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Signal Digitizing – Sampling and Coding

The conversion of an analog signal to digital form involves a twofold approximation. Firstly, in the time domain, the signal function $s(t)$ is replaced by its values at integral time increments T and is thus converted to $s(nT)$. This process is called sampling. Secondly, in the amplitude domain, each value of $s(nT)$ is approximated by a whole multiple of an elementary quantity. This process is called quantization. The approximate value thus obtained is then associated with a number. This process is called coding – a term often used to describe the whole process by which the value of $s(nT)$ is transformed into the number representing it.

The effect of these two approximations on the signal will be analyzed in this chapter. To achieve this, two basic tools will be used: Fourier analysis and distribution theory.

1.1 Fourier Analysis

Fourier analysis is a method of decomposing a signal into a sum of individual components which can easily be produced and observed. The importance of this decomposition is that a system's response to the signal can be deduced from these individual components using the superposition principle. These elementary component signals are periodic and complex, so both the amplitude and phase of the systems can be studied. They are represented by a function $s_e(t)$ such that:

$$S_e(t) = e^{j2\pi ft} = \cos(2\pi ft) + j\sin(2\pi ft) \tag{1.1}$$

where f is the inverse of the period – that is, the frequency of the elementary signal.

Since the elementary signals are periodic, clearly, the analysis is simplified when the signal itself is periodic. This case will be examined first, although it is not the most interesting, since a periodic signal is completely determinate and carries practically no information.

1.1.1 Fourier Series Expansion of a Periodic Function

Let $s(t)$ be a function of a periodic variable t with period T – that is, satisfying the relation:

$$s(t+T) = s(t) \tag{1.2}$$

Under certain conditions, this function can be expanded in a Fourier series as:

$$s(t) = \sum_{n=-\infty}^{\infty} C_n e^{j2\pi nt/T} \tag{1.3}$$

Digital Signal Processing: Theory and Practice, Tenth Edition. Maurice Bellanger.

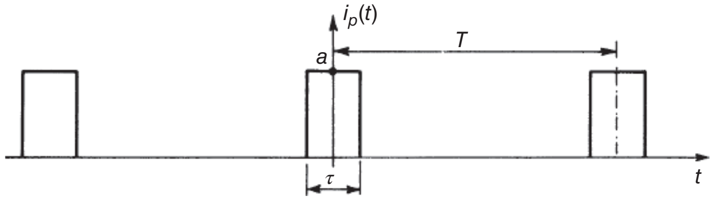


Figure 1.1 Impulse train.

The index n is an integer, and the C_n, called the Fourier coefficients, are defined by:

$$C_n = \frac{1}{T}\int_0^T s(t)e^{-j2\pi nt/T}\mathrm{d}t \tag{1.4}$$

In fact, the Fourier coefficients minimize the square of the difference between the function $s(t)$ and the series (1.3). Expression (1.4) is obtained by taking the derivative with respect to the index n coefficient of the quantity:

$$\int_0^T \left(s(t) - \sum_{m=-\infty}^{\infty} C_m e^{j2\pi mt/T}\right)^2 \mathrm{d}t$$

and setting that derivative to zero.

Example: Figure 1.1 shows an example of a Fourier expansion of a function $i_\mathrm{p}(t)$ composed of a train of impulses, each of width τ and amplitude a, occurring at time intervals T. The time origin is taken as being at the center of an impulse.

The coefficients C_n are given by:

$$C_n = \frac{1}{T}\int_{-\tau/2}^{\tau/2} a e^{-j2\pi nt/T}\mathrm{d}t = \frac{a\tau}{T}\frac{\sin(\pi n\tau/T)}{\pi n\tau/T} \tag{1.5}$$

and the Fourier expansion is:

$$i_\mathrm{p}(t) = \frac{a\tau}{T}\sum_{n=-\infty}^{\infty}\frac{\sin(\pi n\tau/T)}{\pi n\tau/T}e^{j2\pi nt/T} \tag{1.6}$$

The importance of this example for the study of sampled systems is readily apparent.

The properties of Fourier series expansions are given in Ref. [1]. One important property, expressed by the Bessel–Parseval equation, is that power is conserved in the expansion of the signal:

$$\sum_{n=-\infty}^{\infty} |C_n|^2 = \frac{1}{T}\int_0^T |s(t)|^2\,\mathrm{d}t \tag{1.7}$$

The constituent elements resulting from the expansion of a periodic signal have frequencies which are integer multiples of $1/T$ (the inverse of the period). They form a discrete set in the space of all frequencies. In contrast, if the signal is not periodic, the Fourier components form a continuous domain in the frequency space.

1.1.2 Fourier Transform of a Function

Let $s(t)$ be a function of t. Under certain conditions, one can write:

$$s(t) = \int_{-\infty}^{\infty} S(f)e^{j2\pi ft}\,\mathrm{d}f \tag{1.8}$$

where

$$S(f) = \int_{-\infty}^{\infty} S(f)e^{j2\pi ft}\,\mathrm{d}t \tag{1.9}$$

The function $S(f)$ is the Fourier transform of $s(t)$. More commonly, $S(f)$ is called the spectrum of signal $s(t)$.

Example: To calculate the Fourier transform $I(f)$ of an isolated pulse $i(t)$ of width τ and amplitude a, centered on the time origin (Figure 1.2):

$$I(f) = \int_{-\infty}^{\infty} i(t)e^{-j2\pi ft}\,\mathrm{d}t = a\int_{-\tau/2}^{\tau/2} e^{-j2\pi ft}\,\mathrm{d}t$$

$$I(f) = a\tau\frac{\sin(\pi f\tau)}{\pi f\tau} \tag{1.10}$$

Figure 1.3 represents the function $I(f)$. This will be used frequently in this book. It is important to note that it will be zero for nonzero frequencies which are whole multiples of the inverse of the impulse width. A table of this function is given in Appendix 1.

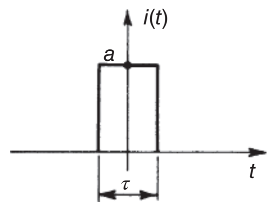


Figure 1.2 Isolated impulse.

This example clearly shows the correspondence between the Fourier coefficients and the spectrum. In effect, by comparing equations (1.6) and (1.10), it can be verified that, apart from the factor $1/T$, the coefficients of the Fourier series expansion of an impulse train correspond to the values of the spectrum of the isolated impulse at frequencies which are whole multiples of the inverse of the period of the impulses.

In the case of a nonperiodic function, there is an expression similar to the Bessel–Parseval relation, but this time the energy in the signal is conserved, instead of the power:

$$\int_{-\infty}^{\infty} |S(f)|^2\mathrm{d}f = \int_{-\infty}^{\infty} |s(t)|^2\mathrm{d}t \tag{1.11}$$

Let $s'(t)$ be the derivative of the function $s(t)$; its Fourier transform $S_\mathrm{d}(f)$ is given by:

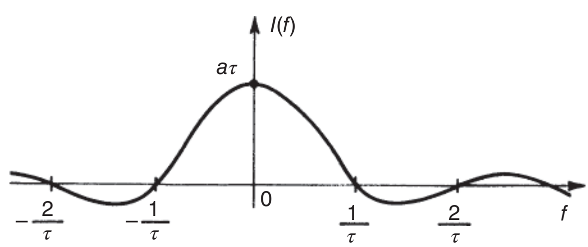
$$S_\mathrm{d}(f) = \int_{-\infty}^{\infty} e^{-j2\pi ft}s'(t)\mathrm{d}t = j2\pi fS(f) \tag{1.12}$$

Thus, taking the derivative of a signal leads to multiplying its spectrum by $j2\pi f$.

One essential property of the Fourier transform (in fact, the main reason for its use) is that it transforms a convolution into a simple product. Consider two time functions, $x(t)$ and $h(t)$, with Fourier transforms $X(f)$ and $H(f)$, respectively. The convolution $y(t)$ is defined by:

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(t-\tau)h(\tau)\mathrm{d}\tau \tag{1.13}$$

Figure 1.3 Spectrum of an isolated impulse.



The Fourier transform of this product is:

$$Y(f) = \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} x(t-\tau)h(\tau)d\tau \right) e^{-j2\pi ft}\mathrm{d}t$$

$$Y(f) \int_{-\infty}^{\infty} h(\tau)e^{-j2\pi f\tau}\mathrm{d}\tau \int_{-\infty}^{\infty} x(u)e^{-j2\pi fu}\mathrm{d}u = H(f)X(f)$$

Conversely, it can be shown that the Fourier transform of a simple product is a convolution product.

An interesting result can be derived from the abovementioned properties. Let us consider the Fourier transform $II(f)$ of the function $i^2(t)$. Because of equations (1.10) and (1.13), we obtain:

$$II(f) = I(f) * I(f) = aI(f) \tag{1.14}$$

and therefore,

$$\int_{-\infty}^{\infty} \frac{\sin(\pi\phi\tau)}{\pi\phi\tau} \frac{\sin(\pi(f-\phi)\tau)}{\pi(f-\phi)\tau}\mathrm{d}\phi = \frac{1}{\tau}\frac{\sin(\pi f\tau)}{\pi f\tau}$$

Taking $f = n/\tau$, for any integer n,

$$\int_{-\infty}^{\infty} \frac{\sin(\pi\phi\tau)}{\pi\phi\tau} \cdot \frac{\sin(\pi(\phi\tau - n))}{\pi(\phi\tau - n)}\mathrm{d}\phi = 0 \tag{1.15}$$

Thus, the functions sin $\pi(x - n)/[\pi(x - n)]$, with n being an integer, forms a set of orthogonal functions.

The definition and properties of the Fourier transform can be extended to multivariate functions. Let $s(x_1, x_2, \ldots, x_n)$ be a function of n real variables. Its Fourier transform is a function $S(\lambda_1, \lambda_2, \ldots, \lambda_n)$ defined by:

$$S(\lambda_1, \lambda_2, \ldots, \lambda_n) = \iint_{\mathbb{R}^n} \cdots \int s(x_1, x_2, \ldots, x_n) \times \boldsymbol{e}^{-\mathbf{j2}\pi(\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_n x_n)}\mathbf{d}x_1\mathbf{d}x_2 \cdots \mathbf{d}x_n \tag{1.16}$$

If the function $s(x_1, x_2, \ldots, x_n)$ is separable – that is, if:

$$s(x_1, x_2, \ldots, x_n) = s(x_1)s(x_2)\cdots s(x_n)$$

then:

$$S(\lambda_1, \lambda_2, \ldots, \lambda_n) = S(\lambda_1)S(\lambda_2)\cdots S(\lambda_n)$$

The variables x_i $(1 \leqslant i \leqslant n)$ often represent distances (for example, for two dimensions), and in that case, λ_i are called spatial frequencies.

1.2 Distributions

Mathematical distributions constitute a formal mathematical representation of the physical distributions found in experiment [1].

1.2.1 Definition

A distribution D is defined as a continuous linear function in the vector space $\mathscr{D}$ of functions defined in $\mathbb{R}^n$, indefinitely differentiated, and having a bounded support.

With each function φ belonging to $\mathscr{D}$, the distribution D associates a complex number $D(\varphi)$ which will also be denoted by $\langle D, \varphi\rangle$, with the properties:

(1) $D(\varphi_1 + \varphi_2) = D(\varphi_1) + D(\varphi_2)$.
(2) $D(\lambda\varphi) = \lambda D(\varphi)$ where λ is a scalar.
(3) If φ_j converges to φ when j tends toward infinity, the sequence $D(\varphi_j)$ converges to $D(\varphi)$.

Examples:

(i) If $f(t)$ is a function which is summable over any bounded ensemble, it defines a distribution D_f by:

$$\langle D_f, \phi\rangle = \int_{-\infty}^{\infty} f(t)\phi(t)\mathrm{d}t \tag{1.17}$$

(ii) If φ' denotes the derivative of φ, the function:

$$\langle D, \phi\rangle = \int_{-\infty}^{\infty} f(t)\phi'(t)\mathrm{d}t = \langle f, \phi\rangle \tag{1.18}$$

is also a distribution.

(iii) The Dirac distribution δ is defined by:

$$\langle \delta, \phi\rangle = \phi(0) \tag{1.19}$$

The Dirac distribution δ at a real point x is defined by:

$$\langle \delta(t-x), \phi\rangle = \phi(x) \tag{1.20}$$

This distribution is said to represent a mass of +1 at the point x.

(iv) Consider a pulse $i(t)$ of duration τ, with amplitude $a = 1/\tau$, centered on the origin. It defines a distribution D_i:

$$\langle D_i, \phi\rangle = \frac{1}{\tau}\int_{-\tau/2}^{\tau/2} \phi(t)\mathrm{d}t$$

For very small values of τ, this becomes:

$$\langle D_i, \phi\rangle \simeq \phi(0)$$

that is, the Dirac distribution can be regarded as the limit of the distribution D_i when τ tends toward 0.

1.2.2 Differentiation of Distributions

The derivative $\partial D/\partial t$ of a distribution D is defined by the relation:

$$\left\langle \frac{\partial D}{\partial t}, \phi\right\rangle = -\left\langle D, \frac{\partial \phi}{\partial t}\right\rangle \tag{1.21}$$

To illustrate this, consider the Heaviside function Y, or single-step function, which is zero when $t < 0$ and +1 if $t \geqslant 0$:

$$\left\langle \frac{\partial Y}{\partial t}, \phi\right\rangle = -\left\langle Y, \frac{\partial \phi}{\partial t}\right\rangle = -\int_{0}^{\infty} \phi'(t)\mathrm{d}t = \phi(0) = \langle \delta, \phi\rangle \tag{1.22}$$

As a result, the discontinuity in Y appears in the derivative as a point of unit mass.

This example illustrates the considerable practical interest of the notion of a distribution, which means that a number of the concepts and properties of continuous functions can be extended to discontinuous functions.

1.2.2.1 The Fourier Transform of a Distribution

By definition, the Fourier transform of a distribution D is a distribution denoted by FD such that:

$$\langle FD, \phi \rangle = \langle D, F\phi \rangle \tag{1.23}$$

By applying this definition to distributions with a point support, we obtain:

$$\langle F\delta, \phi \rangle = \langle \delta, F\phi \rangle = \int_{-\infty}^{\infty} \phi(t)\mathrm{d}t = \langle 1, \phi \rangle \tag{1.24}$$

Consequently, $F\delta = 1$. Similarly, $F\delta(t - a) = e^{-\mathrm{j}2\pi fa}$.

A case which is fundamental to the study of sampling is that of the set (u) of Dirac distributions shifted by T and such that:

$$u(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT) \tag{1.25}$$

This set is a distribution of unit mass points separated on the abscissa by whole multiples of T. Its Fourier transform is:

$$Fu = \sum_{n=-\infty}^{\infty} e^{-\mathrm{j}2\pi fnT} = U(f) \tag{1.26}$$

and it can be shown that this sum is a point distribution.

A simple demonstration can be obtained from the Fourier series development of the function $i_{\mathrm{p}}(t)$, formed by the set of separate pulses of duration T, with width τ, and amplitude $1/\tau$, centered on the time origin.

One can consider $u(t)$ as the limit of $i_{\mathrm{p}}(t)$ as τ tends toward zero:

$$u(t) = \lim_{\tau \to 0} i_p(t)$$

and by referring to relation (1.6), we find that:

$$\lim_{\tau \to 0} i_{\mathrm{p}}(t) = \frac{1}{T} \sum_{n=-\infty}^{\infty} e^{\mathrm{j}2\pi nt/T}$$

The following fundamental property is demonstrated in Ref. [2].

The Fourier transform of the time distribution, represented by unit mass points separated by whole multiples of T *is a frequency distribution of points of mass* $1/T$ *separated by whole multiples of* $1/T$.

That is:

$$U(f) = \sum_{n=-\infty}^{\infty} e^{\mathrm{j}\pi fnT} = \frac{1}{T} \sum_{n=-\infty}^{\infty} \delta\left(f - \frac{n}{T}\right) \tag{1.27}$$

This result will be used when studying signal sampling. The property of the Fourier transform whereby it exchanges convolution and multiplication applies equally to distributions.

Before considering the influence of the sampling and quantizing operations on the signal, it is useful to discuss the characteristics of those signals which are most often studied.

1.3 Some Commonly Studied Signals

A signal is defined as a function of time $s(t)$. This function can be either an analytic expression or the solution of a differential equation, in which case the signal is said to be deterministic.

1.3.1 Deterministic Signals

Sine waves are the most frequently used signals of this type. For example,

$$s(t) = A\ \cos(\omega t + \alpha)$$

where A is the amplitude, $\omega = 2\pi f$ is the angular frequency, and α is the phase of the signal.

Signals of this type are easy to reproduce and recognize at different points of a system. They allow the various characteristics to be visualized in a simple way. Moreover, as mentioned above, they serve as the basis for the decomposition of any deterministic signal through the Fourier transform.

If the system is linear and invariant in time, it can be characterized by its frequency response $H(\omega)$. For each value of the frequency, $H(\omega)$ is a complex number whose modulus is the amplitude of the response. By convention, the function $\varphi(\omega)$ such that:

$$H(\omega) = |H(\omega)|e^{-\mathrm{j}\phi(\omega)} \tag{1.28}$$

is defined as the phase. This convention allows the group delay $\tau(\omega)$, a positive function for real systems, to be expressed as:

$$\tau(\omega) = \frac{\mathrm{d}\phi}{\mathrm{d}\omega} \tag{1.29}$$

The group delay refers to transmission lines on which different frequencies of the signal propagate at different speeds, which leads to energy dispersion in time. As an illustration of the concept, let us consider two close frequencies $\omega \pm \Delta\omega$ and the corresponding phases per unit length $\varphi \pm \Delta\varphi$. The sum signal is expressed by:

$$s(t) = \cos[(\omega + \Delta\omega)t - (\varphi + \Delta\varphi)] + \cos[(\omega - \Delta\omega)t - (\varphi - \Delta\varphi)]$$

or

$$s(t) = 2\cos(\omega t - \varphi)\cos(\Delta\omega t - \Delta\varphi)$$

This is a modulated signal, and there is no dispersion if the two factors in the above expression undergo the same delay per unit length – that is, $\Delta\varphi/\Delta\omega$ is constant. Thus, the group delay characterizes the dispersion imposed on the signal energy by a transmission line or any equivalent system.

If the sinusoidal signal $s(t)$ is applied to the system, then an output signal $s_r(t)$ is obtained such that:

$$s_r(t) = A|H(\omega)|\cos[\omega t + \alpha - \phi(\omega)] \tag{1.30}$$

Once again, this is a sinusoidal signal, and comparison with the applied signal reveals the response of the system. The importance of this procedure (for example, for test operations) can readily be appreciated.

Deterministic signals, meanwhile, do not give a good representation of real signals because they do not carry any information except by the fact of their presence. Real signals are generally represented by a random function $s(t)$. For testing and analyzing systems, random signals are also used, but they must have particular characteristics which do not present undue complications for their generation and use. A study of such signals is given in Vol. 2 of Ref. [2].

1.3.2 Random Signals

A random signal is defined at each time, t, by a probability law for its amplitude. This law can be expressed by a probability density $p(x, t)$ defined by:

$$p(x,t) = \lim_{\Delta x \to 0} \frac{\text{Prob}[x \leq s(t) \leq x + \Delta x]}{\Delta x} \tag{1.31}$$

It is stationary if its statistical properties are independent of time – that is, if the probability density is independent of time:

$$p(x,t) = p(x)$$

It is of second-order if it possesses a first-order moment called the mean value, which is the mathematical expectation of $s(t)$, denoted by $E[s(t)]$ and defined by:

$$m_1(t) = E[s(t)] = \int_{-\infty}^{\infty} xp(x,t)\mathrm{d}x \tag{1.32}$$

and a second-order moment called the covariance:

$$E[s(t_1)s(t_2)] = m_2(t_1,t_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1x_2p(x_1,x_2;t_1,t_2)\mathrm{d}x_1\mathrm{d}x_2$$

where $p(x_1,x_2;t_1,t_2)$ is the probability density of a pair of random variables $[s(t_1),s(t_2)]$.

The stationarity can be limited to the moments of first and second order; then, the signal is said to be stationary of order 2 or stationary in the wider sense. For such a signal:

$$E[s(t)] = \int_{-\infty}^{\infty} xp(x)\mathrm{d}x = m_1$$

The independence of time is translated for the probability density $p(x_1, x_2, t_1, t_2)$ as follows:

$$p(x_1,x_2;t_1,t_2) = p(x_1,x_2;0,t_2-t_1) = p(x_1,x_2;\tau)$$

where,

$$\tau = t_2 - t_1$$

Only the difference between the two observation times is involved:

$$E[s(t_1)s(t_2)] = m_2(\tau) \tag{1.33}$$

The function $r_{xx}(\tau)$, such that:

$$r_{xx}(\tau) = E[s(t)s(t-\tau)] \tag{1.34}$$

is called the autocorrelation function of the signal.

A random signal $s(t)$ also has a time mean m_T. This is a random variable defined by:

$$m_\mathrm{T} = \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} s(t)\mathrm{d}t \tag{1.35}$$

The ergodicity of this average illustrates that it takes a particular value k with probability 1. For a stationary signal, the ergodicity of the time average implies equality with the average of the amplitudes at a given instant. In effect, we use the expectation of the variable m_T:

$$E[m_\mathrm{T}] = k = E\left[\lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} s(t)\mathrm{d}t\right] = \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} E[s(t)]\mathrm{d}t = m_1$$

This result has important consequences in practice, as it provides a means of finding the statistical properties of the signal at a given instant from observation over the time period. The ergodicity of the covariance in the stationary case is also very interesting because it leads to the relation:

$$r_{xx}(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} s(t)s(t-\tau)\mathrm{d}t \tag{1.36}$$

The autocorrelation function $r_{xx}(\tau)$ of the signal $s(t)$ is fundamental for the study of ergodic second-order stationary signals. Its principal properties are:

(1) It is an even function:

$$r_{xx}(\tau) = r_{xx}(-\tau)$$

(2) Its maximum is at the origin and corresponds to the power P of the signal:

$$r_{xx}(0) = E[s^2(t)] = P$$

(3) The power spectral density is the Fourier transform of the autocorrelation function:

$$\Phi_{xx}(f) = \int_{-\infty}^{\infty} r_{xx}(\tau)e^{j2\pi f\tau}\mathrm{d}\tau = 2\int_{0}^{\infty} r_{xx}(\tau)\cos(2\pi f\tau)\mathrm{d}\tau$$

In effect, $r_{xx}(\tau) = s(\tau)^{*}s(-\tau)$, and if $S(f)$ denotes the Fourier transform of $s(t)$, we obtain:

$$\Phi_{xx}(f) = S(f)\overline{S(f)} = |S(f)|^2 \tag{1.37}$$

This last property is physically translated by the fact that the more rapidly the signal varies (that is, the more its spectrum tends toward high frequencies), the narrower its autocorrelation function becomes. In the limit case, the signal is purely random, and the function becomes zero for $\tau \neq 0$. This signal is called white noise and is described by:

$$r_{xx}(\tau) = P\delta$$

The spectral density is then constant:

$$\Phi_{xx}(f) = P$$

In fact, such a signal has no physical reality since its power is infinite, but it does offer a useful mathematical model for signals with a spectral density that is virtually constant over a wide frequency band.

1.3.3 Gaussian Signals

Among the probability distributions that can be considered for a signal $s(t)$, one particular category is of special interest – normal or Gaussian distributions. In effect, normal random distributions retain their normal character under any linear operation, such as convolution through a certain distribution, filtering, differentiation, or integration. These random distributions are also frequently used for modeling real signals and for testing systems.

A random variable x is said to be Gaussian if its probability law has a density $p(x)$ which follows the normal or Gaussian law:

$$p(x) = \frac{1}{\sigma\sqrt{(2\pi)}} e^{-(x-m)^2/(2\sigma^2)} \tag{1.38}$$

The parameter m is the mean of the variable x; the variance σ^2 is the second-order moment of the centered random variable $(x-m)$; and σ is the standard deviation. The variable $(x-m)/\sigma$ has a zero average and a unit standard deviation. A tabulation is given in Appendix 2.

A random variable is characterized by its amplitude probability law and also by its moments m_n, such that:

$$m_n = \int_{-\infty}^{\infty} x^n p(x) \mathrm{d}x \tag{1.39}$$

These moments are the coefficients of the series expansion of the function $F(u)$, called the characteristic function of the random variable x and defined by:

$$F(u) = \int_{-\infty}^{\infty} e^{\mathrm{j}ux} p(x) \mathrm{d}x \tag{1.40}$$

It is the inverse Fourier transform of the probability density $p(x)$ which is expressed by:

$$p(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\mathrm{j}ux} F(u) \mathrm{d}u \tag{1.41}$$

On the basis of equation (1.40), the following series expansion is obtained:

$$F(u) = \sum_{n=0}^{\infty} \frac{(ju)^n}{n!} m_n \tag{1.42}$$

and for a centered Gaussian variable:

$$F(u) = e^{-\sigma^2 u^2/2} \tag{1.43}$$

The normal law can be generalized to multidimensional random variables [3]. The characteristic function of a k-dimensional Gaussian variable $x(x_1, x_2, \ldots, x_k)$ is expressed by:

$$F(u_1, u_2, \ldots, u_k) = \exp\left(-\frac{1}{2} \sum_{i=1}^{k} \sum_{j=1}^{k} r_{ij} u_i u_j \right) \tag{1.44}$$

where

$$r_{ij} = E(x_i x_j)$$

The probability density is obtained through Fourier transformation. For two dimensions, we get:

$$p(x_1, x_2) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{(1-r^2)}} \exp\left\{ -\frac{1}{2(1-r^2)} \left[\frac{x_1^2}{\sigma_1^2} - \frac{2rx_1x_2}{\sigma_1\sigma_2} + \frac{x_2^2}{\sigma_2^2} \right] \right\} \tag{1.45}$$

where r is the correlation coefficient:

$$r = \frac{E(x_1 x_2)}{\sigma_1 \sigma_2}$$

A random signal $s(t)$ is said to be Gaussian if, for any set of k times t_i $(1 \leqslant i \leqslant k)$, the k-dimensional random variable $s = [s(t_1), \ldots, s(t_k)]$ is Gaussian. According to equation (1.44), the probability law of that variable is completely defined by the autocorrelation function $r_{xx}(\tau)$ of the signal $s(t)$.

Example: The signal defined by the equations:

$$r_{xx}(\tau) = \sigma^2 e^{-|\tau|/(RC)} \tag{1.46}$$

$$p(x) = \frac{1}{\sigma\sqrt{(2\pi)}} e^{-x^2/(2\sigma^2)} \tag{1.47}$$

is an approximation to white Gaussian noise which is used in the analysis of systems or for modeling signals. It is a stationary signal with a zero average and a spectral density which is not strictly constant, but which corresponds to a uniform distribution filtered by an *RC*-type low-pass filter. It is obtained by amplifying the thermal noise generated at the terminals of a resistor.

The Gaussian distribution can be obtained from a uniform distribution. Let $p(y)$ be the Rayleigh distribution:

$$p(y) = \frac{y}{\sigma^2} e^{-(y^2/2\sigma^2)}; \quad y \geqslant 0 \tag{1.48}$$

and $p(x)$ the uniform distribution over the interval [0,1]. Changing variables so that:

$$p(x)\mathrm{d}x = p(y)\mathrm{d}y$$

one gets:

$$p(y) = p(x)\frac{\mathrm{d}x}{\mathrm{d}y} = \frac{\mathrm{d}x}{\mathrm{d}y} = \frac{y}{\sigma^2}\mathrm{e}^{-(y^2/2\sigma^2)}$$

and therefore,

$$x = e^{-(y^2/2\sigma^2)}; \quad y = \sigma\sqrt{\left[21\mathrm{n}\left(\frac{1}{x}\right)\right]} \tag{1.49}$$

The Gaussian distribution is obtained by considering two independent variables, x and y, and the variable z, given by:

$$z = y\ \cos\ 2\pi x \tag{1.50}$$

With the help of the variable:

$$z' = y\ \sin\ 2\pi x$$

the following equation is obtained:

$$p(z,z')\mathrm{d}z\,\mathrm{d}z' = p(z)p(z')\mathrm{d}z\,\mathrm{d}z' = p(y)p(x)\mathrm{d}x\,\mathrm{d}y = p(z)p(z')y\,\mathrm{d}y2\pi\mathrm{d}x$$

Hence:

$$p(z)p(z') = \frac{1}{2\pi}\frac{1}{\sigma^2}e^{-(y^2/2\sigma^2)} = \frac{1}{2\pi\sigma^2}e^{-(z^2+z'^2/2\sigma^2)}$$

and finally,

$$p(z) = \frac{1}{\sigma\sqrt{2\pi}}e^{-(z^2/2\sigma^2)}$$

The method is often used to generate digital Gaussian signals.

1.3.3.1 Peak Factor of a Random Signal

A random signal is defined by a probability law of its amplitude, such that this amplitude is often not bounded. This is the case for Gaussian signals, as can be seen from equation (1.38).

Signal processing can only be performed for a limited range of amplitudes, and restrictions are necessary. An important parameter is the peak factor defined as the ratio of a certain amplitude A_m to the standard deviation σ. By convention, the amplitude A_m is that value which is exceeded for not more than 10^{-5} of the total time. This relationship is often expressed in decibels (dB) by f_c, where:

$$f_\mathrm{c} = 20\log_{10}(A_\mathrm{m}/\sigma) \tag{1.51}$$

Following that convention, the peak factor for a Gaussian signal is 12.9 dB, and when this definition is applied to a sinusoidal signal, a peak factor of 3 dB results.

A stationary model used to represent telephone signals is formed by the random signal whose amplitude probability density obeys the following exponential, or Laplace, distribution:

$$p(x)\frac{1}{\sigma\sqrt{2}}e^{-\sqrt{2}|x|/\sigma} \tag{1.52}$$

The peak factor rises, in this case, to 17.8 dB.

In conclusion, ergodic second-order stationary functions characterized by an amplitude probability distribution and an autocorrelation function can be used to model the majority of signals of interest. They are widely used in system analysis.

In addition to the other possibilities for representing signals, it is important to have some sort of global measure, so as to be able, for example, to follow a signal through the processing system. Such a measure can be obtained by defining norms on the function which represents the signal.

1.4 The Norms of a Function

A norm is a positive real function which satisfies the relations:

$$||x|| \geq 0; \quad k||x|| = ||kx||$$

where k is real and positive.

One category of norms that is frequently employed is the set called L_p norms [4]. The L_p norm of a continuous function $s(t)$ defined over the interval [0,1] is denoted by $||s||_p$ and defined by:

$$||s||_p = \left[\int_0^1 |s(t)|^p \mathrm{d}t\right]^{1/p} \tag{1.53}$$

Three values of p are of particular interest:

(1) $p = 1$:

$$||s||_1 = \int_0^1 |s(t)|\mathrm{d}t \tag{1.53a}$$

(2) $p = 2$:

$$||s||_2^2 = \int_0^1 |s(t)|^2 \, \mathrm{d}t \tag{1.53b}$$

which is the expression for the energy of the signal $s(t)$.

(3) $p = \infty$:

$$||s||_\infty = \max_{0 \leqslant t \leqslant 1} |s(t)| \tag{1.53c}$$

This is Chebyshev's norm.

Norms are also used in approximation techniques to measure the discrepancy between a function $f(x)$ and the function $F(x)$ being approximated. The approximation is made by the least squares method if the norm L_2 is used and, in Chebyshev's sense, if the norm L_∞ is used.

The L_p norms can be generalized by introducing a real, positive weighting function of $p(x)$. The weighted L_p norm of the difference function $f(x) - F(x)$ is then written:

$$\|f(x) - F(x)\|_p = \left[\int_0^1 |f(x) - F(x)|^p p(x)\mathrm{d}x\right]^{1/p} \tag{1.53d}$$

These expressions are applied when calculating filter coefficients and in realization problems – in particular for the scaling of internal data in memories and for noise estimation.

1.5 Sampling

Sampling consists of representing a signal function $s(t)$ by its value $s(nT)$ taken at whole multiples of the time interval T, called the sampling period. Such an operation can be analyzed in a simple and concise way by using distribution theory. In fact, by definition, the distribution of unit masses at whole multiple points on the axis, with period T, associates with the function $s(t)$ the ensemble of its values $s(nT)$, where n is a whole number. Conforming to the notation given earlier, this distribution is denoted by $u(t)$ and is written:

$$u(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

The sampling operation affects the spectrum $S(f)$ of the signal. Considering the fundamental relation (1.27), it appears that the spectrum $U(f)$ of the distribution $u(t)$ is formed of lines of amplitude $1/T$ at frequencies which are whole multiples of the sampling frequency $f_s = 1/T$. Thus, $u(t)$ is expressed as a sum of elementary sinusoidal signals, collectively called carriers:

$$u(t) = \frac{1}{T}\sum_{n=-\infty}^{\infty} e^{j2\pi nt/T} \tag{1.54}$$

Hence, the set of values for the signal $s(nT)$ corresponds to the product with the signal $s(t)$ of the ensemble of component signals which make up $u(t)$. That is, the operation of sampling is an amplitude modulation of the signal by an infinite number of carriers with frequencies which are whole multiples of the sampling frequency $f_s = 1/T$. Consequently, the sampled signal spectrum includes the function $S(f)$, called the baseband, and other images or sidebands, which correspond to the translation of the baseband by whole multiples of the sampling frequency. The operation of sampling and its influence on the signal spectrum are represented in Figure 1.4.

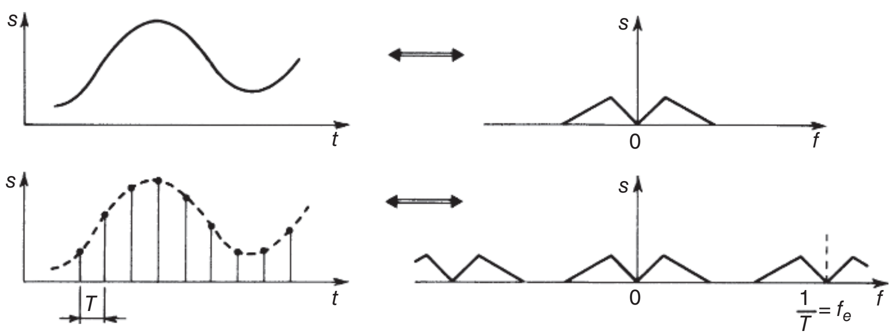


Figure 1.4 The spectral incidence of sampling.

The sampled signal spectrum $S_s(f)$ is expressed as the convolution of $S(f)$ with $U(_f)$, so that:

$$S_s(f) = 1/T \sum_{n=-\infty}^{\infty} S(f - n/T) \tag{1.55}$$

It is important to note that the function $S_s(f)$ is periodic – that is, the sampling has introduced a periodicity into the frequency space, which constitutes a fundamental characteristic of the sampled signals.

The sampling operation, as described above, which is called ideal sampling, may seem rather unrealistic in that it would be difficult in practice to manipulate or reconstitute an instantaneous signal value. Real samplers, or circuits which reconstitute samples, all possess a certain aperture time. In fact, it can be shown that sampling, or the reconstitution of samples by pulses having a given width, simply introduces a modification of the signal spectrum.

In effect, when sampling a signal $x(t)$ by a set of pulses separated by period T, with width τ and amplitude a, it is quite possible for a quantity σ_n to be collected in the nth period, and this is written:

$$\sigma_n = a \int_{nT-\tau/2}^{nT+\tau/2} s(t)\mathrm{d}t$$

This quantity expresses the result of the convolution of the signal $s(t)$ by the elementary pulse $i(t)$. The function which results in this case at the sampling time nT is the function $s*i$. That is, the sampled signal does not have $S(f)$ for its spectrum, but the product:

$$S_p(f) = U(f)S(f) = \frac{1}{T} \sum_{n=-\infty}^{\infty} S\left(\frac{n}{T}\right) \delta\left(f - \frac{n}{T}\right)$$

Similar reasoning applies in the case of reconstitution of samples with a duration τ. In fact, it is the convolution product of the samples $s(nT)$ with the elementary pulse $i(t)$ which is reconstituted. Thus, we have the proposition:

Sampling or the reconstitution of samples by pulses with width τ can be treated as ideal sampling or ideal reconstitution, provided that the signal spectrum is multiplied by the spectrum of the elementary pulse train.

In practice, however, whenever τ is small in comparison with the period T, the correction is negligible.

1.6 Frequency Sampling

The type of sampling considered above is carried out on a temporal basis. Nevertheless, the general characteristics are also applicable to frequency sampling.

Let us calculate the spectrum of a periodic function $s_p(t)$ with period T. Such a function can be regarded as resulting from the convolution product of the function $s(t)$ which takes values of $s_p(t)$ over one period and is zero elsewhere, and the point distribution $u(t)$. This results in the following relation between the Fourier transforms:

$$S_p(f) = U(f)S(f) = \frac{1}{T} \sum_{n=-\infty}^{\infty} S\left(\frac{n}{T}\right) \delta\left(f - \frac{n}{T}\right) \tag{1.56}$$

where we again meet the coefficients of the Fourier series expansion of the function $s_p(t)$. The case where $s(t)$ is a square pulse is represented in Figure 1.5.

It is apparent that the spectrum of the periodic function $s_p(t)$ is a set of lines forming a sample of the spectrum of one period of the function. Sampling in frequency space corresponds to a periodicity in the time space. This is a useful interpretation for digital analysis of spectra.

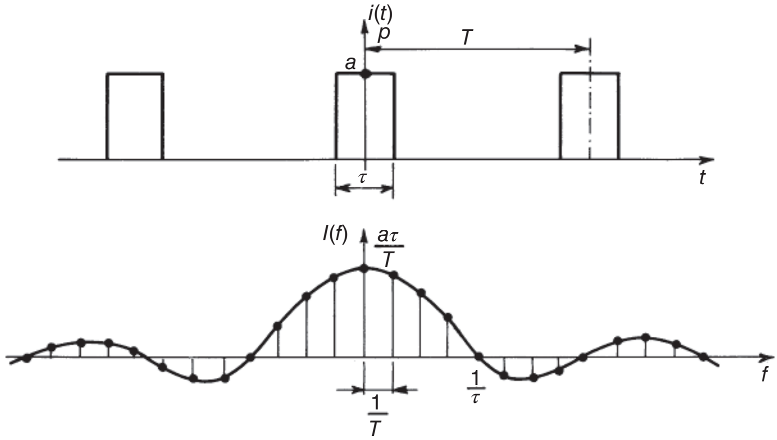


Figure 1.5 The spectrum of an impulse train.

1.7 The Sampling Theorem

This theorem establishes the conditions under which the set of samples of a signal correctly represents that signal. A signal is said to be correctly represented by the set of its samples taken with the periodicity T, if it is possible, from this set of values, to completely reconstitute the original signal.

The sampling has introduced a periodicity of the spectrum in frequency space. To restore the original signal, this periodicity must be suppressed – that is, the image bands must be eliminated. This can be achieved using a low-pass filter with a transfer function $H(f)$ which is equal to $1/f_s$ up to the frequency $f_s/2$ and is zero at higher frequencies. At the output of such a filter, a continuous signal appears, which can be expressed as a function of the values $s(nT)$. The square wave impulse response $h(t)$ of the filter is written, using equation (1.10), as:

$$h(t) = \frac{\sin(\pi t/T)}{\pi t/T}$$

The signal at the output of the filter, $s(f)$, corresponds to the convolution product of the set $s(nT)$ with the function $h(t)$. It is written as:

$$s(t) = \int_{-\infty}^{\infty} \left[\sum_{n=-\infty}^{\infty} s(\theta)\delta(\theta - nT) \right] \frac{\sin\ \pi(t-\theta)/T}{\pi(t-\theta)/T} \mathrm{d}\theta$$

and hence,

$$s(t) = \sum_{n=-\infty}^{\infty} s(nT) \frac{\sin \pi(t/T - n)}{\pi(t/T - n)} \tag{1.57}$$

This is the formula for interpolating signal values at points sited between the samples. It can be verified that it reproduces $s(nT)$ for multiples of the period T, and the reconstitution process is represented in Figure 1.6.

In order for the calculated signal $s(t)$ to be identical to the original signal, the spectrum $S(f)$ has to be identical to the spectrum of the original signal. As shown in Figure 1.6, this condition is satisfied if and only if the original spectrum does not contain any components with frequencies greater than or equal to $f_s/2$.

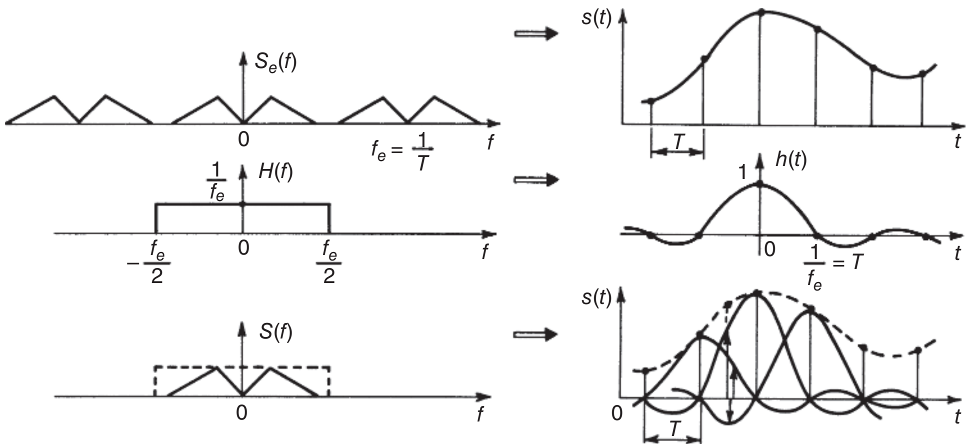


Figure 1.6 Reconstruction of the signal after sampling.

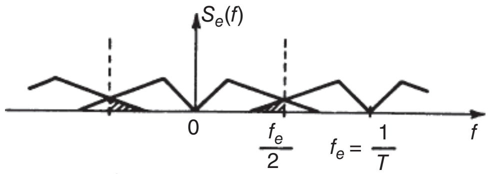


Figure 1.7 Spectrum aliasing.

If this is not the case, and the image bands overlap the baseband, as shown in Figure 1.7, a foldover distortion or aliasing is introduced and the restoring filter produces a signal that is different from the original one. From this, the sampling theorem or Shannon's theorem is derived:

A signal which does not contain any component with a frequency equal to or greater than a value f_m is completely determined by the set of its values at regularly spaced intervals of period $T = 1/(2f_m)$.

Thus, the sampling frequency of a signal is determined by the upper limit of its frequency band. In practice, the signal band is generally limited by filtering to a value below $f_s/2$ before sampling at frequency f_s, in order that the restoring filter can be practically realizable.

It is interesting to note that the sampling frequency is determined by the bandwidth of the signal. The reconstitution illustrated in Figure 1.6 was for a low-frequency signal with which a low-pass filter was associated. It is apparent that the same reasoning can also be applied to a signal occupying a restricted region in frequency space by using an associated band-pass filter. In particular, this property is applicable to modulated signals and is used in certain types of digital filters.

The result given at the end of Section 1.1.2 enables sampling to be presented from another viewpoint. Equation (1.57) shows that sampling corresponds to decomposition of the signal $s(t)$ in accordance with the set of orthogonal functions $\frac{\sin \pi(t/T-n)}{\pi(t/T-n)}$ and Shannon's theorem simply expresses the condition for this set to form a signal decomposition basis (several decomposition bases may exist).

1.8 Sampling of Sinusoidal and Random Signals

The properties given above can be clearly illustrated by sampling sinusoidal signals, whose features are of use in numerous applications.

1.8.1 Sinusoidal Signals

Let the signal $s(t) = \cos(2\pi ft + \varphi)$, with $0 \leqslant \varphi \leqslant \pi/2$, be sampled with the period $T = 1/f_s = 1$. The samples are given by the set $s(n)$ such that:

$$s(n) = \cos(2\pi fn + \phi)$$

If the ratio $f/f_s = f$ is a rational number, this becomes:

$$f = N_1/N_2 \quad \text{with } N_1 \text{ and } N_2 \text{ as integers}$$

Then,

$$s(n + N_2) = \cos[2\pi f(n + N_2) + \phi] = s(n)$$

The set exhibits the periodicity N_2 and comprises at most N_2 different numbers. On the other hand, since the sampling frequency is more than twice the signal frequency, $N_1/N_2 < \frac{1}{2}$. The ensemble of N_2 different samples permits the representation of a number of sinusoidal signals equal to the largest whole number less than $N_2/2$. For example, if $N_2 = 8$, with the ensemble of numbers – 2 cos $(2\pi n/8 + \varphi)$, $(n = 0,1, \ldots,7)$, it is possible to represent samples of three sinusoidal signals:

$$2\cos\left(2\pi\frac{N_1}{8}t + \phi\right) \quad \text{with} \quad N_1 = 1,2,3$$

Figure 1.8 represents this sampling for $\varphi = 0$, and in this particular case, four numbers are sufficient: ± 2 and $\pm\sqrt{2}$.

If we then add to the three sinusoidal signals in Figure 1.8 the continuous signal with the value 1, and the oscillating signal cos (πt) with frequency $\frac{1}{2}$, and amplitude 1, the result is that when the composite signal is sampled, zero values appear. This is true except for points which are multiples of 8, where the value 8 is obtained as shown in Figure 1.9(a). The spectrum of this sum is obtained directly by applying the relation:

$$\cos x = \frac{1}{2}(e^{jx} + e^{-jx})$$

It is formed of lines with an amplitude of 1 at frequencies which are multiples of $\frac{1}{8}$ (Figure 1.9(b)). This spectrum has already been studied in Section 1.2, and equation (1.27) again applies.

Spectrum analyzers and digital frequency synthesizers use the fact that it is possible to produce a range of sinusoidal signals from a limited set of numbers which are stored, for example, in a computer memory.

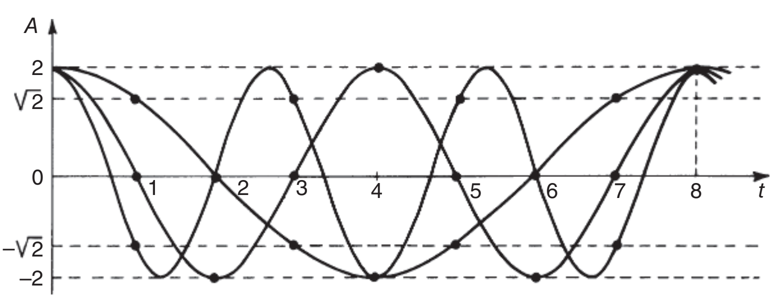


Figure 1.8 Sampling the signals cos $(2\pi N/8t)$.

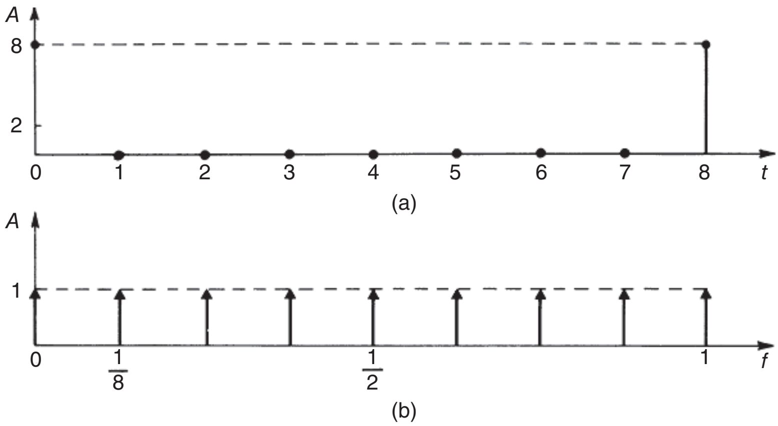


Figure 1.9 (a) Sampling the signal: $s(t) = 1 + 2\sum_{n=1}^{3} \cos(2\pi n/8t) + \cos(\pi t)$. (b) corresponding spectrum.

1.8.2 Discrete Random Signals

If the random signal $s(t)$ is sampled with the unit period $T = 1$, a discrete random signal results, which, by definition, has the same probability distribution of amplitude. The results obtained for the continuous case can be applied to the discrete one, particularly for the random signals which are second-order stationary and ergodic [5].

Thus, the autocorrelation function of the discrete signal $s(n)$ is the sequence $r(n)$, such that:

$$r(n) = E[s(i)s(i-n)] \tag{1.58}$$

This is the sampling of the autocorrelation function $r_{xx}(\tau)$ of the continuous random signal defined by expression (1.34). Its Fourier transform gives the power spectrum density $\Phi_d(f)$ of the discrete signal, which is related to the spectral density $\Phi_{xx}(f)$ of the continuous signal by a relation similar to equation (1.55); that is,

$$\Phi_d(f) = \frac{1}{T}\sum_{n=-\infty}^{\infty} \Phi_{xx}\left(f - \frac{n}{T}\right) \tag{1.59}$$

If the sampling frequency is not high enough, or if the spectrum $\Phi_{xx}(f)$ spans an infinite domain, aliasing takes place.

The hypothesis of ergodicity for the discrete signal $s(n)$ leads to the relation:

$$r(n) = \lim_{N\to\infty} \frac{1}{2N+1}\sum_{i=-N}^{N} s(i)s(i-n) \tag{1.60}$$

This relation gives the opportunity to extend the concept of autocorrelation function to deterministic signals. Then, for a periodic signal with period N_0, the autocorrelation function is the sequence $r(n)$ given by:

$$r(n) = \frac{1}{N_0}\sum_{i=0}^{N_0-1} s(i)s(i-n) \tag{1.61}$$

It is a periodic sequence with the same period.

Example:

$$s(n) = A \sin\left(2\pi \frac{n}{N_0}\right)$$

$$r(n) = \frac{1}{N_0}\sum_{i=0}^{N_0} A^2 \sin\left(2\pi \frac{i}{N_0}\right) \sin 2\pi\left(\frac{i-n}{N_0}\right)$$

$$r(n) = \frac{A^2}{2} \cos\left(2\pi \frac{n}{N_0}\right)$$

The period of $r(n)$ is N_0 and $r(0)$ is the signal power.

A discrete random signal can also be defined directly. For example, if $r(n) = 0$ for $n \neq 0$, the signal $s(n)$ is a discrete white noise, and the spectral density is constant over the frequency interval $\left[-\frac{1}{2}, \frac{1}{2}\right]$. This signal represents a physical reality – it is a sequence of noncorrelated random variables, and it can be obtained through an algorithm which produces statistically independent numbers.

1.8.3 Discrete Noise Generation

It has been shown in Section 1.8.1 that the generation of digital sinusoidal signals is particularly simple. Such signals can be used to simulate noise, for example, by addition of a large number of sine waves of different frequencies and constant amplitudes, having random or pseudorandom phases. This approach can lead to particularly efficient realizations, like that which has been standardized for measuring equipment in digital telephone transmission and is as follows:

A pseudorandom sequence is created, which is a periodic sequence of $2^N - 1$ bits comprising approximately as many "zeros" as "ones" and which simulates a random sequence in which the bits are independent and have the probability $\frac{1}{2}$ of having a value of 0 or 1 (or in order to center the variables, a value of $\pm\frac{1}{2}$).

If such a set is filtered (filtering in fact consists of weighted summation), the numbers obtained after filtering obey a probability distribution which approximates the normal distribution.

Pseudorandom sequences are studied in Ref. [6] and can be obtained easily using a suitably looped N-bit shift register. Figure 1.10 gives an example used in measuring equipment where $N = 17$. The generator polynomial is written:

$$g(x) = 1 + x^3 + x^{17} \tag{1.62}$$

The sequence contains $2^N - 1 = 131071$ bits, and it is periodic with a period $T = (2^N - 1)\tau$, where $\tau = 1/f_H$ denotes the clock period. The spectrum is formed of lines $1/T$ apart. For $f_H = 370$ kHz, the distance between the lines is 2.8 Hz and there are 36 lines per 100 Hz.

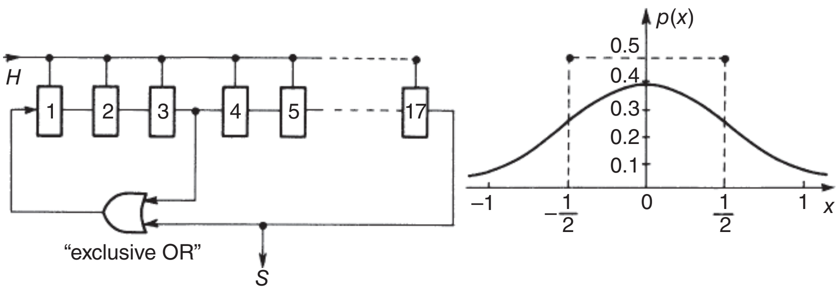


Figure 1.10 Pseudorandom generator and the probability distribution after filtering.

By applying to this set a narrowband filter which passes only the band 450–550 Hz, an approximately Gaussian signal is obtained. The signal has a peak factor of 10.5 dB and is an excellent test signal for digital transmission equipment. If the filtering is performed numerically, the set of numbers obtained can be used to test digital processing equipment.

1.9 Quantization

Quantization is the approximation of each signal value $s(t)$ by a whole multiple of an elementary quantity q which is called the quantizing step. If q is constant for all signal amplitudes, the quantization is said to be uniform. This operation is carried out by passing the signal through a device which has a staircase characteristic and produces the signal $s_q(t)$, as shown in Figure 1.11 for $q = 1$.

The way in which the approximation is made defines the centering of this characteristic. For example, the diagram represents the case (called rounding), where each value of the signal between $\left(n - \frac{1}{2}\right) q$ and $\left(n + \frac{1}{2}\right) q$ is rounded to nq. This approximation minimizes the power of the error signal. It is also possible to have approximation by default, which is obtained by truncation and consists of approximating every value between nq and $(n + 1)q$ by nq; the characteristic is therefore displaced by $q/2$ toward the right on the abscissa.

The effect of this approximation is to superimpose on the original signal an error signal $e(t)$ called the quantizing distortion or, more commonly, the quantizing noise. Thus:

$$s(t) = s_q(t) + e(t) \tag{1.63}$$

The case of rounding is illustrated in Figure 1.12. The amplitudes at odd multiples of $q/2$ are called the decision amplitudes. The amplitude of the error signal lies between $-q/2$ and $q/2$ and power measures the degradation undergone by the signal.

When the variations in the signal are large relative to the quantizing step – that is, when quantization has been carried out sufficiently finely – the error signal is equivalent to a set of elementary signals which are each formed from a straight-line segment (Figure 1.13). The power of such an

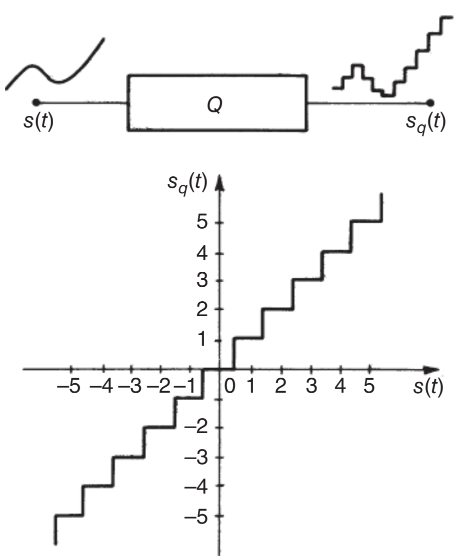


Figure 1.11 The quantization operation.

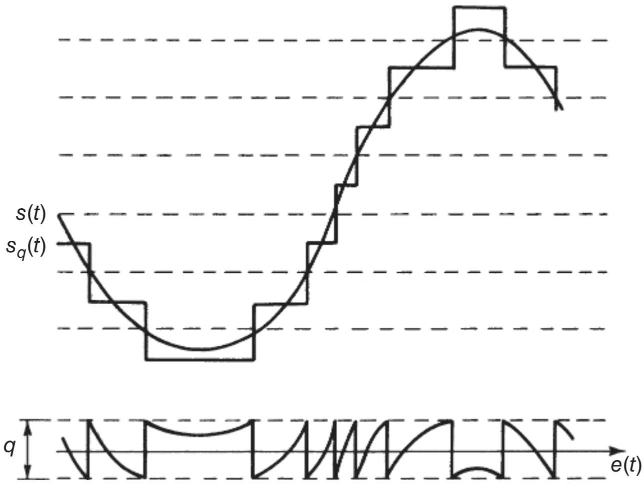


Figure 1.12 Quantization error.

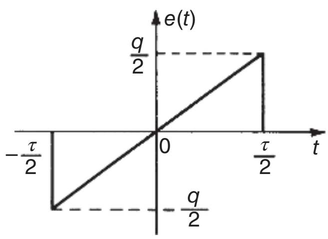


Figure 1.13 Elementary error signal.

elementary signal of width τ is written:

$$B = \frac{1}{\tau}\int_{-\tau/2}^{\tau/2} e^2(t)\mathrm{d}t = \frac{1}{\tau}\left(\frac{q}{\tau}\right)^2 \int_{-\tau/2}^{\tau/2} t^2\mathrm{d}t = \frac{q^2}{12} \tag{1.64}$$

The value obtained in this way, $B = q^2/12$, is a satisfactory estimate of the power of the quantizing noise in the majority of actual cases.

The spectral distribution of the error signal is more difficult to discern. The spectrum of the elementary error signal in Figure 1.13, $E_\tau(f)$, can be derived from its derivative. Using expressions (1.22) and then (1.12), the following is obtained:

$$E_\tau(f) = \frac{1}{\mathrm{j}2\pi f}q\left[\frac{\sin(\pi f\tau)}{\pi f\tau} - \cos(\pi f\tau)\right] \tag{1.65}$$

It appears that the majority of the energy is found around the frequency $1/\tau$. Under these conditions, the spectral distribution of the error signal depends both on the slope of the elementary signal – that is, on the statistical distribution of the derivative $s'(t)$ of the signal – and on the size of the step q relative to the signal. Reference [7] gives the calculation of this spectrum for a noise signal and shows the spread as a function of frequency, when the quantizing step is sufficiently small, to be on a range several hundred times the width of the signal band. If the signal to be quantized is not random, the spectrum of the error signal will be concentrated on certain frequencies – for example, the harmonics of a sinusoidal signal.

When converting an analog signal into digital form, quantization occurs, along with the sampling, as the two operations are carried out in succession. While sampling is normally carried out first, it is equally valid to carry out the quantization first and the sampling second, at a frequency f_s which is usually a little over twice the bandwidth of the signal. Under these conditions, the error signal often has a spectrum which extends beyond the sampling frequency, and since it is actually the sum of the signal and the error signal which is sampled, aliasing occurs, and the whole energy of the error signal is recovered in the frequency band $(-f_s/2, f_s/2)$. In the majority of cases, the spectral energy density of the quantizing noise is constant, and the following statement can be made:

The noise produced during uniform quantization with a step q has a power which is generally expressed by $B = q^2/12$ and shows a constant spectral distribution in the frequency band $(-f_s/2, f_s/2)$.

It should be noted that the quantization of small signals (those with amplitude of the same order of magnitude as step q) depends critically on the centering of the characteristic. For example, with the centering in Figure 1.11, a sinusoidal signal with an amplitude of less than $q/2$ is totally suppressed. It is possible, nevertheless, to suitably code these small signals by superimposing on them a large-amplitude auxiliary signal which is removed later in the process. Thus, the coding introduces limits on the small amplitudes of the signal. Equally, however, other limits appear in the case of large amplitudes, as will be seen below.

1.10 The Coding Dynamic Range

The signal which is sampled and quantized in amplitude is represented by a set of numbers which are almost always in binary form. If each number has N bits, the maximum number of quantized amplitudes that it is possible to represent is 2^N. Thus, the range of amplitudes that can be coded is subject to a twofold limitation – at low values, it is limited by the quantum q, and at larger values by $2^N q$. Any amplitude that exceeds this value cannot be represented and the signal is clipped. This results in degradation, so that, for example, if the signal is sinusoidal, harmonic distortion is introduced.

If the range of amplitudes to be coded covers the domain $[-A_m, A_m]$, we get:

$$A_m = 2^N q/2 \tag{1.66}$$

and, with rounding, the error signal $e(t)$ is:

$$|e(t)| \leq A_m 2^{-N}$$

By definition, the peak power of a coder is the power of that sinusoidal signal which has the maximum possible amplitude, A_m, which the coder will pass without clipping. It is expressed by:

$$P_c = \frac{1}{2}\left[\frac{2^N q}{2}\right]^2 = 2^{2N-3} q^2$$

Figure 1.14 illustrates this signal together with the quantizing step and the decision amplitudes.

The coding dynamic range is defined as the ratio between this peak power and the power of the quantizing noise; this is, in fact, the maximum value of the signal-to-noise ratio (SNR) for a sinusoidal signal with uniform coding. The following formula expresses this dynamic range:

$$P_c/B = (S/B)_{\max} = 2^{2N-3} \cdot 12 = \frac{3}{2} \cdot 2^{2N}$$

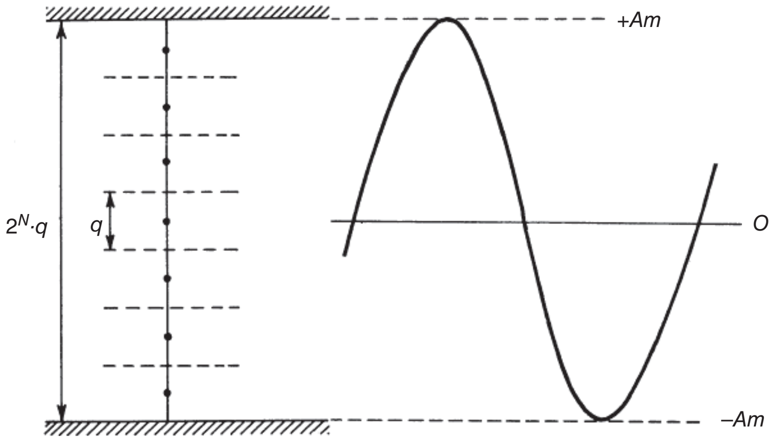


Figure 1.14 Peak power of the coder.

This can be expressed more conveniently in dB as:

$$P_c/B = 6.02N + 1.76\text{dB} \tag{1.67}$$

This formula, which is of great practical use, relates the number of bits in the coding to the range of amplitudes which can be coded.

The signal to be coded is, in general, not sinusoidal. However, it is still possible to treat this case if an equivalent peak power can be defined for the signal, which can then be taken as the peak power of the coder. This occurs, for example, with multiplexed telephone signals. The case of random Gaussian signals is of particular importance because they conveniently represent a large number of signals which are encountered in practice. The maximum amplitude of the coder must then be correctly positioned relative to the signal amplitude so that the distortion introduced by peak limitation remains within the imposed limits.

It is evident from the table given in Appendix 2 that the probability is less than 10^{-3} that a signal exceeds 3.4σ when it has a zero mean and a power of σ^2. Figure 1.15 gives an example of coding where $\sigma = q$. It would appear that the probability of clipping is less than 5×10^{-4} for the chosen parameters.

Finally, in order to obtain the maximum SNR, it is necessary to consider the peak SNR expressed by:

$$(\text{SNR})_c = \frac{A_{\text{m}}^2 q^2}{12} = 3.2^N$$

and subtract the peak factor F_c. Thus, the general expression for the maximum SNR is:

$$(\text{SNR})_{\text{max}} = 6.02\,N + 4.77 - F_c\,dB \tag{1.67bis}$$

This result is used not only in specifying analog signal coders but also in digital processing for determining memory sizes and scaling internal data.

The coding dynamic range for a given number of bits can be considerably increased if coding is carried out with a quantizing step which varies with the amplitude of the signal. This is called nonlinear coding. Numerous variations can be envisaged, but of particular importance is the 13-segment coding law which has been standardized by the International Telegraph and Telephone Consultative Committee (CCITT) for the coding of signals in telecommunications networks [8].

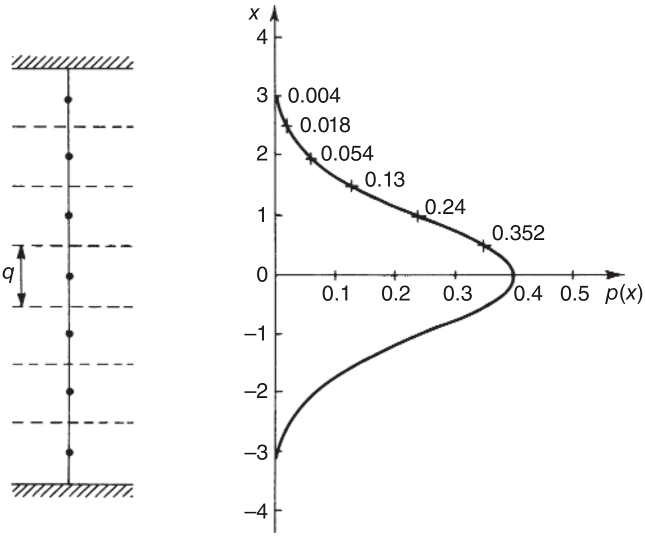


Figure 1.15 Coding of a Gaussian signal.

1.11 Nonlinear Coding with the 13-segment A-law

In nonlinear coding using the 13-segment A-law, the positive and negative amplitudes which are to be coded are divided into 7 ranges, each with a quantizing step which is related to an elementary step q by some power of 2. This operation can be regarded as resulting from linear coding which was preceded by a compression in which the signal x is transformed into the signal y according to the following relations:

$$y = \text{sign}(x)\frac{1+\log A\,|\,x\,|}{1+\log \quad A} \qquad \text{for} \quad \frac{1}{A} \leqslant |\,x\,| \leqslant 1$$

$$y = \text{sign}(x)\frac{A\,|\,x\,|}{1+\log A} \qquad \text{for} \quad 0 \leqslant |\,x\,| \leqslant \frac{1}{A} \tag{1.68}$$

The parameter A controls the dynamic range increase; the value used is $A = 87.6$.

The compression characteristic of the 13-segment A-law is shown in Figure 1.16.

The operation can be described as follows:

$$\text{if } 0 < x < \frac{1}{64}, \qquad \text{then } y = 16x$$

$$\frac{1}{64} \leqslant |\,x\,| \leqslant \frac{1}{32} \qquad y = 8x + \frac{1}{8}$$

$$\frac{1}{32} \leqslant |\,x\,| \leqslant \frac{1}{16} \qquad y = 4x + \frac{1}{4}$$

$$\frac{1}{16} \leqslant |\,x\,| \leqslant \frac{1}{8} \qquad y = 2x + \frac{3}{8}$$

$$\frac{1}{8} \leqslant |\,x\,| \leqslant \frac{1}{4} \qquad y = x + \frac{1}{2}$$

$$\frac{1}{4} \leqslant |\,x\,| \leqslant \frac{1}{2} \qquad y = \frac{1}{2}x + \frac{5}{8}$$

$$\frac{1}{2} \leqslant |\,x\,| \leqslant 1 \qquad y = \frac{1}{4}x + \frac{3}{4}$$

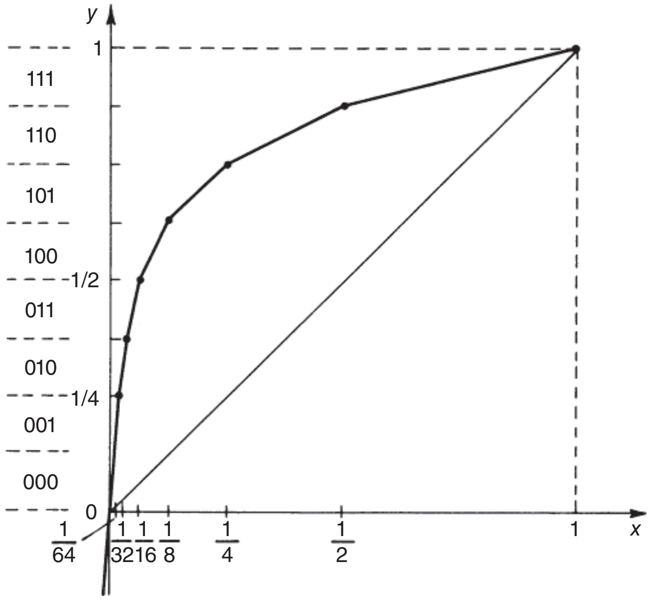


Figure 1.16 The 13-segment compression A-law.

This characteristic causes seven straight-line segments to appear in both the positive and negative quadrants. As the two segments which encompass the origin are colinear, the characteristic has a total of 13 segments.

Since the quantization of the y amplitudes is carried out with the quantum q, quantization of the x amplitudes near the origin is based on a quantum $q/16$ – that is, the dynamic of the coder is increased by 24 dB. Amplitudes close to unity are less well quantized as the step is multiplied by 4. The power of the quantization noise is a function of the signal amplitude – it is necessary to calculate an average for each value, and this interferes with the statistics of the signal.

Figure 1.17 gives the SNR for a Gaussian signal as a function of the signal level after coding into 8 bits for linear and nonlinear coding. The reference level for the signal (0 dB) is the peak power of the coder, and it is clearly apparent that the dynamic is extended by nonlinear coding. For low amplitudes, quantization in fact corresponds to 12 bits. In practice, the signal can be coded by linear quantization into 12 bits, followed by a process which is very close to the conversion of an integer into a floating-point number:

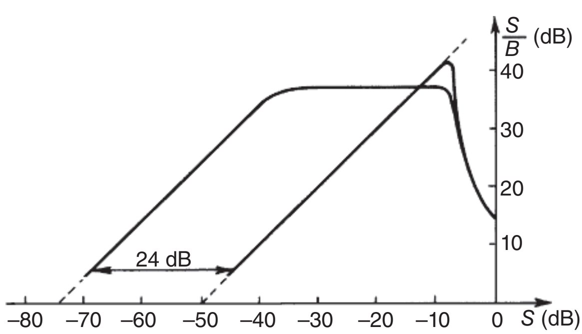


Figure 1.17 Eight-bit linear and nonlinear coding of a Gaussian signal.

For example, the 12-bit number

+0 0 0 1 0 1 1 0 1 1 0

corresponds to the 8-bit one

+1 0 0 0 1 1 0

after application of the compression law.

The three bits which follow the sign give the code for the exponent; the four following bits give the position within the segment or the mantissa. The difference in the conversion from a whole number to a floating-point number appears in the vicinity of the origin.

The implementation can use an array of gates arranged in parallel, or a shift register combined with a 3-bit counter, for serial realization. Equally, one could use memories holding the conversion table.

Another nonlinear coding law is also widely used in telecommunications – the 15-segment μ-law, which follows the relation:

$$y = \mathrm{sign}(x)\frac{\log(1+\mu|x|)}{\log(1+\mu)} \quad \text{for } -1 \leqslant x \leqslant 1 \tag{1.69}$$

The compression parameter is $\mu = 255$.

1.12 Optimal Coding

The coding can be improved when the probability distribution $p(x)$ of the signal amplitude is known. For a given number of bits N, an optimal quantizing characteristic can be found, which minimizes the total quantizing distortion.

The signal amplitude range is divided into $M = 2^N$ subsets (x_{i-1},x_i) with $-(M/2)+1 \leqslant i \leqslant M/2$ and every subset is represented by a value $y_{i,}$ as shown in Figure 1.18. The optimization consists of determining the set of values x_i and y_i which minimize the error signal power E^2 expressed by:

$$E^2 = \sum_{i=-(M/2)+1}^{M/2} \int_{x_i-1}^{x_i} (x-y_i)^2 p(x)\mathrm{d}x$$

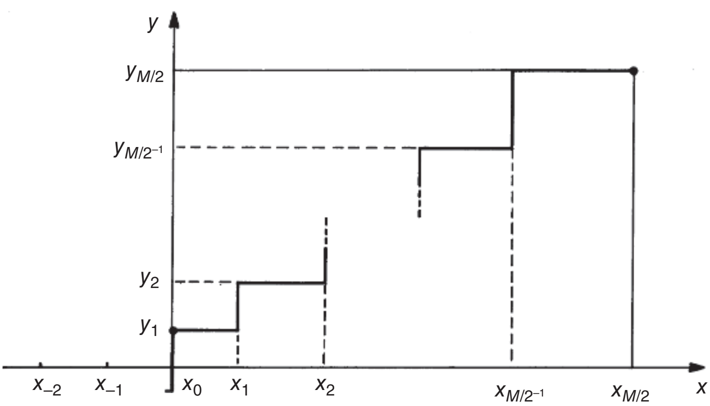


Figure 1.18 Optimal quantization characteristic.

Taking the derivative with respect to x_i and y_i, it can be shown that the following relations must hold:

$$x_i = \frac{1}{2}(y_i + y_{i+1}) \quad \text{for} -\frac{M}{2}+1 \leqslant i \leqslant \frac{M}{2}-1$$
$$\int_{x_i-0}^{x_i}(x-y_i)p(x)\mathrm{d}x = 0 \quad \text{for} \quad \in -\frac{M}{2}+1 \leqslant i \leqslant \frac{M}{2}$$
$$p(x_{m/2}) = p(x_{-M/2}) = 0 \tag{1.70}$$

These relations lead to the determination of the quantizing characteristic. If $p(x)$ is an even function, $x_0 = 0$, and an iterative procedure is used, starting with an *a priori* choice for y_1. If relation (1.70) is not satisfied for $M/2$, another initial choice is made for y_1 and so on [9].

Table 1.1 gives the error signal power obtained with a Gaussian signal of unit power, for several numbers of bits N, for the optimal coding and for uniform coding with the best scaling of the quantizing characteristic [9]. Table 1.2, taken from Reference [10], gives, for the same conditions, the values which correspond to a signal probability density following expression (1.48).

The coding optimization can also be carried out with respect to the information content, by introducing the concept of entropy H defined as follows [2, Vol. 3]:

$$H = -\sum_i p_i \log_2(p_i) \tag{1.71}$$

with

$$-\frac{M}{2}+1 \leqslant i \leqslant \frac{M}{2},$$

and where p_i designates the probability that the signal is in the amplitude subrange represented by y_i.

Considering that:

$$\sum_i p_i = 1$$

Table 1.1 Coding of unitary Gaussian signal.

N **E**	**1**	**2**	**3**	**4**	**5**
Optimal coding	0.3634	0.1175	0.03454	0.0095	0.0025
Uniform coding	0.3634	0.1188	0.03744	0.01154	0.00349
Entropy H	1	1.911	2.825	3.765	4.730

Table 1.2 Coding of unitary Laplacian signal.

N E^2	**1**	**2**	**3**	**4**	**5**
Optimal coding	0.5	0.1765	0.0548	0.0954	0.00414
Uniform coding	0.5	0.1963	0.0717	0.0254	0.0087

the entropy is zero when the amplitude is concentrated in a single subrange. It is maximal when the signal amplitude is uniformly distributed when it takes the value $H_{\max}$ equal to the number of bits N of the coder:

$$H_{\max} = \log_2 \ M = N \tag{1.72}$$

In fact, the entropy measures the difference between a given distribution and the uniform distribution. The quantizing characteristic which maximizes the entropy is that which leads to amplitude subranges corresponding to a uniform probability distribution.

The last row in Table 1.1 shows that for a Gaussian signal, the quantizing law which minimizes the error signal power leads to entropy values close to the maximum N.

1.13 Quantity of Information and Channel Capacity

The results obtained for sampling and quantization can be used, inversely, to evaluate the quantity of information carried by a signal or to determine the capacity of a transmission channel.

A real channel of bandwidth f_m can carry $2f_m$ independent samples per second, as shown in Figure 1.4, by replacing τ with $2f_m$. The quantity of information per sample depends on the relative powers of the useful signal, the noise, and their amplitude distributions.

An important particular case is the Gaussian channel [11].

Let us assume a set of M symbols of N bits each is to be transmitted by a channel in the presence of white Gaussian noise of power $B = \sigma_b^2$.

In a M-dimension hyperspace, the M symbols occupy the volume of a hypersphere V_M defined by:

$$V_M = \int_o^R r^{M-1}\,\mathrm{d}r \int_{i=1,..,M-1} \cdots \int f(\theta_i)\,\mathrm{d}\theta_i = \frac{R^M}{M} F_\theta \tag{1.73}$$

If a uniform distribution of the symbols in the hypersphere with radius R is assumed, the energy of the corresponding signal is:

$$E_S = \frac{1}{V_M} \int_0^R r^2 r^{M-1} \mathrm{d}r \int_{i=1} \cdots \int_{M-1} f(\theta_i)\mathrm{d}\theta_i = R^2 \frac{M}{M+2} \tag{1.74}$$

The quantity of transmitted information is $M{\cdot}N$ bits. In the hypersphere, it is possible to associate with each set of bits a volume V_s expressed by:

$$V_s = \frac{V_M}{2^{MN}} = \frac{1}{M} F(\theta) \left(\frac{R}{2^N}\right)^M \tag{1.75}$$

Now, a M-component noise with energy $E_b = M\sigma_b^2$ is assigned to each set of bits. When M tends toward infinity, the point representing the noise in the hypersphere comes close to a sphere with radius $\sigma_b^2\sqrt{M}$ and centered on the point representing the set of bits. In fact, for M Gaussian random variables $b(n)$, the variable $r = \sqrt{\sum_{n=1}^{M} b^2(n)}$ has the first-order moment $m_1 = \sigma_b\sqrt{\frac{M^2}{M+1}}$ and its variance $m_2 - m_1^2$ tends toward zero when M tends toward infinity. The volume of the sphere is:

$$V_b = \frac{(\sqrt{M}\,\sigma_b)^M}{M} F(\theta) \tag{1.76}$$

The condition for the absence of transmission errors is that the volume of the sphere be included in the volume assigned to each set of bits, which leads to:

$$\sqrt{M}\,\sigma_b < \frac{R}{2^N} \tag{1.77}$$

However, when M tends toward infinity, according to (1.74), R^2 represents the energy of the sum of the signal with power S and the noise:

$$R^2 = M\left(S + \sigma_b^2\right) \tag{1.78}$$

Then, there are no transmission errors if the following inequality is satisfied:

$$2^{2N} < \frac{S + \sigma_b^2}{\sigma_b^2} \tag{1.79}$$

Hence:

$$N < \frac{1}{2}\log_2\left(1 + \frac{S}{\sigma_b^2}\right) \tag{1.80}$$

Assuming a real channel of bandwidth W and no distortion, the symbols may be emitted at the rate $2W$, and the asymptotic capacity of the channel, in bits per second, is:

$$C = W\log_2\left(1 + \frac{S}{B}\right) \tag{1.81}$$

The assumptions are worth emphasizing:

– distortion-free channel
– white Gaussian noise
– infinite transmission delay

In practice, channel equalization and error-correcting codes make it achievable to approach that limit with finite transmission delay.

1.14 Binary Representations

There are several ways of establishing the correspondence between the set of quantized amplitudes and the set of binary numbers which must represent them. As the signals to be coded generally have both positive and negative amplitudes, the preferred representations are those which preserve the sign information. The following are the most usual representations:

(1) Sign and absolute value
(2) Offset binary
(3) 1s-complement
(4) 2s-complement

Table 1.3 defines them for three bits.

The most useful representations for analog-to-digital conversion are sign and absolute value, and offset binary. The remaining two representations are mostly used in arithmetic circuits. As mentioned in Section 1.11, nonlinear coding brings a considerable increase of the dynamic range.

Table 1.3 Binary representation for linear code.

Number	Sign and value	Offset binary	1s complement	2s complement
+3	0 1 1	1 1 1	0 1 1	0 1 1
+2	0 1 0	1 1 0	0 1 0	0 1 0
+1	0 0 1	1 0 1	0 0 1	0 0 1
+0	0 0 0	1 0 0	0 0 0	0 0 0
–0	1 0 0	–	1 1 1	–
–1	1 0 1	0 1 1	1 1 0	1 1 1
–2	1 1 0	0 1 0	1 0 1	1 1 0
–3	1 1 1	0 0 1	1 0 0	1 0 1
		(0 0 0)		(1 0 0)

Digital processing machines, and particularly general-purpose ones, often use floating-point representations in which each number has three parts: the sign bit, the mantissa, and the exponent. The mantissa represents the fractional part, and the exponent is a power of the base number, for example, with base 10: $+0.719 \times 10^5$.

The dynamic range extension comes from the multiplicative effect introduced by the exponent. For example, in base 2 for a 6-bit exponent and 16-bit mantissa, the dynamic range is: $2^{2^6} \times 2^{16} = 2^{80} \simeq 10^{24}$ – that is, 24 decimal numbers. Additional gain is achieved by choosing a base which is, itself, a power of two, such as 8 or 16, leading to octal or hexadecimal operations (Figure 1.19).

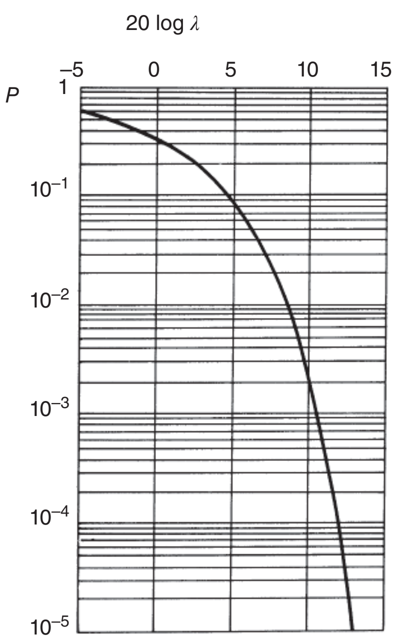


Figure 1.19 The reduced normal distribution: P as a function of 20 log λ.

1.A Appendix 1: The Function $I(x)$

The set of values:

$$I(n) = \frac{\sin(\pi(n/20))}{\pi(n/20)}$$

for $0 \leqslant n \leqslant 159$ when $n = k + 20N$, is given in the table below.

k	$N = 0$	$N = 1$	$N = 2$	$N = 3$	$N = 4$	$N = 5$	$N = 6$	$N = 7$
0	1	0	0	0	0	0	0	0
1	0.99589	−0.04742	0.02429	−0.01633	0.01229	−0.00986	0.00823	−0.00706
2	0.98363	−0.08942	0.04684	−0.03173	0.02399	−0.01929	0.01613	−0.01385
3	0.96340	−0.12566	0.06721	−0.04588	0.03482	−0.02806	0.02350	−0.02021
4	0.93549	−0.15591	0.08504	−0.05847	0.04455	−0.03598	0.03018	−0.02599
5	0.90032	−0.18006	0.10004	−0.06926	0.05296	−0.04287	0.03601	−0.03105
6	0.85839	−0.19809	0.11196	−0.07804	0.05989	−0.04850	0.04088	−0.03528
7	0.81033	−0.21009	0.12069	−0.08466	0.06520	−0.05301	0.04466	−0.03859
8	0.75683	−0.21624	0.12614	−0.08904	0.06880	−0.05606	0.04730	−0.04091
9	0.69865	−0.21682	0.12832	−0.09113	0.07065	−0.05769	0.04874	−0.04220
10	0.63662	−0.21221	0.12732	−0.09095	0.07074	−0.05787	0.04897	−0.04244
11	0.57162	−0.20283	0.12329	−0.08856	0.06910	−0.05665	0.04800	−0.04164
12	0.50455	−0.18921	0.11643	−0.08409	0.06581	−0.05406	0.04587	−0.03983
13	0.43633	−0.17189	0.10702	−0.07770	0.06099	−0.05020	0.04265	−0.03707
14	0.36788	−0.15148	0.09538	−0.06960	0.05479	−0.04518	0.03844	−0.03344
15	0.30011	−0.12862	0.08185	−0.06002	0.04739	−0.03914	0.03335	−0.02904
16	0.23387	−0.10394	0.06682	−0.04924	0.03989	−0.03298	0.02751	−0.02399
17	0.17001	−0.07811	0.05071	−0.03753	0.02980	−0.02470	0.02110	−0.01841
18	0.10929	−0.05177	0.03392	−0.02522	0.02007	−0.01667	0.01426	−0.01245
19	0.05242	−0.02544	0.01688	−0.01261	0.01006	−0.00837	0.00716	−0.00626

1.B Appendix 2: The Reduced Normal Distribution

$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$		$P = \frac{2}{\sqrt{2\pi}} \int_\lambda^\infty e^{-x^2/2} dx$		$P = \frac{2}{\sqrt{2\pi}} \int_\lambda^\infty e^{-x^2/2} dx$	
X	$10^5 \cdot f(x)$	100 P	λ	λ	100 P
0	39,894	100	0	0	100
0.2	39,104	95	0.0627	0.2	84.148
0.4	36,827	90	0.1257	0.4	68.916
0.6	33,322	85	0.1891	0.6	54.851
0.8	28,969	80	0.2533	0.8	42.371
1	24,197	75	0.3186	1	31.731

$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$		$P = \frac{2}{\sqrt{2\pi}} \int_{\lambda}^{\infty} e^{-x^2/2} dx$		$P = \frac{2}{\sqrt{2\pi}} \int_{\lambda}^{\infty} e^{-x^2/2} dx$	
X	**$10^5 \cdot f(x)$**	**100 P**	**λ**	**λ**	**100 P**
1.2	19,419	70	0.3853	1.2	23.014
1.4	14,973	65	0.4538	1.4	16.151
1.6	11,092	60	0.5244	1.6	10.960
1.8	7,895	55	0.5978	1.8	7.186
2	5,399	50	0.6745	2	4.550
2.2	3,547	45	0.7554	2.2	2.781
2.4	2,239	40	0.8416	2.4	1.640
2.6	1,358	35	0.9346	2.6	0.932
2.8	792	30	1.0364	2.8	0.511
3	443	25	1.1503	3	0.270
3.2	238	20	1.2816	3.2	0.137
3.4	123	15	1.4395	3.4	0.067
3.6	61	10	1.6449	3.6	0.032
3.8	29	5	1.9600	3.8	0.014
4	13	1	2.5758	4	0.006
4.2	5.9	0.1	3.2905	4.5	0.00068
4.4	2.5	0.01	3.8906	5	0.000057
4.6	1	0.001	4.4172	5.5	0.000004
		0.0001	4.8916		
		0.00001	5.3267		

Approximation for large values of λ:

$$P \approx \frac{3}{4} \frac{1}{\lambda} e^{-\lambda/2}$$

Exercises

1.1 Consider the Fourier series expansion of the periodic function $i(t)$ of period T, which is zero throughout the period except for the range $-\tau/2 \leqslant t \leqslant \tau/2$, where it has a value of 1.
Give the value of the coefficients for $\tau = T/2$ and $\tau = T/3$.
Verify that the expansion leads to $i(0) = 1$ and draw the function when the expansion is limited to 5 terms.

1.2 Analyze the sampling at the frequency f_s of the signal $s(t) = \sin(\pi f_s t + \varphi)$ when φ varies from 0 to $\pi/2$.
Examine the reconstitution of this signal from the samples.

1.3 Calculate the amplitude distortion introduced into a signal reconstituted by pulses with a width of half the sampling period.

1.4 A signal occupies the frequency band $[f_1, f_2]$. What conditions should be imposed on the frequency f_1 so that this signal can be sampled directly at a frequency between f_2 and $2f_2$?

1.5 Analyze the sampling of a signal given by:

$$s_i(t) = \sum_{n=1}^{3} \sin\left(2\pi \frac{n}{8}\frac{t}{T}\right)$$

and compare it with that of the signal:

$$s_r(t) = \sum_{n=1}^{3} \cos\left(2\pi \frac{n}{8}\frac{t}{T}\right)$$

Show by studying the spectra that the combination of the two sets of samplings forms the sampling of a complex signal.

1.6 Let $s(t)$ be the signal defined by:

$$s(t) = 1 + 2\sum_{k=1}^{3} \cos\left(2\pi\frac{kt}{8} + \varphi_k\right) + \cos(\pi t + \varphi_4)$$

This signal is sampled with the period $T = 1$. What is the maximum value of $s(n)$, where n is a whole number? Show that there exists a set of values φ_k $(k = 1,2,3,4)$ which minimizes the maximum value of $s(n)$. Can this property be generalized?

1.7 A digital frequency synthesizer is constructed from read-only memory of 16 kbits with an access time of 500 ns. Knowing that the numbers which represent the samplings of sinusoidal signals total 8 bits, what are the characteristics of the synthesizer, and the frequency range and increment step that can be obtained?

1.8 What is the probability distribution of the amplitudes of the sinusoidal signal:

$$s(t) = A\ \cos\left(2\pi\frac{t}{T}\right)$$

Give its autocorrelation function. Give the autocorrelation function of a stationary random Gaussian function whose spectrum has a uniform distribution in the frequency band (f_1, f_2).

1.9 Calculate the spectrum of a set of impulses with width $T/2$, separated by T, the occurrence of each pulse having probability p. In particular, examine the case where $p = 1/2$.
What happens to the spectrum if these pulses form a pseudorandom sequence with a length $2^4-1 = 15$ produced by a 4-bit shift register, following the polynomial $g(x) = x^4 + x + 1$?

1.10 A sinusoidal signal with frequency 1050 Hz is sampled at 8 kHz and coded into 10 bits. What is the maximum value of the SNR? What is the value of the signal-to-quantization-noise ratio measured in the frequency band 300–500 Hz? What are the values if the sampling frequency is increased to 16 kHz?

1.11 The sinusoidal signal $\sin(2\pi t/8 + \varphi)$ with $0 \leqslant \varphi \leqslant \pi/2$ is sampled with period $T = 1$ and coded into 5 bits.
In the case where $\varphi = 0$, calculate the power and the spectrum of the quantization noise. How does this spectrum appear as a function of the phase φ?

1.12 Consider a coding scale in which the quantizing step has a value q. What is the quantization of the signal $s_1(t) = \alpha q \sin(\omega_1 t)$ for $-1 \leqslant \alpha \leqslant 1$, as a function of the centering of the quantization characteristic? Show the envelope of the restored signal after decoding and narrow filtering around the frequency ω_1.
The signal $s_2(t) = 10q \sin \omega_2 t$ is superimposed on $s_1(t)$. Show the envelope of the restored signal under these conditions.

1.13 Assume a Gaussian signal is to be coded. How many bits would be required to have the signal-to-quantization-noise ratio greater than 50 dB? Can this number be reduced if signal clipping is allowed for 1% of this time?

1.14 The signal $s(t) = A \sin(2\pi \cdot 810t)$ is coded into 8 bits. If the quantization step is q, trace the curve which shows the signal-to-quantization-noise ratio as a function of the amplitude A when this amplitude varies from q to $2^7 q$. Sketch the corresponding curve for nonlinear coding following the 13-segment A-law.

1.15 Calculate the limits of the amplitude subranges for the optimal 2-bit coding of a unit Gaussian signal.

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