1

Material Bodies and Kinematics

1.1 Introduction

Many important biological structures can be considered as continuous, and many of these can be regarded as one-dimensional and straight. Moreover, it is not uncommon to observe that, whenever these structures deform, grow, sustain heat and undergo chemical reactions, they remain straight. Let us look at some examples.

Tendons. One of the main functions of tendons is to provide a connection between muscles (made of relatively soft tissue) and bone (hard tissue). Moreover, the deformability of tendinous tissue and its ability to store and release elastic energy are important for the healthy performance of human and animal activities, such as walking, running, chewing and eye movement. Tendons are generally slender and straight. Figure 1.1 shows a human foot densely populated by a network of tendons and ligaments. The Achilles tendon connects the calcaneus bone with the gastrocnemius and soleus muscles located in the lower leg.

Muscle components. Most muscles are structurally too complex to be considered as one-dimensional entities. On the other hand, at some level of analysis, muscle fibres and their components down to the myofibril and sarcomere level can be considered as straight one-dimensional structural elements, as illustrated schematically in Figure 1.2.

Hair. Figure 1.3 shows a skin block with follicles and hair. When subjected to tensile loads, hair can be analysed as a one-dimensional straight structure.

One of the questions that continuum mechanics addresses for these and more complex structures is the following: what is the mechanism of transmission of load? The general answer to this question is: *deformation*. It took millennia of empirical familiarity with natural and human-made structures before this simple answer could be arrived at. Indeed, the majestic Egyptian pyramids, the beautiful Greek temples, the imposing Roman arches, the overwhelming Gothic cathedrals and many other such structures were conceived, built and utilized without any awareness of the fact that their deformation, small as it might be, plays a crucial role in the process of transmission of load from one part of the structure to another. In an intuitive picture, one may say that the deformation of a continuous structure is the reflection of the change in atomic distances at a deeper level, a change that results in the development of internal forces in response to the applied external loads. Although this

The Elements of Continuum Biomechanics, First Edition. Marcelo Epstein.

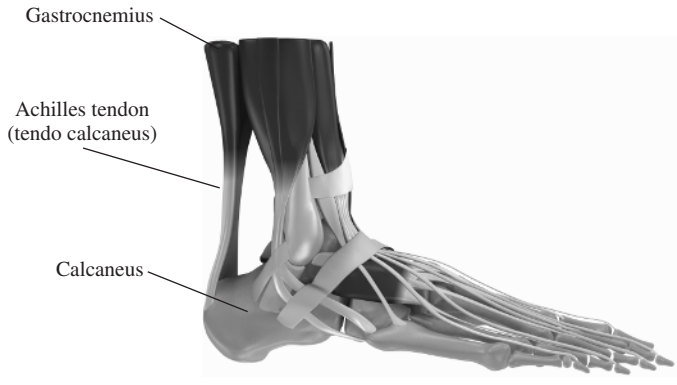

Figure 1.1 Tendons and ligaments in the human foot

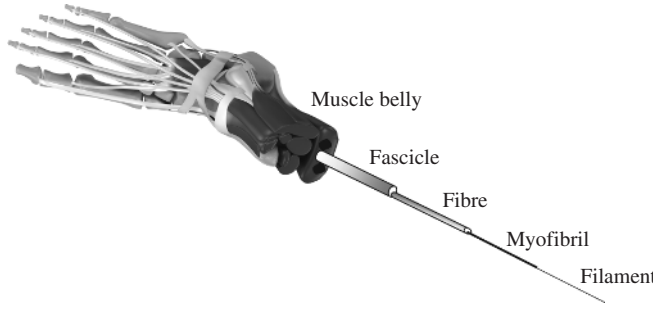

Figure 1.2 A skeletal muscle and its components

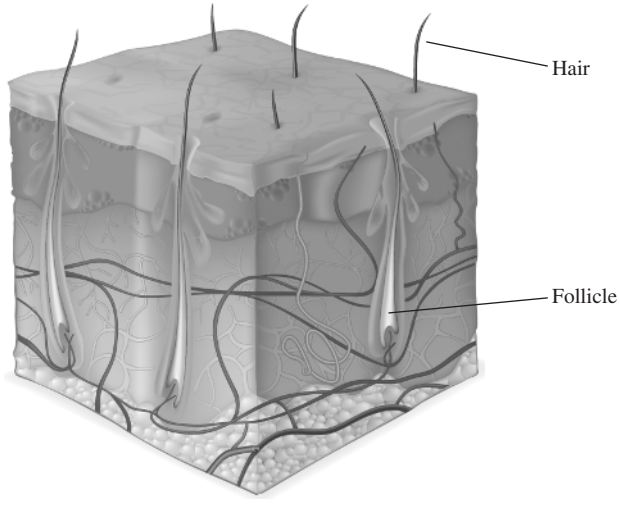

Figure 1.3 Skin cube with follicles and hair

naïve model should not be pushed too far, it certainly contains enough physical motivation to elicit the general picture and to be useful in many applications.

Once the role of the deformation has been recognized, continuum mechanics tends to organize itself in a tripartite fashion around the following questions:

1. How is the deformation of a continuous medium described mathematically?
2. What are the physical laws applicable to all continuous media?
3. How do different materials respond to various external agents?

This subdivision of the discipline is not only paedagogically useful, but also epistemologically meaningful. The answers to the three questions just formulated are encompassed, respectively, under the following three headings:

1. continuum kinematics;
2. physical balance laws;
3. constitutive theory.

From the mathematical standpoint, *continuum kinematics* is a direct application of the branch of mathematics known as differential geometry. In the one-dimensional context implied by our examples so far, all that needs to be said about differential geometry can be summarily absorbed within the realm of elementary calculus and algebra. For this to be the case, it is important to bear in mind not only that the structures considered are essentially one-dimensional, but also that they remain straight throughout the process of deformation.

The *physical balance laws* that apply to all continuous media, regardless of their material constitution, are mechanical (balance of mass, linear momentum, angular momentum) and thermodynamical (balance of energy, entropy production). In some applications, electromagnetical, chemical and other laws may be required. The fact that all these laws are formulated over a continuous entity, rather than over a discrete collection of particles, is an essential feature of continuum mechanics.

Finally, by not directly incorporating the more fundamental levels of physical discourse (cellular, molecular, atomic, subatomic), continuum mechanics must introduce phenomenological descriptors of material behaviour. Thus, tendon responds to the application of forces differently from muscle or skin. In other words, geometrically identical pieces of different materials will undergo vastly different deformations under the application of the same loads. One may think that the only considerations to be borne in mind in this respect are purely experimental. Nevertheless, there are some principles that can be established *a priori* on theoretical grounds, thus justifying the name of *constitutive theory* for this fundamental third pillar of the discipline. In particular, the introduction of *ideal material models*, such as elasticity, viscoelasticity and plasticity, has proven historically useful in terms of proposing material responses that can be characterized by means of a relatively small number of parameters to be determined experimentally.

1.2 Continuous versus Discrete

A pendulum, an elastic spring, a shock absorber: are they to be considered as continuous entities? The answer to this question depends ultimately on the level of description adopted. Leaving aside the deeper fact that bodies are made of a very large, though finite,

number of particles, it is clear that, at a more mundane phenomenological level, the three entities just mentioned could be considered as fully-fledged three-dimensional continuous bodies. Moreover, if a spring is slender and straight, it might be appropriate to analyse it as a one-dimensional structure of the kind discussed in the previous section. A tendon, in fact, can be regarded as an example of a spring-like biological structure.

On the other hand, as we know from many encounters with an elastic spring attached to a mass in physics textbooks, there is a different sense in which words such as 'spring' or 'damper' can be used. To highlight the main difference between a 'real' spring and the physics textbook spring (which may be called a 'spring element'), we observe that in the former the mass and the elastic quality are smoothly distributed over the length of the spring. In other words, these properties are specified as some *functions of position* along the length of the spring. Concomitantly, the deformation is expressed in terms of some *displacement field* over the same domain. Physically, this implies that phenomena such as the propagation of sound waves become describable in this context. In contrast, the physics textbook spring element is characterized by just three numbers: its total mass (concentrated at one end), its relaxed length and its stiffness. The deformation is given by a single number, namely, the total instantaneous length. In a model of this kind, therefore, the description of any phenomenon distributed throughout the structure has been irrevocably sacrificed.

Example 1.2.1 Sonomicrometry. Introduced in the 1950s in biomechanical applications, *sonomicrometry* is a length-measuring technique based on the implantation of piezoelectric crystals at two points of an organ and the measurement of the time elapsed between the emission and the reception of an ultrasound signal from one crystal to the other. In particular, sonomicrometry has been used to measure the real-time elongation of muscle fibres during locomotion. Clearly, although for many purposes the muscle fibre can be considered as a simple spring element, the technique of sonomicrometry relies on the fact that sound waves propagate along the fibre and, therefore, on the fact that its properties are smoothly distributed.

From the foregoing, it appears that a fundamental distinction should be drawn between discrete and continuous mechanical systems. Although the intuitive understanding of the meaning of these terms can easily be illustrated by means of examples, as shown in Figure 1.4, the details are surprisingly technical. A cursory reading of the following formal, yet not completely rigorous, characterization should be sufficient for our purposes.

Definition 1.2.2 A *mechanical system* consists of an *underlying set* M and a finite number n of (real-valued) *state variables*:

$$\psi_i = \psi_i(p,t), \quad i = 1,2,\ldots,n, \tag{1.1}$$

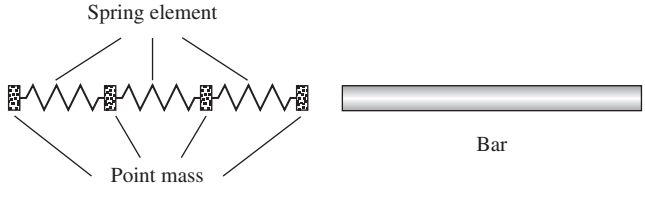

Figure 1.4 Discrete and continuous mechanical systems

where $p \in M$ and t denotes time. The underlying set is also called the *material set* of the system. Its elements are called *material points* or *particles*. (Some basic terminology on sets is reviewed in Window 1.1.)

Window 1.1 Some terminology and notation
Regarding a *set* informally as a collection of objects, called *elements* or *members* of the set, the symbol '$\in$' stands for the expression 'belongs to' or 'is an element of'. Thus, '$p \in A$' means 'p is an element of the set A', or 'p belongs to A'. A set C is a *subset* of the set A if every element of C is also an element of A. This relation is indicated as $C \subset A$. In particular, every set is a subset of itself.

The *Cartesian product* $A \times B$ of two sets, A and B, is the set whose elements are all the ordered pairs (a, b), where $a \in A$ and $b \in B$. The symbol $\mathbb{R}$ represents the *real line*, which is the collection of all real numbers. The set $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ is the *real plane*, consisting of all ordered pairs of real numbers. The *n-dimensional real space* $\mathbb{R}^n = \underbrace{\mathbb{R} \times \ldots \times \mathbb{R}}_{n}$ is constructed similarly by induction.

The physical meaning of the state variables ψ_i depends on the system at hand. They may represent displacements, velocities, temperature, stress, enthalpy, and so on. The time interval of interest, $t_0 \leq t \leq t_1$, is a subset of the real line $\mathbb{R}$. We assume that, within this interval, each state variable is a continuous (and usually also differentiable) function of the time variable t for each fixed point p of the underlying set M. It is only the nature of this set that differentiates between discrete and continuous systems.

Definition 1.2.3 A mechanical system is said to be *discrete* if its underlying set is finite or countable.

Recall that a collection is finite (countable) if it can be put in a one-to-one correspondence with a finite (infinite) subset of the natural numbers. The evolution in time of a discrete mechanical system is usually described by a system of ordinary differential equations (ODEs).

If a system is not discrete, its underlying set M is infinite and non-countable. This result, in and of itself, is not enough for a non-discrete system to be continuous. A classical counterexample is described in Window 1.2. To make matters more precise, let us declare that all the mechanical systems of interest in our present context have a material set M that is a subset of a one-dimensional *material universe* consisting of a copy $\mathcal{M}$ of the real line $\mathbb{R}$ (Figure 1.5). In this primeval material universe there are no pre-assigned length scales. The only properties that matter are the topological ones, such as the notion of open interval.

Definition 1.2.4 A *continuous one-dimensional mechanical system* is a system whose underlying material subset is an open interval $\mathcal{B}$ of the material universe $\mathcal{M}$. Such a material set $\mathcal{B}$ is called a *one-dimensional body* or a *one-dimensional material continuum*.

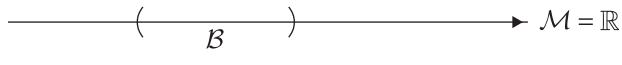

Figure 1.5 The material universe and a material body

Window 1.2 A non-discrete set that is not a continuum

The *Cantor set*, named after Georg Cantor (1845–1916), one of the founders of modern set theory, is obtained by starting from a closed finite interval of the line and deleting its open middle third, thus obtaining two separate closed intervals, each of length equal to one-third of the original segment. The process is repeated on each of these intervals, and so on *ad infinitum*. The result of the first few steps is shown in Figure 1.6.

It can be shown that what is left is a collection of points as numerous as the original segment. In other words, there exists a one-to-one correspondence between the Cantor set and the set of real numbers (which is not countable).

Figure 1.6 Generation of the Cantor set

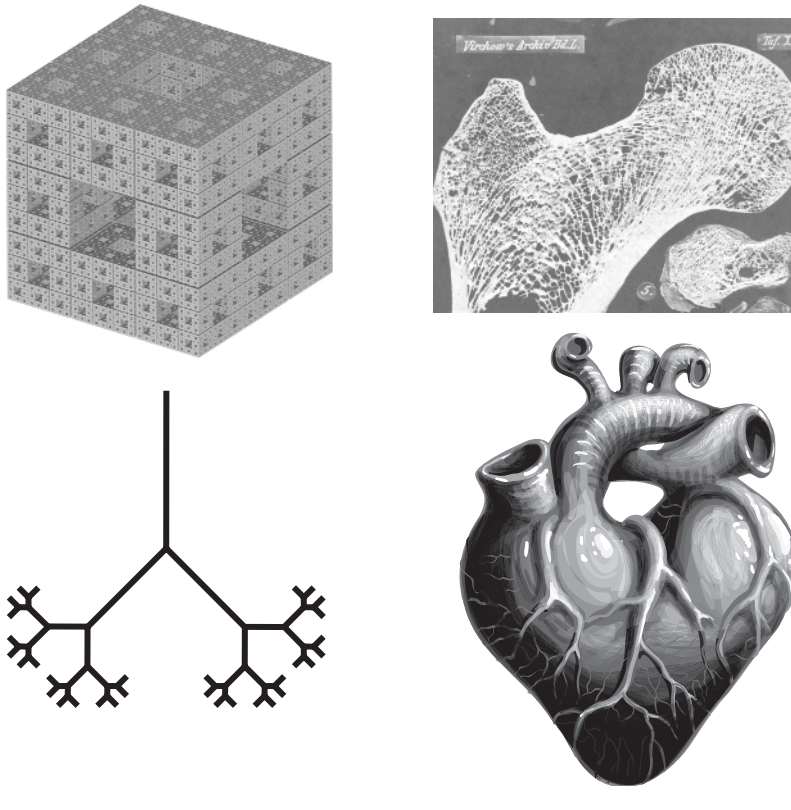

Figure 1.7 Fractals and biological applications

The Cantor set is an example of a self-similar *fractal*. Although originally introduced as curious mathematical entities, self-similar fractals (such as the Koch snowflake, the Sierpiński carpet, the Menger sponge and the fractal tree) can be treated as viable mechanical systems, called *fractal bodies*, to model biological structures, as suggested in Figure 1.7. Fractal bodies are neither discrete systems nor material continua according to our definitions.

A similar approach will be used in Section 5.5 to define a three-dimensional material body $\mathcal{B}$ as a connected open subset of $\mathbb{R}^3$, namely, of the three-dimensional material universe. The material universe does not presuppose any particular values for the mass or for any material properties whatsoever. Thus, a material body $\mathcal{B}$ can be regarded as just a continuous collection of particles. In contrast with the discrete counterpart, we observe that the state variables ψ_i now become *fields* defined over $\mathcal{B}$. For this reason it may be said that continuum mechanics belongs to the realm of physical *field theories*. The governing equations are, in general, partial differential equations (PDEs) rather than ODEs, as was the case in discrete systems.

Remark 1.2.5 We have followed the traditional definition of a continuous material body as an *open* subset of the material universe, thus automatically excluding its boundary (which, in the one-dimensional case, consists of at most two points). In other words, we are just concerned with the interior points of a physical material body. The inclusion of the boundary plays an important role when effects such as surface tension are of interest.

1.3 Configurations and Deformations

The material body itself, defined as a subset of the abstract material universe, is inaccessible to measurement. It acts as some kind of Platonic idea which manifests itself imperfectly in the world of experience. These manifestations, in the case of continuum mechanics, are called *configurations*. Thus, a configuration is a map κ of the material body into the physical space (which, in our one-dimensional context, is the real line):[1]

$$\kappa : \mathcal{B} \to \mathbb{R}. \tag{1.2}$$

To qualify as a configuration, however, this map cannot be arbitrary. The first restriction we will impose is *continuity*. Physically, continuity means that we do not want the body to be broken into separate pieces, but to remain as a connected unit. The second condition that we are going to impose corresponds to the physical notion of *impenetrability of matter*. In other words, we want to avoid a situation whereby two different material points occupy simultaneously the same position in space. From the mathematical point of view, this condition is equivalent to the requirement that the map be *injective*, or *one-to-one*. We summarize these conditions in the following definition.

Definition 1.3.1 A *configuration* of a (one-dimensional) body $\mathcal{B}$ is a continuous one-to-one map of $\mathcal{B}$ into $\mathbb{R}$.

[1] Some basic terminology on maps is reviewed in Window 1.3.

Window 1.3 Functions and maps
Given two sets, A and B, a *function* f from A (the *domain*) to B (the *codomain*),

$$f : A \to B,$$

assigns to *every* element of A some element of B. If $p \in A$, we say that the function f *acts* on p to produce the element $f(p) \in B$. A common notation for the action of a function is

$$p \mapsto f(p).$$

We also say that p is *mapped* by f to $f(p)$, the *image* of p. Depending on the context, functions are also called *maps* or *mappings*.

The *range* of the function $f{:}A \to B$ is the subset of the codomain B consisting of all elements $q \in B$ such that, for at least one $p \in A$, $f(p) = q$. For example, the range of the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $(x, y) \mapsto x^2 + y^2$ is the closed positive semi-axis of the real line. If $C \subset A$, we denote by $f(C)$ the subset of B consisting of all the images of elements of C. Thus, the range of a function is precisely $f(A)$.

A function is *onto* or *surjective* if its range equals its codomain. In other words, every element of the codomain is the image of at least one element of the domain. A function $f : A \to B$ is *one-to-one* or *injective* if, for every pair $p_1, p_2 \in A$, $p_1 \neq p_2$ implies $f(p_1) \neq f(p_2)$. Thus, no two different elements of the domain are assigned the same element of the range. A function is *bijective* if it is both one-to-one (injective) and onto (surjective). If $f : A \to B$ is bijective, and only in this case, we can define the *inverse function* $f^{-1} : B \to A$ by assigning to each element $q \in B$ the unique element $p \in A$ such that $f(p) = q$. Noting that every function is clearly surjective over its range $f(A) \subset B$, we can always define the restricted inverse $f^{-1} : f(A) \to A$ of any injective function f.

If $f : A \to B$ and $g : B \to C$ are functions, the *composition* of g with f is the function $g \circ f : A \to C$ defined by

$$p \mapsto (g \circ f)(p) = g(f(p)),$$

for each $p \in A$. Notice that the codomain of f must be a subset of the domain of g for the composition to be defined. The composition of functions enjoys the associative property: $h \circ (g \circ f) = (h \circ g) \circ f$, which justifies the less precise notation $h \circ g \circ f$.

We note that a configuration can also be understood as the introduction of a coordinate system in the body. Indeed, a formula such as equation (1.2), under the conditions of continuity and injectivity, assigns to each point of the body a unique real number within an open interval of the real line. Based on this observation, it turns out to be convenient to arbitrarily choose a configuration as a reference, so that quantities defined over the material body can be explicitly expressed in terms of coordinate formulas. We will denote this *reference configuration* by some special symbol such as κ_0.

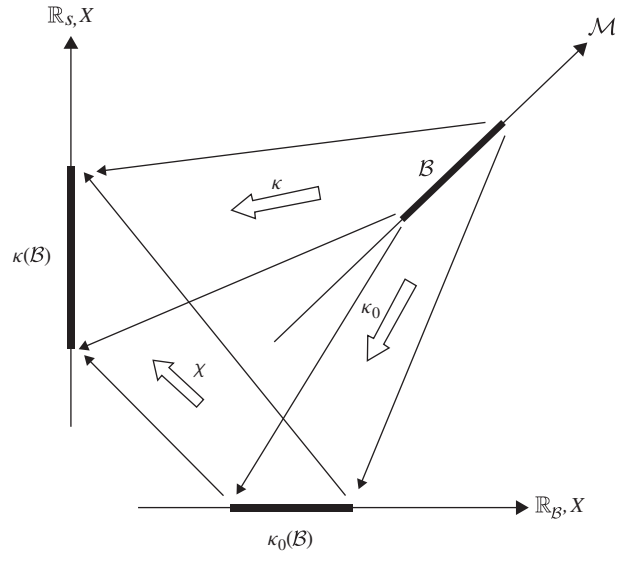

Figure 1.8 Configurations and deformation

Definition 1.3.2 Given two configurations, κ_0 and κ, the *deformation* of the second relative to the first is defined as the composition:

$$\chi = \kappa \circ \kappa_0^{-1}. \tag{1.3}$$

It is important to notice that the inverse function κ_0^{-1} is well defined over the image $\kappa_0(\mathcal{B})$, since configurations are by definition one-to-one. Figure 1.8 depicts the notion of deformation. It is customary to refer to a configuration other than the reference as the *current* or *spatial* configuration. The deformation is thus the map that assigns to each point of the reference configuration a point in the current configuration.

Conceptually, it is convenient to distinguish between the copy of $\mathbb{R}$ where the reference configuration lives and the copy of $\mathbb{R}$ where the current configurations take place. In Figure 1.8 we denote these two copies by $\mathbb{R}_{\mathcal{B}}$ and $\mathbb{R}_{\mathcal{S}}$, respectively.

A nice way to visualize the difference between the reference space $\mathbb{R}_{\mathcal{B}}$ and the physical space $\mathbb{R}_{\mathcal{S}}$ in a biomechanical context is to imagine that the former represents the space where samples are prepared for experiment. For, example, a tendon or a muscle fibre may initially lie on a table where markers and electrodes are placed. This is the world of $\mathbb{R}_{\mathcal{B}}$. Once the preparation has been accomplished, the specimen is placed between, say, the jaws of a tensioning device. This is the world of $\mathbb{R}_{\mathcal{S}}$, where deformations take place.

The usefulness of a fixed (albeit arbitrarily chosen) reference configuration resides in the fact that all configurations can be referred to it so that the notion of configuration can be advantageously replaced by the notion of deformation, thereby making coordinate expressions explicitly available to describe the physical phenomena at hand. Denoting the running coordinate in the reference configuration κ_0 by X and in the current configuration κ by x, the deformation χ is expressed in coordinates by some function

$$x = x(X). \tag{1.4}$$

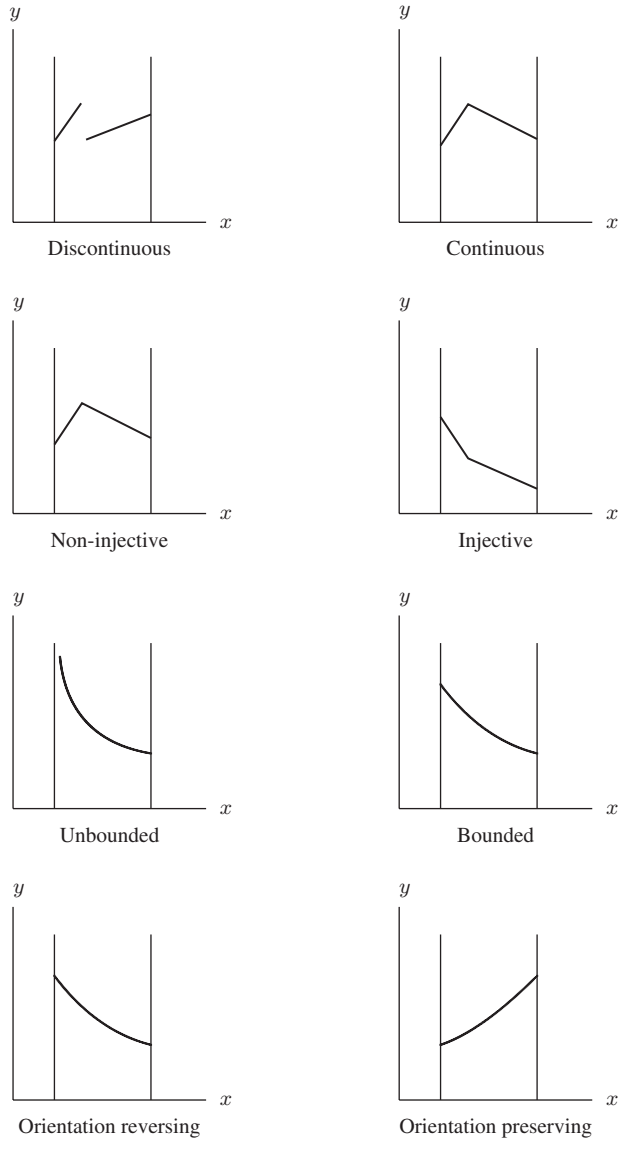

Figure 1.9 Some features of functions of a single variable

This function is continuous and one-to one. For physical reasons, we will impose two further restrictions. The first restriction stems from the fact that we would like deformations to be *orientation preserving* (so that what we call left and right in the reference configuration is consistent with what we call left and right in all other configurations). The second restriction is slightly more technical. We will assume that all deformations (and their inverses) are differentiable functions and that the derivatives are bounded.[2] Figure 1.9 illustrates various properties of functions that have been mentioned above.

[2] This condition can be further relaxed by not requiring differentiability and replacing the boundedness by a Lipschitz condition.

Although in principle we could adopt different measuring units in $\mathbb{R}_{\mathcal{B}}$ and $\mathbb{R}_{\mathcal{S}}$, we will assume that the same units (inches, millimetres, etc.) are used in both. As far as the material universe $\mathcal{M}$ is concerned, it is devoid of units since, as may be recalled, it just represents all possible collections of particles and their topological properties (such as the notions of open and closed intervals) without any metric connotation. For practical purposes, one may identify the body with any one of its reference configurations, thus avoiding the need for the material universe, with the proviso that the reference configuration is an arbitrary choice.

1.4 The Deformation Gradient

Having assumed the deformation to be differentiable, it is natural to consider the meaning of its derivative.

Definition 1.4.1 Given a (differentiable) deformation $x = x(X)$, the *deformation gradient* at a body point X_0 is defined as the derivative

$$F = \left.\frac{\partial x}{\partial X}\right|_{X=X_0}. \tag{1.5}$$

The reason for using partial derivatives is that, as we will soon establish, the deformation is in general considered also a function of time. Since the derivative can be evaluated at each and every body point, the deformation gradient can be considered as a function of X. We note that by the assumption of preservation of orientation the deformation is always a monotonically increasing function, whence we obtain

$$F > 0. \tag{1.6}$$

Notice that, by virtue of its definition, the deformation gradient is non-dimensional.

We now explore the physical meaning of the deformation gradient. In this regard, it is useful to adopt Leibniz's view of the notion of derivative (at a point X_0) as a linear function that maps increments dX of the independent variable into increments dx of the dependent variable according to the formula

$$dx = \left.\frac{\partial x}{\partial X}\right|_{X=X_0} dX. \tag{1.7}$$

The derivative is thus a local linear approximation to the function, as shown in Figure 1.10. In our context, we write

$$dx = F\, dX. \tag{1.8}$$

If we regard the differential dX as a small piece of the body around a point X_0 in the reference configuration, the deformation gradient tells us the (approximate) size $dx = F\, dX$ of the piece made up of the same material points in the current configuration, as shown in Figure 1.10. The two pieces will be of the same size if, and only if, it so happens that, at the point in question, $F = 1$. We may say, therefore, that a unit value of the deformation gradient corresponds to a rigid deformation of a small neighbourhood of the point. If the deformation gradient is greater than 1, we have an increase in length (or *extension*). A deformation gradient less than 1 corresponds to a decrease in length (or *contraction*).

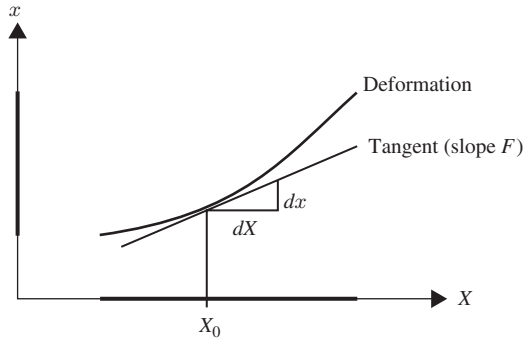

Figure 1.10 The deformation gradient

1.5 Change of Reference Configuration

It is important to bear in mind that the deformation gradient depends on *both* the current and the reference configuration. We want to investigate what happens to the deformation gradient when a different choice of reference configuration is made. The conceptual scheme is illustrated in Figure 1.11. Let κ_0 and κ_1 denote two different reference configurations and let X and Y denote the coordinates in the respective referential spaces $\mathbb{R}_{\mathcal{B}_0}$ and $\mathbb{R}_{\mathcal{B}_1}$. Then, the *change of reference configuration* κ_{10} from κ_0 to κ_1 is defined by the composition

$$\kappa_{10} = \kappa_1 \circ \kappa_0^{-1}, \tag{1.9}$$

or, in coordinates, by some function

$$Y = Y(X). \tag{1.10}$$

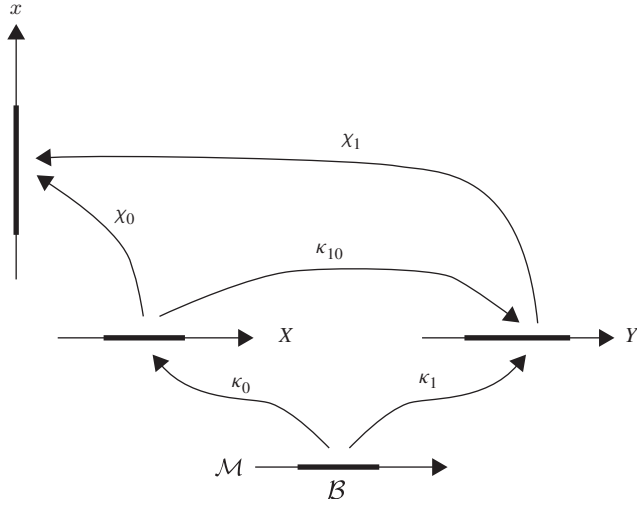

Figure 1.11 Change of reference configuration

This function is smooth, orientation preserving and smoothly invertible. As such, we may calculate the *gradient of the change of reference* at X as the (positive) scalar

$$H = \frac{dY}{dX}. \tag{1.11}$$

Let us denote the deformations relative to κ_0 and κ_1 by χ_0 and χ_1, respectively. They are related by the composition

$$\chi_0 = \chi_1 \circ \kappa_{10}, \tag{1.12}$$

as can be inferred from Figure 1.11. In terms of coordinate representations, these deformations are given by some functions

$$x = x(X) \tag{1.13}$$

and

$$x = \hat{x}(Y), \tag{1.14}$$

respectively, where we have placed a hat over the second function to indicate a different functional relationship. Using equation (1.10), we may write

$$x = \hat{x}(Y) = \hat{x}(Y(X)) = x(X), \tag{1.15}$$

as the relation between the two deformations of the same current configuration. We are interested in the relation between the deformation gradients F and $\hat{F}$ of the deformations $x(X)$ and $\hat{x}(Y)$, respectively. Taking the derivative of equation (1.15) with respect to X, we obtain

$$F = \frac{\partial x}{\partial X} = \frac{\partial \hat{x}}{\partial Y}\,\frac{dY}{dX} = \hat{F}H. \tag{1.16}$$

In short, the chain rule of differentiation provides us with the result that the deformation gradients with respect to different reference configurations are related by multiplication with the gradient of the change of reference configuration. We notice that, since H cannot vanish (it is, in fact, always positive according to our assumption of preservation of orientation), it makes sense to write equation (1.16) alternatively as

$$\hat{F} = FH^{-1}. \tag{1.17}$$

We may thus establish that the arbitrariness in the choice of reference configuration is governed by equation (1.15) in the sense that all kinematical measurements (such as the deformation gradient) will be necessarily related by a strict transformation rule (such as equation (1.16)).

1.6 Strain

A notion closely related to the deformation gradient is that of *strain*. One of the reasons for introducing this notion is that we would like to have a quantity that vanishes whenever the neighbourhood of a point undergoes a rigid deformation (rather than attaining a unit value, as the deformation gradient does). An easy way to tackle this issue is to define the

strain as

$$e = F - 1. \tag{1.18}$$

An alternative and illuminating way to write this definition is obtained by using Leibniz's notation for the derivative. We have

$$e = F - 1 = \frac{dx - dX}{dX}. \tag{1.19}$$

In other words, the strain is a measure of the relative elongation, namely, the change in length divided by the original length (in the reference configuration) of a small piece of material. Notice that, since F is positive, the range of variation of e is the interval $(-1, \infty)$, where the lower limit corresponds to the collapse of a segment to a point.

The measure of strain e just defined, also known as the *engineering strain*, is not the only possibility. In three-dimensional problems (when the deformation gradient is represented not by just a number but by a matrix) it turns out that a more convenient and legitimate measure of strain is obtained by considering first a squared measure of the deformation gradient. In our one-dimensional context this measure, called the (right) *Cauchy–Green* scalar, is defined as

$$C = F^2. \tag{1.20}$$

Note that this number preserves the above mentioned property of the deformation gradient in the sense that it attains a unit value if and only if the neighbourhood preserves its length (i.e., undergoes a rigid deformation). Thus, we can define a new measure of strain by means of the formula:

$$E = \frac{1}{2}(C - 1). \tag{1.21}$$

This strain measure is called the *Lagrangian strain*. As desired, it vanishes for rigid deformations. The presence of the factor $\frac{1}{2}$ requires some further justification. It stems from the following condition to be imposed on all strain measures.

Condition 1.6.1 Small strains. For *small strains*, all strain measures must reduce to the engineering strain e.

Recalling that all strain measures are non-dimensional (or imposing this as a restriction) and that for rigid deformations all strain measures must vanish, a *small strain* is, by definition, a number with an absolute value that is very small compared to the number 1 or, more precisely, a number whose square is very small when compared with its absolute value. Typically, a small strain is of the order of 10^{-2} or smaller. Small strains in biomechanical applications are typical of hard tissue, such as bone, tooth and hair under tension. Soft tissues, such as skin, muscle and connective tissue, can undergo much larger strains. Let us check whether the condition just proposed is verified by the Lagrangian strain given by equation (1.21). Using this equation in conjunction with equations (1.18) and (1.20), we can write

$$E = \frac{1}{2}(C - 1) = \frac{1}{2}(F^2 - 1) = \frac{1}{2}(F + 1)(F - 1) = \frac{1}{2}(e + 2)e = e + \frac{1}{2}e^2. \tag{1.22}$$

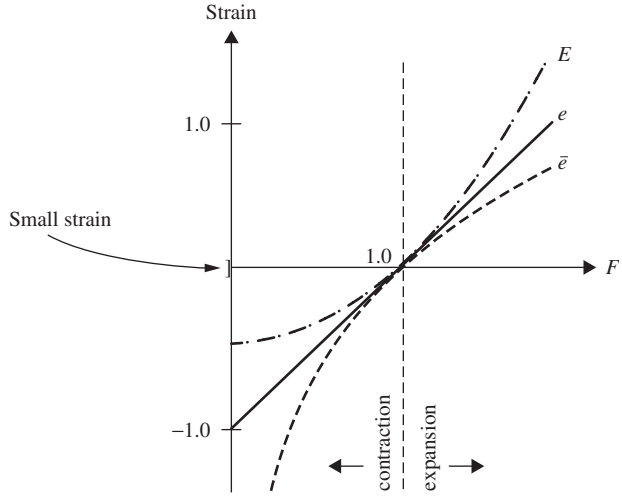

Figure 1.12 Various strain measures

For small strains the last (quadratic) term is negligible compared to e, so that we have that $E \approx e$. In this derivation, the factor $\frac{1}{2}$ plays a decisive role.

Another useful measure of strain is the *logarithmic strain*, given by

$$\bar{e} = \ln F, \tag{1.23}$$

where 'ln' denotes the natural logarithm. Notice that this definition makes sense, since F is strictly positive. Moreover, the logarithmic strain vanishes for rigid deformations, as expected for a measure of strain. The range of variation of the logarithmic strain is the whole line $(-\infty, \infty)$. To check the fulfilment of the small-strain condition, we recall that the natural logarithm function has a convergent Taylor expansion around the unit argument given by

$$\ln F = F - \frac{1}{2}F^2 + O(F^3), \tag{1.24}$$

which proves that the small-strain condition is indeed satisfied by the logarithmic strain. Clearly, an equivalent way to formulate the small-strain condition is to require that all measures of strain (defined analytically in terms of the deformation gradient) must have a unit derivative at $F = 1$. There are many other (in fact, infinite) possible measures of strain satisfying this condition, the choice being a matter of convenience. Figure 1.12 illustrates this fact for the three measures of strain discussed in this section.

1.7 Displacement

It may seem surprising that the concept of displacement, sometimes thought of as being the fundamental kinematic field, has not arisen yet in our treatment and, more particularly, that it was not necessary to invoke it in order to define the important notion of strain. What we have called a configuration may also be called a *placement*, namely a positioning of each point of a body in space. A deformation is thus the passage from one placement

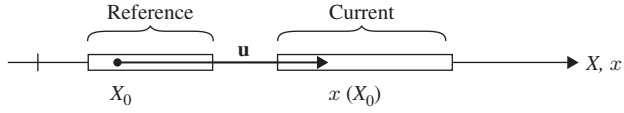

Figure 1.13 Displacement vector

(the reference configuration) to another (the current configuration) and one might be tempted, if only for linguistic reasons, to call it a *dis-placement*. But the usual concept of displacement requires the introduction of additional geometrical structure into the picture, a structure that, as already remarked, does not need to be assumed at all, except as a matter of convenience.

More specifically, to introduce the notion of displacement, we need to immerse both the reference space $\mathbb{R}_{\mathcal{B}}$ and the physical space $\mathbb{R}_{\mathcal{S}}$ in a common all-encompassing space. Alternatively, we may choose to identify $\mathbb{R}_{\mathcal{B}}$ with $\mathbb{R}_{\mathcal{S}}$ and adopt a common coordinate origin and the same orientation. When so doing, the referential and spatial coordinate axes, X and x, become identical, while keeping the distinction of their roles as labels, respectively, of points in the reference and in the current configurations. Adopting this policy, the displacement of a point X_0 is, by definition, the vector $\mathbf{u}$ issuing from X_0 and ending at $x(X_0)$, as shown in Figure 1.13.

Since in a one-dimensional context (recalling that $\mathbb{R}_{\mathcal{B}}$ has been identified with $\mathbb{R}_{\mathcal{S}}$) a vector is represented by a single scalar, positive if pointing forward and negative if backward, we conclude that the displacement associated with a given deformation is the scalar field $u(X)$ defined by the formula

$$u(X) = x(X) - X. \tag{1.25}$$

Denoting the *displacement gradient* by ∇u, we obtain the following equations:

$$\nabla u = \frac{\partial u}{\partial X} = F - 1, \quad e = \nabla u, \quad E = \nabla u + \frac{1}{2}(\nabla u)^2. \tag{1.26}$$

In particular, we observe that the engineering strain is identical to the displacement gradient.[3]

1.8 Motion

A *motion* is a collection of configurations smoothly parametrized by time. In terms of coordinates, a motion is represented by a function of two variables,

$$x = x(X, t), \tag{1.27}$$

where the first argument extends over a fixed reference configuration and the second over some closed time interval $[t_0, t_1]$. The assumption of smoothness can be relaxed, but we will always assume that the motion is at least twice differentiable with respect to the time variable.

[3] For those even superficially acquainted with the more general setting, it hardly needs to be remarked that, in a three-dimensional context, the situation is more involved and that the very definition of large engineering strains is cast into doubt (see Chapter 5).

Recalling that X plays the role of labelling body points, we define the *trajectory of a body point* X_0 as the curve

$$x_0(t) = x(X_0, t). \tag{1.28}$$

We are using the term 'curve' in the sense of a *parametrized* curve. In our spatially one-dimensional setting the image of the curve is clearly contained within the straight line $\mathbb{R}_{\mathcal{S}}$.

Motions can be regarded both as collections of configurations and as collections of trajectories. Graphically speaking, if we represent the motion in equation (1.27) as a surface, this surface can be combed in two ways, namely, by means of trajectories or by means of configurations. Notice that, while the configurations are monotonically increasing curves, this is not necessarily the case for the trajectories (a particle may reverse the sense of its motion, for example). Figure 1.14 is a Mathematica®-generated plot of the motion

$$x = 0.4X(X+1)(1+\sin t) + X, \quad 0 < X < 1, \quad 0 < t < 2\pi. \tag{1.29}$$

This might be the motion of a piece of tendon of unit reference length, one of whose ends is fixed while the other is attached to a reciprocating motor. The figure clearly shows how the surface representing the motion may be spanned by either one of two families of curves, representing trajectories ($X =$ constant) or configurations ($t =$ constant).

We define the *velocity* and the *acceleration* at a point X and at time t as the derivatives

$$v(X,t) = \frac{\partial x(X,t)}{\partial t} \tag{1.30}$$

and

$$a(X,t) = \frac{\partial v(X,t)}{\partial t} = \frac{\partial^2 x(X,t)}{\partial t^2}. \tag{1.31}$$

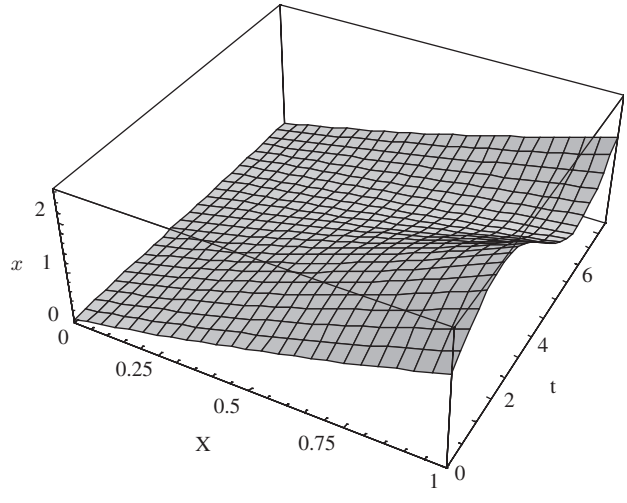

Figure 1.14 A motion $x = x(X,t)$

As before, since the variable X plays the role of labelling body points, we may also say that these expressions represent the velocity and acceleration of the material particle X. Letting X and t vary over their domains of definition, we obtain the *velocity* and *acceleration fields*. These fields are defined so far over the reference configuration. Notice that in these definitions the motion can be replaced by the displacement as a function of X and t.

Exercise 1.8.1 Even when we identify the referential and spatial axes, the reference configuration need not be a configuration actually occupied by the body at any point of time. (a) Show that, for the motion defined by equation (1.29), the spatial configuration does indeed coincide with the reference configuration at specific times. What are these times? (b) Propose a small modification of equation (1.29) to produce a different motion for which the spatial configuration never coincides with the reference configuration.

Exercise 1.8.2 For the motion defined by equation (1.29): (a) Obtain the engineering strain and the Lagrangian strain. (b) Find the time ranges for which these two measures of strain agree over the whole range of the body within a tolerance of 1% . (c) What is the maximum absolute value of the strain in those time ranges?

Exercise 1.8.3 For the motion (1.29), it is proposed to change the reference configuration to that occupied by the body at time $t = 0$. (a) Denoting by Y the corresponding reference coordinate, rewrite the equation of motion with respect to the new reference configuration. What is the range of Y occupied by the body? (b) Repeat Exercise 1.8.2 for the new description of the motion. Are the time ranges the same? Why, or why not?

Exercise 1.8.4 Yet another change of reference configuration is suggested. It consists of stretching the original reference configuration by means of a constant engineering strain of 0.2. Denoting the new reference coordinate by Z, rewrite the equation of motion with respect to the new reference configuration. Comment on the possible agreement of the engineering and the Lagrangian strains in this case. Does the body ever occupy the suggested reference configuration?

1.9 The Lagrangian and Eulerian Representations of Fields

We have already introduced several kinematic fields: the motion itself, the deformation gradient, the various strain fields, the displacement, the velocity and the acceleration. At any fixed instant of time, these fields are defined over the body in the reference configuration, namely, over the open interval $\kappa_0(\mathcal{B})$ of $\mathbb{R}_{\mathcal{B}}$. Letting time vary over its domain of interest $[t_0, t_1]$, we obtain what may be called a body–time field, that is, a function of X and t defined over the rectangle with sides $\kappa_0(\mathcal{B})$ and $[t_0, t_1]$ or, more precisely, over the Cartesian product $\kappa_0(\mathcal{B}) \times [t_0, t_1]$. Any such function $\psi(X, t)$ will be called a *Lagrangian* (or *referential*) *field*. In addition to the fields already introduced, we will encounter other fields of interest, such as temperature fields, stress fields, and so on.

A Lagrangian field assigns a value of some physical quantity to each material particle at each instant of time. But suppose now that, instead of adopting this point of view, we place measurement devices (such as thermometers) at many fixed locations spread over the physical space $\mathbb{R}_{\mathcal{S}}$ and record their readings as time passes. Collecting the results of these measurements at each spatial location x and at each time t, we obtain a function $\psi(x, t)$. A field defined in this manner is called an *Eulerian* (or *spatial*) *field*.

Remark 1.9.1 The space-time domain of an Eulerian field. There are two possibilities that may arise as the body moves. At any given location x of the measuring device and at any given instant of time t, there either happens to be a material particle or not. If the former is the case, we obtain a reading $\psi(x,t)$. Otherwise, we either do not obtain any reading or the reading is physically meaningless, since there is no particle associated with that place and time. Except for the case in which the body is the whole real line, this situation is likely to arise at some points and at some instants. Collecting the results of the measuring devices, at each instant of time t we certainly obtain a field, namely a function of x defined over the current configuration, that is, the open interval $\chi(\kappa_0(\mathcal{B}),t)$ of $\mathbb{R}_{\mathcal{S}}$. This open interval, however, is a function of time. In other words, the field $\psi(x,t)$ is no longer defined over a rectangle, but over a more sinuous two-dimensional figure, as illustrated in Figure 1.15 for the motion described in equation (1.29).

Our immediate interest is in establishing a precise connection between the Lagrangian and Eulerian versions of the field associated with the same physical variable. In the case of temperature, for instance, the Lagrangian version would correspond to having thermocouples glued to various body particles which, thus, accompany the body in its motion. The Eulerian version corresponds to placing the thermocouples at fixed spatial positions. Leaving aside the practical details of such experimental arrangements, it is clear that both thought experiments make perfect physical sense. Given a Lagrangian field $\psi = \psi(X,t)$, we denote the corresponding Eulerian field by $\hat{\psi}(x,t)$. This notational distinction is not always necessary, since it is usually clear in each case which the independent variables are. Nevertheless, we want to emphasize the different functional dependence for the sake of clarity at this junction. The functions $\psi = \psi(X,t)$ and $\hat{\psi}(x,t)$ are related via the motion by the following equation:

$$\hat{\psi}(x,t) = \hat{\psi}(x(X,t),t) = \psi(X,t). \tag{1.32}$$

A field is called *steady* or *stationary* if the function $\hat{\psi}(x,t)$ happens to be independent of time.

It is often necessary to evaluate the gradient of a given field, namely, the derivative of the field with respect to the space-like variable. It follows from equation (1.32) that the

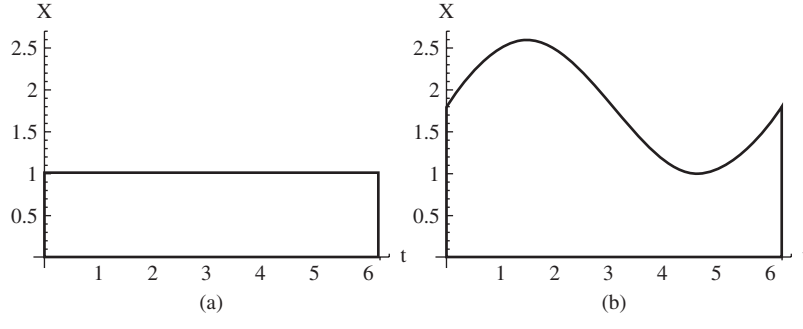

Figure 1.15 Domains of definition of a (a) Lagrangian field (b) and corresponding Eulerian field for the motion of Figure 1.14

referential gradient and the *spatial gradient* are related by

$$\frac{\partial \psi(X,t)}{\partial X} = \frac{\partial \hat{\psi}(x,t)}{\partial x}\frac{\partial x(X,t)}{\partial X} = \frac{\partial \hat{\psi}(x,t)}{\partial x}F(X,t). \tag{1.33}$$

In terms of the gradient notation already used in equation (1.26), we can rewrite equation (1.33) as follows:

$$\nabla \psi = \nabla \hat{\psi}\ F. \tag{1.34}$$

Alternatively, the designations Grad and grad are often used to denote, respectively, the referential and spatial gradients of fields. In terms of this notation, equation (1.34) can be written as

$$\text{Grad } \psi = (\text{grad } \psi)F, \tag{1.35}$$

where the 'hat' has been dropped, since there should not be any room for confusion.

Exercise 1.9.2 An unsteady temperature field has been measured by means of thermocouples placed at fixed spatial positions. Over the domain represented in Figure 1.15(b), the experimental results have been approximated by the formula

$$\theta = (300 + 20x)(1 + 0.1\cos 0.2t). \tag{1.36}$$

Knowing the motion (1.29), express the temperature as a Lagrangian field over the reference configuration.

Exercise 1.9.3 Calculate the Eulerian velocity field of the motion (1.29) by first obtaining the Lagrangian field and then explicitly inverting the equation of motion as $X = X(x,t)$.

1.10 The Material Derivative

A similar treatment of derivatives with respect to the time variable gives rise to the important notion of the *material derivative* or, more precisely, *material time derivative*. Using, as before, the chain rule of differentiation on equation (1.33), we obtain

$$\frac{\partial \psi(X,t)}{\partial t} = \frac{\partial \hat{\psi}(x,t)}{\partial x}\frac{\partial x(X,t)}{\partial t} + \frac{\partial \hat{\psi}(x,t)}{\partial t} = \frac{\partial \hat{\psi}(x,t)}{\partial t} + \frac{\partial \hat{\psi}(x,t)}{\partial x}v(X,t), \tag{1.37}$$

where $v(X,t)$ is the referential version of the velocity field. Equation (1.37) can also be written as

$$\frac{\partial \psi(X,t)}{\partial t} = \frac{\partial \hat{\psi}(x,t)}{\partial t} + (\text{grad } \psi)\ v(X,t). \tag{1.38}$$

Notice that on the left-hand side of this equation, it is the particle that is held fixed while taking the limit, while on the right-hand side it is the spatial position that is held fixed. To avoid the need to explicitly write the independent variables and/or indicate by a hat the different functions involved, the following convention is used for time derivatives: if the particle is held fixed, then the time derivative is denoted by $\frac{D}{Dt}$, while the notation $\frac{\partial}{\partial t}$ is

retained whenever the spatial position is kept fixed. In terms of this convenient notation, equation (1.38) can be written as

$$\frac{D\psi}{Dt} = \frac{\partial \psi}{\partial t} + (\text{grad } \psi)\, v. \tag{1.39}$$

The derivative $\frac{D\psi}{Dt}$ of a field ψ with respect to time while keeping the material particle fixed (whether calculated directly from the Lagrangian representation or indirectly by the right-hand side of equation (1.39)) is known as the material derivative of the field. For a stationary field, equation (1.39) yields the following interesting result:

$$\left.\frac{D\psi}{Dt}\right|_{\text{stationary}} = (\text{grad } \psi)\, v. \tag{1.40}$$

This means that the 'correction' term between the spatial and referential partial derivatives has to do with the spatial gradient and with the velocity. This corrective term is also known as *advection*.

It may prove instructive to provide an intuitive picture of the meaning of the advection term appearing in the calculation of the material derivative out of purely spatial data. Imagine that two observers are placed at two nearby fixed positions in the laboratory separated by a small space interval Δx, as shown in Figure 1.16. They are charged with observing some phenomenon, such as the flow of blood through an artery narrowed by atherosclerosis, and are supposed to record the readings (perhaps temperatures or velocities) of instruments fixed at their respective stations as time goes on. If the field happens to be stationary, the readings will be constant in time. Let these constant readings be ψ and $\psi + d\psi$. A particle (a blood cell, say) that passes in front of the first observer at time t with velocity v will pass in front of the second observer at time $t + \frac{\Delta x}{v}$. This means that, in going from the first observer to the second in an interval of time $\Delta t = \frac{\Delta x}{v}$, this particle undergoes an increase in the quantity ψ precisely equal to the value $\Delta\psi$. The rate of change of ψ at the particle in question is therefore given by

$$\left.\frac{\Delta\psi}{\Delta t}\right|_{\text{particle}} = \frac{\Delta\psi}{\frac{\Delta x}{v}} = \frac{\Delta\psi}{\Delta x}\, v. \tag{1.41}$$

In the limit as $\Delta x \to 0$ we recover equation (1.40).

A particularly interesting application of the concept of material derivative is provided by the calculation of the acceleration. Assume that the velocity field (of a blood flow, for

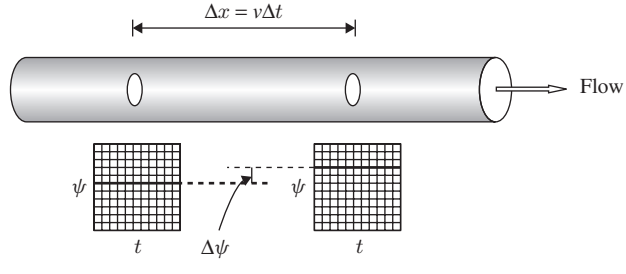

Figure 1.16 Meaning of the advection term

example) has been obtained experimentally as a *spatial* field, namely as

$$v = v(x,t). \tag{1.42}$$

This is the standard situation in fluids, where it is impractical, if not impossible, to follow the motion of particles over long periods of time. How do we obtain the value of the acceleration of the particle instantaneously located at a spatial position x? Since the acceleration is precisely the rate of change of the velocity *following the particle*, the answer to this question is given precisely by the material derivative of the velocity, namely,

$$a(x,t) = \frac{Dv}{Dt}. \tag{1.43}$$

Using equation (1.39), we obtain

$$a(x,t) = \frac{\partial v}{\partial t} + v \text{ grad } v. \tag{1.44}$$

Thus, the acceleration field in its Eulerian version contains an advection term which is fundamentally non-linear in the velocity.

Exercise 1.10.1 Obtain the material derivative of the temperature field (1.36) under the motion (1.29) in two different ways: by using the result of Exercise 1.9.2 and, independently, by using the result of Exercise 1.9.3 and equation (1.39). Verify that both results are in agreement with each other. Notice that the second method would be the only possibility available if the Eulerian velocity field alone were known, rather than the motion itself. The Eulerian velocity field may be available through experimental measurements, just like the given temperature field.

1.11 The Rate of Deformation

An important question that arises in physics is whether or not a quantity that can be measured has an intrinsic physical meaning independent of the observer making the measurement. Clearly, this question is also of great importance in continuum mechanics. There is, however, an additional problem of a somewhat similar nature peculiar to continuum mechanics, a problem that can be traced back to the introduction of the concept of reference configuration. Consider, for example, the fundamental notion of deformation gradient. By its very definition as a map from a reference configuration into physical space, it is clear that the deformation gradient depends on the reference configuration adopted. In fact, equation (1.16) shows explicitly how the deformation gradient is affected by a change of reference configuration. This dependence is transferred automatically to the various measures of strain. The velocity, on the other hand, although obviously dependent on the observer's state of motion, is independent of the reference configuration.

With these ideas in mind, we consider the material derivative of the deformation gradient and investigate how it is affected by a change of reference configuration. Starting from equation (1.16), and bearing in mind that a change of reference configuration is independent of time, we obtain

$$\frac{DF}{Dt} = \frac{D\hat{F}}{Dt}H. \tag{1.45}$$

We conclude that the material derivative of the deformation gradient still depends on the reference configuration, much in the same way as the deformation gradient itself. Consider, however, the combination

$$L = \frac{DF}{Dt}F^{-1}. \tag{1.46}$$

It is not difficult to verify that, by virtue of equations (1.16) and (1.45), this expression is independent of the reference configuration. We call this quantity L the *rate of deformation* or the *stretching*.[4]

In trying to justify this terminology, we observe that, if one were to adopt (as one certainly may) the configuration at time t as a reference configuration, the deformation gradient at any particle at time t would be equal to 1, and the deformation gradient at the same particle and at a nearby time $t + \Delta t$ would be very close to 1, say $1 + \Delta F$. Thus, in the limit as $\Delta t \to 0$ the ratio $\frac{\Delta F}{\Delta t}$ delivers L. It can be said, then, that the rate of deformation is the material derivative of the deformation gradient when the present configuration is adopted instantaneously as a reference configuration. Clearly, this is an intrinsic measure of the rate of change of the strain, regardless of which measure of stain is adopted, since all measures of strain coincide for deformation gradients very close to 1.

It is interesting that there is another way to arrive at the same concept. We have already encountered the spatial velocity gradient grad v as it appeared, for example, in equation (1.44), and later to appear in other contexts. Using equation (1.35), we obtain

$$\begin{aligned} \text{grad } v &= (\text{Grad v})F^{-1} = \frac{\partial}{\partial X}\frac{\partial x(X,t)}{\partial t}F^{-1} \\ &= \frac{\partial}{\partial t}\frac{\partial x(X,t)}{\partial X}F^{-1} = \frac{\partial F(X,t)}{\partial t}F^{-1} = \frac{DF}{Dt}F^{-1} = L. \end{aligned} \tag{1.47}$$

In this derivation, we use the fact that mixed *partial* derivatives are commutative. This is in general not true when the material derivative is expressed in terms of the spatial variable. It is for this reason that we were careful to indicate the independent variables involved at each of the steps of the derivation. We have arrived at the important kinematic result stating that the rate of deformation is equal to the spatial velocity gradient, which is usually readily amenable to direct measurement.

Exercise 1.11.1 Show that the rate of deformation L equals the material derivative of the logarithmic strain.

Exercise 1.11.2 Obtain the velocity gradient of the motion (1.29) by differentiation of the result of Exercise 1.9.3. Verify that this result is in agreement with equation (1.46) regardless of which reference configuration is used, whether it is the original one, the ones suggested in Exercises 1.8.3 of 1.8.4 or, for that matter, any other reference configuration.

1.12 The Cross Section

Always faithful to our policy of focusing on one-dimensional bodies, we must recognize that in many applications it may prove convenient to restore, at least in part,

[4] Although we try to preserve in this one-dimensional setting many of the notational features of the three-dimensional counterpart (such as carefully distinguishing between left and right multiplication and replacing the division by a scalar with multiplication by its reciprocal), there are some important differences attributable mainly to the fact that in the one-dimensional setting there are no rotations.

the three-dimensionality of the body by means of the notion of the *cross section*. Our one-dimensional bodies are essentially three-dimensional material entities in which all the 'action' takes place along a straight line, but the material situated in planes perpendicular to this line will also be affected by the process of deformation. We may say that the notion of the cross section is a price to be paid in exchange for the relative simplicity of a one-dimensional theory vis-à-vis the complexity of a fully-fledged three-dimensional setting. Denoting by $\hat{S}(X)$ the area of the cross section in the reference configuration, its counterpart in the current configuration will be some function

$$\hat{s} = \hat{s}(x,t). \tag{1.48}$$

This function will have to be prescribed by an *ad hoc* kinematic assumption representing the physical situation at hand.

We consider two important examples. The first example is, in fact, a very common occurrence in biomechanics whenever the tissue under study is, within experimental tolerance, *incompressible*. Incompressibility (or volume preservation) arises very often in applications because many biological tissues are made of a large proportion of water, which is nearly incompressible. Equating the referential and current volumes of an infinitesimally thin slice, we obtain

$$\hat{S}(X)dX = \hat{s}(x(X,t),t)\, dx, \tag{1.49}$$

which can also be written, with some abuse of notation, as

$$\hat{s} = \frac{\hat{S}}{F}. \tag{1.50}$$

We can render this condition independent of the reference configuration by taking the material derivative. The result is

$$\frac{D\hat{s}}{Dt} = -\hat{s}\, L. \tag{1.51}$$

Exercise 1.12.1 Prove equation (1.51) by differentiation of (1.50).

The second example corresponds to the flow of a fluid within a tapered rigid pipe or to a solid confined compression test. In this case, the cross sections at the spatial locations are prescribed by the shape of the pipe as some function $f(x)$ independent of time. We have, therefore,

$$\hat{s}(x,t) = f(x), \tag{1.52}$$

or, taking the material derivative,

$$\frac{D\hat{s}}{Dt} = \frac{df}{dx}\, v. \tag{1.53}$$

For a pipe of constant cross section, this condition reduces to

$$\frac{D\hat{s}}{Dt} = 0, \tag{1.54}$$

as expected.

Exercise 1.12.2 By invoking equations (1.51) and (1.53), prove that, if a fluid flowing within a rigid pipe is incompressible, the product $f(x)\, v(x,t)$ is independent of position x along the pipe. Justify this result on intuitive grounds. Notice that, due to the self-imposed limitations of a one-dimensional theory, the longitudinal velocity v is tacitly assumed to be constant throughout each cross section.

Exercise 1.12.3 A rigid pipe of known variable cross section $s = s(x,t)$ carries water between an inlet at $x = 0$ and an outlet at $x = L$. Express the acceleration of the water $a = a(x,t)$ in terms of the shape of the pipe and the velocity $v_L(t) = v(L,t)$ measured at the outlet.