\right|^2 - W_{pot}\left(\vec{r}\right) = \frac{1}{2}m\dot{r} - W_{pot}\left(\vec{r}\right) \tag{1.57}$$

where $\left|\frac{d\vec{r}}{dt}\right| = \dot{r}$ is the particle velocity and $W_{pot}\left(\vec{r}\right)$ stands for potential energy (in the absence of electromagnetic field).

The corresponding action F related to Lagrangian (1.57) is given by integral

$$F = \int_{t_1}^{t_2} L dt \tag{1.58}$$

Now varying action (1.58)

$$\delta F = 0 \tag{1.59}$$

yields Newton's second law:

$$m\frac{d^2\vec{r}}{dt^2} = -\nabla W_{pot}\left(\vec{r}\right) \tag{1.60}$$

where the right-hand side represents a force acting on the particle.

Now, according to the symmetry of the Lagrangian stemming from Noether's theorem [2, 3, 9], a total time derivative may be added to Lagrangian (1.57) without changing the equation of motion [3].

Thus, choosing an arbitrary differentiable scalar function $q\Lambda\left(\vec{r},t\right)$, it can be written as

$$L' = L + q\frac{d\Lambda}{dt} \tag{1.61}$$

where q stands for the electric charge.

Now, a new action F' can be defined as follows

$$F' = \int_{t_1}^{t_2} L'dt \tag{1.62}$$

and using (1.61), it follows

$$F' = \int_{t_1}^{t_2}\left(L + q\frac{d\Lambda}{dt}\right)dt = \int_{t_1}^{t_2} Ldt + \int_{t_1}^{t_2} q\frac{d\Lambda}{dt}dt \tag{1.63}$$

Furthermore, one obtains

$$F' = F + q\left\{\Lambda\left[\vec{r}(t_2),t_2\right] - \Lambda\left[\vec{r}(t_1),t_1\right]\right\} \tag{1.64}$$

As the end points of the interval are fixed

$$\delta\vec{r} = 0 \tag{1.65}$$

performing the variation of functional F' simply yields

$$\delta F' = \delta F \tag{1.66}$$

Therefore, the Lagrangian L and L' are equivalent.

For the point particle, the charge density can be written as

$$\rho\left(\vec{R},t\right) = q\delta\left[\vec{R} - \vec{r}(t)\right] \tag{1.67}$$

where δ [ ] is the Dirac delta function.

In the next step, the current density can be expressed as charge in motion, i.e.

$$\vec{J}\left(\vec{R},t\right) = q\frac{d\vec{r}(t)}{dt}\delta\left[\vec{R}-\vec{r}(t)\right] \tag{1.68}$$

Now, taking into account the total differential of scalar function $\Lambda\left(\vec{r},t\right)$

$$d\Lambda = \frac{\partial\Lambda}{\partial x}dx + \frac{\partial\Lambda}{\partial y}dy + \frac{\partial\Lambda}{\partial z}dz + \frac{\partial\Lambda}{\partial t}dt = \nabla\Lambda\cdot d\vec{r} + \frac{\partial\Lambda}{\partial t}dt \tag{1.69}$$

total time derivative is obtained as

$$\frac{d\Lambda}{dt} = \nabla\cdot\Lambda\frac{d\vec{r}}{dt} + \frac{\partial\Lambda}{\partial t} = \frac{\partial\Lambda}{\partial n}\vec{r}_0\frac{d\vec{r}}{dt} + \frac{\partial\Lambda}{\partial t} \tag{1.70}$$

and the Lagrangian can be written as follows

$$L' = L + q\left[\frac{\partial\Lambda}{\partial n}\frac{dr}{dt} + \frac{\partial\Lambda}{\partial t}\right] \tag{1.71}$$

Integrating over volume V yields

$$\int_V \rho\left(\vec{R},t\right)dV = \int_V q\delta\left[\vec{R}-\vec{r}(t)\right]dV \tag{1.72}$$

and one simply has

$$q = \int_V \rho\left(\vec{R},t\right)dV \tag{1.73}$$

Finally, Lagrangian (1.61) takes the form

$$L' = L + \int_V \rho\left(\vec{R},t\right)\frac{d\vec{r}}{dt}\cdot\nabla\Lambda\left(\vec{R},t\right)dV + \int_V \rho\left(\vec{R},t\right)\frac{\partial\Lambda\left(\vec{R},t\right)}{\partial t}dV \tag{1.74}$$

which, taking into account (1.68), becomes

$$L' = L + \int_V \vec{J}\left(\vec{R},t\right)\cdot\nabla\Lambda\left(\vec{R},t\right)dV + \int_V \rho\left(\vec{R},t\right)\frac{\partial\Lambda\left(\vec{R},t\right)}{\partial t}dV \tag{1.75}$$

If V tends to infinity and current is assumed to vanish at infinity after performing integration by parts, it follows

$$L' = L - \int_V \nabla \cdot \vec{J}\left(\vec{R},t\right)\Lambda\left(\vec{R},t\right)dV + \frac{d}{dt}\int_V \rho\left(\vec{R},t\right)\Lambda\left(\vec{R},t\right)dV - \int_V \frac{\partial \rho\left(\vec{R},t\right)}{\partial t}\Lambda\left(\vec{R},t\right)dV \tag{1.76}$$

which now can be written as

$$L' = L - \int_V \left[\nabla \cdot \vec{J}\left(\vec{R},t\right) - \frac{\partial \rho\left(\vec{R},t\right)}{\partial t}\right]\Lambda\left(\vec{R},t\right)dV + \frac{d}{dt}\int_V \rho\left(\vec{R},t\right)\Lambda\left(\vec{R},t\right)dV \tag{1.77}$$

Now Lagrangian L' can be equivalent to (1.57) if the second term of the right-hand side is equal to zero.

According to the fundamental lemma of variational calculus, as $\Lambda\left(\vec{R},t\right)$ is an arbitrary space-dependent function, the second integral term from the right-hand side vanishes identically if the following condition is satisfied

$$\nabla \cdot \vec{J}\left(\vec{R},t\right) - \frac{\partial \rho\left(\vec{R},t\right)}{\partial t} = 0 \tag{1.78}$$

Expression (1.78) is a well-known equation of continuity relating to charge and current density, respectively.

Therefore, one of the basic equations of classical electromagnetics is derived from the gauge invariance of classical mechanics, i.e. from the symmetry property of the Lagrangian.

Namely, the gauge invariance of the Lagrangian implies conservation of electric charge.

It is worth noting that most of standard textbooks used for various courses in electromagnetics start from Maxwell's equations and the continuity equation is usually derived from Maxwell's equations (which are considered rigorous mathematical expressions of previously discovered laws as a consequence of experiments) via certain mathematical manipulations.

This chapter exploits the approach that goes the other way around by exploiting the symmetry properties of the Lagrangian, similar to the approach presented in [3, 6–8].

It is rather worth stressing that within the framework of such an approach, electromagnetic potentials are regarded as more fundamental entities than in the case in which one considers corresponding equation of motions instead of Lagrangian. Namely, in latter approach, starting from the electric and magnetic fields being gauge invariant, potentials are no more than auxiliary functions, i.e. pure mathematical constructs with no physical meaning.

1.5 Lorentz Force from Gauge Invariance

In the next step, Lagrangian (1.75) is rewritten by introducing vector function

$$\vec{A} = \nabla\Lambda\left(\vec{R}, t\right) \tag{1.79}$$

and scalar function

$$\varphi = -\frac{\partial\Lambda\left(\vec{R}, t\right)}{\partial t} \tag{1.80}$$

which now gives new Lagrangian

$$L_i = L + \int_V \vec{J}\left(\vec{R}, t\right) \cdot \vec{A}\left(\vec{R}, t\right) dV - \int_V \rho\left(\vec{R}, t\right) \varphi\left(\vec{R}, t\right) dV \tag{1.81}$$

Therefore, current density $\vec{J}\left(\vec{R}, t\right)$ is coupled to the vector potential $\vec{A}\left(\vec{R}, t\right)$, while charge density is coupled to the scalar potential $\varphi\left(\vec{R}, t\right)$ [3, 4].

Therefore, integral terms in (1.81) describe interaction of the charged particle with the electromagnetic field potentials.

Again, to obtain an equivalent Lagrangian, the total time derivative is added

$$L_i{}' = L_i + q\frac{d\Lambda}{dt} \tag{1.82}$$

and the obtained Lagrangian is

$$L_i' = L + \int_V \vec{J} \cdot \vec{A} dV - \int_V \rho\varphi dV + q\frac{d\Lambda}{dt} \tag{1.83}$$

which can be also written as

$$L' = L + \int_V \vec{J} \cdot \vec{A}' dV - \int_V \rho\varphi' dV \tag{1.84}$$

If the potentials are defined as follows [2, 3]

$$\vec{A}' = \vec{A} + \nabla\Lambda\left(\vec{R}, t\right) \tag{1.85}$$

$$\varphi' = \varphi - \frac{\partial \Lambda\left(\vec{R}, t\right)}{\partial t} \tag{1.86}$$

The vector and scalar potentials in (1.83) correspond to the additional terms in (1.75).

Expressions (1.85, 1.86) are referred to as gauge transformations added to ensure invariance of the Lagrangian under the addition of a total time derivative.

The corresponding equation of motions of the particle interacting with electromagnetic field can now be obtained from the least action principle.

The total action for the charged particle F_i can be written as

$$F_i = \int_{t_1}^{t_2} L_i dt \tag{1.87}$$

Varying functional (1.87) and taking into account

$$\frac{d\vec{A}}{dt} = \frac{\partial \vec{A}}{\partial x}\frac{dx}{dt} + \frac{\partial \vec{A}}{\partial y}\frac{dy}{dt} + \frac{\partial \vec{A}}{\partial z}\frac{dz}{dt} + \frac{\partial \vec{A}}{\partial t} = \left(\vec{v} \cdot \nabla\right)\vec{A} + \frac{\partial \vec{A}}{\partial t} \tag{1.88}$$

leads to the following equation of motion [3]

$$m\frac{d^2\vec{r}}{dt^2} = -\nabla W_{pot}\left(\vec{r}\right) + q\left(-\nabla\varphi - \frac{\partial \vec{A}}{\partial t}\right) + q\frac{d\vec{r}}{dt} \times \left(\nabla \times \vec{A}\right) \tag{1.89}$$

representing Newton's second law with additional forces due to the existence of the electromagnetic field.

Note that (1.89) is equivalent to expression

$$m\frac{d^2\vec{r}}{dt^2} = -\nabla W_{pot}\left(\vec{r}\right) + q\left(-\nabla\varphi' - \frac{\partial \vec{A}'}{\partial t}\right) + q\frac{d\vec{r}}{dt} \times \left(\nabla \times \vec{A}'\right) \tag{1.90}$$

The vector and scalar potentials are not unique and consequently not measurable physical quantities aiming as they are mathematical constructs and do not represent physical fields [1, 3, 4].

However, the electric field and magnetic field can be defined in terms of potentials A and φ as they are invariant under gauge transformations (1.85) and (1.86) as follows [3]:

$$\vec{B} = \nabla \times \vec{A} \tag{1.91}$$

$$\vec{E} = -\nabla\varphi - \frac{\partial \vec{A}}{\partial t} \tag{1.92}$$

where B denotes the magnetic field and E stands for the electric field.

It can be concluded that electric and magnetic fields are obtained from the equations of motion for charge particles and they are gauge invariant.

The total equation of motion can be finally written as follows

$$m\frac{d^2\vec{r}}{dt^2} = -\nabla W_{pot}\left(\vec{r}\right) + q\vec{E} + q\frac{d\vec{r}}{dt} \times \vec{B} \tag{1.93}$$

Equation (1.93) represents Newton's second law of motion with the Lorentz force included, i.e. it can be written as

$$m\frac{d^2\vec{r}}{dt^2} = \vec{F}_{mech} + \vec{F}_{EM} \tag{1.94}$$

where $\vec{F}_{mech}$ is the force acting to the particle due to the potential energy

$$\vec{F}_{mech} = -\nabla W_{pot} \tag{1.95}$$

while the Lorentz force due to the interaction of particles with the electromagnetic field is

$$\vec{F}_{EM} = q\left(\vec{E} + \vec{v} \times \vec{B}\right) \tag{1.96}$$

Therefore, (1.96) is the second fundamental equation of electromagnetics derived from the gauge invariance starting from action in classical mechanics.

Namely, according to the gauge invariance, vector potential can be changed without any effect on the charged particle. This can be referred to as gauge transformation.

It is also worth noting that Lorentz force (1.96), being a basis of classical particle electromagnetics, cannot be obtained from Maxwell's equations for stationary media, while (1.78, 1.91, 1.92) can be readily derived from Maxwell's equation, which is available elsewhere, e.g. in [4].

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