1

Matrix Two-Person Games

If you must play, decide upon three things at the start: the rules of the game, the stakes, and the quitting time.

–Chinese proverb

Everyone has a plan until you get hit.

–Mike Tyson, Heavyweight Boxing Champ,1986–1990, 1996

You can't always get what you want.

–Mothers everywhere

Be sober, be vigilant; because your adversary the devil, as a roaring lion, walketh about, seeking whom he may devour.

–1 Peter 5:8

1.1 What Is Game Theory?

We should start by making sure we understand what this subject is about. Game theory as a branch of **optimization theory** is lots of things, not all of which are covered in this book. Game theory studied here is roughly the math/science of how to make decisions in situations in which

- You might have incomplete information, or the situation may involve some randomness you don't control.
- There may be other players whose goals are different from yours. These can be **opponents** whose aims are opposite yours, or collaborators whose aims are consistent with yours (as in **cooperative games**), or relevant actors whose aims may simply differ from yours.
- You are trying to maximize some kind of outcome to yourself called your **utility** or **payoff**—this could be a probability of winning, an amount of money, your happiness—really anything.

Game Theory: An Introduction, Third Edition. E. N. Barron.

Figure 1.1 The Card Players by Paul Cezanne. Everett Collection/Shutterstock.

When random elements are part of the game, like the dealing of cards or the chances a dictator will attack, you can't predict exactly what will happen, but you try to make things work out as best as you can on average. There are many examples of games in this category. To name just a few:

- **Economic**: Companies trying to make decisions such as setting production limits, who likely have only partial information on the effects of their choices, and where there may be competitors whose choices are not known.
- **Military**: It seems pretty clear that strategic decisions such as troop deployments in a combat zone involve both uncertainty and opponents whose actions are not known.
- **Actual** games: Some standard games played for fun or profit fall into this category. Probably the best example is poker, which we will spend some time thinking about. Of course the full game of poker is way too complicated to analyze, and we will only do so for very specialized and simple games whose rules will be specified precisely.

We will study many examples of all of these, and many have very important and impactful consequences in the real world.

Another main branch of game theory is combinatorial game theory. This would include games like chess, go, life, etc.—in theory you could work out every possibility and find the absolute perfect moves, but it is just too complicated. But, using game theory, modern computers can defeat any grand master in chess and any combinatorial game for which computer programmers have already tackled the problem. We will touch on a few games in this category of combinatorial games, but it is not the focus of this book. If you picked this up to be a better chess player read a chess book. The game theory in this book has a lot to say about poker, however, and some professional poker players use and have developed game theory for poker.

1.2 Motivating Examples

In this section we will present three simple games to provide some insight into why game theory is interesting and important and has attracted the attention of many scientists (and gamblers).

1.2.1 Three Card Poker

Here is a very simplified model of a game of poker. The rules are as follows.

- Three possible cards are involved in this game: Ace, Queen, and King. There are two players Actor, known as player I, and Responder, known as player II. They each must ante \$1 to play the game.
- Actor will be dealt either an Ace (A) or a queen (Q), the two possibilities being equally likely. Actor knows the card dealt but Responder does not.
- Responder will always be dealt a King (K). Both players know this.
- Actor may choose to pass (equivalently, fold) or bet. If Actor chooses to pass, she loses \$1 no matter what card she has.
- If Actor bets, then Responder must choose to call or fold. If Responder folds then Actor wins the pot, i.e., Actor wins \$1 from Responder. If Responder calls, the highest card wins the pot, which now contains \$2 from each player. So if Actor has an Ace, Responder wins \$2, and if Actor has a Queen, Responder loses \$2.

You might think the strategy for Actor should be to fold with a Queen and bet with an Ace, but this may not be the case: Actor may want to bluff by betting on a Queen at least some of the time, and indeed we see poker players bluffing in real life. It will be very helpful to draw a diagram, called a game tree, of each player's choices and the resulting payoffs. These game trees will be studied extensively in Chapter 3.

It is important to note that in Figure 1.2 a dotted line connecting two nodes means that the two nodes are in the same **information set**, labeled 2:1 in this case. What that means is that player II, Responder, knows that player I, Actor, has made a bet but player II does not know what card she holds, i.e., player I has arrived at the information set 2:1 at one of the two nodes, but player II doesn't know which one. **This models information available to player II** and is a very important part of game theory.

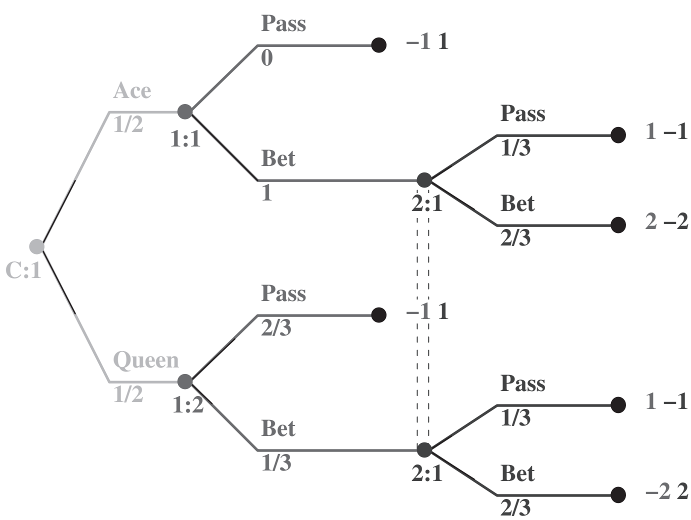

Figure 1.2 A simple poker game.

We cannot (easily) figure out exactly what should happen from the tree. Since Responder does not immediately know if Actor has Ace or Queen, she doesn't know how to respond to a bet. If Actor has Ace she wants to fold, but if Actor has a Queen, she prefers to call. Because of that, Actor does not immediately know what to do if she has a Queen. If Responder is planning to fold, Actor should bet and win \$1 instead of losing \$1. But if Responder would call a bet, Actor should fold, and accept the loss of only \$1 instead of losing \$2. Notice that the numbers at the end of the branches represent the payoffs to each player if each branch is followed.

To analyze the game, we convert the tree to a matrix. To do that, we need to understand what the possible strategies are. **A strategy for a player is a plan for what to do starting from the beginning of the game to the last possible move.** You can imagine then why strategies can be so complicated for a full poker game like 5 card stud, or a game of chess. Here is a set of strategies for our game that works for Actor.

PP : pass on Ace and pass on Queen
PB : pass on Ace and bet on Queen
BP : bet on Ace and pass on Queen
BB : bet on Ace and bet on King

These strategies take into account the fact that Actor must have a plan on what to do if she gets an Ace or a Queen.

For Responder it is simpler: She only has to make a choice if there is a bet since she plays second, and there are only two choices, call or fold. Her strategies are

c : if there is a bet, call
f : if there is a bet, fold

We can now calculate the **expected payoff** to Actor for each pair of strategies, getting a description of this as a matrix game, with Actor playing rows:

$$\begin{array}{c|cc} & c & f \\ \hline PP & -1 & -1 \\ PB & -\frac{3}{2} & 0 \\ BP & \frac{1}{2} & 0 \\ BB & 0 & 1 \end{array}.$$

The expected payoff is a calculation to see how much a player can expect to win or lose on each play on average if the game is played many times. This is a consequence of what mathematicians call the **law of averages** to account for payoffs depending on chance outcomes. To see how this is calculated for our problem, take for example BP versus c. Then

- If Actor has an Ace she bets and Responder calls; Actor has the higher card so she wins \$2.
- If Actor has a Queen, she passes; so she loses \$1.

Each of these possibilities happens with probability $\frac{1}{2}$, so the expected outcome is

$$E(BP, c) = \frac{1}{2}2 + \frac{1}{2}(-1) = \frac{1}{2}.$$

Similarly, PB versus c means Actor folds with an Ace and bets with a Queen, while Responder always bets. The expected payoff to Actor is $E(PB, c) = \frac{1}{2}(-1) + \frac{1}{2}(-2) = -\frac{3}{2}$. The rest of the entries you can verify.

What would each player do? Remember that the numbers in the matrix represent the expected payoff to Actor. Responder obtains the negative of the payoffs to Actor. This is the hallmark of a zero-sum game.

For Actor, it is clear that *PP* would be a bad strategy since Actor can always do better by playing *BB* or *BP*. This means we could essentially drop *PP* from consideration. For the same reason *PB* should be dropped because Actor would be foolish to play it instead of *BP* or *BB*. This means the matrix can actually be reduced to

$$\begin{array}{c|cc} & c & f \\ \hline BP & \frac{1}{2} & 0 \\ BB & 0 & 1 \end{array}.$$

Now it is not so clear what to do for either player. If Actor decides to play *BP*, then Responder would play f; knowing that, Actor would instead play *BB* but then Responder would play c. This reasoning never ends until game theory enters the stage.

Let's assume that Actor is not going to always do the same thing but will randomize between *BP* and *BB*. After all, Actor and Responder would be exceedingly bad poker players if they always did the same thing. What does randomize mean? It means that Actor will play *BP* with probability, say p, and then *BB* with probability $1 - p$. Naturally, we then have to calculate Actor's expected payoff. We're going to do it first assuming Responder plays c and then if Responder plays f.

$$E_c(p) = \frac{1}{2}p + 0(1-p) = \frac{p}{2}, \text{ and } E_f(p) = 0p + 1(1-p) = 1 - p.$$

A way to think about this now is Actor will choose p so that it doesn't matter whether Responder plays c or f. This would say Actor is **indifferent** to Responder's choice and it would give the same expected payoff to Actor. This requires

$$\frac{p}{2} = 1 - p \implies p = \frac{2}{3}.$$

Voila! Actor should play *BP* exactly $\frac{2}{3}$ of the time and *BB*, $\frac{1}{3}$ of the time. Using similar reasoning for Responder, she should call $\frac{2}{3}$ of the time and fold $\frac{1}{3}$ of the time. The **indifference principle**, also called the **equality of payoffs principle**, for Responder says that Responder would choose the probability of playing c, namely q so that Responder is indifferent between Actor playing *BP* or *BB*. This would lead to the equations

$$q\frac{1}{2} = (1-q) \cdot 1 \implies q = \frac{2}{3}.$$

(It is a coincidence here that these probabilities are the same for the two players. This does not always have to be the case.) Thus, in terms of the 4×2 matrix game, the strategies for Actor is to never play *PP* or *PB* and play *BP* with probability $\frac{2}{3}$ and *BB* with probability $\frac{1}{3}$.

But this is a kind of strange way of saying it: if you are playing, and you are looking at a card, what you need to know is, how often should I bet?

- If Actor has a Queen: she should play *BP* exactly $\frac{2}{3}$ of the time, so that two thirds of the time she should pass. Actor should also play *BB* $\frac{1}{3}$ of the time, so bet $\frac{1}{3}$ of the time with a Queen.
- If Actor has an Ace she should bet no matter what.

The upshot is: Actor always bets with the Ace, and bets $\frac{1}{3}$ of the time with a Queen. For Responder, if there is a bet, call with probability $\frac{2}{3}$ and fold with probability $\frac{1}{3}$. This could be implemented, say by flipping an unbalanced coin.

Now we see a very interesting phenomenon that is well known to poker players. Namely, the optimal strategy for Actor has her betting one-third of the time when she has a losing card (Queen). We expected some type of result like this because of the incentive for Actor to bet even when she has a Queen. **Bluffing with positive probability is a part of an optimal strategy when done in the right proportion.**

The **indifference principle** gave us a way of finding the best strategies to play for each player. The reason it works is that when a player doesn't care which strategy the opponent will play, since she gets the same payoff either way, then the player uses a strategy that immunizes against the choices of the opponent. We will see later that this will mean that if either player deviates from the indifference strategy, then the opponent will do better.

Another point to be made about this example is you can see why a complete game of poker with five cards and many rounds of betting, raising, calling, folding, etc., is beyond analysis. Strategies are just too complicated for a complete poker analysis at least with classical computers.

1.2.2 Simplified Baseball

In this focused example of what happens between a pitcher and a batter in baseball, we consider that the pitcher can throw one of two pitches. The batter must guess which of the two pitches he will be faced with. In major league baseball, if the batter guesses wrong, it is almost impossible to hit the ball.

The pitcher, who we label as player I or PI, can throw

i) A fastball
ii) A changeup.

The batter, who we label as player II or PII can guess

i) A fastball is coming
ii) A changeup is coming

Note that batter is forced to make one of these guesses; they can't just wait and see what happens because the pitch comes in so fast. The choices are essentially made simultaneously. Once the players have made their choices, the pitcher pitches and the batter tries to hit the ball. What matters is how likely it is that the batter gets a hit. We will first look at things from the pitcher's point of view, so we record the **probability of the batter being out.** This probability, of course, depends on the two choices the players made, and let's say it depends as follows:

- If Pitcher throws fastball and Batter guesses fastball: 0.6
- If Pitcher throws fastball and Batter guesses changeup: 0.9
- If Pitcher throws changeup, and Batter guesses fastball: 0.8
- If Pitcher throws changeup, and Batter guesses changeup: 0.4

The pitcher's goal is just to maximize the probability of an out, and the batter's goal is to maximize the probability of a hit, which is the same as minimizing the probability of an out. We will arrange this as a matrix. The rows represent the pitcher's possible choices: Fastball (F) and change-up (C). The columns represent the batter's possible choices: Guess fastball (GF) and Guess changeup (GC). The entries are the payoff to the pitcher, which in this case is the probability of an out:

	GF	*GC*
F	0.6	0.9
C	0.8	0.4

The question is, what should the players do? Well, if players are forced to make firm decisions, we can't figure it out. Someone always wants to change. For instance,

- If Pitcher is playing F and Batter is playing GF, then Pitcher would like to change to C, since her payoff goes up from 0.4 to 0.6.

- If Pitcher is playing C and Batter is playing GF, then Batter would like to change to GC—then the outcome switches form 0.8 to 0.4, meaning Batter is less likely to be out.

You can check the other two cases.

What does this mean? We analyze this by having Pitcher choose the pitch to throw first and then letting Batter respond. The possibilities work out as

- If Pitcher plays F. Then Batter could play GF with outcome 0.6 or GC with an outcome 0.9. Hitter wants a low probability of an out, so she plays GF. GF is PII's best response to PI's choice of F.
- If Pitcher plays C. Then Batter could play GF with outcome 0.8 or GC with outcome 0.4. Hitter wants a low probability of an out, so she plays GC. GC is PII's best response to PI's choice of C.

The best Pitcher can do is play F and Batter plays GF, with outcome 0.6. Naturally, in real life, if a pitcher always threw a fastball, every batter would always play GF and the pitcher would be out of the game in no time. But there are other possible strategies for a pitcher. The pitcher can play randomly! Let's use the most obvious random strategy: Throwing a fastball or changeup each with probability 0.5, maybe by flipping a fair coin to decide. Then

- If Batter plays GF, the expected outcome to Pitcher is $0.5 \times 0.6 + 0.5 \times 0.8 = 0.7$
- If Batter plays GC, the expected outcome to Pitcher is $0.5 \times 0.9 + 0.5 \times 0.4 = 0.65$

Of course Batter then would play GC, because she is happier with the outcome 0.65. Well, 0.65 is better than 0.6, so adding randomness helped the Pitcher! And actually we should have known that from the real world: in baseball, pitchers rarely throw the same pitch all the time; they mix them up and keep the batter guessing.

Now the question is, is Pitcher using the best strategy now? I would argue no: Batter is responding by always guessing changeup, so Pitcher can do better by throwing fastball a bit more often. But, exactly how much more often? Well, again we should look back to how we solved the poker example: we should figure out how Batter should respond to every possible Pitcher strategy, and the Pitcher chooses the strategy that works out best when Batter always uses this response.

Figure 1.3 is a graphical representation of the payoffs to each player as a function of p.

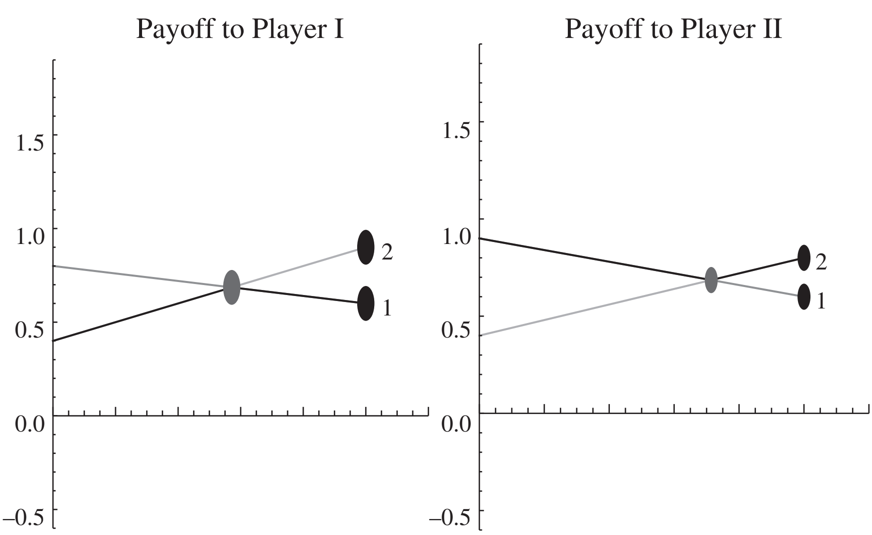

Figure 1.3 Expected payoffs to Pitcher (left) and Batter (right).

The horizontal axis is for the variable p =the probability Pitcher, PI, throws a fastball. So, if $p = 0.3$, this would mean throw a fastball with probability 0.3, and a changeup with probability 0.7. Equivalently, throw a fastball 30% of the time, and a changeup 70% of the time. The vertical axis is the payoff to Pitcher (for the left graph) depending on which line is chosen by Batter and which p is chosen by Pitcher.

If Batter is playing GF, then the payoff to Pitcher is a function of this p:

$$E_{GF}(p) = 0.6p + 0.8(1 - p) = 0.8 - 0.2p.$$

This is line labeled 1 in Figure 1.3 for the payoff to PI. Similarly, if Batter is playing GC, then the payoff is a different function of p:

$$E_{GC}(p) = 0.9p + 0.4(1 - p) = 0.4 + 0.5p.$$

This is line labeled 2 in Figure 1.3 for the payoff to PI. Depending on what Batter does, the payoff to Pitcher is somewhere on either line 1 or line 2, depending on Pitcher's choice of p.

We are forcing Pitcher to choose p first, so Batter will always choose the **lower** of the two lines above that p because Batter wants Pitcher to get the lowest possible payoff. Knowing that, the best thing Pitcher can do is choose the value of p at the intersection of the two lines, which is the value of p for which it doesn't matter what Batter does. This is the value of p for which Pitcher is indifferent to Batter's choices. The indifference principle shows up again.

To find this exact value of p, notice that, at that point, the two possible outcomes must be the same for Batter. This lets us find p by solving an equation!

$$E_{GF}(p) = E_{GC}(p) \Leftrightarrow 0.8 - 0.2p = 0.4 + 0.5p \Leftrightarrow p = \frac{4}{7}.$$

The expected payoff to Pitcher using $p = \frac{4}{7}$ is then $E_{GF}(p) = E_{GC}(p) = 0.6857$. This is clearly better than the expected payoff of 0.65 if $p = 0.5$.

We could then do a similar calculation for PII using the graph on the right of the two lines representing the expected payoff to Batter on the right of the figure, and get that they should play GF with probability $\frac{5}{7}$ and GC with probability $\frac{2}{7}$. Notice that PI wants to get the highest point on the lower of the two lines, while PII wants to get the lowest point on the higher of the two lines. It is then automatic to calculate the expected payoff to batter will be 0.6857. It is not a coincidence that the expected payoff to each player is the same when both players are playing their best strategies.

One of the great applications of game theory to sports is that a team could have an analysis like this for every pitcher and every batter and determine the best strategy for every at bat. In addition, it is possible to see what happens if, say, the pitcher improves his pitches by any amount and then analyze the batter's best strategy to the improvement. This is the content of sports analytics and there is no reason why a similar analysis can't be applied to football, basketball, or any sport.

Here's another example in which it would be foolish to always play the same thing. It will be clear that *randomization of strategies* must be included and is an essential element of games. The simple game of showing fingers illustrates this basic principle.

Example 1.1 ***Evens or Odds:***

In this game, each player decides to show one, two, or three fingers. If the total number of fingers shown is even, player I wins +1 and player II loses −1. If the total number of fingers is odd, player I loses −1, and player II wins +1. The strategies in this game are simple: deciding how many fingers to show. We may represent the payoff matrix as follows:

Evens	Odds		
I/II	1	2	3
1	1	−1	1
2	−1	1	−1
3	1	−1	1

The row player here and throughout this book will always want to maximize his payoff, while the column player wants to **minimize** the payoff to the row player, so that her own payoff is maximized (because it is a zero or constant sum game). The rows are called the **pure strategies** for player I, and the columns are called the **pure strategies** for player II.

The following question arises: How should each player decide what number of fingers to show? If the row player **always** chooses the same row, say, one finger, then player II can **always** win by showing two fingers. No one would be stupid enough to play like that. So what do we do? There is no obvious strategy that will always guarantee a win for either player.

If a player always plays the same strategy, the opposing player can win the game. It seems that the only alternative is for the players to mix up their strategies and play some rows and columns sometimes and other rows and columns at other times. Another way to put it is that the only way an opponent can be prevented from learning about your strategy is if you yourself do not know exactly what pure strategy you will use. That only can happen if you choose a strategy randomly.

Our next example is a simple illustration of strategies in a combinatorial game.

1.2.3 2×2 NIM

The game works as follows: Four pennies are set out in two piles of two pennies each. Player I chooses a pile and then decides to remove one or two pennies from the pile chosen. Then player II chooses a pile with at least one penny and decides how many pennies to remove. Then player I starts the second round with the same rules. When both piles have no pennies, the game ends and the loser is the player who removed the last penny. The loser pays the winner one dollar. Figure 1.4 shows all the possibilities for how this came can progress. When the game is drawn as a tree representing the successive moves, just as we did in the poker example, it is called a game in **extensive form**.

It turns out that we can also represent this using a matrix, like we did with the example in Section 1.2.1, we just need to be very careful about what a strategy should be. As discussed in Section 1.2.1, a strategy should be a complete plan for what to do for the whole game, made before the game starts. Here are the possible strategies for each player in 2×2 Nim:

Strategies for player I
(1) Play (1,2) then, if at (0,2) play (0,1)
(2) Play (1,2) then if at (0,2) play (0,0)
(3) Play (0,2)

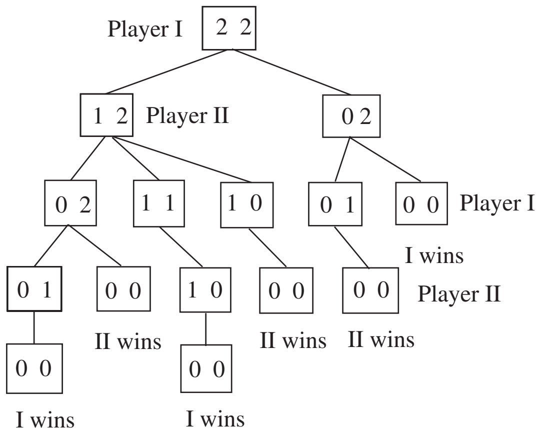

Figure 1.4 2×2 Nim tree.

Strategies for II are more involved:

Strategies for player II
(1) If at (1,2) $\rightarrow$ (0,2); if at (0,2) $\rightarrow$ (0, 1)
(2) If at (1,2) $\rightarrow$ (1,1); if at (0,2) $\rightarrow$ (0,1)
(3) If at (1, 2) $\rightarrow$ (1, 0); if at (0, 2) $\rightarrow$ (0, 1)
(4) If at (1, 2) $\rightarrow$ (0, 2); if at (0, 2) $\rightarrow$ (0, 0)
(5) If at (1, 2) $\rightarrow$ (1, 1); if at (0, 2) $\rightarrow$ (0, 0)
(6) If at (1, 2) $\rightarrow$ (1, 0); if at (0, 2) $\rightarrow$ (0, 0)

Playing strategies for player I against player II results in the payoff matrix for the game, with the entries representing the payoffs to player I.

Player I/Player II	1	2	3	4	5	6
1	1	1	−1	1	1	−1
2	−1	1	−1	−1	1	−1
3	−1	−1	−1	1	1	1

From here it is pretty clear that PII should play strategy 3, and win the dollar no matter what player I does. There will be no need for randomization and, despite the setup being more complicated, this works out more simply than the baseball example.

This game is not very interesting because there is always a winning strategy for player II and it is pretty clear what it is (The former, but not the latter, is pretty much true for any combinatorial game). Why would player I ever want to play this game? There are actually many games like this (tic-tac-toe is an obvious example) that are not very interesting because their outcome (the winner and the payoff) is determined as long as the players play optimally. Chess is not so obvious an example because the number of strategies is so vast that the game cannot, or has not, been analyzed in this way.

One of the main points in this example is the complexity of the strategies. In a game even as simple as tic-tac-toe, the number of strategies is fairly large, and in a game like chess you can forget about writing them all down.

The Nim example is called a **combinatorial game** because both players know every move, and there is no element of chance involved. A separate branch of game theory considers only combinatorial games but this takes us too far afield and the theory of these games is not considered in this book.[1]

1.3 Mathematical Setup

We are now ready to start making things more precise.

1.3.1 Definition of a Matrix Game

For now our games will have only two players, who we call player I (PI) and player II (PII). The game works by

- PI makes a choice between n possible strategies, labeled $1, \dots, n$.
- PII simultaneously makes a choice between m strategies, labeled $1, \dots, m$,
- Once the pair of strategies, i for PI and j for PII is chosen, PI gets a payoff of a_{ij}, and PII gets a payoff of $-a_{ij}$.

The game is determined by the numbers a_{ij}, and we can arrange these numbers in a matrix. These numbers are called the **payoffs to player I** and the matrix is called the **payoff** or **game matrix**. This is a zero sum game, because no matter what strategies are chosen, the sum of the two payoffs to the two players is 0. In a **zero sum game**, whatever one player wins the other loses, and we only need to record the payout to I. The result is

	Player II			
Player I	Strategy 1	Strategy 2	$\cdots$	Strategy m
Strategy 1	a_{11}	a_{12}	$\cdots$	a_{1m}
Strategy 2	a_{21}	a_{22}		a_{2m}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
Strategy n	a_{n1}	a_{n2}	$\cdots$	a_{nm}

By agreement we place player I as the row player and player II as the column player. We also agree that the numbers in the matrix represent the payoff to player I. In a zero sum game the payoffs to player II would be the negative of those in the matrix so we don't have to record both of those. Of course, if player I has some payoff which is negative, then player II would have a positive payoff.

Summarizing, a **two-person zero sum game** in matrix form means that there is a matrix $A = (a_{ij}), i = 1, \dots, n, j = 1, \dots, m$ of real numbers so that if player I, the row player, chooses to play row i and player II, the column player, chooses to play column j, then the payoff to player I is a_{ij} and the payoff to player II is $-a_{ij}$. Both players want to choose strategies that will maximize their individual payoffs. In the matrix, player I will always be the row player and she wants to maximize, while player II will be the column player and she wants to minimize the payoff to player I.

A **constant sum matrix game**, meaning the sum of the two payoffs for each player is the same no matter what strategies are chosen. Constant sum games can also be reduced to the zero sum case: if the sum of the payoffs is C then the game is equivalent to a new game where you start by giving PI the payoff C, then play a new game where PII's payoff is the negative of PI's payoff. So for our purposes constant sum games will work out the same as zero sum games.

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1 The interested reader can see the book *Winning Ways for your Mathematical Plays* (Academic Press, 1982) by Elwyn R. Berlekamp, John H. Conway, and Richard K. Guy.

For example, the simplified baseball game in Section 1.2.2 is governed by the matrix $A = \begin{bmatrix} 0.6 & 0.9 \\ 0.8 & 0.4 \end{bmatrix}$. Pitcher's payoff was the probability of an out and Batter's payoff was the probability of a hit, and these always add up to 1, not to zero. This is an example of a constant sum game with $C = 1$.

In this section we will focus on finding the solution of a game, and on defining what we mean when we say players are playing their best strategies. The starting point here is that we force I to play first. Then it is possible to figure out how each player should play. We will then turn it around and force II to play first, and see what happens there.

We look at a game with matrix $A = (a_{ij})$ from player I's viewpoint. Player I assumes that player II is playing her best, so II chooses a column j so as to

$$\text{Minimize}\, a_{ij} \text{ over } j = 1, \ldots, n$$

for any given row i. Then player I can guarantee that she can choose the row i that will maximize this. So **player I** can **guarantee** that in the **worst possible situation** she can get at least

$$\boxed{v^- \equiv \max_{i=1,\ldots,n} \min_{j=1,\ldots,m} a_{ij},}$$

and we call this number v^- the **lower value of the game**. It is also called player I's gain floor.

Next, consider the game from II's perspective. Player II assumes that player I is playing her best, so that I will choose a row so as to

$$\text{Maximize}\, a_{ij} \text{ over } i = 1, \ldots, n$$

for any given column $j = 1, \ldots, m$. **Player II** can therefore choose her column j so as to **guarantee** a loss of no more than

$$\boxed{v^+ \equiv \min_{j=1,\ldots,m} \max_{i=1,\ldots,n} a_{ij},}$$

and we call this number v^+ the **upper value of the game**. It is also called player II's loss ceiling.

In summary, v^- represents the least amount that player I can be guaranteed to receive and v^+ represents the largest amount that player II can guarantee can be lost. This description makes it clear that we should always have $v^- \leq v^+$.

Here is how to find the upper and lower values for any given matrix in a two-person zero sum game with a finite number of strategies for each player.

For each row of the game matrix, find the minimum payoff in each column and write it in a new additional last column. Then the lower value is the largest number in that last column, that is, the maximum over rows of the minimum over columns. Similarly, in each column find the maximum of the payoffs (written in the last row). The upper value is the smallest of those numbers in the last row. Here's the procedure:

$$\begin{array}{cccc|c}
a_{11} & a_{12} & \cdots & a_{1m} & \longrightarrow \min_j a_{1j} \\
a_{21} & a_{22} & \cdots & a_{2m} & \longrightarrow \min_j a_{2j} \\
\vdots & \vdots & \cdots & \vdots & \\
a_{n1} & a_{n2} & \cdots & a_{nm} & \longrightarrow \min_j a_{nj} \\
\hline
\downarrow & \downarrow & \cdots & \downarrow & \\
\max_i a_{i1} & \max_i a_{i2} & \cdots & \max_i a_{im} & v^- = \text{largest min} \\
 & & & & v^+ = \text{smallest max}
\end{array}$$

Here is the precise definition of the value of a game in pure strategies.

Definition 1.3.1 *A matrix game with matrix* $A_{n\times m} = (a_{ij})$ *has the lower value lower value*

$$v^- \equiv \max_{i=1,\dots,n} \min_{j=1,\dots,m} a_{ij}.$$

and the upper value

$$v^+ \equiv \min_{j=1,\dots,m} \max_{i=1,\dots,n} a_{ij},$$

The **game has a value in pure strategies** *if* $v^- = v^+$, *and we write it as* $v = v(A) = v^+ = v^-$.

One way to look at the value of a game is as a handicap. This means that if the value v is positive, player I should pay player II the amount v in order to make it a **fair game**, with $v = 0$. If $v < 0$, then player II should pay player I the amount $-v$ in order to even things out for player I before the game begins. If a game did have $v^+ = v^-$ it would mean that the players should always play a particular row and column. These games, which may be fairly complicated are not that interesting. Think of Baseball, or Poker, when it is known exactly what the players will do.

Example 1.2 Let's see if there is a value for the examples Poker, Baseball, and 2×2 Nim. First for NIM,

$$\begin{array}{cccccc|cc}
1 & 1 & -1 & 1 & 1 & -1 & \longrightarrow & \min = -1 \\
-1 & 1 & -1 & -1 & 1 & -1 & \longrightarrow & \min = -1 \\
-1 & -1 & -1 & 1 & 1 & 1 & \longrightarrow & \min = -1 \\
\hline
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & & \boxed{v^- = -1} \\
\max = 1 & \max = 1 & \max = -1 & \max = 1 & \max = 1 & \max = 1 & \boxed{v^+ = -1} &
\end{array}$$

We see that v^- = largest min $= -1$ and v^+ = smallest max $= -1$. This says that $v^+ = v^- = -1$, and so 2×2 NIM has value $v = -1$.

For Poker, our matrix was

$$\begin{array}{c|cc}
 & c & f \\
\hline
PP & 0 & 0 \\
PB & -\frac{1}{2} & 1 \\
BP & \frac{1}{2} & 0 \\
BB & 0 & 1
\end{array}.$$

We calculate $v^- = \max\{0, -\frac{1}{2}, 0, 0\} = 0$, and $v^+ = \min\{\frac{1}{2}, 1\} = \frac{1}{2}$. This game does not have a value in pure strategies.

For Baseball we had the matrix

$$\begin{array}{c|cc}
 & GF & GC \\
\hline
F & 0.6 & 0.9 \\
C & 0.8 & 0.4
\end{array}$$

and we calculate $v^- = \max\{0.6, 0.4\} = 0.6$, $v^+ = \min\{0.8, 0.9\} = 0.8$. This game also does not have a value in pure strategies.

We have mentioned that the most that player I can be guaranteed to win should be less than (or equal to) the most that player II can be guaranteed to lose, (i.e., $v^- \le v^+$), Here is a quick verification of this fact.

Claim: $v^- \le v^+$. For any column j we know that for any fixed row i, $\min_j a_{ij} \le a_{ij}$, and so taking the max of both sides over rows, we obtain

$$v^- = \max_i \min_j a_{ij} \le \max_i a_{ij}.$$

This is true for any column $j = 1, \dots, m$. The left side is just a number (i.e., v^-) independent of i as well as j, and it is smaller than the right side for any j. But this means that $v^- \le \min_j \max_i a_{ij} = v^+$, and we are done.

1.3.2 Saddle Points: What It Means to be Optimal

Now here is a precise definition of a (pure) saddle point involving only the payoffs, which basically tells the players what to do in order to obtain the value of the game when $v^+ = v^-$.

Definition 1.3.2 *A saddle point in pure strategies is a pair of strategies row i^* for PI and column j^* for PII such that, if the players choose (i^*, j^*), then neither one wants to change to another row or column. Equivalently, $a_{i^*j^*}$ is the max in its column and min in its row. Written mathematically,*

$$a_{ij^*} \le a_{i^*j^*} \le a_{i^*j}, \textit{ for all rows } i = 1, \dots, n \textit{ and columns } j = 1, \dots, m. \tag{1.3.1}$$

In words, (i^*, j^*) is a saddle point if when player I deviates from row i^*, but II still plays j^*, then player I will get less. Also, if player II deviates from column j^* but I sticks with i^*, then player I will do better.

You can spot a **saddle point in a matrix** (if there is one) as the entry which is **simultaneously the smallest in a row and largest in the column.** A matrix may have none, one, or more than one saddle point. here's a condition which guarantees at least one saddle.

Lemma 1.3.3 *A game will have a saddle point in pure strategies if and only if*

$$v^- = \max_i \min_j a_{ij} = \min_j \max_i a_{ij} = v^+. \tag{1.3.2}$$

Proof: If (1.3.1) is true, then

$$v^+ = \min_j \max_i a_{ij} \le \max_i a_{ij^*} \le a_{i^*j^*} \le \min_j a_{i^*j} \le \max_i \min_j a_{ij} = v^-.$$

But $v^- \le v^+$ always, and so we have equality throughout and $v = v^+ = v^- = a_{i^*j^*}$.

On the other hand, if (1.3.2) holds, then

$$v^+ = \min_j \max_i a_{ij} = \max_i \min_j a_{ij} = v^-.$$

Let j^* be such that $v^+ = \max_i a_{ij^*}$ and i^* such that $v^- = \min_j a_{i^*j}$. Then

$$a_{i^*j} \ge v^- = v^+ \ge a_{ij^*}, \text{for any } i = 1, \dots, n,\ j = 1, \dots, m.$$

In addition, taking $j = j^*$ on the left, and $i = i^*$ on the right, gives $a_{i^*j^*} = v^+ = v^-$. This satisfies the condition for (i^*, j^*) to be a saddle point. ■

When a saddle point exists in pure strategies (which will happen if $v^+ = v^-$), (1.3.1) says that if any player deviates from playing her part of the saddle, then the other player can take advantage and improve his payoff. In this sense, each part of a saddle is a **best response** to the other.

In the game of NIM we know there is a saddle point and it occurs at column 3, but any row. This shows that saddle points are not necessarily unique.

We now know that $v^+ \geq v^-$ is always true. We also know how to play if $v^+ = v^-$. The issue is what do we do if $v^+ > v^-$. That will be a topic we study in the next section.

Problems

1.1 There are 100 bankers lined up in each of 100 rows. Pick the richest banker in each row. Javier is the poorest of those. Pick the poorest banker in each column. Raoul is the richest of those. Who is richer: Javier or Raoul?

1.2 Suppose that two companies are both thinking about introducing competing products into the marketplace. They choose the time to introduce the product, and their choices are 1 month, 2 months, or 3 months. The payoffs correspond to market share.

I/II	1	2	3
1	0.5	0.6	0.8
2	0.4	0.5	0.9
3	0.2	0.4	0.5

For instance, if player I introduces the product in 3 months and player II introduces it in 2 months, then it will turn out that player I will get 40% of the market. The companies want to introduce the product in order to maximize their market share. This is also a constant sum game. Find v^+, v^- and a saddle point if any.

1.3 In a Nim game start with 4 pennies. Each player may take 1 or 2 pennies from the pile. Suppose player I moves first. The game ends when there are no pennies left and the player who took the last penny pays 1 to the other player.

(a) Draw the game as we did in 2×2 Nim.
(b) Write down all the strategies for each player and then the game matrix.
(c) Find v^+, v^-. Would you rather be player I or player II?

1.4 In the game rock–paper–scissors both players select one of these objects simultaneously. The rules are as follows: paper beats rock, rock beats scissors, and scissors beats paper. The losing player pays the winner \$1 after each choice of object. If both choose the same object the payoff is 0.

(a) What is the game matrix?
(b) Find v^+ and v^- and determine whether a saddle point exists in pure strategies, and if so, find it.

1.5 Each of two players must choose a number between 1 and 5. If a player's choice = opposing player's choice +1, she loses \$2; if a player's choice $\geq$ opposing player's choice +2, she wins \$1. If both players choose the same number the game is a draw.

(a) What is the game matrix?

(b) Find v^+ and v^- and determine whether a saddle point exists in pure strategies, and if so, find it.

1.6 Each player displays either one or two fingers and simultaneously guesses how many fingers the opposing player will show. If both players guess either correctly or incorrectly, the game is a draw. If only one guesses correctly, he wins an amount equal to the total number of fingers shown by both players. Each pure strategy has two components: the number of fingers to show, the number of fingers to guess. Find the game matrix, v^+, v^-, and optimal pure strategies if they exist.

1.7 Let x be an unknown number and consider the matrices

$$A = \begin{bmatrix} 0 & x \\ 1 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 1 \\ x & 0 \end{bmatrix}.$$

Show that no matter what x is, each matrix has a pure saddle point.

1.8 If we have a game with matrix A and we modify the game by adding a constant C to every element of A, call the new matrix $A + C$, is it true that $v^+(A + C) = v^+(A) + C$?

(a) If it happens that $v^-(A + C) = v^+(A + C)$, will it be true that $v^-(A) = v^+(A)$, and conversely?

(b) What can you say about the optimal pure strategies for $A + C$ compared to the game for just A?

1.9 Consider the square game matrix $A = (a_{ij})$ where $a_{ij} = i - j$ with $i = 1, 2, \ldots, n$, and $j = 1, 2, \ldots, n$. Show that A has a saddle point in pure strategies. Find them and find $v(A)$.

1.10 Player I chooses 1, 2, or 3 and player II guesses which number I has chosen. The payoff to I is |I's number − II's guess|. Find the game matrix. Find v^- and v^+.

1.11 In a baseball example player I, the batter, expects the pitcher (player II) to throw a fastball (F), a slider(S), or a curveball(C). This is the game matrix.

I/II	F	C	S
F	0.30	0.25	0.20
C	0.26	0.33	0.28
S	0.28	0.30	0.33

Find v^+ and v^-.

1.12 In a football game the offense has two strategies: run or pass. The defense also has two strategies: defend against the run, or defend against the pass. A possible game matrix is

$$A = \begin{bmatrix} 3 & 6 \\ x & 0 \end{bmatrix}.$$

This is the game matrix with the offense as the row player I. The numbers represent the number of yards gained on each play. The first row is run, the second is pass. The first column is defend the run and the second column is defend the pass. Assuming that $x > 0$, find the value of x so that this game has a saddle point in pure strategies.

1.4 Mixed Strategies

In the Baseball example and the Poker example we found that they do not have a saddle point in pure strategies. This means that if a player always plays a row (or column) all the time, the other player can take advantage of it. The only way to avoid this is if a player randomizes her plays. This implies that not only does the opponent not know which row (or column) will be played with certainty, but the same will be true of the player who is choosing the row (or column). The players will be mixing up their choices of rows and columns, i.e., they will choose mixed strategies.

1.4.1 Definition of Mixed Strategies

Definition 1.4.1 *A **mixed strategy** is a vector $X = (x_1, \dots, x_n)$ for player I and $Y = (y_1, \dots, y_m)$ for player II, where*

$$x_i \geq 0, \quad \sum_{i=1}^{n} x_i = 1 \ \text{ and } y_j \geq 0, \quad \sum_{j=1}^{m} y_j = 1.$$

The components x_i represent the probability that row i will be used by player I, so $x_i = Prob$ $(I$ uses row $i)$, and y_j the probability column j will be used by player II, that is, $y_j = Prob(II$ uses row $j)$. The set of mixed strategies with k components is denoted by

$$S_k \equiv \left\{ (z_1, z_2, \dots, z_k) \mid z_i \geq 0, i = 1, 2, \dots, k, \sum_{i=1}^{k} z_i = 1 \right\}.$$

In this terminology, a mixed strategy for player I is any element $X \in S_n$ and for player II any element $Y \in S_m$.

If player I uses the mixed strategy $X = (x_1, \dots, x_n) \in S_n$ then she will use row i on each play of the game with probability x_i. Every pure strategy is also a mixed strategy by choosing all the probability to be concentrated at the row or column that the player wants to always play. For example, if player I wants to always play row 3, then the mixed strategy she would choose is $X = (0, 0, 1, 0, \dots, 0)$. Therefore, allowing the players to choose mixed strategies permits many more choices, and the mixed strategies make it possible to mix up the pure strategies used. The set of mixed strategies contains the set of all pure strategies in this sense, and it is a generalization of the idea of strategy.

Now, if the players use mixed strategies the payoff can be calculated only in the **expected** sense. That means the game payoff will represent what each player can expect to receive and will actually receive on average only if the game is played many, many times. More precisely, we calculate as follows.

Definition 1.4.2 *Given a choice of mixed strategy $X \in S_n$ for player I and $Y \in S_m$ for player II, the* **expected payoff** *to player I of the game is*

$$\begin{aligned} E(X, Y) &= \sum_{i=1}^{n} \sum_{j=1}^{m} a_{ij} \ \ Prob(I \text{ uses } i \ \text{ and } II \text{ uses } \ j) \\ &= \sum_{i=1}^{n} \sum_{j=1}^{m} a_{ij} \ \ Prob(I \text{ uses } i) Prob(II \text{ uses } \ j) \\ &= \sum_{i=1}^{n} \sum_{j=1}^{m} x_i a_{ij} y_j = X\ A\ Y^T. \end{aligned}$$

In a zero sum two-person game the expected payoff to player II would be $-E(X, Y)$. *The independent choice of strategy by each player justifies the fact that*

$$Prob(I \text{ uses } i \text{ and } II \text{ uses } j) = Prob(I \text{ uses } i)Prob(II \text{ uses } j).$$

Notation 1.4.3 *For an* $n \times m$ *matrix* $A = (a_{ij})$ *we denote the jth column vector of* A *by* A_j *and the ith row vector of* A *by* ${}_iA$. *So*

$$A_j = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{nj} \end{bmatrix} \text{ and } {}_iA = \left(a_{i1}, a_{i2}, \dots, a_{im}\right).$$

If player I decides to use the pure strategy $X = (0, \dots, 0, 1, 0, \dots, 0)$ *with row i used 100% of the time and player II uses the mixed strategy* Y, *we denote the expected payoff by* $E(i, Y) = {}_iA \cdot Y^T$. *Similarly, if player II decides to use the pure strategy* $Y = (0, \dots, 0, 1, 0, \dots, 0)$ *with column j used 100% of the time, we denote the expected payoff by* $E(X, j) = XA_j$. *We may also write*

$$\boxed{E(i, Y) = {}_iA \cdot Y^T = \sum_{j=1}^{m} a_{ij} y_j, \quad E(X, j) = X\, A_j = \sum_{i=1}^{n} x_i a_{ij}, \text{ and } E(i, j) = a_{ij}.}$$

Notice too that

$$\boxed{E(X, Y) = \sum_{i=1}^{n} x_i E(i, Y) = \sum_{j=1}^{m} E(X, j) y_j.}$$

In $E(i, Y)$ *we say that row i is played against* Y *and in* $E(X, j)$ *we say* X *is played against column j.*

1.4.2 Optimal Mixed Strategies

In the matrix zero sum game the goals are that player I wants to maximize her expected payoff and player II wants to minimize the expected payoff to I using mixed strategies.

We may define the **upper and lower values of the mixed game** as

$$\boxed{v^+ = \min_{Y \in S_m} \max_{X \in S_n} XAY^T, \quad \text{and } v^- = \max_{X \in S_n} \min_{Y \in S_m} XAY^T.}$$

We will see shortly, however, that this is really not needed because it is always true that $v^+ = v^-$ when we allow mixed strategies. Of course, we have seen that this is not true when we permit only pure strategies.

We can define what we mean by a saddle point in mixed strategies.

Definition 1.4.4 *A* **saddle point in mixed strategies** *is a pair* (X^*, Y^*) *of probability vectors* $X^* \in S_n, Y^* \in S_m$, *which satisfies*

$$E(X, Y^*) \le E(X^*, Y^*) \le E(X^*, Y), \quad \forall\ (X \in S_n, Y \in S_m). \tag{1.4.1}$$

If player I decides to use a strategy other than X^* *but player II still uses* Y^*, *then I receives an expected payoff smaller than that obtainable by sticking with* X^*. *A similar statement holds for player II. So* (X^*, Y^*) *is an* **equilibrium** *in this sense.*

We will see in the next section that a saddle point in mixed strategies always exists. That's the good news! But first we deal with the more practical question of actually finding these saddle points. And before that comes an even simpler question: how do you check that a pair of mixed strategies (X^*, Y^*) really is a saddle point, once you think you've found it? This is an issue because Definition 1.4.4 would require you to check the outcome of infinitely many possible strategies, since the sets S_n, S_m are infinite. The following theorem says it is enough to check only the pure strategies.

Theorem 1.4.5 *A pair of mixed strategies X^*, Y^* for a zero sum matrix game is a saddle point if and only if*

$$E(i, Y^*) \le E(X^*, Y^*) \le E(X^*, j)$$

for all $i = 1, 2, \ldots, n, j = 1, 2, \ldots, m$.

Proof: If (X^*, Y^*) is a saddle point, then the inequalities hold by substituting $X = (0, 0, \ldots, 1, 0, 0)$, $Y = (0, 1, 0, \ldots, 0)$ with 1 in the ith place for X and 1 in the jth place for Y, into (1.4.1). To show the other direction, assume the inequalities of the theorem hold. We must show that (1.4.1) holds for all (mixed) X and Y. So, choose a strategy $X = (x_1, \ldots, x_n)$. Then

$$E(X, Y^*) = \sum_{i=1}^{n} x_i E(i, Y^*) \le \sum_{i=1}^{n} x_i E(X^*, Y^*) = E(X^*, Y^*),$$

where the middle inequality holds by the condition in the theorem, and the last equality because $\sum_{i=1}^{n} x_i = 1$ since $X = (x_1, \ldots, x_n)$ is a probability vector,

Now choose a strategy $Y = (y_1, \ldots, y_m)$. Then

$$E(X^*, Y) = \sum_{j=1}^{m} y_j E(X^*, j) \ge \sum_{j=1}^{m} y_j E(X^*, Y^*) = E(X^*, Y^*),$$

where again the middle inequality holds by the condition in the theorem, and the last equality because $\sum_{j=1}^{m} y_j = 1$ since $Y = (y_1, \ldots, y_m)$ is a probability vector. ■

Example 1.3 Consider the matrix game defined by

$$A = \begin{bmatrix} -1 & 3 & 2 \\ 3 & 1 & 1 \\ 2 & -2 & 1 \end{bmatrix}.$$

We claim that $X* = (\frac{2}{5}, \frac{3}{5}, 0), Y^* = (\frac{1}{5}, 0, \frac{4}{5})$ is a saddle point in mixed strategies. We don't know how to find this yet but we can check that it is a saddle point by verifying the inequalities in the theorem. First,

$$E(X^*, Y^*) = [\tfrac{2}{5}, \tfrac{3}{5}, 0] \begin{bmatrix} -1 & 3 & 2 \\ 3 & 1 & 1 \\ 2 & -2 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{5} \\ 0 \\ \frac{4}{5} \end{bmatrix} = \frac{7}{5}.$$

Now check what happens if each player switches to a pure strategy:

$$E(1, Y^*) = \tfrac{7}{5}, E(2, Y^*) = \tfrac{7}{5}, E(3, Y^*) = \tfrac{6}{5}. \text{ All} \le \tfrac{7}{5}$$

$$\implies \text{PI does not want to change from } X^*.$$

$$E(X^*, 1) = \tfrac{7}{5}, E(X^*, 2) = \tfrac{9}{5}, E(X^*, 3) = \tfrac{7}{5}. \text{ All} \ge \tfrac{7}{5}$$

$$\implies \text{PII does not want to change from } Y^*.$$

In fact, we can do this calculation in a somewhat easier way by calculating X^*A:

$$\left[\frac{2}{5}, \frac{3}{5}, 0\right]\begin{bmatrix} -1 & 3 & 2 \\ 3 & 1 & 1 \\ 2 & -2 & 1 \end{bmatrix} = \left[\frac{7}{5}, \frac{9}{5}, \frac{7}{5}\right],$$

Against X^* player II can only choose between these three values, so PII can't get lower than $\frac{7}{5}$. Similarly,

$$\begin{bmatrix} -1 & 3 & 2 \\ 3 & 1 & 1 \\ 2 & -2 & 1 \end{bmatrix}\begin{bmatrix} \frac{1}{5} \\ 0 \\ \frac{4}{5} \end{bmatrix} = \begin{bmatrix} \frac{7}{5} \\ \frac{7}{5} \\ \frac{6}{5} \end{bmatrix}.$$

Against Y^*, player I can only choose between these three values, and PI can't get more than $\frac{7}{5}$. This says (X^*, Y^*) is indeed a saddle point in mixed strategies.

Some of the observations in the previous example are important enough that we should state them:

Theorem 1.4.6 *Let $A = (a_{ij})$ be an $n \times m$ game with value $v(A)$. Let $X^* \in S_n$ be a strategy for player I and $Y^* \in S_m$ be a strategy for player II. Then*

1. $v(A) \geq \min_j E(X^*, j)$.
2. $v(A) \leq \max_i E(i, Y^*)$.
3. *If $\min_j E(X^*, j) = \max_i E(i, Y^*)$, then (X^*, Y^*) is a saddle point.*
4. *if v is any number and (X^*, Y^*) are strategies satisfying $E(i, Y^*) \leq v \leq E(X^*, j)$ for all i, j, then $v = value(A)$ and (X^*, Y^*) is a saddle point.*

The last statement is not obvious because it depends on a result that is the major result of game theory, namely the von Neumann minimax theorem. In fact, up to this point we have assumed saddle points exist and we are working toward finding them under this assumption. But, we have to prove they exist and moreover that if they exist then the game must have a value $v(A)$ in mixed strategies. All this is left for later in this chapter.

The next corollary says that only one player really needs to play mixed strategies in order to calculate $v(A)$. It also says that $v(A)$ is always between the pure strategy lower and upper values v^- and v^+.

Corollary 1.4.7

$$v(A) = \min_{Y \in S_m} \max_{1 \leq i \leq n} E(i, Y) = \max_{X \in S_n} \min_{1 \leq j \leq m} E(X, j).$$

In addition, the value of the game with mixed strategies is always between the lower and upper values with pure strategies, i.e., $v^- = \max_i \min_j a_{ij} \leq v(A) \leq \min_j \max_i a_{ij} = v^+$.

Be aware of the fact that not only are the min and max in the corollary being switched (and there is no reason to think this can be done in general) but also the sets over which the min and max are taken are changing. The second part of the corollary is immediate from the first part since

$$v(A) = \min_{Y \in S_m} \max_{1 \leq i \leq n} E(i, Y) \leq \min_{1 \leq j \leq m} \max_{1 \leq i \leq n} a_{ij} = v^+$$

and

$$v(A) = \max_{X \in S_n} \min_{1\le j\le m} E(X,j) \ge \max_{1\le i\le n} \min_{1\le j\le m} a_{ij} = v^-.$$

Remember that $\min_{S_m} \le \min_j$ because mixed strategies S_m always contain all pure strategies. Similarly, $\max_i \le \max_{S_n}$. In the next example we will see how to use the last statement of the theorem.

Example 1.4 Evens and Odds: In the game of evens or odds each player shows 1, 2, or 3 fingers. If the total number of fingers shown is even, player I is paid 1 by player II. If the total number is odd, player I pays player II 1. The game matrix is

I/II	1	2	3
1	1	−1	1
2	−1	1	−1
3	1	−1	1

We calculate that $v^- = -1$ and $v^+ = +1$, so this game does not have a saddle point using only pure strategies. But it does have a value and saddle point using mixed strategies. Suppose that v is the value of this game and $(X^* = (x_1, x_2, x_3), Y^* = (y_1, y_2, y_3))$ is a saddle point. According to Theorem 1.4.6 these quantities should satisfy

$$E(i, Y^*) = {}_iA\ Y^{*T} = \sum_{j=1}^{3} a_{ij}y_j \le v \le E(X^*, j) = X^*A_j = \sum_{i=1}^{3} x_i a_{ij},$$
$$i = 1, 2, 3,\ j = 1, 2, 3.$$

Using the values from the matrix, we have the system of inequalities

$$y_1 - y_2 + y_3 \le v, -y_1 + y_2 - y_3 \le v, \quad \text{and } y_1 - y_2 + y_3 \le v,$$
$$x_1 - x_2 + x_3 \ge v, -x_1 + x_2 - x_3 \ge v, \quad \text{and } x_1 - x_2 + x_3 \ge v.$$

Let's go through finding the strategy X^*. We are looking for numbers x_1, x_2, x_3, and v satisfying $x_1 - x_2 + x_3 \ge v, -x_1 + x_2 - x_3 \ge v$, as well as $x_1 + x_2 + x_3 = 1$ and $x_i \ge 0, i = 1, 2, 3$. But then $x_1 = 1 - x_2 - x_3$, and we reduce to finding x_1, x_2, and v so that

$$1 - 2x_2 \ge v \quad \text{and} - 1 + 2x_2 \ge v \implies -v \ge 1 - 2x_2 \ge v \implies v \le 0.$$

Similarly, looking for $y_1, y_2, y_3 = 1 - y_1 - y_2$ we get

$$1 - 2y_2 \le v \quad \text{and} - 1 + 2y_2 \le v \implies v \ge 1 - 2y_2 \ge -v \implies v \ge 0.$$

This means we must have $v = 0$. Then $x_2 = y_2 = \frac{1}{2}$. This would force $x_1 + x_3 = \frac{1}{2}$ as well.

Instead of substituting for x_1, substitute $x_2 = 1 - x_1 - x_3$ hoping to be able to find x_1 or x_3. You would see that we would once again get $x_1 + x_3 = \frac{1}{2}$. Something is going on with x_1 and x_3, and we don't seem to have enough information to find them. But we can see from the matrix that it doesn't matter whether player I shows one or three fingers! The payoffs in all cases are the same. This means that row 3 (or row 1) is a redundant strategy and we might as well drop it. (We can say the same for column 1 or column 3.) If we drop row 3 we perform the same set of calculations but we quickly find that $x_2 = \frac{1}{2} = x_1$. Now we have our candidates for the saddle points and value, namely, $v = 0, X^* = \left(\frac{1}{2}, \frac{1}{2}, 0\right)$ and also, in a similar way $Y^* = \left(\frac{1}{2}, \frac{1}{2}, 0\right)$. Check that with these candidates the inequalities of Theorem 1.4.6 are satisfied and so they are the actual value and saddle.

However, it is important to remember that with all three rows and columns, the theorem does not give a single characterization of the saddle point. Indeed, there are an infinite number of saddle points, $X^* = (x_1, \frac{1}{2}, \frac{1}{2} - x_1), 0 \le x_1 \le \frac{1}{2}$ and $Y^* = (y_1, \frac{1}{2}, \frac{1}{2} - y_1), 0 \le y_1 \le \frac{1}{2}$. This makes sense because whether player I or player II shows 1 or 3 fingers doesn't really matter as long as 2 fingers is shown 50% of the time. Nevertheless, there is only one value for this, or any matrix game, and it is $v = 0$ in the game of odds and evens.

Later we will see that the theorem gives a method for solving any matrix game if we pair it up with another theory, namely, linear programming, which is a way to optimize a linear function over a set with linear constraints. Linear programming will accommodate the more difficult problem of solving a system of inequalities.

The next theorem give us a way of solving games algebraically without having to solve inequalities.

Theorem 1.4.8 ***(Equilibrium Theorem)*** *If Y is optimal for II and $y_j > 0$, then $E(X,j) =$ value(A) for any optimal mixed strategy X for I. Similarly, if X is optimal for I and $x_i > 0$, then $E(i, Y) =$ value(A) for any optimal Y for II. In symbols,*

$$y_j > 0 \implies E(X,j) = v(A), \text{ and } x_i > 0 \implies E(i, Y) = v(A).$$

Thus, if any optimal mixed strategy for a player has a strictly positive probability of using a row or a column, then that row or column played against any optimal opponent strategy will yield the value. Secondly, if there are two or more rows, say i and k, with positive probability of being played, then $E(i, Y) = E(k, Y)$ since they are both equal to $v(A)$. A similar statement holds for columns. This would say that player II is indifferent to player I playing row i or row k. Therefore, this theorem is also called the **indifference principle**. We have already seen how useful this is in Section 1.2.1.

Proof: If it happens that (X^*, Y^*) are optimal and there is a component of $X^* = (x_1, \dots, x_k^*, \dots, x_n)$, say, $x_k^* > 0$ but $E(k, Y^*) < v(A)$, then multiplying both sides of $E(k, Y^*) < v(A)$ by x_k^* yields $x_k^* E(k, Y^*) < x_k^* v(A)$. Now, it is always true that for any row $i = 1, 2, \dots, n$,

$$E(i, Y^*) \le v(A), \text{ which implies that } x_i E(i, Y^*) \le x_i v(A).$$

But then, by adding, we get

$$\sum_{i=1, i\neq k}^{n} x_i E(i, Y^*) + x_k^* E(k, Y^*) = \sum_{i=1}^{n} x_i E(i, Y^*) < \sum_{i=1}^{n} x_i v(A) = v(A).$$

We see that, under the assumption $E(k, Y^*) < v(A)$, we have

$$v(A) = E(X^*, Y^*) = \sum_{i=1}^{n}\sum_{j=1}^{m} x_i a_{ij} y_j = \sum_{i=1}^{n} x_i E(i, Y^*) < v(A),$$

which is a contradiction. But this means that if $x_k^* > 0$ we must have $E(k, Y^*) = v(A)$.

Similarly, suppose $E(X^*, j) > v(A)$ where $Y^* = (y_1^*, \dots, y_j^*, \dots, y_m^*), y_j^* > 0$. Then

$$v(A) = E(X^*, Y^*) = \sum_{\ell=1}^{m} E(X^*, \ell) y_\ell^* > v(A) y_j^* + \sum_{\ell=1, \ell\neq j}^{m} v(A) y_\ell^* = v(A)$$

again a contradiction. Hence $E(X^*, j) > v(A) \implies y_j^* = 0$. ∎

Example 1.5 For an illustration of the use of the theorem let's consider the matrix $A = \begin{bmatrix} 3 & 2 \\ -2 & 3 \end{bmatrix}$. If we assume $Y = (y_1, y_2)$ with both $y_j > 0$, then we know

$$E(X,1) = 3x_1 - 2x_2 = v \text{ and } E(X,2) = 2x_1 + 3x_2 = v.$$

This says $3x_1 - 2x_2 = 2x_1 + 3x_2$. Since $X = (x_1, x_2)$ is a strategy, $x_1 + x_2 = 1$ so that $x_1 = \frac{5}{6}, x_2 = \frac{1}{6}, v = \frac{13}{6}$. Similarly,

$$3y_1 + 2y_2 = -2y_1 + 3y_2 \implies y_1 = \tfrac{1}{6}, y_2 = \tfrac{5}{6}.$$

The next remark summarizes several important and useful results to find optimal strategies and the value of a game.

Remark 1.4.9 ***Properties of Optimal Strategies***

1. *A number v is the value of the game and (X, Y) is a saddle point if and only if $E(i, Y) \le v \le E(X, j), i = 1, \dots, n, j = 1, \dots, m$.*
2. *If X is a strategy for player I and $value(A) \le E(X, j), j = 1, \dots, m$, then X is optimal for player I.*
3. *If Y is a strategy for player II and $value(A) \ge E(i, Y), i = 1, \dots, m$, then Y is optimal for player II.*
4. *If X is any optimal strategy for player I and $E(X, j) > value(A)$ for some column j, then for any optimal strategy Y for player II, we must have $y_j = 0$. Player II would never use column j in any optimal strategy for player II. Similarly, if Y is any optimal strategy for player II and $E(i, Y) <$ value(A), then any optimal strategy X for player I must have $x_i = 0$. If row i for player I gives a payoff when played against an optimal strategy for player II strictly below the value of the game, then player I would never use that row in any optimal strategy for player I. In symbols, if $(X = (x_i), Y = (y_j))$ is optimal, then*

$$E(X, j) > v(A) \implies y_j = 0, \text{ and } E(i, Y) < v(A) \implies x_i = 0.$$

5. *If player I has more than one optimal strategy, then player I's set of optimal strategies is a convex, closed, and bounded set. Also, if player II has more than one optimal strategy, then player II's set of optimal strategies is a convex, closed, and bounded set.*

1.4.3 Best Response Strategies

If you are playing a game and you determine, in one way or another, that your opponent is using a particular strategy, or is assumed to use a particular strategy, then what should you do? To be specific, suppose that you are player I and you know, or simply assume, that player II is using the mixed strategy Y, optimal or not for player II. In this case you should play the mixed strategy X that maximizes $E(X, Y)$. This strategy that you use would be a **best response** to the use of Y by player II. The best response strategy to Y may not be the same as what you would use if you knew that player II were playing optimally; that is, it may not be a part of a saddle point. Here is the precise definition.

Definition 1.4.10 *A mixed strategy X^* for player I is a* **best response strategy** *to the strategy Y for player II if it satisfies*

$$\max_{X \in S_n} E(X, Y) = \sum_{i=1}^{n} \sum_{j=1}^{m} x_i^* a_{ij} y_j = E(X^*, Y).$$

A mixed strategy Y^* for player II is a best response strategy to the strategy X for player I if it satisfies

$$\min_{Y \in S_n} E(X, Y) = \sum_{i=1}^{n} \sum_{j=1}^{m} x_i a_{ij} y_j^* = E(X, Y^*).$$

Incidentally, if (X^*, Y^*) is a saddle point of the game, then X^* is the best response to Y^*, and vice versa. Unfortunately, knowing this doesn't provide a good way to calculate X^* and Y^* because they are **both** unknown at the start.

Example 1.6 Consider the 3×3 game

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}.$$

The saddle point is $X^* = \left(0, \frac{1}{2}, \frac{1}{2}\right) = Y^*$ and $v(A) = 1$. Now suppose that player II, for some reason, thinks she can do better by playing $Y = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{2}\right)$. What is an optimal response strategy for player I?

Let $X = (x_1, x_2, 1 - x_1 - x_2)$. Calculate

$$E(X, Y) = XAY^T = -\frac{x_1}{4} - \frac{x_2}{2} + \frac{5}{4}.$$

We want to maximize this as a function of x_1, x_2 with the constraints $0 \le x_1, x_2 \le 1$. We see right away that $E(X, Y)$ is maximized by taking $x_1 = x_2 = 0$ and then necessarily $x_3 = 1$. Hence, the best response strategy for player I if player II uses $Y = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{2}\right)$ is $X^* = (0, 0, 1)$. Then, using this strategy, the expected payoff to I is $E(X^*, Y) = \frac{5}{4}$, which is larger than the value of the game $v(A) = 1$. So that is how player I should play if player II decides to deviate from the optimal Y. This shows that any deviation from a saddle could result in a better payoff for the opposing player. However, if one player knows that the other player will not use her part of the saddle, then the best response may not be the strategy used in the saddle. In other words, if (X^*, Y^*) is a saddle point, the best response to $Y \neq Y^*$ may not be X^*, but some other X, even though it will be the case that $E(X^*, Y) \ge E(X^*, Y^*)$.

Because $E(X, Y)$ is linear in each strategy when the other strategy is fixed, the best response strategy for player I will usually be a pure strategy. For instance, if Y is given, then $E(X, Y) = ax_1 + bx_2 + cx_3$, for some values a, b, c that will depend on Y and the matrix. The maximum payoff is then achieved by looking at the largest of a, b, c, and taking $x_i = 1$ for the x multiplying the largest of a, b, c, and the remaining values of $x_j = 0$. How do we know that? Well, to show you what to do in general, we will show that

$$\begin{aligned} &\max\{ax_1 + bx_2 + cx_3 \mid x_1 + x_2 + x_3 = 1,\ x_1, x_2, x_3 \ge 0\} \\ &\quad = \max\{a, b, c\}. \end{aligned} \tag{1.4.2}$$

Suppose that $\max\{a, b, c\} = c$ for definiteness. Now, by taking $x_1 = 0, x_2 = 0, x_3 = 1$, we get

$$\begin{aligned} &\max\{ax_1 + bx_2 + cx_3 \mid x_1 + x_2 + x_3 = 1,\ x_1, x_2, x_3 \ge 0\} \\ &\qquad \ge a \cdot 0 + b \cdot 0 + c \cdot 1 = c. \end{aligned}$$

On the other hand, since $x_1 + x_2 + x_3 = 1$, we see that

$$ax_1 + bx_2 + c(1 - x_1 - x_2) = x_1(a - c) + x_2(b - c) + c \le c,$$

since $a - c < 0, b - c < 0$ and $x_1, x_2 \ge 0$. So, we conclude that

$$c \ge \max\{ax_1 + bx_2 + cx_3 \mid x_1 + x_2 + x_3 = 1,\ x_1, x_2, x_3 \ge 0\} \ge c,$$

and this establishes (1.4.2). This shows that $X^* = (0, 0, 1)$ is a best response to Y.

It is possible to get a mixed strategy best response but only if some or all of the coefficients a, b, c are equal. For instance, if $b = c$, then

$$\max\{ax_1 + bx_2 + cx_3 \mid x_1 + x_2 + x_3 = 1,\ x_1, x_2, x_3 \geq 0\} = \max\{a, c\}.$$

To see this, suppose that $\max\{a, c\} = c$. We compute

$$\begin{aligned}
&\max\{ax_1 + bx_2 + cx_3 \mid x_1 + x_2 + x_3 = 1,\ x_1, x_2, x_3 \geq 0\}\\
&\quad = \max\{ax_1 + c(x_2 + x_3) \mid x_1 + x_2 + x_3 = 1\}\\
&\quad = \max\{ax_1 + c(1 - x_1) \mid 0 \leq x_1 \leq 1\}\\
&\quad = \max\{x_1(a - c) + c \mid 0 \leq x_1 \leq 1\}\\
&\quad = c.
\end{aligned}$$

This maximum is achieved at $X^* = (0, x_2, x_3)$ for any $x_2 + x_3 = 1, x_2 \geq 0, x_3 \geq 0$, and we see that we can get a mixed strategy as a best response.

In general, **if one of the strategies, say, Y is given and known**, then

$$\max_{X \in S_n} \sum_{i=1}^{n} x_i \left(\sum_{j=1}^{m} a_{ij} y_j \right) = \max_{1 \leq i \leq n} \left(\sum_{j=1}^{m} a_{ij} y_j \right).$$

In other words,

$$\max_{X \in S_n} E(X, Y) = \max_{1 \leq i \leq n} E(i, Y).$$

We proved this in the proof of Theorem 1.4.6.

Best response strategies are frequently used when we assume that the opposing player is Nature or some nebulous player that we think may be trying to oppose us, like the market in an investment game, or the weather. The next example helps to make this more precise.

Example 1.7 Suppose that player I has some money to invest with three options: stock (S), bonds (B), or certificates of deposit (CDs). The rate of return depends on the state of the market for each of these investments. Stock is considered risky, bonds have less risk than stock, and CDs are riskless. The market can be in one of three states: good (G), neutral (N), or bad (B), depending on factors such as the direction of interest rates, the state of the economy, prospects for future growth. Here is a possible game matrix in which the numbers represent the annual rate of return to the investor who is the row player:

I/II	G	N	B
S	12	8	−5
B	4	4	6
CD	5	5	5

The column player is the market. This game does not have a saddle in pure strategies. (Why?) If player I assumes that the market is the opponent with the goal of minimizing the investor's rate of return, then we may look at this as a two-person zero sum game. On the other hand, if the investor thinks that the market may be in any one of the three states with equal likelihood, then the market will play the strategy $Y = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$, and then the investor must choose how to respond to that; that is, the investor seeks an X^* for which $E(X^*, Y) = \max_{X \in S_3} E(X, Y)$, where $Y = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$. Of course, the investor uses the same procedure no matter which Y she thinks the market will use. The investor can use this game to compare what will happen under various Y strategies.

This problem may actually be solved fairly easily. If we assume that the market is an opponent in a game then the value of the game is $v(A) = 5$, and there are many optimal strategies, one of which is $X^* = (0, 0, 1), Y^* = (0, \frac{1}{2}, \frac{1}{2})$. If instead $Y = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, then the best response for player I is $X = (0, 0, 1)$, with payoff to I equal to 5. If $Y = (\frac{2}{3}, 0, \frac{1}{3})$, the best response is $X = (1, 0, 0)$, that is, invest in the stock if there is a $\frac{2}{3}$ probability of a good market. The payoff then is $\frac{19}{3} > 5$.

It may seem odd that the best response strategy in a zero sum two person game is usually a pure strategy, but it can be explained easily with a simple example. Suppose that someone is flipping a coin that is not fair—say heads comes up 75% of the time. On each toss you have to guess whether it will come up heads or tails. How often should you announce heads in order to maximize the percentage of time you guess correctly? If you think it is 75% of the time, then you will be correct $75 \times 75 = 56.25\%$ of the time! If you say heads **all the time**, you will be correct 75% of the time, and that is the best you can do.

Here is a simple game with an application of game theory to theology!

Example 1.8 Blaise Pascal constructed a game to show that belief in God is the only rational strategy. The model assumes that you are player I, and you have two possible strategies: believe or don't believe. The opponent is taken to be God, who either plays God exists, or God doesn't exist. (Remember, God can do anything without contradiction.) Here is the matrix, assuming $\alpha, \beta, \gamma > 0$:

You/God	God exists	God doesn't exist
Believe	α	$-\beta$
Don't believe	$-\gamma$	0

If you believe and God doesn't exist, then you receive $-\beta$ because you have foregone evil pleasures in the belief God exists. If you don't believe and God exists, then you pay a price $-\gamma$. Pascal argued that this would be a big price. If you believe and God exists, then you receive the amount α, from God, and Pascal argued this would be a very large amount of spiritual currency.

To solve the game, we first calculate v^+ and v^-. Clearly $v^+ = 0$ and $v^- = \max(-\beta, -\gamma) < 0$, so this game does not have a saddle point in pure strategies unless $\beta = 0$ or $\gamma = 0$. If there is no loss or gain to you if you play don't believe, then that is what you should do, and God should play **not exist**. In this case the value of the game is zero.

Assuming that none of α, β, or γ is zero, we will have a mixed strategy saddle. Let $Y = (y, 1 - y)$ be an optimal strategy for God. Then it must be true that

$$E(1, Y) = \alpha y - \beta(1 - y) = v(A) = -\gamma y = E(2, Y).$$

These are the two equations for a mixed strategy from the fact we are assuming $0 < y < 1$. Solving, we get that $y = \beta/(\alpha + \beta + \gamma)$. The optimal strategy for God is

$$Y = \left(\frac{\beta}{\alpha + \beta + \gamma}, \frac{\alpha + \gamma}{\alpha + \beta + \gamma} \right)$$

and the value of the game to you is

$$v(A) = \frac{-\gamma\beta}{\alpha + \beta + \gamma} < 0.$$

Your optimal strategy $X = (x, 1 - x)$ must satisfy

$$E(X, 1) = \alpha x - \gamma(1 - x) = -\beta x = E(X, 2) \implies$$

$$x = \frac{\gamma}{\alpha + \beta + \gamma}, \text{ and } X = \left(\frac{\gamma}{\alpha + \beta + \gamma}, \frac{\alpha + \beta}{\alpha + \beta + \gamma} \right).$$

Pascal argued that if γ, the penalty to you if you don't believe and God exists is loss of eternal life, represented by a very large number. In this case, the percent of time you play believe, $x = \gamma/(\alpha + \beta + \gamma)$ should be fairly close to 1, so you should play believe with high probability. For example, if $\alpha = 10,\ \beta = 5, \gamma = 100$, then $x = 0.87$.

From God's point of view, if γ is a very large number and this is a zero sum game, God would then play doesn't exist with high probability!! It may not make much sense to think of this as a zero sum game, at least not theologically. Maybe we should just look at this like a best response for you, rather than as a zero sum game.

So, let's suppose that God plays the strategy $Y^0 = (\frac{1}{2}, \frac{1}{2})$. What this really means is that you think that God's existence is as likely as not. What is your best response strategy? For that, we calculate $f(x) = E(X, Y^0)$, where $X = (x, 1-x), 0 \le x \le 1$. We get

$$f(x) = x\,\frac{\alpha + \gamma - \beta}{2} - \frac{\gamma}{2}.$$

The maximum of $f(x)$ over $x \in [0, 1]$ is

$$f(x^*) = \begin{cases} \dfrac{\alpha - \beta}{2}, & \text{at } x^* = 1 \text{ if } \alpha + \gamma > \beta; \\ -\dfrac{\gamma}{2}, & \text{at } x^* = 0 \text{ if } \alpha + \gamma < \beta; \\ -\dfrac{\gamma}{2}, & \text{at any } 0 \le x \le 1 \text{ if } \alpha + \gamma = \beta. \end{cases}$$

Pascal argued that γ would be a very large number (and so would α) compared to β. Consequently, the best response strategy to Y^0 would be $X^* = (1, 0)$. Any rational person who thinks that God's existence is as likely as not would choose to play believe.

Are we really in a zero sum game with God? Probably not because it is unlikely that God loses if you choose to believe. A more likely opponent in a zero sum game like this might be Satan. Maybe the theological game is nonzero sum so that both players can win (and both can lose). In any case, a nonzero sum model of belief will be considered later.

1.4.4 Dominated Strategies

The goal is to find the best strategies. One way to approach this is to get rid of obviously bad strategies. Here is one way to determine strategies that can be eliminated: if one row k is worse than another row i no matter what player II does, then player I should never play the strategy k, as i would definitely be a better choice. And there is a similar argument to remove a column if there is another column that is always better for player II. Let's make this a definition.

Definition 1.4.11

1. *Row i weakly dominates row k if* $a_{ij} \ge a_{kj}$ *for all* $j = 1, 2, \dots, m$. *We say row i strictly dominates row k if all of these inequalities are strict.*
2. *Column j weakly dominates column k if* $a_{ij} \le a_{ik}$, $i = 1, 2, \dots, n$. *We say column j strictly dominates row k if all of these inequalities are strict.*
3. *If a row or column is strictly dominated, then it can be dropped from the matrix.*

Remarks

1. We may also drop rows or columns that are **non-strictly dominated** but the resulting matrix may not result in all the saddle points for the original matrix. Finding all of the saddle points is not what we are concerned with so we will use non-strict dominance in what follows.
2. A row that is dropped because it is **strictly dominated** is played in a mixed strategy with probability 0. But a row that is dropped because it is equal to another row may not have probability 0 of being played. For example, suppose that we have a matrix with three rows and row 2 is the same as row 3. If we drop row 3, we now have two rows and the resulting optimal strategy will look like $X^* = (x_1, x_2)$ for the reduced game. Then for the original game the optimal strategy could be $X^* = (x_1, x_2, 0)$ or $X^* = (x_1, \frac{x_2}{2}, \frac{x_2}{2})$, or in fact $X^* = (x_1, \lambda x_2, (1-\lambda)x_2)$ for any $0 \le \lambda \le 1$. The set of all optimal strategies for player I would consist of all $X^* = (x_1, \lambda x_2, (1-\lambda)x_2)$ for any $0 \le \lambda \le 1$, and this is the most general description. A duplicate row is a redundant row and may be dropped to reduce the size of the matrix. But you must account for redundant strategies.

Another way to reduce the size of a matrix, which is more subtle, is to drop rows or columns by dominance through a convex combination of other rows or columns. If a row (or column) is (strictly) dominated by a convex combination of other rows (or columns), then this row (column) can be dropped from the matrix. If, for example, row k is dominated by a convex combination of two other rows, say, p and q, then we can drop row k. This means that if there is a constant $\lambda \in [0, 1]$ so that

$$a_{kj} \le \lambda a_{pj} + (1-\lambda)a_{qj}, \quad j = 1, \ldots, m,$$

then row k is dominated and can be dropped. Of course, if the constant $\lambda = 1$, then row p dominates row k and we can drop row k. If $\lambda = 0$ then row q dominates row k. More than two rows can be involved in the combination.

For columns, the column player wants small numbers, so column k is dominated by a convex combination of columns p and q if

$$a_{ik} \ge \lambda a_{ip} + (1-\lambda)a_{iq}, \quad i = 1, \ldots, n.$$

The reason convex combination dominance works is because when we use mixed strategies dominance can be established because the probabilities of playing two or more dominating rows would make the probability of playing the dominated row zero.

It may be hard to spot a combination of rows or columns that dominate, but if there are suspects, the next example shows how to verify it.

Example 1.9 Consider the 3×4 game

$$A = \begin{bmatrix} 10 & 0 & 7 & 4 \\ 2 & 6 & 4 & 7 \\ 5 & 2 & 3 & 8 \end{bmatrix}.$$

It seems that we may drop column 4 right away because every number in that column is larger than each corresponding number in column 2. So now we have

$$\begin{bmatrix} 10 & 0 & 7 \\ 2 & 6 & 4 \\ 5 & 2 & 3 \end{bmatrix}.$$

There is no obvious dominance of one row by another or one column by another. However, we suspect that row 3 is dominated by a convex combination of rows 1 and 2. If that is true, we must have, for some $0 \leq \lambda \leq 1$, the inequalities

$$5 \leq \lambda(10) + (1-\lambda)(2),\ \ 2 \leq 0(\lambda) + 6(1-\lambda),\ \ 3 \leq 7(\lambda) + 4(1-\lambda).$$

Simplifying, $5 \leq 8\lambda + 2, 2 \leq 6 - 6\lambda, 3 \leq 3\lambda + 4$. But this says any $\frac{3}{8} \leq \lambda \leq \frac{2}{3}$ will work. So, there is a λ that works to cause row 3 to be dominated by a convex combination of rows 1 and 2, and row 3 may be dropped from the matrix (i.e., an optimal mixed strategy will play row 3 with probability 0). Remember, to ensure dominance by a convex combination, all we have to show is that there are λ's that satisfy all the inequalities. We don't actually have to find them. So now the new matrix is

$$\begin{bmatrix} 10 & 0 & 7 \\ 2 & 6 & 4 \end{bmatrix}.$$

Again there is no obvious dominance, but it is a reasonable guess that column 3 is a bad column for player II and that it might be dominated by a combination of columns 1 and 2. To check, we need to have

$$7 \geq 10\lambda + 0(1-\lambda) = 10\lambda,\ \text{ and } 4 \geq 2\lambda + 6(1-\lambda) = -4\lambda + 6.$$

These inequalities require that $\frac{1}{2} \leq \lambda \leq \frac{7}{10}$, which is okay. So there are λ's that work, and column 3 may be dropped. Finally, we are down to a 2×2 matrix

$$\begin{bmatrix} 10 & 0 \\ 2 & 6 \end{bmatrix}.$$

This game may be solved by assuming that each row and column will be used with positive probability and then solving the system of equations (see Theorem 1.9). The answer is that the value of the game is $v(A) = \frac{30}{7}$ and the optimal strategies for the original game are $X^* = (\frac{2}{7}, \frac{5}{7}, 0)$ and $Y^* = (\frac{3}{7}, \frac{4}{7}, 0, 0)$.

Example 1.10 This example shows that me may lose solutions when we reduce by non-strict dominance. Consider the game with matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}.$$

Again there is no obvious dominance but it's easy to see that row 1 is dominated (non-strictly) by a convex combination of rows 2 and 3. In fact $a_{1j} = \frac{1}{2}a_{2j} + \frac{1}{2}a_{3j}, j = 1, 2, 3$. So if we drop row 1 we next see that column 1 is dominated by a convex combination of columns 2 and 3 and may be dropped. This leaves us with the reduced matrix $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$. The solution for this reduced game is $v = 1, X^* = (\frac{1}{2}, \frac{1}{2}) = Y^*$ and consequently a solution of the original game is $v(A) = 1, X^* = (0, \frac{1}{2}, \frac{1}{2}) = Y^*$. However, it is easy to verify that there is in fact a pure saddle point for this game given by $X^* = (1, 0, 0), Y^* = (1, 0, 0)$ and this is missed by using non-strict domination. Dropping rows or columns that are strictly dominated, however, means that dropped row or column is never played.

Problems

1.13 Suppose A is a 2×3 matrix and A has a saddle point in pure strategies. Show that it must be true that either one column dominates another, or one row dominates the other, or both. Then find a matrix A which is 3×3 and has a saddle point in pure strategies, but no row dominates another and no column dominates another.

1.14 Consider the game with payoff matrix $\begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}$.

(a) Find the upper and lower value, v^+ and v^-.
(b) Show that there is no saddle point in pure strategies.
(c) Find a saddle point in mixed strategies (you can use the graphing method or the principle of indifference… which in the end is the same calculation).
(d) Find the value of the game allowing mixed strategies.

1.15 Solve the game with payoff matrix

$$\begin{bmatrix} 10 & 3 \\ 1 & 9 \\ 7 & 4 \end{bmatrix}.$$

That is, you need to find a pair of optimal strategies, and calculate the expected payoff to player I if both players play optimally.

1.16 Solve the game with the payoff matrix below. That is, find a saddle point in mixed strategies, and the expected payout to PI if the players both play optimally.

$$\begin{bmatrix} 4 & 6 & 5 & 6 \\ 3 & 7 & 8 & 5 \\ 7 & 4 & 7 & 3 \\ 4 & 7 & 6 & 5 \end{bmatrix}.$$

Hint: First eliminate (weakly) dominated strategies as much as possible!

1.17 Consider the following gambling game: There are two players, PI and PII. They each secretly put either \$1 or \$2 in their hands. They simultaneously open their hands, and let the money fall in the pot. If the total amount of money is even (so \$2 or \$4) PI wins and takes it. If it is odd (so, \$3) PII wins and gets the pot.

(a) Describe the possible pure strategies.
(b) Set up a payoff matrix for PI. Hint: there will be negative numbers, since PI sometimes loses money.
(c) Solve the game (i.e., find a saddle point, which could involve mixed strategies).
(d) Is this game fair? That is, does either player have an advantage? If yes, who? Explain.

1.18 An entrepreneur, named Victor, outside Laguna beach can sell 500 umbrellas when it rains and 100 when the sun is out along with 1000 pairs of sunglasses. Umbrellas cost him \$5 each and sell for \$10. Sunglasses wholesale for 2\$ and sell for \$5. The vendor has 2500 dollars to buy the goods. Whatever he doesn't sell is lost as worthless at the end of the day.

(a) Assume Victor's opponent is the weather set up a payoff matrix with the elements of the matrix representing his net profit.
(b) Suppose Victor hears the weather forecast and there is a 30% chance of rain. What should he do?

1.19 You're in a bar and a stranger comes to you with a new pickup strategy.[1] The stranger proposes that you each call Heads or Tails. If you both call Heads, the stranger pays you \$3. If both call Tails, the stranger pays you \$1. If the calls aren't a match, then you pay the stranger \$2.
(a) Formulate this as a two person game and solve it.
(b) Suppose the stranger decides to play the strategy $\tilde{Y} = \left(\frac{1}{3}, \frac{2}{3}\right)$. Find a best response and the expected payoff.

1.20 Suppose that the batter in the baseball game Problem 1.11 hasn't done his homework to learn the percentages in the game matrix. So, he uses the strategy $X^* = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$. What is the pitcher's best response strategy?

1.21 In general, if we have two payoff functions $f(x, y)$ for player I and $g(x, y)$ for player II, suppose that both players want to maximize their own payoff functions with the variables that they control. Then $y^* = y^*(x)$ is a best response of player II to x if

$$g(x, y^*(x)) = \max_y g(x, y), \quad y^* \in \arg\max_y g(x, y)$$

and $x^* = x^*(y)$ is a best response of player I to y if

$$f(x^*(y), y) = \max_x f(x, y), \quad x^* \in \arg\max_x f(x, y)$$

Recall that, for example, $\arg\max_x f(x, y)$ is the set of all $x's$ where the maximum of $f(x, y)$ as a function of x is achieved.
(a) Find the best responses if $f(x, y) = (C - x - y)x$ and $g(x, y) = (D - x - y)y$, where C and D are constants.
(b) Solve the best responses and show that the solution x^*, y^* satisfies $f(x^*, y^*) \geq f(x, y^*)$ for all x, and $g(x^*, y^*) \geq g(x^*, y)$ for all y.

1.22 Consider the following matrix game:

$$\begin{bmatrix} 2 & 4 & 1 & 5 \\ 1 & 0 & 7 & 4 \\ 0 & 9 & 2 & 1 \\ 5 & 6 & 3 & 4 \end{bmatrix}.$$

Five students claim to have solved the game. They give the following optimal strategies:

1. $X = (0, \frac{1}{4}, 0, \frac{3}{4}), Y = (\frac{1}{2}, 0, \frac{1}{2}, 0)$.
2. $X = (0, \frac{1}{2}, 0, \frac{1}{2}), Y = (\frac{1}{2}, 0, \frac{1}{2}, 0)$.
3. $X = (1, 0, 0, 0), Y = (0, 1, 0, 0)$.
4. $X = (0, \frac{1}{4}, 0, \frac{3}{4}), Y = (0, 0, 0, 1)$.
5. $X = (0, \frac{1}{4}, 0, \frac{3}{4}), Y = (\frac{1}{3}, 0, \frac{1}{3}, \frac{1}{3})$.

Which, if any, of them are correct?

1 This question appeared in a column by Marilyn Vos Savant in Parade Magazine on March 31, 2002.

1.5 The Indifference Principle and Completely Mixed Games

The Equilibrium Theorem 1.4.8, also known as the indifference principle, is a key way to find optimal strategies when we know that two or more rows or columns will be used. An important class of games is when all the rows or columns of a game will be played. Such games are called **completely mixed**.

Precisely, a game is called completely mixed if there is a mixed saddle point where both players are using all strategies with non-zero probability. In that case the indifference principle gives a way to find the saddle point. We will illustrate this with some examples.

Example 1.11 Hide and Seek: We consider a game where PII chooses a place to hide, PI chooses a place to look for PII, and PI wins if she finds PII—but the winning amount can vary. For instance, finding PII under the bed may be worth 50 points, in the closet worth 30, and behind the curtain worth 20. This would become the matrix game

$$A = \begin{bmatrix} 50 & 0 & 0 \\ 0 & 30 & 0 \\ 0 & 0 & 20 \end{bmatrix}.$$

We claim that, in this case, both players should use all strategies with some positive probability. First notice that, for any completely mixed strategy X for PI, and any response Y by PII, the expected outcome $E(X, Y) > 0$ is positive. This shows that the game has positive value $v(A) > 0$. But for any non-completely mixed strategy for PI, PII can just hide in the place PI is never looking, and get outcome 0. Next notice that if PII is not playing a totally mixed strategy neither should PI—they should not look where PII never hides! This means it is impossible for a non-completely mixed PII strategy to be part of a saddle point as well.

A formal way to see that the optimal strategies must be completely mixed is to use Remark 1.4.9. If $v(A)$ is the value of the game and Y is optimal for PII, then

$$v(A) \geq \max_{i=1,2,3} E(i, Y) = \max\{50y_1, 30y_2, 20y_3\}.$$

For any Y, optimal or not, this says $v(A) > 0$. Similarly, by Remark 1.5.9, for any optimal X for PI, $v(A) \leq \min\{50x_1, 30x_2, 20x_3\}$. If any component of X, say x_2 for instance, has $x_2 = 0$, then the minimum is 0. That would imply $v(A) \leq 0$, which we already know is not true. So any optimal strategy for PI must be completely mixed. Next, if $X^* = (x_1, x_2, x_3)$ is completely mixed and optimal for PI and $Y^* = (y_1, y_2, y_3)$ is optimal for PII with, say $y_2 = 0$, then, assuming $y_1 > 0, y_3 > 0$,

$$v(A) = E(X^*, Y^*) = 50x_1y_1 + 20x_3y_3 < 50y_1(x_1 + \tfrac{x_2}{2}) + 20y_3(x_3 + \tfrac{x_2}{2}),$$

which would say that there is a better strategy for PI than X^*, namely, $(x_1 + \frac{x_2}{2}, 0, x_3 + \frac{x_2}{2})$. This contradiction shows that no component of Y^* can be zero.

Now we know that both players use all strategies, and so we can write down a system of equations by the indifference principle to solve to find optimal strategy $X^* = (x_1, x_2, 1 - x_1 - x_2), Y^* = (y_1, y_2, 1 - y_1 - y_2)$.

Using PI's indifference

$$50y_1 = 30y_2 = 20(1 - y_1 - y_2).$$

This is really the system of two equations

$$50y_1 = 30y_2. \quad \text{and} \quad 30y_3 = 20(1 - y_1 - y_2).$$

Solving, $y_1 = \frac{6}{31}, y_2 = \frac{10}{31}$. This gives $1 - y_1 - y_2 = \frac{15}{31}$, so the optimal strategy is

$$Y^* = \left(\frac{6}{31}, \frac{10}{31}, \frac{15}{31}\right) \approx (0.194, 0.323, 0.484).$$

A similar calculation gives

$$X^* = \left(\frac{6}{31}, \frac{10}{31}, \frac{15}{31}\right).$$

In fact, in this type of hide and seek game it is always true that the seeker should look in each location with the same probability that the hider hides there. Notice that PI seeks with the highest probability the location with the lowest payoff for PI. The expected payoff to PI is

$$E_I(X^*, Y^*) = X^* A\ Y^{*T} = \frac{300}{31} = 9.6774.$$

Example 1.12 Now let's make some slight changes to the outcomes for the previous example. Say the matrix is

$$A = \begin{bmatrix} 50 & 1 & -1 \\ -1 & 30 & 2 \\ 3 & 1 & 20 \end{bmatrix}.$$

Since the game was totally mixed before, it seems like a reasonable guess that it is totally mixed now. Using that guess, the indifference equations give:

$$50y_1 + y_2 - (1 - y_1 - y_2) = -y_1 + 30y_2 + 2(1 - y_1 - y_2) = 3y_1 + y_2 + 20(1 - y_1 - y_2)$$

and

$$50x_1 - x_2 + 3(1 - x_1 - x_2) = x_1 + 30x_2 + (1 - x_1 - x_2) = -x_1 + 2x_2 + 20(1 - x_1 - x_2).$$

Solving we get

$$x_1 \simeq 0.18, x_2 \simeq 0.32, y_1 \simeq 0.21, y_2 \simeq 0.32,$$

so the solution (to two significant digits) is

$$X^* = (0.18, 0.32, 0.50), \qquad Y^* = (0.21, 0.32, 0.47).$$

These strategies are similar to but not identical to those in the previous example, which seems reasonable. The expected payoff to player I is $E(X^*, Y^*) = 10.34$.

We made a guess here, so we should now verify that this is a solution by checking that it is in fact a saddle point. But, since the strategies we found are valid, the indifference equations we solved show that any response to X^* gives the same payoff to PII (and similarly any response to Y^* gives the same payoff to PI). Thus we have in fact shown that this is a saddle point, and that the game is totally mixed. You might ask what would happen if the game was not totally mixed—we will see in the next example.

Example 1.13 Consider now the game with payout matrix

$$A = \begin{bmatrix} 3 & -1 & 0 \\ -2 & 1 & -1 \\ 2 & 0 & 2 \end{bmatrix}.$$

We don't see any obvious reason either player should avoid any of their three strategies, so we could again guess that the game is completely mixed and set up indifference equations. This time the solution is $x_1 = \frac{1}{4}, x_2 = \frac{1}{2}, y_1 = \frac{1}{2}, y_2 = \frac{3}{2}$, so

$$X^* = \left(\frac{1}{4}, \frac{1}{2}, \frac{1}{4}\right) \qquad Y^* = \left(\frac{1}{2}, \frac{3}{2}, -1\right).$$

But this is impossible: Y^* is supposed to be a probability vector, it cannot have a negative entry! So, this is not a valid saddle point, and the game is not totally mixed.

One thing to note: $X^* = (\frac{1}{4}, \frac{1}{2}, \frac{1}{4})$ looks like a valid strategy for PI, but it is still not optimal. The only saddle point in this game is in fact

$$X^* = \left(0, \frac{2}{5}, \frac{3}{5}\right) \qquad Y^* = \left(\frac{1}{5}, \frac{4}{5}, 0\right).$$

Even if you are only interested in the optimal strategy for one player, it is not enough for the equations to work out for them. You really need to solve for both players before you are sure the solution works.

Example 1.14 It can happen that games have many saddle points, some mixed and some not. Consider the game with matrix

$$A = \begin{bmatrix} -2 & 2 & -1 \\ 1 & 1 & 1 \\ 3 & 0 & 1 \end{bmatrix}.$$

You can directly check that $X^* = (0, 1, 0), Y^* = (0, 0, 1)$ is a pure saddle point, and $v(A) = 1$ because the entry in the second row and third column is simultaneously the highest in the column and the lowest in the row.

Now that we know the value, we can write down a system of inequalities to describe the possible optimal strategies for each player. For PI, they must use a strategy (x_1, x_2, x_3) where each of PII's responses gives at least the outcome 1. So,

$$\begin{aligned} -2x_1 + x_2 + 3x_3 &= E(X, 1) \geq 1 \\ 2x_1 + x_2 &= E(X, 2) \geq 1 \\ -x_1 + x_2 + x_3 &= E(X, 3) \geq 1 \end{aligned}$$

and of course $x_1 + x_2 + x_3 = 1$. The only solution to this system is $(x_1, x_2, x_3) = (0, 1, 0)$—you can check this by computer, or by subbing $x_3 = 1 - x_1 - x_2$ and graphing the inequalities as we do for PII's strategies below.

By the same logic, PII must play a strategy (y_1, y_2, y_3) with $y_1 + y_2 + y_3$ and

$$y_1 + y_2 + y_3 \leq 1, -2y_1 + 2y_2 - y_3 \leq 1, 3y_1 + y_3 \leq 1.$$

We replace $y_3 = 1 - y_1 - y_2$ and then get a graph of the region of points satisfying all the inequalities in (y_1, y_2) space in Figure 1.5.

Now lots of points work! In particular, $Y = (0.15, 0.5, 0.35)$ will give an optimal strategy for player II. So for this example, PI's only optimal strategy is pure, but PII has many optimal strategies, including completely mixed strategies.

Note that this game is not considered completely mixed because that requires a completely mixed optimal strategy for both players.

Remark

Question: When an optimal strategy for a player has a zero component, can you drop that row or column and then solve the reduced matrix?

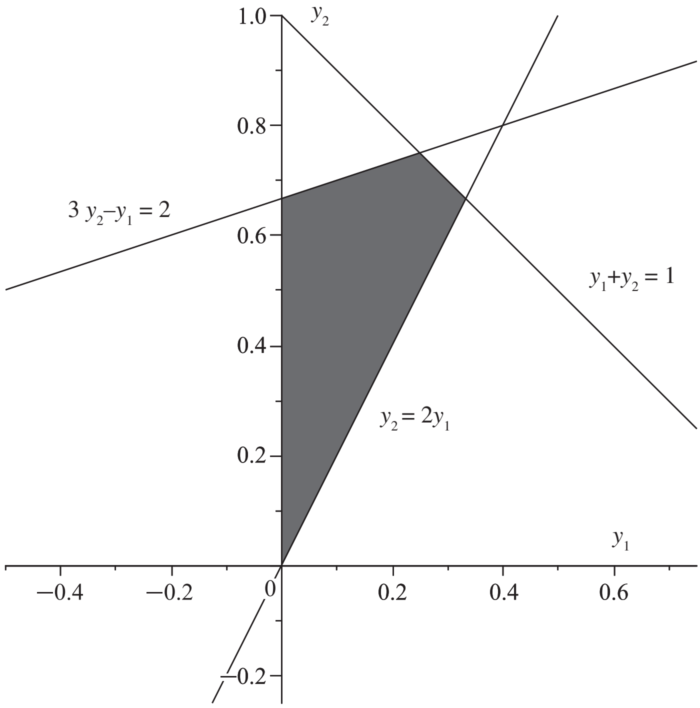

Figure 1.5 Optimal strategy set for Y.

Answer: No, in general.
Consider the game with matrix

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}.$$

Then $X^* = (0, 1, 0), Y^* = (\frac{1}{2}, 0, \frac{1}{2})$ is a saddle point and $v(A) = 0$. Since player I never used strategy 1, you might think we could drop row 1 from the game, If we drop row 1 (because the optimal strategy for player I has $x_1 = 0$) to get the matrix

$$B = \begin{bmatrix} 0 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}.$$

But then $\overline{X} = (1, 0), \overline{Y} = (1, 0, 0)$ is a saddle point for B, but $\overline{Y}$ is **NOT** part of any saddle point for the original game. The original game has multiple equilibria, such as

$$X^* = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right), \quad Y^* = \left(\frac{1}{2}, 0, \frac{1}{2}\right).$$

1.5.1 2 × 2 Games

In this easily solved case, each player has exactly two strategies, so the matrix and strategies look like

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \text{ player I: } X = (x, 1 - x); \text{ player II: } Y = (y, 1 - y).$$

For any mixed strategies we have $E(X,Y) = XAY^T$, which, written out, is

$$E(X,Y) = xy(a_{11} - a_{12} - a_{21} + a_{22}) + x(a_{12} - a_{22}) + y(a_{21} - a_{22}) + a_{22}.$$

Now here is the theorem giving the solution of this game.

Theorem 1.5.1 *In the 2×2 game with matrix A,* **assume that there are no pure optimal strategies.** *If we set*

$$x^* = \frac{a_{22} - a_{21}}{a_{11} - a_{12} - a_{21} + a_{22}}, \qquad y^* = \frac{a_{22} - a_{12}}{a_{11} - a_{12} - a_{21} + a_{22}},$$

then $X^* = (x^*, 1 - x^*)$, $Y^* = (y^*, 1 - y^*)$ are optimal mixed strategies for players I and II, respectively. The value of the game is

$$v(A) = E(X^*, Y^*) = \frac{a_{11}a_{22} - a_{12}a_{21}}{a_{11} - a_{12} - a_{21} + a_{22}}.$$

Remarks

1. The main assumption you need before you can use the formulas is that the game does **not** have a pure saddle point. If it does, you find it by checking $v^+ = v^-$, and then finding it directly. You don't need to use any formulas. Also, when we write down these formulas, it had better be true that $a_{11} - a_{12} - a_{21} + a_{22} \neq 0$, but if we assume that there is no pure optimal strategy, then this must be true. In other words, it isn't difficult to check that when $a_{11} - a_{12} - a_{21} + a_{22} = 0$, then $v^+ = v^-$ and that violates the assumption of the theorem.
2. For a preview of the next section where the game matrix has an inverse, we may write the formulas as

$$X^* = \frac{(1\ 1)A^*}{(1\ 1)A^* \begin{bmatrix} 1 \\ 1 \end{bmatrix}} \quad \text{and} \quad Y^{*T} = \frac{A^* \begin{bmatrix} 1 \\ 1 \end{bmatrix}}{(1\ 1)A^* \begin{bmatrix} 1 \\ 1 \end{bmatrix}} \quad \mathit{value}(A) = \frac{\det(A)}{(1\ 1)A^* \begin{bmatrix} 1 \\ 1 \end{bmatrix}},$$

where

$$A^* = \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix} \text{ and } \det(A) = a_{11}a_{22} - a_{12}a_{21}.$$

Recall that the inverse of a 2×2 matrix is found by swapping the main diagonal numbers, and putting a minus sign in front of the other diagonal numbers, and finally dividing by the determinant of the matrix. A^* is exactly the first two steps, but we don't divide by the determinant. The matrix we get is defined even if the matrix A doesn't have an inverse. Remember, however, that we need to make sure that the matrix doesn't have optimal pure strategies first.

Notice too, that if $\det(A) = 0$, the value of the game is zero.

Where do the formulas come from? By Theorem 1.4.6 since there is a saddle point which is completely mixed, we know that $E(X^*, 1) = v = E(X^*, 2)$ and $E(1, Y^*) = v = E(2, Y^*)$, when (X^*, Y^*) is a saddle point. This gives the equations

$$a_{11}x_1 + a_{21}(1 - x_1) = a_{12}x_1 + a_{22}(1 - x_1) \text{ and } a_{11}y_1 + a_{12}(1 - y_1) = a_{21}y_1 + a_{22}(1 - y_1).$$

Solving these equations results in the formulas in the statement.

Example 1.15 In the game with $A = \begin{bmatrix} -2 & 5 \\ 2 & 1 \end{bmatrix}$ we see that $v^- = 1 < v^+ = 2$, so there is no pure saddle for the game. If we apply the formulas, we get the mixed strategies $X^* = (\frac{1}{8}, \frac{7}{8})$ and $Y^* = (\frac{1}{2}, \frac{1}{2})$ and the value of the game is $v(A) = \frac{3}{2}$. Notice here that

$$A^* = \begin{bmatrix} 1 & -5 \\ -2 & -2 \end{bmatrix}$$

and $\det(A) = -12$. The matrix formula for player I gives

$$X^* = \frac{(1\ 1)A^*}{(1\ 1)A^*(1\ 1)^T} = \left(\frac{1}{8}, \frac{7}{8}\right).$$

1.5.2 Completely Mixed Games and Invertible Matrix Games

In this section[1] we will solve the class of games in which the matrix A is square, say, $n \times n$ and invertible, so $\det(A) \neq 0$ and A^{-1} exists and satisfies $A^{-1}A = AA^{-1} = I_{n\times n}$. The matrix $I_{n\times n}$ is the $n \times n$ identity matrix consisting of all zeros except for ones along the diagonal. For the present let us suppose that

> Player I has an optimal strategy that is **completely mixed**, specifically, $X = (x_1, \dots, x_n)$, and $x_i > 0, i = 1, 2, \dots, n$. So player I plays every row with positive probability.

By the Equilibrium Theorem 1.4.8 we know that this implies that every optimal Y strategy for player II, must satisfy

$$E(i, Y) = {}_iAY^T = value(A), \text{ for every row } i = 1, 2, \dots, n.$$

Y played against any row will give the value of the game. If we write $J_n = (1\ 1\ \dots 1)$ for the row vector consisting of all 1s, we can write

$$AY^T = v(A)J_n^T = \begin{bmatrix} v(A) \\ \vdots \\ v(A) \end{bmatrix}. \tag{1.5.1}$$

Now, if $v(A) = 0$, then $AY^T = 0J_n^T = 0$, and this is a system of n equations in the n unknowns $Y = (y_1, \dots, y_n)$. It is a homogeneous linear system. Since A is invertible this would have the one and only solution $Y^T = A^{-1}0 = 0$. But that is impossible if Y is a strategy (the components must add to 1). So, if this is going to work, the value of the game cannot be zero, and we get, by multiplying both sides of (1.5.1) by A^{-1}, the following equation:

$$A^{-1}A\ Y^T = Y^T = v(A)A^{-1}J_n^T.$$

This gives us Y if we knew $v(A)$. How do we get that? The extra piece of information we have is that the components of Y add to 1 (i.e., $\sum_{j=1}^n y_j = J_nY^T = 1$). So

$$J_nY^T = 1 = v(A)J_nA^{-1}J_n^T,$$

and therefore

$$v(A) = \frac{1}{J_nA^{-1}J_n^T} \text{ and then } Y^T = \frac{A^{-1}J_n^T}{J_nA^{-1}J_n^T}.$$

1 This involves some knowledge of basic linear algebra—see the appendix for the essentials of what we use.

We have found the only candidate for the optimal strategy for player II assuming that every component of X is greater than 0. However, if it turns out that this formula for Y has at least one $y_j < 0$, something would have to be wrong; specifically, it would **not** be true that X was completely mixed because that was our hypothesis. But, if it turns out that $y_j \geq 0$ for every component, we could try to find an optimal X for player I by the exact same method. This would give us

$$X = \frac{J_n A^{-1}}{J_n A^{-1} J_n^T}.$$

This method will work if the formulas we get for X and Y end up satisfying the condition that they are strategies. If either X or Y has a negative component, then it fails. But notice that the strategies do not have to be completely mixed as we assumed from the beginning, only bona fide strategies.

Here is a summary of what we know.

Theorem 1.5.2 *Assume that*

1. $A_{n\times n}$ has an inverse A^{-1}.
2. $J_n A^{-1} J_n^T \neq 0$.
3. $v(A) \neq 0$.

Set $X = (x_1, \dots, x_n)$, $Y = (y_1, \dots, y_m)$, *and*

$$v \equiv \frac{1}{J_n A^{-1} J_n^T}, \qquad Y^T = \frac{A^{-1} J_n^T}{J_n A^{-1} J_n^T}, \qquad X = \frac{J_n A^{-1}}{J_n A^{-1} J_n^T}.$$

If $x_i \geq 0, i = 1, \dots, n$ *and* $y_j \geq 0, j = 1, \dots, n$, *we have that* $v = v(A)$ *is the value of the game with matrix* A *and* (X, Y) *is a saddle point in mixed strategies.*

Now the point is that when we have an invertible game matrix, we can always use the formulas in the theorem to calculate the number v and the vectors X and Y. If the result gives vectors with nonnegative components, then, by the Proposition 1.4.8, v must be the value, and (X, Y) is a saddle point. Notice that, directly from the formulas, $J_n Y^T = 1$ and $X J_n^T = 1$, so the components given in the formulas will automatically sum to 1; it is only the sign of the components that must be checked.

Here is a simple direct verification that (X, Y) is a saddle and v is the value assuming $x_i \geq 0, y_j \geq 0$. Let $Y' \in S_n$ be any mixed strategy and let X be given by the formula $X = \frac{J_n A^{-1}}{J_n A^{-1} J_n^T}$. Then, since $J_n^T Y' = 1$, we have

$$\begin{aligned} E(X, Y') = X\,AY'^T &= \frac{1}{J_n A^{-1}\, J_n^T} J_n A^{-1} A Y'^T \\ &= \frac{1}{J_n A^{-1}\, J_n^T} J_n Y'^T \\ &= \frac{1}{J_n A^{-1}\, J_n^T} = v. \end{aligned}$$

Similarly, for any $X' \in S_n$, $E(X', Y) = v$, and so (X, Y) is a saddle and v is the value of the game.

Incidentally, these formulas match the formulas when we have a 2×2 game with an invertible matrix because then $A^{-1} = (1/\det(A))A^*$.

In order to guarantee that the value of a game is not zero, we may add a constant to every element of A that is large enough to make all the numbers of the matrix positive. In this case the value of the new game could not be zero. Since $v(A + b) = v(A) + b$, where b is the constant added to every element, we can find the original $v(A)$ by subtracting b. Adding the constant to all the elements of A

will not change the probabilities of using any particular row or column; that is, the optimal mixed strategies are not affected by doing that.

Even if our original matrix A does not have an inverse, if we add a constant to all the elements of A, we get a new matrix $A + b$ and the new matrix **may** have an inverse (of course, it may not as well). We may have to try different values of b. Here is an example.

Example 1.16 Consider the matrix

$$A = \begin{bmatrix} 0 & 1 & -2 \\ 1 & -2 & 3 \\ -2 & 3 & -4 \end{bmatrix}.$$

This matrix has negative and positive entries, so it's possible that the value of the game is zero. The matrix does not have an inverse because the determinant of A is 0. So, let's try to add a constant to all the entries to see if we can make the new matrix invertible. Since the largest negative entry is -4, let's add 5 to everything to get

$$A + 5 = \begin{bmatrix} 5 & 6 & 3 \\ 6 & 3 & 8 \\ 3 & 8 & 1 \end{bmatrix}.$$

This matrix does have an inverse given by

$$B = \frac{1}{80}\begin{bmatrix} 61 & -18 & -39 \\ -18 & 4 & 22 \\ -39 & 22 & 21 \end{bmatrix}.$$

Next we calculate using the formulas $v = 1/(J_3 B J_3^T) = 5$, and

$$X = v(J_3 B) = \left(\frac{1}{4}, \frac{1}{2}, \frac{1}{4}\right) \quad \text{and } Y = \left(\frac{1}{4}, \frac{1}{2}, \frac{1}{4}\right).$$

Since both X and Y are strategies (they have nonnegative components), the theorem tells us that they are optimal and the value of our original game is $v(A) = v - 5 = 0$.

The next example shows what can go wrong.

Example 1.17 Let

$$A = \begin{bmatrix} 4 & 1 & 2 \\ 7 & 2 & 2 \\ 5 & 2 & 8 \end{bmatrix}.$$

Then it is immediate that $v^- = v^+ = 2$ and there is a pure saddle $X^* = (0, 0, 1), Y^* = (0, 1, 0)$. If we try to use Theorem 1.5.2, we have $\det(A) = 10$, so A^{-1} exists and is given by

$$A^{-1} = \frac{1}{5}\begin{bmatrix} 6 & -2 & -1 \\ -23 & 11 & 3 \\ 2 & -\frac{3}{2} & \frac{1}{2} \end{bmatrix}.$$

If we use the formulas of the theorem, we get

$$v = -1,\ X = v(A)(J_3 A^{-1}) = \left(3, -\frac{3}{2}, -\frac{1}{2}\right),$$

and

$$Y = v(A^{-1} J_3^T)^T = \left(-\frac{3}{5}, \frac{9}{5}, -\frac{1}{5}\right).$$

Obviously these are completely messed up (i.e., wrong). The problem is that the components of X and Y are not nonnegative even though they do sum to 1.

Example 1.18 Hide and Seek Revisited: Suppose that we have $a_1 > a_2 > \cdots > a_n > 0$. The game matrix is

$$A = \begin{bmatrix} a_1 & 0 & 0 & \cdots & 0 \\ 0 & a_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_n \end{bmatrix}.$$

Because $a_i > 0$ for every $i = 1, 2, \ldots, n$, we know $\det(A) = a_1 a_2 \ldots a_n > 0$, so A^{-1} exists. It seems pretty clear that a mixed strategy for both players should use each row or column with positive probability; that is, we think that the game is completely mixed. In addition, since $v^- = 0$ and $v^+ = a_n$, the value of the game satisfies $0 \le v(A) \le a_n$. Choosing $X = (\frac{1}{n}, \ldots, \frac{1}{n})$ we see that

$$\min_Y XAY^T = \min_Y \left(y_1 \frac{a_1}{n} + \cdots + y_n \frac{a_n}{n} \right) = \frac{a_n}{n} > 0,$$

so that $v(A) = \max_X \min_Y XAY^T > 0$. This isn't a fair game for player II. It is also easy to see that

$$A^{-1} = \begin{bmatrix} \frac{1}{a_1} & 0 & 0 & \cdots & 0 \\ 0 & \frac{1}{a_2} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{1}{a_n} \end{bmatrix}.$$

Then, we may calculate from Theorem 1.5.2 that

$$v(A) = \frac{1}{J_n A^{-1} J_n^T} = \frac{1}{\frac{1}{a_1} + \frac{1}{a_2} + \cdots \frac{1}{a_n}},$$

$$X^* = v(A) \left(\frac{1}{a_1}, \frac{1}{a_2}, \ldots, \frac{1}{a_n} \right) = Y^*.$$

The strategies X^*, Y^* are legitimate strategies and they are optimal. Notice that for any $i = 1, 2, \ldots, n$, we obtain

$$1 < a_i \left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} \right),$$

so that $v(A) < \min(a_1, a_2, \ldots, a_n) = a_n$ and we have verified that $0 < v(A) < a_n$.

1.5.3 An Application: Optimal Target Choice and Defense

Optimal target choice and defense obviously has a clear military application and is the reason why this problem was first studied. Maybe a not so obvious application is when a predator company seeks to takeover one of n prey companies.

Suppose a company, player I, is seeking to takeover one of n companies which have decreasing values to player I, namely, $a_1 > a_2 > \cdots > a_n > 0$. These companies are managed by a private equity firm which can defend exactly one of the companies from takeover. Suppose that an attack made on company i has probability $1 - p$ of being taken over if it is defended, and will definitely be taken over if it is attacked by not being defended. Player I can attack exactly one of the companies and player II can choose to defend exactly one of them. If an attack is made on an undefended company, the payoff to I is a_i. If an attack is made on a defended company, I's payoff is $(1-p)a_i$. The payoff matrix is

I/II	1	2	3	$\cdots$	n
1	$(1-p)a_1$	a_1	a_1	$\cdots$	a_1
2	a_2	$(1-p)a_2$	a_2	$\cdots$	$\vdots$
3	a_3	a_3	$(1-p)a_3$	$\cdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
n	a_n	a_n	a_n	$\cdots$	$(1-p)a_n$

Let's start with $n=3$. We want to find the value and optimal strategies.

Since $\det(A) = a_1a_2a_3p^2(3-p) > 0$ the invertible matrix theorem can be used if $J_3A^{-1}J^T \neq 0$ and the resulting strategies are legit. We have, after dividing by $a_1a_2a_3$,

$$v(A) = \frac{1}{J_3A^{-1}J_3^T} = \frac{(3-p)\,a_1a_2a_3}{a_1a_2 + a_1a_3 + a_2a_3} = \frac{(3-p)}{\frac{1}{a_3} + \frac{1}{a_2} + \frac{1}{a_1}}$$

and $X^* = v(A)\,J_3A^{-1}$, after simplifying resulting in

$$X^* = \left(\frac{\frac{1}{a_1}}{\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3}}, \frac{\frac{1}{a_2}}{\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3}}, \frac{\frac{1}{a_3}}{\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3}} \right).$$

The formula for $Y^{*T} = v(A)A^{-1}J_3^T$ gives

$$Y^* = \left(\frac{a_3 + a_2 + (p-2)\,a_1}{p\left(a_1 + a_2 + a_3\right)}, \frac{a_3 + a_1 + (p-2)\,a_2}{p\left(a_1 + a_2 + a_3\right)}, \frac{a_1 + a_2 + (p-2)\,a_3}{p\left(a_1 + a_2 + a_3\right)} \right).$$

Since $p-2<0$, Y^* may not be a legitimate strategy for player II. Nevertheless, player I's strategy is legitimate and is easily verified to be optimal by showing that $E(X^*, j) \geq v(A), j = 1, 2, 3$.

For example if we take $p = 0.1, a_i = 4 - i, i = 1, 2, 3$, the matrix is

$$A = \begin{bmatrix} 0.27 & 3 & 3 \\ 2 & 0.18 & 2 \\ 1 & 1 & 0.9 \end{bmatrix}$$

and we calculate from the formulas above,

$$v(A) = 1.58,\ X^* = (0.18, 0.27, 0.54),\ Y^* = (4.72, 2.09, -5.81),$$

which is obviously not a strategy for player II. The invertible theorem gives us the right thing for v and X^* but not for Y^*. Interestingly enough, PI attacks the highest value company with the lowest probability.

Another approach is needed because we can't seem to use this to find Y^*. The idea is still to use part of the invertible theorem for $v(A)$ and X^* but then to do a direct derivation of Y^*. We will follow an argument due to Dresher [5].[1]

Because of the structure of the payoffs it seems reasonable that an optimal strategy for player II (and I as well) will be of the form $Y = (y_1, \dots, y_k\ 0, 0, 0)$, that is, II will not defend and therefore use the columns $k+1, \dots, n$, for some $k > 1$. We need to find the nonzero y_j's and the value of k.

1 A Rand corporation mathematician and game theorist (1911–1992).

We have the optimal strategy X^* for player I and we know that $x_1 > 0, x_2 > 0$. By the indifference principle and Remark 1.5.9 we then know that $E(1, Y^*) = v, E(2, Y^*) = v$. Written out, this is

$$E(1, Y^*) = (1-p)\, a_1 y_1 + a_1 y_2 + \cdots + a_1 y_k = v,$$
$$E(2, Y^*) = a_2 y_1 + (1-p)\, a_2 y_2 + \cdots + a_2 y_k = v.$$

Using the fact that $\sum_{j=1}^{k} y_j = 1$, we have

$$a_1 (1-p) y_1 + a_1 \left(\sum_{j=2}^{k} y_j \right) = a_1 \left(1 - p y_1\right),$$

$$a_2 y_1 + a_2 (1-p) y_2 + a_2 \left(\sum_{j=3}^{k} y_j \right) = a_2 \left(1 - p y_2\right),$$

and $a_2(1 - py_2) = a_1(1 - py_1)$. Solving for y_2 we get

$$y_2 = \frac{1}{p}\left[1 - \frac{a_1}{a_2}\left(1 - p y_1\right)\right].$$

In general, the same argument using the fact that $x_1 > 0$ and $x_i > 0, i = 2, 3, \ldots, k$ results in all the $y_j's, j = 2, 3, \ldots, k$ in terms of y_1:

$$y_j = \frac{1}{p}\left[1 - \frac{a_1}{a_j}\left(1 - p y_1\right)\right], \quad j = 2, 3, \ldots, k.$$

The index $k > 1$ is yet to be determined and remember we are setting $y_j = 0, j = k+1, \ldots, n$. Again using the fact that $\sum_{j=1}^{k} y_j = 1$, we obtain after some algebra the formula

$$y_j = \begin{cases} \frac{1}{p}\left[1 - \dfrac{k-p}{a_j G_k}\right], & \text{if } j = 1, 2, \ldots, k; \\ 0, & \text{otherwise.} \end{cases}$$

Here, $G_k = \sum_{i=1}^{k} \frac{1}{a_i}$.

Now note that if we use the results from the invertible matrix theorem for v and X^*, we may define

$$v(A) = \frac{(k-p)}{G_k},$$

and

$$X^* = \left(\frac{\frac{1}{a_1}}{G_k}, \frac{\frac{1}{a_2}}{G_k}, \ldots, \frac{\frac{1}{a_k}}{G_k}, 0, 0, \ldots, 0 \right),$$

and X^* is still a legitimate strategy for player I and $v(A)$ will still be the value of the game for any value of $k \le n$. You can easily verify that $E(X^*, j) = v(A)$ for each column j. Now we have the candidates in hand we may verify that they do indeed solve the game.

Theorem 1.5.3 *Set $G_k = \sum_{i=1}^{k} \frac{1}{a_i}$. Notice that $G_{k+1} \ge G_k$. Let k^* be the value of i giving $\max_{1 \le i \le n} \frac{i-p}{G_i}$. Then,*

$$X^* = \left(x_1, \ldots, x_n\right), \quad \textit{where } x_i = \begin{cases} \dfrac{1}{a_i G_{k^*}}, & \textit{if } i = 1, 2 \ldots, k^* \\ 0, & \textit{if } i = k^* + 1, \ldots, n \end{cases}$$

is optimal for player I, and $Y^* = (y_1, y_2, \ldots, y_n)$,

$$y_j = \begin{cases} \frac{1}{p}\left(1 - \dfrac{k^* - p}{a_j G_{k^*}}\right), & \text{if } j = 1, 2, \ldots, k^* \\ 0, & \text{otherwise,} \end{cases}$$

is optimal for player II. In addition, $v(A) = \frac{k^*-p}{G_{k^*}}$ *is the value of the game.*

One point of the theorem is that PI should only attack the companies $1, 2, \ldots, k^*$ and PII should only defend those companies. It is, of course true that it is possible $k^* = n$ in which case both optimal strategies are completely mixed and all companies should be attacked and defended in some proportion.

Proof: The only thing we need to verify is that Y^* is a legitimate strategy for player II and is optimal for that player.

First, let's consider the function $f(k) = k - a_k G_k$. We have

$$f(k+1) - f(k) = k + 1 - a_{k+1}G_{k+1} - k + a_k G_k = 1 - a_{k+1}G_{k+1} + a_k G_k.$$

Since $a_k > a_{k+1}$,

$$f(k+1) - f(k) = 1 - a_{k+1}G_{k+1} + a_k G_k > 1 - a_{k+1}\left(G_k - G_{k+1}\right) > 1.$$

We conclude that $f(k+1) > f(k)$, that is, f is a strictly increasing function of $k = 1, 2, \ldots$. Also $f(1) = 1 - a_1 G_1 = 0$ and we may set $f(n+1) =$ any big positive integer. We can then find exactly one index k^* so that $f(k^*) \le p < f(k^*+1)$ and this will be the k^* in the statement of the theorem. We will show later that it is characterized as stated in the theorem.

Now for this k^*, we have

$$k^* - a_{k^*}G_{k^*} \le p < k^* + 1 - a_{k^*+1}G_{k^*+1} \Rightarrow a_{k^*+1} < \frac{k^* - p}{G_{k^*}} \le a_{k^*}.$$

Since the $a_i's$ decrease, we get

$$\frac{k^*-p}{G_{k^*}} \le a_i, \quad i = 1, 2 \ldots, k^*,$$
$$\frac{k^*-p}{G_{k^*}} > a_i, \quad i = k^*+1, k^*+2 \ldots, n.$$

Since

$$E(i, Y^*) = \begin{cases} \frac{k^*-p}{G_{k^*}} = v(A), & \text{if } i = 1, 2, \ldots, k^*, \\ a_i < v(A), & \text{if } i = k^*+1, \ldots, n, \end{cases}$$

this proves by Theorem 1.4.8 and Remark 1.5.9 that Y^* is optimal for player II. Notice also that $y_j \ge 0$ for all $j = 1, 2, \ldots, n$ since by definition

$$y_j = \begin{cases} \frac{1}{p}\left(1 - \dfrac{k^* - p}{a_j G_{k^*}}\right), & \text{if } j = 1, 2, \ldots, k^*, \\ 0, & \text{otherwise,} \end{cases}$$

and we have shown that

$$\frac{k^* - p}{G_{k^*}} \le a_j, j = 1, 2, \ldots, k^*.$$

Hence, Y^* is a legitimate strategy.

Finally, we show that k^* can also be characterized as follows: k^* is the value of i achieving $\max_{1\le i\le n} \frac{i-p}{G_i} = \frac{k^*-p}{G_{k^*}}$. To see why, the function $\varphi(k) = \frac{k-p}{G_k}$ has a maximum value at some value $k = \ell$. That means $\varphi(\ell+1) \le \varphi(\ell)$ and $\varphi(\ell-1) \le \varphi(\ell)$. Using the definition of φ, we get

$$\ell - a_\ell G_\ell \le p \le \ell + 1 - a_{\ell+1} G_{\ell+1},$$

which is what k^* must satisfy. Consequently, $\ell = k^*$ and we are done. ∎

In the problems you will calculate X^*, Y^*, and v for specific values of p, a_i.

Example 1.19 Consider the optimal target choice and defense problem with $n = 3$, $p = \frac{1}{2}$, and $a_1 = 7, a_2 = 5, a_3 = 3$. The matrix becomes

$$A = \begin{bmatrix} \frac{7}{2} & 7 & 7 \\ 5 & \frac{5}{2} & 5 \\ 3 & 3 & \frac{3}{2} \end{bmatrix}.$$

Then, we should calculate the values $\varphi(k) = \frac{k-p}{G_k}$, which decides companies that are worth attacking. First, $G_1 = \frac{1}{7}, G_2 = \frac{1}{7} + \frac{1}{5} = 0.3429, G_3 = \frac{1}{7} + \frac{1}{5} + \frac{1}{3} = 0.6762$. Next,

$$\varphi(1) = \frac{1 - \frac{1}{2}}{\frac{1}{7}} = 3.5, \quad \varphi(2) = \frac{2 - \frac{1}{2}}{\frac{1}{7} + \frac{1}{5}} = 4.375, \quad \varphi(3) = \frac{3 - \frac{1}{2}}{\frac{1}{7} + \frac{1}{5} + \frac{1}{3}} \simeq 3.697.$$

This is largest for $k^* = 2$, so PI should only attack the first two companies, and should do so in proportion to $\frac{1}{a_i}$. So, $X^* = \frac{1}{G_{k^*}}(\frac{1}{7}, \frac{1}{5}, 0)$. This results in

$$X^* = \left(\frac{5}{12}, \frac{7}{12}, 0\right).$$

Now for the defender, they should never defend company 3. Using the formula in the theorem we have

$$y_1 = 2\left(1 - \frac{2 - \frac{1}{2}}{7\frac{12}{35}}\right) = \frac{3}{4}, \qquad y_2 = 2\left(1 - \frac{2 - \frac{1}{2}}{5\frac{12}{35}}\right) = \frac{1}{4}.$$

Therefore $Y^* = (\frac{3}{4}, \frac{1}{4}, 0)$. Furthermore, $v(A) = \frac{35}{8}$.

Another way to find Y^* is to look at the 2×2 game which remains after company 3 is ignored by both players. This has matrix

$$A = \begin{bmatrix} \frac{7}{2} & 7 \\ 5 & \frac{5}{2} \end{bmatrix}.$$

Solving gives $Y^* = \left(\frac{3}{4}, \frac{1}{4}, 0\right)$.

Notice that, perhaps unexpectedly, PI is most likely to attack the company of middle value. This is actually the typical situation: the attacking company never attacks targets below some cutoff, is most likely to attack the targets of moderate value, and attacks the highest value targets less often. The defender's strategy is more obvious: she defends the highest value targets the most often.

Problems

1.23 In a simplified analysis of a football game suppose that the offense can only choose a pass or run, and the defense can choose only to defend a pass or run. Here is the matrix in which the payoffs are the average yards gained:

	Defense	
Offense	Run	Pass
Run	1	8
Pass	10	0

The offense's goal is to maximize the average yards gained per play. Find $v(A)$ and the optimal strategies using the explicit formulas. Check your answers by solving graphically as well.

1.24 Suppose that an offensive pass against a run defense now gives 12 yards per play on average to the offense (so the 10 in the previous matrix changes to a 12). Believe it or not, the offense should pass less, not more. Verify that and give a game theory (not math) explanation of why this is so.

1.25 Does the same phenomenon occur for the defense? To answer this, compare the original game in Problem 1.23 to the new game in which the defense reduces the number of yards per run to 6 instead of 8 when defending against the pass. What happens to the optimal strategies?

1.26 If the matrix of the game has an inverse, the formulas can be simplified to

$$X^* = \frac{(1\ 1)A^{-1}}{(1\ 1)A^{-1}\begin{bmatrix}1\\1\end{bmatrix}}, \quad Y^{*T} = \frac{A^{-1}\begin{bmatrix}1\\1\end{bmatrix}}{(1\ 1)A^{-1}\begin{bmatrix}1\\1\end{bmatrix}}, \quad v(A) = \frac{1}{(1\ 1)A^{-1}\begin{bmatrix}1\\1\end{bmatrix}}$$

where $A^{-1} = \frac{1}{\det(A)}\begin{bmatrix}a_{22} & -a_{12}\\ -a_{21} & a_{11}\end{bmatrix}$. This requires that $\det(A) \neq 0$. Construct an example to show that even if A^{-1} does not exist, the original formulas with A^{-1} replaced by A^* still hold but now $v(A) = 0$.

1.27 Solve the 2×2 games using the formulas:

$$(\mathbf{a})\begin{bmatrix}4 & -3\\ -9 & 6\end{bmatrix}; \qquad (\mathbf{b})\begin{bmatrix}8 & 99\\ 29 & 6\end{bmatrix}; \qquad (\mathbf{c})\begin{bmatrix}-32 & 4\\ 74 & -27\end{bmatrix}.$$

1.28 Give an example to show that when optimal pure strategies exist, the formulas in the theorem won't work.

1.29 Let $A = (a_{ij}), i, j = 1, 2$. Show that if $a_{11} + a_{22} = a_{12} + a_{21}$, then $v^+ = v^-$ or, equivalently, there are optimal pure strategies for both players. This means that if you end up with a zero denominator in the formula for $v(A)$, it turns out that there had to be a saddle in pure strategies for the game and hence the formulas don't apply from the outset.

1.30 There are two presidential candidates, Harry and Tom, who will choose which states they will visit to garner votes. The states are of different values: Let's say State 1 is worth 10, State 2 worth 12, and State 3 worth 14. If they go to different states, the each gain the benefit for that state. So, for instance, if tom goes to state 1 and Harry to state 3, the outcome for tom is $+6 - 10 = -4$. But, Tom is a better debater than Harry: if they go to the same state, Tom gets half of the benefit from visiting that state, and Harry gets nothing.

(a) Set this up as a matrix game.

(b) Show that, Harry plays the mixed strategy $Y = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$, then whatever Tom does their expected outcome is at most $\frac{13}{3}$. Conclude that the value of this game is at most $\frac{13}{3}$.

(c) If Harry plays a strategy which is not completely mixed, then Tom has a response giving them an outcome of at least 5—explain! Using this and the previous part, conclude that Harry's optimal strategy is completely mixed.

(d) Using the fact that Harry's optimal strategy is completely mixed, set up and solve a system of equations giving Tom's optimal strategy X^*.

(e) In the previous part you should have found that Tom's optimal strategy is also completely mixed. Using this, set up and solve a system of equations giving Harry's optimal strategy Y^*.

(f) Calculate the expected outcome $E(X^*, 2)$.

(g) Calculate the expected outcome $E(1, Y^*)$.

(h) Calculate value of this game. Or, if you can see how, explain why you already know the value and just write it down.

1.31 Consider Theorem 1.5.3 and let $p = 0.5, a_1 = 9, a_2 = 7, a_3 = 6, a_4 = 1$ Find $v(A), X^*, Y^*$, solving the game. Find the value of k^* first.

1.32 Consider the matrix game

$$A = \begin{bmatrix} 3 & 5 & 3 \\ 4 & -3 & 2 \\ 3 & 2 & 3 \end{bmatrix}.$$

Show that there is a saddle in pure strategies at $(1, 3)$ and find the value. Verify that $X^* = (\frac{1}{3}, 0, \frac{2}{3}), Y^* = (\frac{1}{2}, 0, \frac{1}{2})$ is also an optimal saddle point. Does A have an inverse? Find it and use the formulas in the theorem to find the optimal strategies and value.

1.33 Solve the following games:

$$\textbf{(a)} \begin{bmatrix} 2 & 2 & 3 \\ 2 & 2 & 1 \\ 3 & 1 & 4 \end{bmatrix}; \quad \textbf{(b)} \begin{bmatrix} 5 & 4 & 2 \\ 1 & 5 & 3 \\ 2 & 3 & 5 \end{bmatrix}; \quad \textbf{(c)} \begin{bmatrix} 4 & 2 & -1 \\ -4 & 1 & 4 \\ 0 & -1 & 5 \end{bmatrix}.$$

(d) Use dominance to solve the following game even though it has no inverse:

$$A = \begin{bmatrix} -4 & 2 & -1 \\ -4 & 1 & 4 \\ 0 & -1 & 5 \end{bmatrix}.$$

1.34 To underscore that the formulas can be used only if you end up with legitimate strategies, consider the game with matrix

$$A = \begin{bmatrix} 1 & 5 & 2 \\ 4 & 4 & 4 \\ 6 & 3 & 4 \end{bmatrix}.$$

(a) Does this matrix have a saddle in pure strategies? If so find it.
(b) Find A^{-1}.
(c) Without using the formulas, find an optimal **mixed** strategy for player II with two positive components.
(d) Use the formula to calculate Y^*. Why isn't this optimal? What's wrong?

1.35 Show that $value(A + b) = value(A) + b$ for any constant b, where, by $A + b = (a_{ij} + b)$, is meant A plus the matrix with all entries $= b$. Show also that (X, Y) is a saddle for the matrix $A + b$ if and only if it is a saddle for A.

1.36 Derive the formula for $X = \frac{J_n A^{-1}}{J_n A^{-1} J_n^T}$, assuming the game matrix has an inverse. Follow the same procedure as that in obtaining the formula for Y.

1.37 A magic square game has a matrix in which each row has a row sum that is the same as each of the column sums. For instance, consider the matrix

$$A = \begin{bmatrix} 11 & 24 & 7 & 20 & 3 \\ 4 & 12 & 25 & 8 & 16 \\ 17 & 5 & 13 & 21 & 9 \\ 10 & 18 & 1 & 14 & 22 \\ 23 & 6 & 19 & 2 & 15 \end{bmatrix}.$$

This is a magic square of order 5 and sum 65. Find the value and optimal strategies of this game and show how to solve any magic square game.

1.38 Why is the hide-and-seek game in Example 1.18 called that? Determine what happens in the hide-and-seek game if there is at least one $a_k = 0$.

1.39 Solve the hide and seek game in Example 1.18 with matrix $A_{n\times n} = (a_{ij}), a_{ii} = \frac{i}{i+1}, i = 1, 2, \ldots, n$, and $a_{ij} = 0, i \neq j$. Find the general solution and give the exact solution when $n = 5$.

1.40 For the game with matrix

$$\begin{bmatrix} -1 & 0 & 3 & 3 \\ 1 & 1 & 0 & 2 \\ 2 & -2 & 0 & 1 \\ 2 & 3 & 3 & 0 \end{bmatrix},$$

we determine that the optimal strategy for player II is $Y = (\frac{3}{7}, 0, \frac{1}{7}, \frac{3}{7})$. We are also told that player I has an optimal strategy X which is completely mixed. Given that the value of the game is $\frac{9}{7}$, find X.

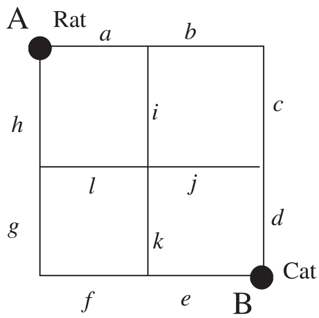

Figure 1.6 Maze for Cat versus Rat.

1.41 A triangular game is of the form

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix}.$$

(a) Find conditions under which this matrix has an inverse.
(b) Now consider

$$A = \begin{bmatrix} 1 & -3 & -5 & 1 \\ 0 & 4 & 4 & -2 \\ 0 & 0 & 8 & 3 \\ 0 & 0 & 0 & 50 \end{bmatrix}.$$

Solve the game by finding $v(A)$ and the optimal strategies.

1.42 Suppose an evader (called Rat) is forced to run a maze entering at point A (see Figure 1.6). A pursuer (called Cat) will also enter the maze at point B. Rat and Cat will run exactly four segments of the maze and the game ends. If Cat and Rat ever meet at an intersection point of segments at the same time, Cat wins +1 (Rat) and the Rat loses −1 because it is zero sum, while if they never meet during the run, both Cat and the Rat win 0. In other words, if Cat finds Rat, Cat gets +1 and otherwise, Cat gets 0. We are looking at the payoffs from Cat's point of view, who wants to maximize the payoffs, while Rat wants to minimize them. Figure 1.6 shows the setup.
The strategies for Rat consist of all the choices of paths with four segments that Rat can run. Similarly, the strategies for Cat will be the possible paths it can take. With four segments it will turn out to be a 16×16 matrix. Using dominance to eliminate bad strategies the game reduces to the 3×3 game

Cat/Rat	*abcj*	*aike*	*hlia*
dcbi	1	0	1
djke	0	1	1
ekjd	1	1	0

Now solve the game.

1.43 In tennis two players can choose to hit a ball left, center, or right of where the opposing player is standing. Name the two players I and II and suppose that I hits the ball, while II anticipates where the ball will be hit. Suppose that II can return a ball hit right 90% of the time, a ball hit left 60% of the time, and a ball hit center 70% of the time. If II anticipates incorrectly, she can return the ball only 20% of the time. Score a return as +1 and a ball not returned as −1. Find the game matrix and the optimal strategies.

1.6 Finding Saddle Points in General

In this section we will see that we can completely solve all matrix games using several methods. We begin by generalizing the graphical method we introduced in the Baseball example from Section 1.2.2 to any $2 \times n$ or $m \times 2$ game. Then we use a powerful method called linear programming to solve any matrix game but which mostly requires a computer. First semester Calculus can also be used to find saddle points and we illustrate this method as well. Finally, the special case of symmetric games is considered.

1.6.1 Graphical Methods

Specifically, we will consider all $2 \times m$ and $n \times 2$ matrix games, so all games where one of the players has only 2 possible strategies. We begin by doing a 2×2 example, following the method we used in the earlier baseball example, but now with somewhat more formal language.

Example 1.20

$$A = \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}.$$

The first step is that we must check whether there are pure optimal strategies because if there are, then we can't use the graphical method. It only finds the strategies that are mixed, but we don't need any fancy methods to find the pure ones. Since $v^- = 2$ and $v^+ = 3$, we know the optimal strategies must be mixed. Now we use Theorem 1.4.6 to find the optimal strategy $X = (x, 1-x), 0 < x < 1$, for player I, and the value of the game $v(A)$. Here is how it goes.

Playing X against each column for player II, we get

$$E(X,1) = XA_1 = x + 3(1-x) \quad \text{and} \quad E(X,2) = XA_2 = 4x + 2(1-x).$$

We now plot each of these functions of x on the same graph in Figure 1.7. Each plot will be a straight line with $0 \le x \le 1$.

Now, here is how to analyze this graph from player I's perspective. First, the point at which the two lines intersect is $(x^* = \frac{1}{4}, v = \frac{10}{4})$. If player I chooses an $x < x^*$, then the best I can receive is on the highest line when x is on the left of x^*. In this case the line is $E(X,1) = x + 3(1-x) > \frac{10}{4}$. Player I will receive this higher payoff only if player II decides to play column 1. But player II will also have this graph and II will see that if I uses $x < x^*$ then II should definitely **not** use column 1 but should use column 2 so that I would receive a payoff on the **lower** line $E(X,2) < \frac{10}{4}$. In other words, I will see that II would switch to using column 2 if I chose to use any mixed strategy with $x < \frac{1}{4}$. What if I uses an $x > \frac{1}{4}$? The reasoning is similar; the best I could get would happen if player II chose to use column 2 and then I gets $E(X,2) > \frac{10}{4}$. But I cannot assume that II will play stupidly. Player II will see that if player I chooses to play an $x > \frac{1}{4}$, II will choose to play column 1 so that I receives some payoff on the line $E(X,1) < \frac{10}{4}$.

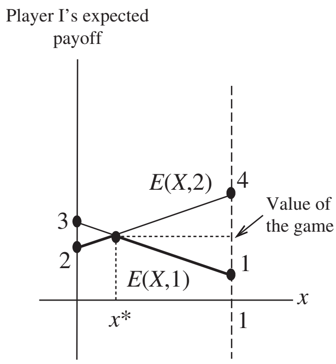

Figure 1.7 X against each column for player II.

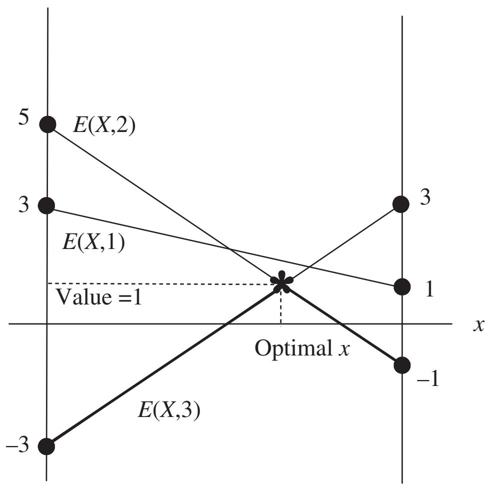

Figure 1.8 Mixed for player I versus player II's columns.

Conclusion: player I, assuming that player II will be doing her best, will choose to play $X = (x^*, 1 - x^*) = \left(\frac{1}{4}, \frac{3}{4}\right)$ and then receive exactly the payoff $= v(A) = \frac{10}{4}$. Player I will rationally choose the maximum minimum. The minimums are the bold lines and the maximum minimum is at the intersection, which is the highest point of the bold lines. Another way to put it is that player I will choose a mixed strategy so that she will get $\frac{10}{4}$ no matter what player II does, and if II does not play optimally, player I can get more than $\frac{10}{4}$.

What should player II do? Before you read the examples, try to figure it out.

Example 1.21 This method actually works just as well in the $2 \times m$ case. Consider the matrix

$$A = \begin{bmatrix} 1 & -1 & 3 \\ 3 & 5 & -3 \end{bmatrix}.$$

Consider the graph for player I first.

Looking at Figure 1.8, we see that the optimal strategy for I is the x value where the two lower lines intersect and yields $X^* = \left(\frac{2}{3}, \frac{1}{3}\right)$. Also, $v(A) = E(X^*, 3) = E(X^*, 2) = 1$. To find the optimal strategy for player II, we see that II will never use the first column, since they can figure out Player 1 will play $\left(\frac{2}{3}, \frac{1}{3}\right)$, and this is not the best response. In general, we may drop the columns (or rows in some cases) not used to get the optimal intersection point—Often that is true because the unused rows are dominated or convex-dominated (as column 1 is here) but not always. Both the second and

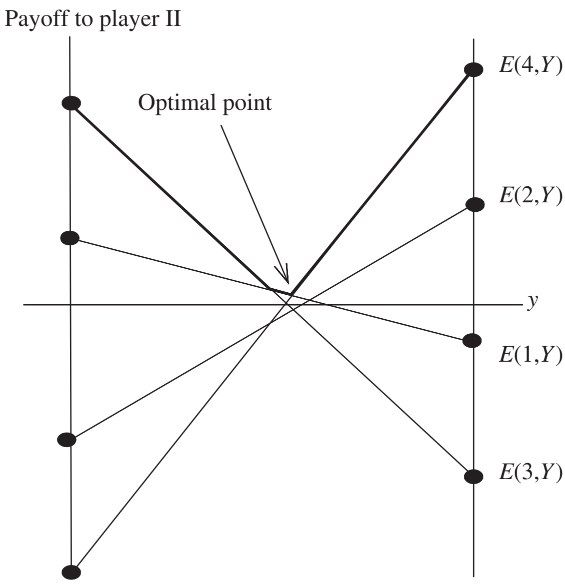

Figure 1.9 Mixed for player II versus 4 rows for player I.

third columns are best responses, so Player 2 should play some combination of these. This means we have reduce to the subgame with the first column removed:

$$\begin{bmatrix} -1 & 3 \\ 5 & -3 \end{bmatrix}.$$

Now this has no pure equilibrium, so we can solve for player II's strategy using the indifference principle from Theorem 1.4.8. We end up solving

$$-q + 3(1 - q) = 5q - 3(1 - q)$$

giving $q = \frac{1}{2}$. Thus player II's optimal strategy for this subgame is $\left(\frac{1}{3}, \frac{2}{3}\right)$. But this is not Y^*, since the original game has 3 strategies for II! We need to record that II never plays the first column. This means $Y^* = \left(0, \frac{1}{2}, \frac{1}{2}\right)$.

Example 1.22 The $n \times 2$ case is only a little different, but we do have to be careful because we must work from the other player's point of view, which changes some calculations. Let's consider

$$A = \begin{bmatrix} -1 & 2 \\ 3 & -4 \\ -5 & 6 \\ 7 & -8 \end{bmatrix}$$

We first check that this has no pure saddle points, so we know we must look for a mixed saddle point. Let Player II use a strategy $Y = (y, 1 - y)$, and graph the payoffs $E(i, Y), i = 1, 2, 3, 4,$: Figure 1.9.

You can see the difficulty with solving games graphically; you have to be very accurate with your graphs. Carefully reading the information, it appears that the optimal strategy for Y will be determined at the intersection point of $E(4, Y) = 7y - 8(1 - y)$ and $E(1, Y) = -y + 2(1 - y)$. This occurs at the point $y^* = \frac{5}{9}$ and the corresponding value of the game will be $v(A) = \frac{1}{3}$. The optimal strategy for player II is $Y^* = \left(\frac{5}{9}, \frac{4}{9}\right)$.

Since this uses only rows 1 and 4, we may now drop rows 2 and 3 to find the optimal strategy for player I. Considering the matrix using only rows 1 and 4, we now calculate $E(X, 1) = -x + 7(1 - x)$ and $E(X, 2) = 2x - 8(1 - x)$ which intersect at $\left(x = \frac{5}{6}, v = \frac{1}{3}\right)$. We obtain that row 1 should be used with probability $\frac{5}{6}$ and row 4 should be used with probability $\frac{1}{6}$, so $X^* = \left(\frac{5}{6}, 0, 0, \frac{1}{6}\right)$.

Just to be sure, let's verify that these are indeed optimal using Theorem 1.4.6. We check that $E(i, Y^*) \le v(A) \le E(X^*, j)$ for all rows and columns. This gives

$$\begin{bmatrix} \frac{5}{6} & 0 & 0 & \frac{1}{6} \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 3 & -4 \\ -5 & 6 \\ 7 & -8 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -1 & 2 \\ 3 & -4 \\ -5 & 6 \\ 7 & -8 \end{bmatrix} \begin{bmatrix} \frac{5}{9} \\ \frac{4}{9} \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ -\frac{1}{9} \\ -\frac{1}{9} \\ \frac{1}{3} \end{bmatrix}.$$

Everything checks.

1.6.2 The $n \times m$ Case and Linear Programming

The graphical method is really just a visual way to illustrate the equivalence of payoffs and to see that player I seeks the maximum minimum while player II seeks the minimum maximum graphically. For a general $n \times m$ matrix A we may not be able to use a graph but we can use the same idea to set up an optimization problem. We have seen that

$$v^- = \max_{X \in S_n} \min_j E(X, j) = \max_{X \in S_n} \{v : v = \min_j E(X, j)\}.$$

Let's make a couple of changes to this equation. First, we replace $v = \min_j E(X, j)$ with $v \le E(X, j)$ for all j. Second, we move the definition of S_m into the equation.

$$v^- = \max \left\{ v : \begin{array}{c} \text{for some } X = (x_1, \dots, x_n), v \le E(X, j) \text{ for all } 1 \le j \le m \\ X \text{ satisfies } x_i \ge 0,\ x_1 + \cdots + x_n = 1 \end{array} \right\}.$$

This problem is looking for $n + 1$ numbers $v, x_1, \dots, x_n$ such that a system of linear inequalities is satisfied, and v is as large as possible subject to those constraints. We are looking to solve the optimization problem with linear constraints:

Maximize v

subject to

$v \le E(X, j) = X\, A_j, \quad j = 1, 2, \dots, m$

$x_1 + x_2 + \cdots + x_n = 1, \quad x_i \ge 0.$

There are well-known algorithms for solving a problem like this, such as the **simplex method**. This type of problem can also be solved by any computer algebra system, such as Mathematica. This will return the lower value v^- as well as the X where this is achieved, so player I's optimal strategy X^* emerges.

To find Y^*, you set up the same problem but for v^+:

$$v^+ = \min \left\{ v : \begin{array}{c} \text{for some } Y = (y_1, \dots, y_m), v \ge E(i, Y) \text{ for all } 1 \le i \le n \\ Y \text{ satisfies } y_i \ge 0, y_1 + \cdots + y_m = 1 \end{array} \right\}.$$

Player II's problem is to solve

Minimize v

subject to

$$v \geq E(i, Y) = {}_iAY^T, \quad i = 1, 2, \ldots, n$$
$$y_1 + y_2 + \cdots + y_m = 1, \quad y_j \geq 0.$$

Matrix Games and Linear Programming

Linear programming is an area of optimization theory developed since World War II that is used to find the minimum (or maximum) of a linear function of many variables subject to a collection of linear constraints on the variables. It is extremely important to any modern economy to be able to solve such problems that are used to model many fundamental problems that a company may encounter. For example, the best routing of oil tankers from all of the various terminals around the world to the unloading points is a linear programming problem in which the oil company wants to minimize the total cost of transportation subject to the consumption constraints at each unloading point. But there are millions of applications of linear programming, which can range from the problems just mentioned to modeling the entire US economy. One can imagine the importance of having a very efficient way to find the optimal variables involved. Fortunately, George Dantzig,[1] in the 1940s, because of the necessities of the war effort, developed such an algorithm, called the **simplex method** that will quickly solve very large problems formulated as linear programs.

Mathematicians and economists working on game theory (including von Neumann), once they became aware of the simplex algorithm, recognized the connection.[2] After all, a game consists in minimizing and maximizing linear functions with linear things all over the place. So a method was developed to formulate a matrix game as a linear program so that the simplex algorithm could be applied.

This means that using linear programming, we can find the value and optimal strategies for a matrix game of any size without any special theorems or techniques. In many respects, this approach makes it unnecessary to know any other computational approach. The downside is that in general one needs a computer capable of running the simplex algorithm to solve a game by the method of linear programming. We will show how to set up the Mathematica commands to solve the problem.

A Little Aside on Linear Programming

A linear programming problem is a problem of the **standard form** (called the **primal program**):

$$\text{Minimize } z = \mathbf{c} \cdot \mathbf{x}$$
$$\text{subject to } \mathbf{x}\, A \geq \mathbf{b}, \mathbf{x} \geq 0,$$

where $\mathbf{c} = (c_1, \ldots, c_n)$, $\mathbf{x} = (x_1, \ldots, x_n)$, $A_{n\times m}$ is an $n \times m$ matrix, and $\mathbf{b} = (b_1, \ldots, b_m)$.

The primal problem seeks to minimize a **linear objective function**, $z(\mathbf{x}) = \mathbf{c} \cdot \mathbf{x}$, over a set of constraints (viz., $\mathbf{x} \cdot A \geq \mathbf{b}$) that are also linear. You can visualize what happens if you try to minimize or maximize a linear function of one variable over a closed interval on the real line.

1 George Bernard Dantzig was born on November 8, 1914, and died on May 13, 2005. He is considered the “father of linear programming.” He was the recipient of many awards, including the National Medal of Science in 1975, and the John von Neumann Theory Prize in 1974.

2 In 1947 Dantzig met with von Neumann and began to explain the linear programming model “as I would to an ordinary mortal.” After a while von Neumann ordered Dantzig to “get to the point.” Dantzig relates that in “less than a minute I slapped the geometric and algebraic versions …on the blackboard.” Von Neumann stood up and said “Oh, that.” Dantzig relates that von Neumann then “proceeded to give me a lecture on the mathematical theory of linear programs.” This story is related in the excellent article by Cottle, Johnson, and Wets in the Notices of the A.M.S., March 2007.

The minimum and maximum must occur at an endpoint. In more than one dimension this idea says that the minimum and maximum of a linear function over a variable that is in a convex set must occur on the boundary of the convex set. If the set is created by linear inequalities, even more can be said, namely, that the minimum or maximum must occur at an extreme point, or corner point, of the constraint set. The method for solving a linear program is to efficiently go through the extreme points to find the best one. That is essentially the simplex method.

For any primal there is a related linear program called the **dual program**:

$$\text{Maximize } w = \mathbf{y}\,\mathbf{b}^T$$
$$\text{subject to } A\,\mathbf{y}^T \le \mathbf{c}^T, \mathbf{y} \ge 0.$$

A very important result of linear programming, which is called the **duality theorem**, states that if we solve the primal problem and obtain the optimal objective $z = z^*$, and solve the dual obtaining the optimal $w = w^*$, then $z^* = w^*$. In our game theory formulation to be given next, this theorem will tell us that the two objectives in the primal and the dual will give us the value of the game. Incidentally, the proof of the duality theorem would provide a proof of the theorem that says if each player uses mixed strategies in a matrix game, there is always a value and a saddle point for the game.

Returning to linear programming and matrix games let's work out an example.

Example 1.23 Consider

$$A = \begin{bmatrix} 2 & -2 & 0 \\ -4 & 4 & 4 \\ 1 & 0 & 4 \end{bmatrix}.$$

Then, we get the problem to determine Y^*.

$$\text{Minimize } v$$
$$\text{subject to}$$
$$v \ge 2y_1 - 2y_2, v \ge -4y_1 + 4y_2 + 4y_3, v \ge y_1 + 4y_2, y_1, y_2, y_3 \ge 0,$$
$$y_1 + y_2 + y_3 = 1, y_1 \ge 0, y_2 \ge 0, y_3 \ge 0.$$

Using Mathematica we apply the command

```
Minimize[{v, v>=2y1-2y2,v>=-4y1+4y2+4y3,v>=y1+4y2,
                    y1+y2+y3==1,y1>=0,y2>=0,y3>=0},{y1,y2,y3}]
```

This results in $v = \frac{4}{9}, Y^* = \left(\frac{4}{9}, \frac{5}{9}, 0\right)$. Similarly, the problem to determine X^* is

$$\text{Maximize } v$$
$$\text{subject to}$$
$$v \le 2x_1 - 4x_2 + x_3, v \le -2x_1 + 4x_2, v \le 4x_2 + 4x_3,$$
$$x_1 + x_2 + x_3 = 1,\ x_1 \ge 0, x_2 \ge 0, x_3 \ge 0.$$

We get $v = \frac{4}{9}, x_1 = 0, x_2 = \frac{1}{9}, x_3 = \frac{8}{9}$.

Here's an example using the theory of duels.

Example 1.24 A Nonsymmetric Noisy Duel. Two players engaged in a duel with pistols begin 10 paces apart. They may shoot at paces (10, 6, 2) with accuracies (0.2, 0.4, 1.0) each. We have the following payoffs to player I at the end of the duel:

- If only player I survives, then player I receives payoff a.
- If only player II survives, player I gets payoff $b < a$. This assumes that the survival of player II is less important than the survival of player I.
- If both players survive, they each receive payoff zero.
- If neither player survives, player I receives payoff g.

We will take $a = 1$, $b = \frac{1}{2}$, $g = 0$. Then here is the expected payoff matrix for player I:

I/II	10	6	2
10	0.24	0.6	0.6
6	0.9	0.36	0.70
2	0.9	0.8	0

The pure strategies are (x, y), where $x = 10, 6, 2$ and $y = 10, 6, 2$. The elements of the matrix are obtained from the general formula

$$E(x, y) = \begin{cases} ax + b(1 - x), & \text{if } x < y; \\ ax + bx + (g - a - b)x^2, & \text{if } x = y; \\ a(1 - y) + by, & \text{if } x > y. \end{cases}$$

For example, if we look at

$$\begin{aligned} E(6, 10) &= aProb\,(\text{II misses at } 10) + bProb\,(\text{II kills I at } 10) \\ &= a(1 - 0.2) + b(0.2) \end{aligned}$$

because if player II kills I at 10, then he survives and the payoff to I is b. If player II shoots, but misses player I at 10, then player I is certain to kill player II later (this is where the noisy part enters) and will receive a payoff of a. Similarly,

$$\begin{aligned} E(x, x) &= aProb\,(\text{II misses})\,Prob\,(\text{I hits}) \\ &\quad + bProb\,(\text{I misses})\,Prob\,(\text{II hits}) + gProb\,(\text{I hits})\,Prob\,(\text{II hits}) \\ &= a(1 - x)x + b(1 - x)x + g(x \cdot x). \end{aligned}$$

The Mathematica command to find the solution of the game for player I is

```
Maximize[{v, v <= 0.24 x1 + 0.9 x2 + 0.9 x3,
  v <= 0.6 x1 + 0.36 x2 + 0.8 x3, v <= 0.6 x1 + 0.7 x2,
  x1 + x2 + x3 == 1, x1 >= 0, x2 >= 0, x3 >= 0}, {x1, x2, x3}]
```

This will give us that the value of the game is $v(A) = 0.549$, and the optimal strategy for player I is $X^* = (0.532, 0.329, 0.140)$. By a similar analysis of player II's linear programming problem, we get $Y^* = (0.141, 0.527, 0.331)$. So player I should fire at 10 paces more than half the time, even though they have the same accuracy functions, and a miss is certain death. Player II should fire at 10 paces only about 14% of the time.

Now let's look at an example in military science.

Example 1.25 Colonel Blotto Games: This is a simplified form of a military game in which the leaders must decide how many regiments to send to attack or defend two or more targets. It is an optimal allocation of forces game. In one formulation from reference [20], suppose that there are two opponents (players), which we call Red and Blue. Blue controls four regiments, and Red controls three. There are two targets of interest, say, A and B. The rules of the game say that the player who sends the most regiments to a target will win one point for the win and one point for every regiment captured at that target. A tie, in which Red and Blue send the same number of regiments to a target, gives a zero payoff. The possible strategies for each player consist of the number of regiments to send to A and the number of regiments to send to B, and so they are pairs of numbers. The payoff matrix to Blue is

Blue/Red	$(3,0)$	$(0,3)$	$(2,1)$	$(1,2)$
$(4,0)$	4	0	2	1
$(0,4)$	0	4	1	2
$(3,1)$	1	−1	3	0
$(1,3)$	−1	1	0	3
$(2,2)$	−2	−2	2	2

For example, if Blue plays $(3,1)$ against Red's play of $(2,1)$, then Blue sends three regiments to A while Red sends two. So Blue will win A, which gives $+1$ and then capture the two Red regiments for a payoff of $+3$ for target A. But Blue sends one regiment to B and Red also sends one to B, so that is considered a tie, or standoff, with a payoff to Blue of 0. So the net payoff to Blue is $+3$.

Solving Colonel Blotto with Mathematica: With software like Mathematica available we can easily solve complicated matrix games as we show on the Colonel Blotto matrix

$$A = \begin{bmatrix} 4 & 0 & 2 & 1 \\ 0 & 4 & 1 & 2 \\ 1 & -1 & 3 & 0 \\ -1 & 1 & 0 & 3 \\ -2 & -2 & 2 & 2 \end{bmatrix}.$$

The Mathematica commands to solve this game for player II directly are

```
Y={y1,y2,y3,y4}
Minimize[{v,A.Y<=v,y1+y2+y3+y4==1,y1>=0,y2>=0,y3>=0,y4>=0}
                ,{v,y1,y2,y3,y4}]
gives output
{14/9,{v->14/9,y1->1/30,y2->7/90,y3->8/15,y4->16/45}}
```

We get $Y^* = (\frac{1}{30}, \frac{7}{90}, \frac{8}{15}, \frac{16}{45})$ and $v(A) = \frac{14}{9}$.

For player I,

```
Maximize[{v,X.A >=v,x1+x2+x3+x4+x5==1,x1>=0,x2>=0,x3>=0,x4>=0,
                x5>=0},{v,x1,x2,x3,x4,x5}]
gives output
{14/9,{v->14/9,x1->4/9,x2->4/9,x3->0,x4->0,x5->1/9}}
```

We get $X^* = \left(\frac{4}{9}, \frac{4}{9}, 0, 0, \frac{1}{9}\right)$ and $v(A) = \frac{14}{9}$.

We may also use the Mathematica function **LinearProgramming[c,m,b]** in matrix form to solve this game.

The Mathematica command Mathematica function **LinearProgramming[c,m,b]** solves the following program: Minimize $c \cdot x$ subject to $m \cdot x \geq b, x \geq 0$. To use this command, c is the vector of the coefficients of the variables in the objective. For player II, our coefficient variables are

$c = (0, 0, 0, 0, 1)$ for the objective function $z_{II} = 0y_1 + 0y_2 + 0y_3 + 0y_4 + 1v$. The matrix m is

$$m = \begin{bmatrix} 4 & 0 & 2 & 1 & -1 \\ 0 & 4 & 1 & 2 & -1 \\ 1 & -1 & 3 & 0 & -1 \\ -1 & 1 & 0 & 3 & -1 \\ -2 & -2 & 2 & 2 & -1 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix}.$$

The last column is added for the variable v. The last row is added for the constraint $y_1 + y_2 + y_3 + y_4 = 1$. Finally, the vector b becomes

$$b = \{\{0, -1\}, \{0, -1\}, \{0, -1\}, \{0, -1\}, \{0, -1\}, \{1, 0\}\}.$$

The first number, 0, is the actual constraint, and the second number makes the inequality $\leq$ instead of the default $\geq$. The pair $\{1, 0\}$ in b comes from the equality constraint $y_1 + y_2 + y_3 + y_4 = 1$ and the second zero converts the inequality to equality. So here it is:

```
m = {{4, 0, 2, 1, -1}, {0, 4, 1, 2, -1}, {1, -1, 3, 0, -1},
    {-1, 1, 0, 3, -1}, {-2, -2, 2, 2, -1}, {1, 1, 1, 1, 0}}
c = {0, 0, 0, 0, 1}
b = {{0, -1}, {0, -1}, {0, -1}, {0, -1}, {0, -1}, {1, 0}}
LinearProgramming[c, m, b]
```

Mathematica gives the output

$$\left\{y_1 = \frac{7}{90}, y_2 = \frac{1}{30}, y_3 = \frac{16}{45}, y_4 = \frac{8}{15}, v = \frac{14}{9}\right\}.$$

The solution for player I is found similarly, but we have to be careful because we have to put it into the form for the **LinearProgramming** in Mathematica, which always minimizes and the constraints in Mathematica form are $\geq$. Player I's problem is a maximization problem so the cost vector is

$$c = (0, 0, 0, 0, 0, -1)$$

corresponding to the variables $x_1, x_2, \ldots, x_5$, and $-v$. The constraint matrix is the transpose of the Blotto matrix but with a row added for the constraint $x_1 + \cdots + x_5 = 1$. It becomes

$$q = \begin{bmatrix} 4 & 0 & 1 & -1 & -2 & -1 \\ 0 & 4 & -1 & 1 & -2 & -1 \\ 2 & 1 & 3 & 0 & 2 & -1 \\ 1 & 2 & 0 & 3 & 2 & -1 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}.$$

Finally, the b vector becomes

$$b = \{\{0, 1\}, \{0, 1\}, \{0, 1\}, \{0, 1\}, \{1, 0\}\}.$$

In each pair, the second number 1 means that the inequalities are of the form $\geq$. In the last pair, the second 0 means the inequality is actually $=$. For instance, the first and last constraints are, respectively

$$4x_1 + 0x_2 + x_3 - x_4 - 2x_5 - v \geq 0 \quad \text{and} \quad x_1 + x_2 + x_3 + x_4 + x_5 = 1.$$

With this setup the Mathematica command to solve is simply

LinearProgramming[c, q, b]
with output

$$\left\{x_1 = \frac{4}{9}, x_2 = \frac{4}{9}, x_3 = 0, x_4 = 0, x_5 = \frac{1}{9}, v = \frac{14}{9}\right\}.$$

Naturally, Blue, having more regiments, will come out ahead, and a rational opponent (Red) would capitulate before the game even began. Observe also that with two equally valued targets, it is optimal for the superior force (Blue) to not divide its regiments, but for the inferior force to split its regiments, except for a small probability of doing the opposite.

1.6.3 Using Calculus

Calculus introduces one of the main topics in optimization, namely how to find minima, maxima, and saddle points of functions and we will see how it works to find completely mixed saddle points in games. For simplicity we will only consider 2×2 games because calculations get complicated for bigger games and there are better methods anyway.

Write $f(x, y) = E(X, Y)$, where $X = (x, 1-x)$, $Y = (y, 1-y)$, $0 \le x, y \le 1$. Then

$$\begin{aligned} f(x,y) &= (x, 1-x)\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} y \\ 1-y \end{bmatrix} \\ &= x[y(a_{11}-a_{21}) + (1-y)(a_{12}-a_{22})] + y(a_{21}-a_{22}) + a_{22}. \end{aligned} \tag{1.6.1}$$

By assumption there are no optimal pure strategies, and so the extreme points of f will be found inside the intervals $0 < x, y < 1$. But that means that we may take the partial derivatives of f with respect to x and y and set them equal to zero to find all the possible critical points. The function f has to look like the function depicted in Figure 1.10.

Figure 1.10 is the graph of $f(x, y) = XAY^T$ taking

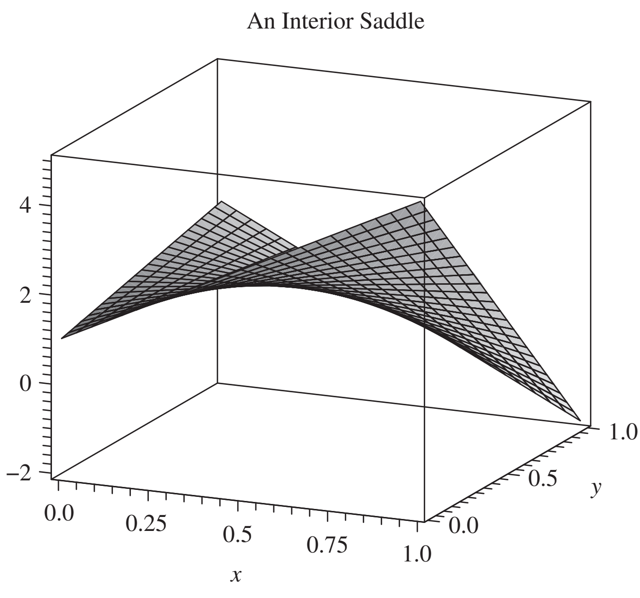

Figure 1.10 Concave in x, convex in y, saddle at $(\frac{1}{8}, \frac{1}{2})$.

$A = \begin{bmatrix} -2 & 5 \\ 2 & 1 \end{bmatrix}$. It is a concave–convex function which has a saddle point at $x = \frac{1}{8}, y = \frac{1}{2}$. You can see now why it is called a **saddle**.

Returning to our general function $f(x, y)$ in (1.6.1), take the partial derivatives and set to zero:

$$\frac{\partial f}{\partial x} = y\alpha + \beta = 0 \text{ and } \frac{\partial f}{\partial y} = x\alpha + \gamma = 0,$$

where we have set

$$\alpha = (a_{11} - a_{12} - a_{21} + a_{22}), \quad \beta = (a_{12} - a_{22}), \quad \gamma = (a_{21} - a_{22}).$$

Notice that if $\alpha = 0$, the partials are never zero (assuming $\beta, \gamma \neq 0$), and that would imply that there are pure optimal strategies (in other words, the min and max must be on the boundary). The existence of a pure saddle is ruled out by assumption. We solve where the partial derivatives are zero to get

$$x^* = -\frac{\gamma}{\alpha} = \frac{a_{22} - a_{21}}{a_{11} - a_{12} - a_{21} + a_{22}} \quad \text{and} \quad y^* = -\frac{\beta}{\alpha} = \frac{a_{22} - a_{12}}{a_{11} - a_{12} - a_{21} + a_{22}},$$

which is the same as our previous result on 2×2 games. How do we know this is a saddle and not a min or max of f? The reason is because if we take the second derivatives, we get the matrix of second partials (called the **Hessian**):

$$H = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} = \begin{bmatrix} 0 & \alpha \\ \alpha & 0 \end{bmatrix}$$

Since $\det(H) = -\alpha^2 < 0$ (unless $\alpha = 0$, which is ruled out) a theorem in elementary calculus says that an interior critical point with this condition must be a saddle.[1]

So the procedure is as follows. Look for pure strategy solutions first (calculate v^- and v^+ and see if they're equal), but if there aren't any, use the formulas to find the mixed strategy solutions.

Example 1.26 In the game with $A = \begin{bmatrix} -2 & 5 \\ 2 & 1 \end{bmatrix}$ we see that $v^- = 1 < v^+ = 2$, so there is no pure saddle for the game. If we apply the formulas, we get the mixed strategies $X^* = \left(\frac{1}{8}, \frac{7}{8}\right)$ and $Y^* = \left(\frac{1}{2}, \frac{1}{2}\right)$ and the value of the game is $v(A) = \frac{3}{2}$. Notice here that

$$A^* = \begin{bmatrix} 1 & -5 \\ -2 & -2 \end{bmatrix}$$

and $\det(A) = -12$. The matrix formula for player I gives

$$X^* = \frac{(1\ 1)A^*}{(1\ 1)A^*(1\ 1)^T} = \left(\frac{1}{8}, \frac{7}{8}\right).$$

1.6.4 Symmetric Games

Symmetric games are important classes of two-person games in which the players can use the exact same set of strategies and any payoff that player I can obtain using strategy X can be obtained by player II using the same strategy $Y = X$. The two players can switch roles, and the game doesn't

1 The calculus definition of a saddle point of a function $f(x, y)$ is a point so that in every neighborhood of the point there are x and y values that make f bigger and smaller than f at the candidate saddle point.

change. Such games can be quickly identified by the rule that $A = -A^T$. Any matrix satisfying this is said to be **skew symmetric**. If we want the roles of the players to be symmetric, then we need the matrix to be skew symmetric.

Why is skew symmetry the correct thing? Well, if A is the payoff matrix to player I, then the entries represent the payoffs to player I and the negative of the entries, or $-A$ represent the payoffs to player II. So player II wants to maximize the column entries in $-A$. This means that from player II's perspective, the game matrix must be $(-A)^T$ because it is always the row player by convention who is the maximizer; that is, A is the payoff matrix to player I and $-A^T$ is the payoff to player II. If we want the payoffs to player II to be the same as the payoffs to player I, then we must have the same game matrices for each player and so $A = -A^T$. If this is the case, the matrix must be square, $a_{ij} = -a_{ji}$, and the diagonal elements of A, namely, a_{ii}, must be 0. We can say more. In what follows it is helpful to keep in mind that for any appropriate size matrices $(AB)^T = B^T A^T$. Furthermore, since XAY^T is a scalar $(XAY^T)^T = XAY^T = YA^T X^T$.

Since the players may change roles without affecting the game, the following theorem seems reasonable.

Theorem 1.6.1 *For any skew symmetric game $v(A) = 0$ and if X^* is optimal for player I, then it is also optimal for player II. In particular, there are saddle points where the players are using the same strategy. These are called* **symmetric saddle points.**

Proof: Suppose X and Y are strategies. Then

$$E(X,Y) = XAY^T = X(-A^T)Y^T = -YAX^T = -E(Y,X).$$

If (X^*, Y^*) is a saddle point $v(A) = E(X^*, Y^*) = -E(Y^*, X^*)$ and for all X, Y,

$$E(X,Y^*) \le E(X^*,Y^*) \le E(X^*,Y) \implies -E(Y^*,X) \le -E(Y^*,X^*) \le -E(Y,X^*)$$

so that $E(Y^*,X) \ge E(Y^*,X^*) \ge E(Y,X^*)$. This says $v(A) = E(Y^*,X^*)$ and (Y^*,X^*) is also a saddle point. We also know $-E(Y^*,X^*) = v(A) = E(Y^*,X^*)$ which makes $v(A) = 0$. ■

Example 1.27 A ubiquitous game is rock–paper–scissors. Each player chooses one of rock, paper, or scissors. The rules are Rock beats Scissors; Scissors beats Paper; Paper beats Rock. We'll use the following payoff matrix:

I/II	Rock	Paper	Scissors
Rock	0	$-a$	b
Paper	a	0	$-c$
Scissors	$-b$	c	0

Assume that $a, b, c > 0$. Then $v^- = \max\{-a, -c, -b\} < 0$ and $v^+ = \min\{a, c, b\} > 0$ so this game does not have a pure saddle point. Since the matrix is skew symmetric we know that $v(A) = 0$ and $X^* = Y^*$ is a saddle point, so we need only find $X^* = (x_1, x_2, x_3)$. We may use several methods to solve this and we will use the results in Remark 1.4.9. We have

$$x_2 a - x_3 b \ge 0, \quad -ax_1 + cx_3 \ge 0, \quad x_1 b - x_2 c \ge 0 \implies x_2 \frac{a}{b} \ge x_3 \ge \frac{a}{c}\frac{c}{b} x_2.$$

Therefore, we have equality throughout, and

$$x_3\left(\frac{c}{a} + \frac{b}{a} + 1\right) = 1 \implies x_3 = \frac{a}{a+b+c}, \quad x_1 = \frac{c}{a+b+c}, \quad x_2 = \frac{b}{a+b+c}.$$

In the standard game $a = b = c = 1$ and the optimal strategy for each player is $X^* = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, which is what you would expect when the payoffs are all the same. But if the payoffs are not the same, say $a = 1, b = 2, c = 3$ then the optimal strategy for each player is $X^* = (\frac{3}{6}, \frac{2}{6}, \frac{1}{6})$.

Example 1.28 Alexander Hamilton was challenged to a duel with pistols by Aaron Burr after Hamilton wrote a defamatory article about Burr in New York. We will analyze one version of such a duel by the two players as a game with the object of trying to find the optimal point at which to shoot. First, here are the rules.

Each pistol has exactly one bullet. They will face each other starting at 10 paces apart and walk toward each other, each deciding when to shoot. In a silent duel a player does not know whether the opponent has taken the shot. In a noisy duel, the players know when a shot is taken. This is important because if a player shoots and misses, he is certain to be shot by the person he missed. The Hamilton–Burr duel will be considered as a silent duel because it is more interesting. We leave as an exercise the problem of the noisy duel.

We assume that each player's accuracy will increase the closer the players are. In a simplified version, suppose that they can choose to fire at 10 paces, 6 paces, or 2 paces. Suppose also that the probability that a shot hits and kills the opponent is 0.2 at 10 paces, 0.4 at 6 paces, and 1.0 at 2 paces. An opponent who is hit is assumed killed.

If we look at this as a zero sum two-person game, the player strategies consist of the pace distance at which to take the shot. The row player is Burr (B), and the column player is Hamilton (H). Incidentally, it is worth pointing out that the game analysis should be done before the duel is actually carried out.

We assume that the payoff to both players is +1 if they kill their opponent, −1 if they are killed, and 0 if they both survive.

So here is the matrix setup for this game.

B/H	10	6	2
10	0	−0.12	−0.6
6	0.12	0	−0.2
2	0.6	0.2	0

To see where these numbers come from, let's consider the pure strategy $(6, 10)$, so Burr waits until 6 to shoot (assuming that he survived the shot by Hamilton at 10) and Hamilton chooses to shoot at 10 paces. Then the expected payoff to Burr is

$$\begin{aligned}&(+1) \cdot \textit{Prob}(\text{H misses at 10}) \cdot \textit{Prob}(\text{Kill H at 6})\\ &\qquad + (-1) \cdot \textit{Prob}(\text{Killed by H at 10}) = 0.8 \cdot 0.4 - 0.2 = 0.12.\end{aligned}$$

The rest of the entries are derived in the same way.

This is a symmetric game with skew symmetric matrix so the value is zero and the optimal strategies are the same for both Burr and Hamilton, as we would expect since they have the same accuracy functions. In this example, there is a pure saddle at position $(3, 3)$ in the matrix, so that $X^* = (0, 0, 1)$ and $Y^* = (0, 0, 1)$. Both players should wait until the probability of a kill is certain.

To make this game a little more interesting, suppose that the players will be penalized if they wait until 2 paces to shoot. In this case we may use the matrix

B/H	10	6	2
10	0	−0.12	1
6	0.12	0	−0.2
2	−1	0.2	0

This is still a symmetric game with skew symmetric matrix, so the value is still zero and the optimal strategies are the same for both Burr and Hamilton. To find the optimal strategy for Burr, we can remember the formulas or, even better, the procedure. So here is what we get knowing that $E(X^*, j) \geq 0, j = 1, 2, 3$:

$$E(X^*, 1) = 0.12\, x_2 - 1 \cdot x_3 \geq 0,$$

$$E(X^*, 2) = -0.12\, x_1 + 0.2\, x_3 \geq 0 \quad \text{and} \quad E(X^*, 3) = x_1 - 0.2\, x_2 \geq 0.$$

These give us

$$x_2 \geq \frac{x_3}{0.12}, \quad \frac{x_3}{0.12} \geq \frac{x_1}{0.2}, \quad \text{and} \quad \frac{x_1}{0.2} \geq x_2,$$

which means equality all the way. Consequently

$$x_1 = \frac{0.2}{0.12 + 1 + 0.2} = \frac{0.2}{1.32}, \quad x_2 = \frac{1}{1.32}, \quad x_3 = \frac{0.12}{1.32}$$

or, $x_1 = 0.15, x_2 = 0.76, x_3 = 0.09$, so each player will shoot, with probability 0.76 at 6 paces.

In the real duel, that took place on July 11, 1804, Alexander Hamilton, who was the first US Secretary of the Treasury and widely considered to be a future president and a genius, was shot by Aaron Burr, who was Thomas Jefferson's vice president of the United States and who also wanted to be president. In fact, Hamilton took the first shot, shattering the branch above Burr's head. Witnesses report that Burr took his shot some 3 or 4 seconds after Hamilton. Whether or not Hamilton deliberately missed his shot is disputed to this day. Nevertheless, Burr's shot hit Hamilton in the torso and lodged in his spine, paralyzing him. Hamilton died of his wounds the next day. Aaron Burr was charged with murder but was later either acquitted or the charge was dropped (dueling was in the process of being outlawed). The duel was the end of the ambitious Burr's political career, and he died an ignominious death in exile. Interestingly, both Burr and Hamilton had been involved in numerous duels in the past. In a tragic historical twist, Alexander Hamilton's son, Phillip, was earlier killed in a duel on November 23, 1801.

Problems

1.44 Consider the game with matrix

$$\begin{bmatrix} 3 & -2 & 4 & 7 \\ -2 & 8 & 4 & 0 \end{bmatrix}.$$

(a) Solve the game using the graphical method.

(b) Find the best response for player I to the strategy $Y = \left(\frac{1}{4}, \frac{1}{2}, \frac{1}{8}, \frac{1}{8}\right)$.

(c) What is II's best response to I's best response?

1.45 Find the matrix for a noisy Hamilton–Burr duel and solve the game.

1.46 Each player displays either one or two fingers and simultaneously guesses how many fingers the opposing player will show. If both players guess correctly or both incorrectly, the game is a draw. If only one guesses correctly, that player wins an amount equal to the total number of fingers shown by both players. Each pure strategy has two components: the number of fingers to show and the number of fingers to guess. Find the game matrix and the optimal strategies.

1.47 This exercise shows that symmetric games are more general than they seem at first and in fact this is the main reason they are important. Assuming that $A_{n\times m}$ is **any** payoff matrix with $value(A) > 0$, define the matrix B that will be of size $(n+m+1)\times(n+m+1)$, by

$$B = \begin{bmatrix} 0 & A & -\vec{1} \\ -A^T & 0 & \vec{1} \\ \vec{1} & -\vec{1} & 0 \end{bmatrix}.$$

The notation $\vec{1}$, for example in the third row and first column, is the $1\times n$ matrix consisting of all 1s. B is a skew symmetric matrix and it can be shown that if $P = (p_1,\dots,p_n,q_1,\dots,q_m,\gamma)$ is an optimal strategy for matrix B, then, setting

$$b = \sum_{i=1}^{n} p_i = \sum_{j=1}^{m} q_j > 0,\quad x_i = \frac{p_i}{b},\quad y_j = \frac{q_j}{b},$$

we have $X = (x_1,\dots,x_n)$, $Y = (y_1,\dots,y_m)$ as a saddle point for the game with matrix A. In addition, $value(A) = \frac{\gamma}{b}$. The converse is also true. Verify all these points with the matrix

$$A = \begin{bmatrix} 5 & 2 & 6 \\ 1 & \frac{7}{2} & 2 \end{bmatrix}.$$

1.48 Following the same procedure as that for player I, look at $E(i,Y), i = 1,2.$ with $Y = (y, 1-y)$. Graph the lines $E(1,Y) = y + 4(1-y)$ and $E(2,Y) = 3y + 2(1-y), 0 \le y \le 1$. Now, how does player II analyze the graph to find Y^*?

1.49 Find the value and optimal X^* for the games with matrices

$$\textbf{(a)} \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix} \qquad \textbf{(b)} \begin{bmatrix} 3 & 1 \\ 5 & 7 \end{bmatrix}$$

What, if anything, goes wrong with (b) if you use the graphical method?

1.50 Solve by the linear programming method the game with the matrix

$$A = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix}$$

1.51 Curly has two safes, one at home and one at the office. The safe at home is a piece of cake to crack and any thief can get into it. The safe at the office is hard to crack and a thief has only a 15% chance of doing it. Curly has to decide where to place his gold bar (worth 1). On the other hand, if the thief hits the wrong place he gets caught (worth −1 to the thief and +1 to Curly). Formulate this as a two person zero sum matrix game and solve it using the graphical method.

1.52 Let z be an unknown number and consider the matrices

$$A = \begin{bmatrix} 0 & z \\ 1 & 2 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & 1 \\ z & 0 \end{bmatrix}$$

(a) Find $v(A)$ and $v(B)$ for any z.

(b) Now consider the game with matrix $A + B$. Find a value of z so that $v(A + B) < v(A) + v(B)$ and a value of z so that $v(A + B) > v(A) + v(B)$. Find the values of $A + B$ using the graphical method. This problem shows that the value is not a linear function of the matrix.

1.53 Suppose that we have the game matrix

$$A = \begin{bmatrix} 13 & 29 & 8 \\ 18 & 22 & 31 \\ 23 & 22 & 19 \end{bmatrix}.$$

Why can this be reduced to $B = \begin{bmatrix} 18 & 31 \\ 23 & 19 \end{bmatrix}$? Now solve the game graphically.

1.54 Two brothers, Curly and Shemp, inherit a car worth 8000 dollars. Since only one of them can actually have the car, they agree they will present sealed bids to buy the car from the other brother. The brother that puts in the highest sealed bid gets the car. They must bid in 1000 dollar units. If the bids happen to be the same, then they flip a coin to determine ownership and no money changes hands. Curly can bid only up to 5000 while Shemp can bid up to 8000.
Find the payoff matrix with Curly as the row player and the payoffs the expected net gain (since the car is worth 8000). Find v^-, v^+ and use dominance to solve the game.

1.55 In the 2×2 Nim game we saw that $v^+ = v^- = -1$. Reduce the game matrix using dominance.

1.56 Consider the matrix game

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}.$$

(a) Find $v(A)$ and the optimal strategies.
(b) Show that $X^* = (\frac{1}{2}, \frac{1}{2})$, $Y^* = (1, 0)$ is not a saddle point for the game even though it does happen that $E(X^*, Y^*) = v(A)$.

1.57 Use the methods of this section to solve the games

$$(\mathbf{a}) \begin{bmatrix} 4 & -3 \\ -9 & 6 \end{bmatrix}, \quad (\mathbf{b}) \begin{bmatrix} 4 & 9 \\ 6 & 2 \end{bmatrix}, \quad (\mathbf{c}) \begin{bmatrix} -3 & -4 \\ -7 & 2 \end{bmatrix}.$$

1.58 Use (convex) dominance and the graphical method to solve the game with matrix
$$A = \begin{bmatrix} 0 & 5 \\ 1 & 4 \\ 3 & 0 \\ 2 & 2 \end{bmatrix}.$$

1.59 The third column of the matrix

$$A = \begin{bmatrix} 0 & 8 & 5 \\ 8 & 4 & 6 \\ 12 & -4 & 3 \end{bmatrix}$$

is dominated by a convex combination. Reduce the matrix and solve the game.

1.60 Four army divisions attack a town along two possible roads. The town has three divisions defending it. A defending division is dug in and hence equivalent to two attacking divisions. Even one division attacking an undefended road captures the town. Each commander must decide how many divisions to attack or defend each road. If the attacking commander captures a road to the town, the town falls. Score 1 to the attacker if the town falls and −1 if it doesn't.

(a) Find the payoff matrix with payoff the attacker's probability of winning the town.
(b) Find the value of the game and the optimal saddle point.

1.61 Consider the matrix game $A = \begin{bmatrix} a_4 & a_3 & a_3 \\ a_1 & a_6 & a_5 \\ a_2 & a_4 & a_3 \end{bmatrix}$, where $a_1 < a_2 < \cdots < a_5 < a_6$. Use dominance to solve the game.

1.62 Aggie and Baggie are fighting a duel each with one lemon meringue pie starting at 20 paces. They can each choose to throw the pie at either 20 paces, 10 paces, or 0 paces. The probability either player hits the other at 20 paces is $\frac{1}{3}$; at 10 paces it is $\frac{3}{4}$ and at 0 paces it is 1. If they both hit or both miss at the same number of paces, the game is a draw. If a player gets a pie in the face, the score is −1, the player throwing the pie gets +1.
Set this up as a matrix game and solve it.

1.63 Consider the game with matrix

$$A = \begin{bmatrix} -2 & 3 & 5 & -2 \\ 3 & -4 & 1 & -6 \\ -5 & 3 & 2 & -1 \\ -1 & -3 & 2 & 2 \end{bmatrix}.$$

Someone claims that the strategies $X^* = \left(\frac{1}{9}, 0, \frac{8}{9}, 0\right)$ and $Y^* = \left(0, \frac{7}{9}, \frac{2}{9}, 0\right)$ are optimal.

(a) Is that correct? Why or why not?
(b) If $X^* = \left(\frac{13}{33}, \frac{5}{33}, 0, \frac{15}{33}\right)$ is optimal and $v(A) = -\frac{26}{33}$, find Y^*.

1.64 In the baseball game Problem 1.11 it turns out that an optimal strategy for player I, the batter, is given by $X^* = (x_1, x_2, x_3) = \left(\frac{2}{7}, 0, \frac{5}{7}\right)$ and the value of the game is $v = \frac{2}{7}$. It is amazing that the batter should never expect a curveball with these payoffs under this optimal strategy. What is the pitcher's optimal strategy Y^*?

1.65 In a football game we use the matrix $A = \begin{bmatrix} 3 & 6 \\ 8 & 0 \end{bmatrix}$. The offense is the row player. The first row is Run, the second is Pass. The first column is Defend against Run, the second is Defend against the Pass.

(a) Use the graphical method to solve this game.
(b) Now suppose the offense gets a better quarterback so the matrix becomes $A = \begin{bmatrix} 3 & 6 \\ 12 & 0 \end{bmatrix}$.
What happens?

1.66 Two players Reinhard and Carla play a number game. Carla writes down a number either 1, 2 or 3. Reinhard chooses a number (again 1, 2 or 3) and guesses that Carla has written down that number. If Reinhard guesses right he wins \$1 from Carla; if he guesses wrong, Carla tells him if his number is higher or lower and he gets to guess again. If he is right, no money changes hands but if he guesses wrong he pays Carla \$1.

(a) Find the game matrix with Reinhard as the row player and find the upper and lower values. A strategy for Reinhard is of the form

[first guess, guess if low, guess if high]

(b) Find the value of the game by first noticing that Carla's strategy 1 and 3 are symmetric as are $[1, -, 2]$, $[3, 2, -]$, and $[1, -, 3]$, $[3, 1-]$ for Reinhard. Then modify the graphical method slightly to solve.

1.67 We have an infinite sequence of numbers $0 < a_1 \le a_2 \le a_3 \le \cdots$. Each of 2 players chooses an integer independent of the other player. If they both happen to choose the same number k, then player I receives a_k dollars from player II. Otherwise, no money changes hand. Assume that $\sum_{k=1}^{\infty} \frac{1}{a_k} < \infty$.

(a) Find the game matrix (it will be infinite). Find v^+, v^-.

(b) Find the value of the game if mixed strategies are allowed and find the saddle point in mixed strategies. Use Theorem 1.4.6.

(c) Assume next that $\sum_{k=1}^{\infty} \frac{1}{a_k} = \infty$. Show that the value of the game is $v = 0$ and every mixed strategy for player I is optimal but there is no optimal strategy for player II.

1.68 **Calculus to Find the Interior Saddle Points:** Consider

$$A = \begin{bmatrix} 4 & -3 & -2 \\ -3 & 4 & -2 \\ 0 & 0 & 1 \end{bmatrix}.$$

A strategy for each player is of the form $X = (x_1, x_2, 1 - x_1 - x_2)$, $Y = (y_1, y_2, 1 - y_1 - y_2)$, so we consider the function $f(x_1, x_2, y_1, y_2) = XAY^T$. Now solve the system of equations

$$f_{x_1} = f_{x_2} = f_{y_1} = f_{y_2} = 0$$

to get X^* and Y^*. If these are legitimate completely mixed strategies, then you can verify that they are optimal and then find $v(A)$. Carry out these calculations for A and verify that they give optimal strategies.

1.69 Show that for any strategy $X = (x_1, \dots, x_m) \in S_m$ and any numbers $b_1, \dots, b_m$, it must be that

$$\max_{X \in S_m} \sum_{i=1}^{m} x_i b_i = \max_{1 \le i \le m} b_i \quad \text{and} \quad \min_{X \in S_m} \sum_{i=1}^{m} x_i b_i = \min_{1 \le i \le m} b_i.$$

1.70 The properties of optimal strategies (Remark 1.4.9) show that $X^* \in S_n$ and $Y^* \in S_m$ are optimal if and only if $\min_j E(X^*, j) = \max_i E(i, Y^*)$. The common value will be the value of the game. Verify this.

1.71 Show that if (X^*, Y^*) and (X^0, Y^0) are both saddle points for the game with matrix A, then so is (X^*, Y^0) and (X^0, Y^*). In fact, show that (X_λ, Y_β) where $X_\lambda = \lambda X^* + (1 - \lambda) X^0$, $Y_\beta = \beta Y^* + (1 - \beta) Y^0$ and λ, β any numbers in $[0, 1]$, is also a saddle point. Thus if there are two saddle points, there are an infinite number.

1.7 Existence of Saddle Points: The Von Neumann Minimax Theorem

So far we have understood what saddle points are, why there are a reasonable definition of optimal strategies, and a number of ways to look for them. We have also asserted that, if we allow mixed strategies, two player matrix games will always have saddle points if we allow mixed strategies. How do we know that for a fact? It will follow from the celebrated von Neumann inimax Theorem, due to John von Neumann.

Remark Before stating the theorem, it is helpful to realize that not every game will have a saddle point. For instance, here is a silly game that does not: two players each pick a number. Whoever picks the highest number wins $1. There cannot possibly be a saddle point because, as soon as you pick a number, you will wish you picked a bigger one. There will have to be some assumptions on games to know that saddle points exist.

1.7.1 Statement of the Minimax Theorem

We start by considering general functions of two variables $f = f(x, y)$, and give the definition of a saddle point for an arbitrary of function f.

Definition 1.7.1 *Let C and D be sets. A function $f : C \times D \to \mathbb{R}$ has at least one saddle point (x^*, y^*) with $x^* \in C$ and $y^* \in D$ if*

$$f(x, y^*) \le f(x^*, y^*) \le f(x^*, y) \quad \text{for all} \quad x \in C, y \in D.$$

We can think of this function as a game: Player I gets to choose a point $x \in C$ and player II gets to choose a point in $y \in D$. The payoff is $f(x, y)$ to Player I. This type of game is called a **continuous game**. We can define upper and lower value by

$$v^+ = \min_{y \in D} \max_{x \in C} f(x, y), \quad \text{and} \quad v^- = \max_{x \in C} \min_{y \in D} f(x, y).$$

You can check as before that $v^- \le v^+$. If $v^+ = v^-$ we say, as usual, that the **game has a value** $v = v^+ = v^-$. The **von Neumann Minimax Theorem** gives conditions on $f, C,$ and D so that the associated game has a value $v = v^+ = v^-$. It will be used to determine what we need to do in matrix games in order to get a value.

Theorem 1.7.2 *Let $f : C \times D \to \mathbb{R}$ be a continuous function. Let $C \subset \mathbb{R}^n$ and $D \subset \mathbb{R}^m$ be convex, closed, and bounded.*[1] *Suppose that as a function of $x, f(x, y)$ is concave and as a function of $y, f(x, y)$ is convex. Then*

$$v^+ = \min_{y \in D} \max_{x \in C} f(x, y) = \max_{x \in C} \min_{y \in D} f(x, y) = v^-.$$

Von Neumann's theorem tells us what we need in order to guarantee that our game has a value. It is critical that we are dealing with a concave–convex function, and that the strategy sets be convex and bounded. The bounded assumption eliminates the silly game of choosing the largest number.

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1 Convex means the line segment connecting any two points in the set is also in the set. Closed means the set contains all its limit points, and Bounded means the set can be jammed inside a large enough ball.

If you're interested in seeing a proof of this major theorem refer to the appendix Section 1.9 of this chapter. There is also a really cool proof of the theorem for matrix games that uses evolutionary game theory at the end of the book.

One observation: Recall the common calculus test for twice differentiable functions of one variable. If $g = g(x)$ is a function of one variable and has at least two derivatives, then g is convex if $g'' \geq 0$ and g is concave if $g'' \leq 0$.

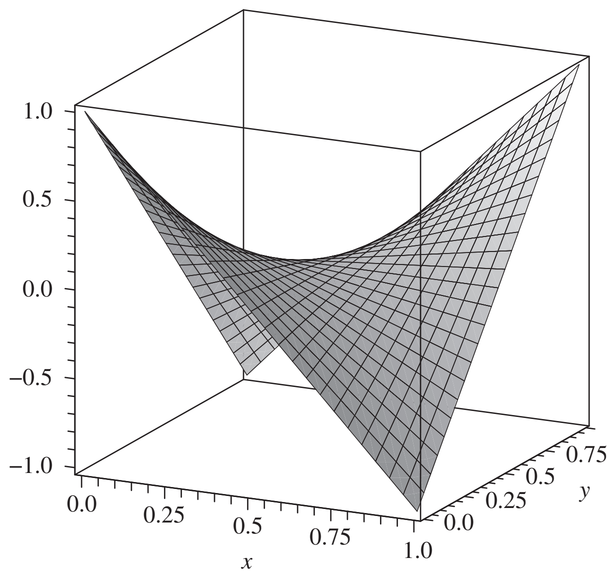
Example 1.29 For an example of the use of von Neumann's theorem, suppose we look at

$$f(x, y) = 4xy - 2x - 2y + 1 \quad \text{on} \quad 0 \leq x, y \leq 1.$$

This function has $f_{xx} = 0 \geq 0, f_{yy} = 0 \leq 0$, so it is convex in y for each x and concave in x for each y. Since $(x, y) \in [0, 1] \times [0, 1]$, and the square is closed and bounded, von Neumann's theorem guarantees the existence of a saddle point for this function. To find it, solve $f_x = f_y = 0$ to get $x = y = \frac{1}{2}$. The Hessian for f, which is the matrix of second partial derivatives, is given by

$$H(f, [x, y]) = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} = \begin{bmatrix} 0 & 4 \\ 4 & 0 \end{bmatrix}.$$

Since $\det(H) = -16 < 0$ we are guaranteed by elementary calculus that $(x = \frac{1}{2}, y = \frac{1}{2})$ is an interior saddle for f. Here is a picture of f:



Incidentally, another way to write our example function would be

$$f(x, y) = (x, 1 - x)A(y, 1 - y)^T, \quad \text{where} \quad A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.$$

Obviously, not all functions will have saddle points. For instance, you will show in an exercise that $g(x, y) = (x - y)^2$ is not concave–convex and in fact does not have a saddle point in $[0, 1] \times [0, 1]$.

1.7.2 Von Neumann's Theorem Guarantees Matrix Games Have Saddle Points

Does a game with matrix A have a saddle point in mixed strategies? von Neumann's minimax theorem tells us the answer is **Yes**.

The minimax theorem basically says that if we have a continuous function $f(x, y)$ which is concave in x and convex in y (called a saddle function, and the variables x and y are members of convex sets, then $\min_y \max_x f(x, y) = \max_y \min_y f(x, y)$. To apply this to a matrix game all we need to do is define the function

$$f(X, Y) \equiv E(X, Y) = XAY^T$$

and the sets S_n for X, and S_m for Y. For any $n \times m$ matrix A, this function is concave in X and convex in Y. Actually, it is even **linear** in each variable when the other variable is fixed. Recall that any linear function is both concave and convex, so our function f is concave–convex and certainly continuous. The second requirement of von Neumann's theorem is that the sets S_n and S_m be convex sets. This is very easy to check and we leave that as an exercise for the reader. These sets S_n and S_m are also closed and bounded. Consequently, we may apply the general minimax theorem to conclude the following.

Theorem 1.7.3 *For any $n \times m$ matrix A, we have*

$$v^+ = \min_{Y \in S_m} \max_{X \in S_n} XAY^T = \max_{X \in S_n} \min_{Y \in S_m} XAY^T = v^-.$$

The common value is denoted $v(A)$, or value(A), and that is the **value of the game**. *In addition, there is at least one saddle point $X^* \in S_n$, $Y^* \in S_m$ so that*

$$E(X, Y^*) \le E(X^*, Y^*) = v(A) \le E(X^*, Y), \quad \textit{forall} \quad X \in S_n, Y \in S_m.$$

Remark The theorem tells us that a saddle point for a matrix game always exists if mixed strategies are used. It does not give us a way to find it. Also, there could be more than one saddle point.

Problems

1.72 Use any method of your choice to solve the games with the following matrices.

$$(\mathbf{a}) \begin{bmatrix} 0 & 3 & 3 & 2 & 4 \\ 4 & 4 & 3 & 1 & 4 \end{bmatrix}; \quad (\mathbf{b}) \begin{bmatrix} 4 & 4 & -4 & -1 \\ -4 & -2 & 4 & 4 \\ 2 & -4 & -1 & -5 \\ -3 & 1 & 0 & -4 \end{bmatrix}; \quad (\mathbf{c}) \begin{bmatrix} 2 & -5 & 3 & 0 \\ -4 & -5 & -5 & -6 \\ 3 & -4 & -1 & -2 \\ 0 & 4 & 1 & 3 \end{bmatrix}$$

1.73 Let $f(x, y) = x^2 + y^2$, $C = D = [-1, 1]$. Find $v^+ = \min_{y \in D} \max_{x \in C} f(x, y)$ and $v^- = \max_{x \in C} \min_{y \in D} f(x, y)$.

1.74 Let $f(x, y) = y^2 - x^2$, $C = D = [-1, 1]$.
(a) Find $v^+ = \min_{y \in D} \max_{x \in C} f(x, y)$ and $v^- = \max_{x \in C} \min_{y \in D} f(x, y)$.
(b) Show that $(0, 0)$ is a pure saddle point for $f(x, y)$.

1.75 Let $f(x, y) = (x - y)^2$, $C = D = [-1, 1]$. Find $v^+ = \min_{y \in D} \max_{x \in C} f(x, y)$ and $v^- = \max_{x \in C} \min_{y \in D} f(x, y)$.

1.76 Solve this game by setting it up as a linear program:

$$\begin{bmatrix} -2 & 3 & 3 & 4 & 1 \\ 3 & -2 & -5 & 2 & 4 \\ 4 & -5 & -1 & 4 & -1 \\ 2 & -4 & 3 & 4 & -3 \end{bmatrix}.$$

In each of the following problems, set up the payoff matrices and solve the games using the linear programming method with both formulations, that is, both with and without transforming variables. You may use Mathematica or any software you want to solve these linear programs.

1.77 Consider the Submarine versus Bomber game. The board is a 3×3 grid in Figure 1.11.

1	2	3
4	5	6
7	8	9

Figure 1.11 3×3 Submarine–Bomber game.

A submarine (which occupies two squares) is trying to hide from a plane which can deliver torpedoes. The bomber can fire one torpedo at a square in the grid. If it is occupied by a part of the submarine, the sub is destroyed (score 1 for the bomber). If the bomber fires at an unoccupied square, the sub escapes (score 0 for the bomber). The bomber should be the row player.

(a) Formulate this game as a matrix and solve it. (Hint: there are 12 pure strategies for the sub and 9 for the bomber.)
(b) Using symmetry the game can be reduced to a 3×2 game. Find the reduction and then solve the resulting game.

1.78 There are two presidential candidates, John and Dick, who will choose which states they will visit to garner votes. Suppose that there are 4 states that are in play, but candidate John only has the money to visit 3 states. If a candidate visits a state, the gain (or loss) in poll numbers is indicated as the payoff in the matrix

John/Dick	State 1	State 2	State 3	State 4
State 1	1	−4	6	−2
State 2	−8	7	2	1
State 3	11	−1	3	−3

Solve the game.

1.79 Assume that we have a silent duel but the choice of a shot may be taken at 10,8,6,4, or 2 paces. The accuracy, or probability of a kill is 0.2, 0.4, 0.6, 0.8, and 1, respectively, at the paces. Set up and solve the game.

1.80 Consider the Cat vs Rat game in Problem 1.42. Suppose each player moves exactly 3 segments. The payoff to the cat is 1 if they meet and 0 otherwise. Find the matrix and solve the game.

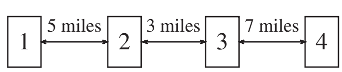
1.81 Drug runners can use three possible methods for running drugs through Florida: small plane, main highway, or back roads. The cops know this, but they can only patrol one of these methods at a time. The street value of the drugs is \$100,000 if the drugs reach New York using the main highway. If they use the back roads, they have to use smaller-capacity cars so the street value drops to \$80,000. If they fly, the street value is \$150,000. If they get stopped, the drugs and the vehicles are impounded, they get fined, and they go to prison. This represents a loss to the drug kingpins of \$90,000, by highway, \$70,000 by back roads, and \$130,000 if by small plane. On the main highway they have a 40% chance of getting caught if the highway is being patrolled, a 30% chance on the back roads, and a 60% chance if they fly a small plane (all assuming that the cops are patrolling that method). Set this up as a zero sum game assuming that the cops want to minimize the drug kingpins gains, and solve the game to find the best strategies the cops and drug lords should use.

1.82 LUC (Loyola University Chicago) is about to play UIC (University of Illinois Chicago) for the state tennis championship. LUC has two players: A and B, and UIC has three players: X,Y,Z. The following facts are known about the relative abilities of the players: X always beats B; Y always beats A; A always beats Z. In any other match each player has a probability $\frac{1}{2}$ of winning. Before the matches, the coaches must decide on who plays who. Assume that each coach wants to maximize the expected number of matches won (these are singles matches and there are 2 games so each coach must pick who will play game 1 and who will play game 2. The players do not have to be different in each game).
Set this up as a matrix game. Find the value of the game and the optimal strategies.

1.83 Player II chooses a number $j \in \{1, 2, \ldots, n\}$, $n \geq 2$, while I tries to guess what it is by guessing an $i \in \{1, 2, \ldots, n\}$. If the guess is correct (i.e., $i = j$) then II pays $+1$ to player I. If $i > j$ player II pays player I the amount b^{i-j} where $0 < b < 1$ is fixed. If $i < j$, player I wins nothing.
(a) Find the game matrix.
(b) Solve the game.

1.84 Consider the asymmetric silent duel in which the two players may shoot at paces $(10, 6, 2)$ with accuracies $(0.2, 0.4, 1.0)$ for player I, but, since II is a better shot, $(0.6, 0.8, 1.0)$ for player II. Given that the payoffs are $+1$ to a survivor and -1 otherwise (but 0 if a draw), set up the matrix game and solve it.

1.85 Suppose that there are four towns connected by a highway as in the following diagram:



Assume that 15% of the total populations of the four towns are nearest to town 1, 30% nearest to town two, 20% nearest to town three, and 35% nearest to town four. There are two superstores, say, I and II, thinking about building a store to serve these four towns. If both stores are in the same town or in two different towns but with the same distance to a town, then I will get a 65% market share of business. Each store gets 90% of the business of the town in which they put the store if they are in two different towns. If store I is closer to a town than II is, then I gets 90% of the business of that town. If store I is farther than store II from a town, store I still gets 40% of that town's business, except for the town II is in. Find the payoff matrix and solve the game.

1.86 Two farmers are having a fight over a disputed six-yard-wide strip of land between their farms. Both farmers think that the strip is theirs. A lawsuit is filed and the judge orders them to submit a confidential settlement offer to settle the case fairly. The judge has decided to accept the settlement offer that concedes the most land to the other farmer. In case both farmers make no concession or they concede equal amounts, the judge will favor farmer II and award her all 6 yards. Formulate this as a constant sum game assuming that both farmers' pure strategies must be the yards that they concede: 0, 1, 2, 3, 4, 5, 6. Solve the game. What if the judge awards three yards if equal concessions?

1.87 Two football teams, B and P, meet for the Superbowl. Each offense can play run right (RR), run left (RL), short pass (SP), deep pass (DP), or screen pass (ScP). Each defense can choose to blitz (BL), defend a short pass (DSP), or defend a long pass (DLP), or defend a run (DR). Suppose that team B does a statistical analysis and determines the following payoffs when they are on defense:

B/P	RR	RL	SP	DP	ScP
BL	−5	−7	−7	5	4
DSP	−6	−5	8	6	3
DLP	−2	−3	−8	6	−5
DR	3	3	−5	−15	−7

A negative payoff represents the number of yards gained by the offense, so a positive number is the number of yards lost by the offense on that play of the game. Solve this game and find the optimal mix of plays by the defense and offense. (**Caution**: This is a matrix in which you might want to add a constant to ensure $v(A) > 0$. Then subtract the constant at the end to get the real value. You do not need to do that with the strategies.)

1.88 Let $a > 0$. Use the graphical method to solve the game in which player II has an infinite number of strategies with matrix

$$\begin{bmatrix} a & 2a & \frac{1}{2} & 2a & \frac{1}{4} & 2a & \frac{1}{6} & \cdots \\ a & 1 & 2a & \frac{1}{3} & 2a & \frac{1}{5} & 2a & \cdots \end{bmatrix}.$$

Pick a value for a and solve the game with a finite number of columns to see what's going on.

1.89 A Latin square game is a square game in which the matrix A is a Latin square. A Latin square of size n has every integer from 1 to n in each row and column.

(a) Solve the game of size 5

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 1 & 5 & 3 \\ 3 & 5 & 4 & 2 & 1 \\ 4 & 1 & 5 & 3 & 2 \\ 5 & 3 & 2 & 1 & 4 \end{bmatrix}.$$

(b) Prove that a Latin square game of size n has $v(A) = \frac{(n+1)}{2}$. You may need the fact that $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$.

1.90 Consider the game with matrix $A = \begin{bmatrix} -3 & -3 & 2 \\ -1 & 3 & -2 \\ 3 & -1 & -2 \\ 2 & 2 & 3 \end{bmatrix}$.

(a) Show that

$$X^* = \left(\frac{3}{8}\left(1 + \frac{\lambda}{3}\right), \frac{5}{16}(1-\lambda), \frac{5}{16}(1-\lambda), \frac{\lambda}{2}\right), \quad Y^* = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{2}\right)$$

is not a saddle point of the game if $0 \le \lambda \le 1$.

(b) Solve the game.

1.91 Two players, Curly and Shemp, are betting on the toss of a fair coin. Shemp tosses the coin, hiding the result from Curly. Shemp looks at the coin. If the coin is heads, Shemp says that he has heads and demands \$1 from Curly. If the coin is tails, then Shemp may tell the truth and pay Curly \$1, or he may lie and say that he got a head and demands \$1 from Curly. Curly can challenge Shemp whenever Shemp demands \$1 to see whether Shemp is telling the truth, but it will cost him \$2 if it turns out that Shemp was telling the truth. If Curly challenges the call and it turns out that Shemp was lying, then Shemp must pay Curly \$2. Find the matrix and solve the game.

1.92 Wiley Coyote[1] is waiting to nab Speedy who must emerge from a tunnel with three exits, A, B, and C. B and C are relatively close together, but far away from A. Wiley can lie in wait for Speedy near an exit, but then he will catch Speedy only if Speedy uses this exit. But Wiley has two other options. He can wait between B and C, but then if Speedy comes out of A he escapes while if he comes out of B or C, Wiley can only get him with probability p. Wiley's last option is to wait at a spot overlooking all three exits, but then he catches Speedy with probability q no matter which exit Speedy uses.

(a) Find the matrix for this game for arbitrary p, q.

(b) Show that using convex dominance, Wiley's options of waiting directly at an exit dominate one or the other or both of his other two options if $p < \frac{1}{2}$ or if $q < \frac{1}{3}$.

(c) Let $p = 3/4, q = 2/5$. Solve the game.

1.93 Left and Right bet either \$1 or \$2. If the total amount wagered is even, then Left takes the entire pot; if it is odd, then Right takes the pot.

(a) Set up the game matrix and analyze this game.

(b) Suppose now that the amount bet is not necessarily 1 or 2: Left and Right choose their bet from a set of positive integers. Suppose the sets of options are the following

1. Left and Right can bet any number in $L = R = \{1, 2, 3, 4, 5, 6\}$;
2. Left and Right can bet any number in $L = R = \{2, 4, 6, 8, 9, 13\}$.

Analyze each of these cases.

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1 Versions of this and the remaining problems appeared on exams at the London School of Economics.

(c) Suppose Left may bet any amount in $L = \{1, 2, 31, 32\}$ and Right any amount in $R = \{2, 5, 16, 17\}$. The payoffs are as follows:
1. If Left $\in \{1, 2\}$ and Right $\in \{2, 5\}$ and Left + Right even $\implies$ Right pays Left the sum of the amounts each bet.
2. If Left $\in \{1, 2\}$ and Right $\in \{2, 5\}$ and Left + Right odd $\implies$ Left pays Right the sum of the amounts each bet.
3. If Left $\in \{31, 32\}$ **or** Right $\in \{16, 17\}$, and Left + Right odd $\implies$ Right pays Left the sum of the amounts each bet.
4. If Left $\in \{31, 32\}$ **or** Right $\in \{16, 17\}$, and Left + Right even $\implies$ Left pays Right the sum of the amounts each bet.

1.94 The evil Don Barzini has imprisoned Tessio and three of his underlings in the Corleone family (total value 4) somewhere in Brooklyn, and Clemenza and his two assistants somewhere in Queens (total value 3). Sonny sets out to rescue either Tessio or Clemenza and their associates; Barzini sets out to prevent the rescue but he doesn't know where Sonny is headed. If they set out to the same place, Barzini has an even chance of arriving first, in which case Sonny returns without rescuing anybody. If Sonny arrives first he rescues the group. The payoff to Sonny is determined as the number of Corleone family members rescued. Describe this scenario as a matrix game and determine the optimal strategies for each player.

1.95 You own two companies Uno and Due. At the end of a fiscal year Uno should have a tax bill of \$3 million, and Due should have a tax bill of \$1 million. By cooking the books you can file tax returns making it look as though you should not have to pay any tax at all for either Uno or Due, or both. Due to limited resources, the IRS (internal revenue service) can only audit one company. If a company with cooked books is audited, the fraud is revealed and all unpaid taxes will be levied plus a penalty of p% of the tax due.
(a) Set this up as a matrix game. Your payoffs will depend on p.
(b) In the case the fine is $p = 50\%$, what strategy should you adopt to minimize your expected tax bill?
(c) For what values of p is complete honesty optimal?

1.96 **A Nonsymmetric Noisy Duel**. We consider a nonsymmetric duel at which the two players may shoot at paces $(10, 6, 2)$ with accuracies $(0.2, 0.4, 1.0)$ each.
- If only player I survives, then player I receives payoff a.
- If only player II survives, player I gets payoff $b < a$. This assumes that the survival of player II is less important than the survival of player I.
- If both players survive, they each receive payoff zero.
- If neither player survives, player I receives payoff g.

We will take $a = 1, b = \frac{1}{2}, g = 0$.
(a) Find the expected payoff matrix for player I.
(b) Solve the game.

1.8 Review Problems

Problems

Give the precise definition, complete the statement, or answer True or False. If False, give the correct statement.

1.97 Complete the statement: In order for (X^*, Y^*) to be a saddle point and v to be the value, it is necessary and sufficient that $E(X^*, j)$ ____________, for all ____________ and $E(i, Y^*)$ ____________ for all ____________.

1.98 State the Von Neumann Minimax theorem for two person zero sum games with matrix A.

1.99 Suppose (X^*, Y^*) is a saddle point, $v(A)$ is the value of the game and $x_k > 0$ for the *kth* component of X^*. What is $E(k, Y^*)$? What is the most you can say if $x_k = 0$ for $E(k, Y^*)$?

1.100 Suppose A is a game matrix and Y^0 is a given strategy for player II. Define what it means for X^* to be a best response strategy to Y^0 for player I.

1.101 Suppose $A_{n\times n}$ has an inverse A^{-1}. What three assumptions do you need to be able to use the formulas

$$v := 1/J_n A^{-1} J_n^T, \quad X^* = vJ_n A^{-1}, \quad Y^* = vA^{-1} J_n^T$$

to conclude that $v = v(A)$ and (X^*, Y^*) are optimal mixed strategies?

1.102 $v(A)$ is the value and (X^*, Y^*) is a Saddle Point of the game with matrix A if and only if….

1.103 $v(A)$ is the value and (i^*, j^*) is a pure saddle point of the game with matrix A if and only if ….

1.104 For any game $v^+ \le v(A) \le v^-$.

1.105 For any matrix game $E(X, Y) = XAY^T$ and the value satisfies $v(A) = \max_X \min_Y E(X, Y) = \min_Y \max_X E(X, Y)$.

1.106 If (X^*, Y^*) is a saddle point for a matrix game, then X^* is a best response strategy to Y^* for player I, but the reverse is not necessarily true.

1.107 If Y is an optimal strategy for player II in a zero sum game and $E(i, Y) < value(A)$ for some row i, then for any optimal strategy X for player I, we must have … .

1.108 The graphical method for solving a $2 \times m$ game won't work if … .

1.109 (X^*, Y^*) is a saddle point if and only if $\min_{j=1,2,\ldots,m} E(X^*, j) = \max_{i=1,2,\ldots,n} E(i, Y^*)$.

1.110 If (Y^0, X^0) is a saddle point for the game A, then $E(Y^0, X) \geq E(Y^0, X^0) \geq E(Y, X^0)$, for all strategies X for player II and Y for player I.

1.111 If $X^* = (x_1^*, \ldots, x_n^*)$ is an optimal strategy for player I and $x_1^* = 0$, then we may drop row 1 when we look for the optimal strategy for player II.

1.112 For a game matrix $A_{n\times m}$ and given strategies $(X^* = (x_1, \ldots, x_n), Y^* = (y_1, \ldots, y_m)$, then $E(X^*, Y^*) = \sum_{i=1}^{n} x_i\, E(i, Y^*) = \sum_{j=1}^{m} y_j\, E(X^*, j)$

1.113 If $i^* = 2, j^* = 3$, is a pure saddle point in the game with matrix $A_{3\times 4}$, then using mixed strategies, $X^* =$ ___________, $Y^* =$ ___________ is a mixed saddle point for A.

1.114 For a 2×2 game with matrix $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, the value of the game is $\frac{\det(A)}{J_2 A^* J_2^T}$, where

$$A^* = \begin{bmatrix} \text{----} & \text{----} \\ \text{----} & \text{----} \end{bmatrix}$$

1.115 The invertible matrix theorem says that if A^{-1} exists, then $v(A) = \frac{1}{J_n A^{-1} J_n^T}$ is the value of the game.

1.116 If $f(x, y) = (x, 1 - x) \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} y \\ 1 - y \end{bmatrix}$, then a mixed saddle point for the game $X^* = (x^*, 1 - x^*), Y^* = (y^*, 1 - y^*), 0 < x^*, y^* < 1$, satisfies

$$\frac{\partial f}{\partial x}(x^*, y^*) = \frac{\partial f}{\partial y}(x^*, y^*) = 0.$$

1.117 If A is a skew symmetric game with $v(A) > 0$, then there is always a saddle point in which $X^* = -Y^*$.

1.118 If A is skew symmetric then for any strategy X for player I, $E(X, X) = 0$.

1.9 Appendix: A Proof of the von Neumann Minimax Theorem

Once von Neumann proved the minimax theorem many different proofs were constructed. The proof we present here uses only elementary properties of convex functions and some advanced calculus (seeDevinatz [4]). At the end of the last chapter is a newer proof using evolutionary game theory.

Theorem 1.9.1 *Let $f : C \times D \to \mathbb{R}$ be a continuous function. Let $C \subset \mathbb{R}^n$ and $D \subset \mathbb{R}^m$ be convex, closed, and bounded.*[1] *Suppose that as a function of $x, f(x, y)$ is concave and as a function of $y, f(x, y)$ is convex. Then*

$$v^+ = \min_{y\in D} \max_{x\in C} f(x, y) = \max_{x\in C} \min_{y\in D} f(x, y) = v^-.$$

1 Convex means the line segment connecting any two points in the set is also in the set. Closed means the set contains all its limit points, and Bounded means the set can be jammed inside a large enough ball.

Proof:

1. Assume first that f is *strictly* concave–convex, meaning that

$$f(\lambda x + (1-\lambda) z, y) > \lambda f(x,y) + (1-\lambda) f(z,y), \quad 0 < \lambda < 1,$$
$$f(x, \mu y + (1-\mu) w) < \mu f(x,y) + (1-\mu) f(x,w), \ 0 < \mu < 1.$$

The advantage of doing this is that for each $x \in C$ there is one and only one $y = y(x) \in D$(y depends on the choice of x) so that

$$f(x, y(x)) = \min_{y \in D} f(x,y) := g(x).$$

This defines a function g: $C \to \mathbb{R}$ that is continuous (since f is continuous on the closed bounded sets $C \times D$and thus is uniformly continuous). Furthermore, $g(x)$ is concave since

$$g(\lambda x + (1-\lambda) z) \geq \min_{y \in D} (\lambda f(x,y) + (1-\lambda) f(z,y)) \geq \lambda g(x) + (1-\lambda) g(z).$$

So, there is a point $x^* \in C$ at which g achieves its maximum:

$$g(x^*) = f(x^*, y(x^*)) = \max_{x \in C} \min_{y \in D} f(x,y).$$

2. Let $x \in C$ and $y \in D$ be arbitrary. Then, for any $0 < \lambda < 1$, we obtain

$$\begin{aligned} f(\lambda x + (1-\lambda) x^*, y) &> \lambda f(x,y) + (1-\lambda) f(x^*, y) \\ &\geq \lambda f(x,y) + (1-\lambda) f(x^*, y(x^*)) \\ &= \lambda f(x,y) + (1-\lambda) g(x^*). \end{aligned}$$

Now take $y = y(\lambda x + (1-\lambda)x^*) \in D$ to get

$$\begin{aligned} g(x^*) &\geq f(\lambda x + (1-\lambda) x^*, y(\lambda x + (1-\lambda) x^*)) = g(\lambda x + (1-\lambda) x^*) \\ &\geq g(x^*)(1-\lambda) + \lambda f(x, y(\lambda x + (1-\lambda) x^*)), \end{aligned}$$

where the first inequality follows from the fact that $g(x^*) \geq g(x)$, $\forall x \in C$. As a result, we have

$$g(x^*)[1-(1-\lambda)] = g(x^*)\lambda \geq \lambda f(x, y(\lambda x + (1-\lambda) x^*)),$$

or

$$f(x^*, y(x^*)) = g(x^*) \geq f(x, y(\lambda x + (1-\lambda) x^*)) \quad \text{for all } x \in C.$$

3. Sending $\lambda \to 0$, we see that $\lambda x + (1-\lambda)x^* \to x^*$ and $y(\lambda x + (1-\lambda)x^*) \to y(x^*)$. We obtain

$$f(x, y(x^*)) \leq f(x^*, y(x^*)) := v, \qquad \text{for any } x \in C.$$

Consequently, with $y^* = y(x^*)$

$$f(x, y^*) \leq f(x^*, y^*) = v, \qquad \forall x \in C.$$

In addition, since $f(x^*, y^*) = \min_y f(x^*, y) \leq f(x^*, y)$ for all $y \in D$, we get

$$f(x, y^*) \leq f(x^*, y^*) = v \leq f(x^*, y), \qquad \forall x \in C, \quad y \in D.$$

This says that (x^*, y^*) is a saddle point and the minimax theorem holds, since

$$\min_y \max_x f(x,y) \leq \max_x f(x, y^*) \leq v \leq \min_y f(x^*, y) \leq \max_x \min_y f(x,y),$$

and so we have equality throughout because the right side is always less than the left side.

4. The last step would be to get rid of the assumption of strict concavity and convexity. Here is how it goes. For $\varepsilon >$, set

$$f_\varepsilon(x,y) \equiv f(x,y) - \varepsilon |\, x|^2 + \varepsilon |\, y|^2, \quad |x|^2 = \sum_{i=1}^{n} x_i^2, \quad |y|^2 = \sum_{j=1}^{m} y_j^2.$$

This function will be strictly concave–convex, so the previous steps apply to f_ε. Therefore, we get a point $(x_\varepsilon, y_\varepsilon) \in C \times D$ so that $v_\varepsilon = f_\varepsilon(x_\varepsilon, y_\varepsilon)$ and

$$f_\varepsilon\left(x, y_\varepsilon\right) \leq v_\varepsilon = f_\varepsilon\left(x_\varepsilon, y_\varepsilon\right) \leq f_\varepsilon\left(x_\varepsilon, y\right), \qquad \forall x \in C, y \in D.$$

Since $f_\varepsilon\left(x, y_\varepsilon\right) \geq f\left(x, y_\varepsilon\right) - \varepsilon |\, x|^2$ and $f_\varepsilon\left(x_\varepsilon, y\right) \leq f\left(x_\varepsilon, y\right) + \varepsilon |y|^2$, we get

$$f\left(x, y_\varepsilon\right) - \varepsilon |x|^2 \leq v_\varepsilon \leq f\left(x_\varepsilon, y\right) + \varepsilon |y|^2, \qquad \forall (x,y) \in C \times D.$$

Since the sets C, D are closed and bounded, we take a sequence $\varepsilon \to 0$, $x_\varepsilon \to x^* \in C$, $y_\varepsilon \to y^* \in D$ and also $v_\varepsilon \to v \in \mathbb{R}$. Sending $\varepsilon \to 0$, we get

$$f(x, y^*) \leq v \leq f(x^*, y), \qquad \forall (x,y) \in C \times D.$$

This says that $v^+ = v^- = v$ and (x^*, y^*) is a saddle point. ■

Bibliographic Notes

The classic reference for the material in this chapter is the wonderful two-volume set of books by S. Karlin[1] [20], one of the pioneers of game theory when he was a member of the Rand Institute. The proof of the von Neumann minimax theorem, which uses only advanced calculus, is original to Karlin. On the other hand, the birth of game theory can be dated to the appearance of the seminal book by von Neumann and Morgenstern, reference[31].

The idea of using mixed strategies in order to establish the existence of a saddle point in this wider class is called **relaxation**. This idea is due to von Neumann and extends to games with continuous strategies, as well as the modern theory of optimal control, differential games, and the calculus of variations. It has turned out to be one of the most far-reaching ideas of the twentieth century.

The graphical solution of matrix games is extremely instructive as to the objectives of each player. Apparently, this method is present at the founding of game theory, but its origin is unknown to the author of this book.

Poker models have been considered from the birth of game theory. Any gambler interested in poker should consult the book by Karlin [20] for much more information and results on poker models. We will also study poker models when we look at continuous games.

The ideas of a best response strategy and viewing a saddle point as a pair of strategies, each of which is a best response to the other, leads naturally in later chapters to a Nash equilibrium for more general games.

Invertible matrices are discussed by Owen [33] and Ferguson [6] and in many other references. Symmetric games were studied early in the development of the theory because it was shown (see Problem 1.47, from Karlin [20]) that any matrix game could be symmetrized so that techniques developed for symmetric games could be used to yield information about general games. Before the advent of high-speed computers, this was mostly of theoretical interest because the symmetrization made the game matrix much larger. Today, this is not much of a problem.

1 June 8, 1924–December 18, 2007. He was was Professor of Mathematics, Stanford University.

The introduction of the Burr–Hamilton duel is an example of a game of timing that is properly considered in the continuous-time setting discussed in later chapters. For the full generality see Karlin's book [20].

The fundamental idea of using linear programming to solve games, and the use of the duality theorem to even prove the existence of value of a matrix game, might have been known to several mathematicians at the same time. The exact originator of this idea is unknown to the present author but possibly may be due to Dantzig, von Neumann, Karlin, or others.

Problem 1.85 is a classic example of the use of game theory in a business problem. Here it is used to determine the location of a business depending on marketing conditions and the location of a competitor's store. Versions of this problem appear throughout the literature, but this version is due to R. Bronson. Problem 1.86 is an example of a dispute resolution problem (refer to Winston's book [44]) presaging the idea of cooperative games considered in Chapter 6. Problem 1.88 is an instructive problem on the use of the graphical method and appears in at least the books by [20] and [16]. The Colonel Blotto game is an early application of game theory to military theory. There is an entire class of games known as Blotto games.

An AI Poem on Gambling

This poem, which is apt for many of the games in this book, was unbelievably generated by ChatGPT in under one second:

In a hazy room, where fortunes loom,
There's a dance with fate, in the
gambler's tomb.
With cards and dice, and a spinning wheel,
Hope and despair intertwine, a surreal ordeal.

The roulette wheel spins, a whirlwind of
chance,
Red or black, the outcome to enhance.
Heartbeats quicken, as the ball takes flight,
Destiny twirls, in the depths of the night.

Blackjack tables beckon, with cards concealed,
Players' hopes congealed, as strategies are
revealed.
Aces and faces, kings and queens collide,
Fortunes shift swiftly, with every turn of
the tide.

Eyes ablaze with a glint of greed,
Players gather, fueled by their need.
Chips clink and jingle, as bets are laid,
Each soul entangled, by the game's charade.

Poker faces hide tales of deceit,
Bluffing and raising, opponents they meet.
Chips rise and fall, with every hand dealt,
As fortunes fluctuate, like cards in a welt.

But amidst the thrill and the alluring spell,
Lies a cautionary tale that history can tell.
For in this dance with luck and desire,
Many find themselves consumed by the fire.

So tread with care, in this realm of chance,
For gambling's allure can swiftly enhance.
Enjoy the game, but know the stakes,
Lest your life and fortune, it overtakes.