IN THIS CHAPTER

» Measuring the area of shapes with classical geometry

» Finding the area of shapes on the *xy*-plane

» Using integration to frame the area problem

Chapter 1

An Aerial View of the Area Problem

In Calculus I, you discovered how to use math to solve a single problem: the *tangent problem* or *slope problem*, which involved finding the slope of a tangent line to a function on the *xy*-graph. Calculus II is also devoted to solving a single problem: the *area problem* — finding the area of a region of the *xy*-graph under a function.

In this chapter, you use a variety of simple methods to frame and solve area problems. First, you use formulas from classical geometry to measure the area of familiar shapes on the *xy*-graph. Next, you discover how to define the area of a region under a function as a definite integral.

To finish up, you put those two skills together, both setting up and solving definite integrals to solve area problems for relatively simple functions.

Ready to get started?

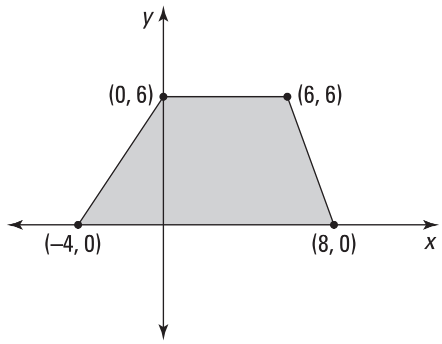
Measuring Area on the *xy*-Graph

People have been calculating the area of shapes for thousands of years. More than 2,000 years ago, Euclid developed a thorough system of geometry that included methods for finding the area of squares, rectangles, triangles, circles, and so on.

Descartes's invention of the *xy*-graph in the early 17th century made the study of geometry possible from an algebraic perspective, laying the foundation for calculus.

But you don't need calculus to measure the area of many basic shapes on the *xy*-graph. Here are a few formulas you know from geometry that are just as useful when measuring area on the *xy*-graph.

Square:	Rectangle:	Triangle:
Area $= s^2$	Area $= bh$	Area $= \frac{1}{2}bh$
Trapezoid:	Circle:	Semi-Circle:
Area $= \frac{b_1 + b_2}{2}h$	Area $= \pi r^2$	Area $= \frac{1}{2}\pi r^2$

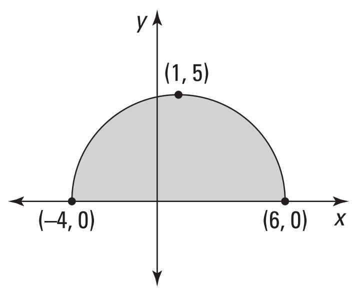


Q. What is the area of the shape in this figure?

EXAMPLE

A. **54.** The figure shows a trapezoid with bases of lengths 12 and 6, and a height of 6. Plug these values into the formula for a trapezoid:

$$\frac{b_1 + b_2}{2}h = \frac{12 + 6}{2}(6) = 54$$



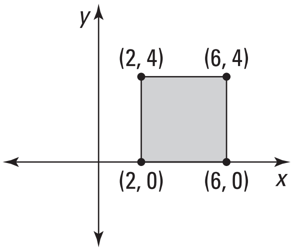
Q. Find the area inside the semi-circle shown in the figure.

EXAMPLE

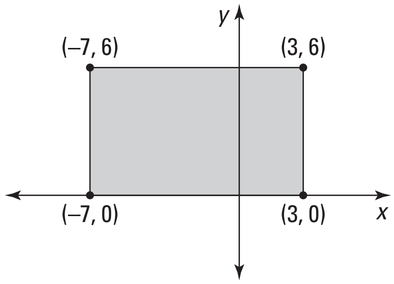
A. $\mathbf{12.5\pi}$**.** The semi-circle has a radius of 5, so plug this value into the formula for a semi-circle: $\frac{1}{2}\pi r^2 = \frac{1}{2}\pi(5)^2 = 12.5\pi$

In each of the following *xy*-graphs, find the area of the shape.

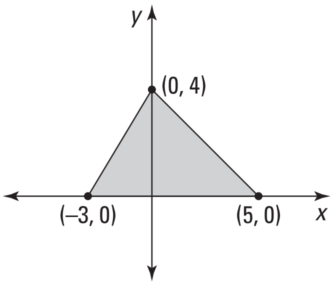
1



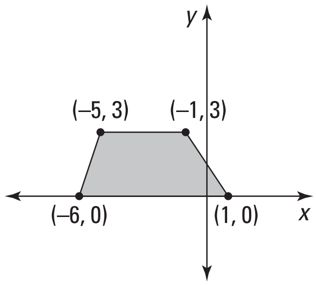
2



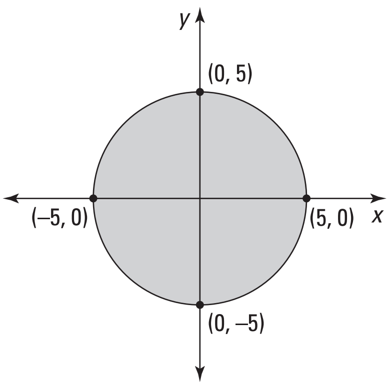
3



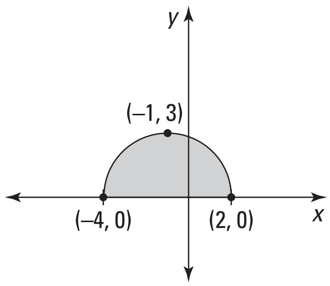
4



5



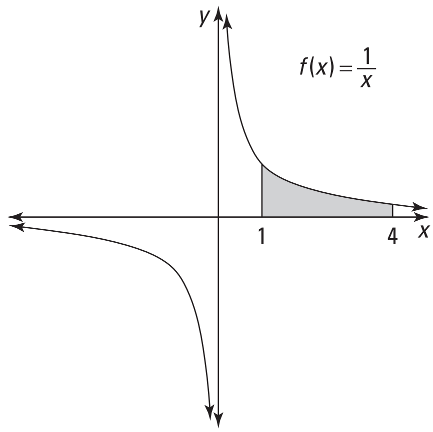
6



Defining Area Problems with the Definite Integral

The *definite integral* provides a systematic way to define an area on the *xy*-graph. For functions that reside entirely above the *x*-axis, a definite integral $\int_a^b f(x)\,dx$ defines an area that lies:

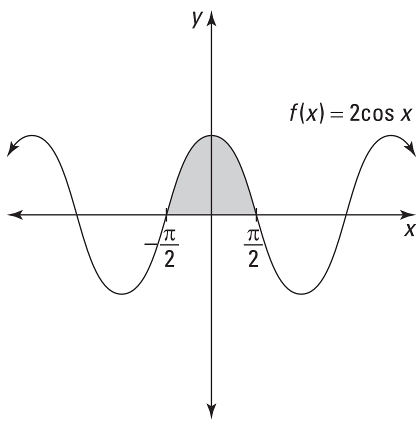
- Vertically between the function *f*(*x*) and the *x*-axis
- Horizontally between two *x*-values, *a* and *b*, called the *limits of integration*



EXAMPLE

Q. What definite integral describes the shaded area in the figure?

A. $\int_1^4 \frac{1}{x}\,dx$**.** The shaded area resides vertically between the function $\frac{1}{x}$ and the *x*-axis, and horizontally between $x = 1$ and $x = 4$.



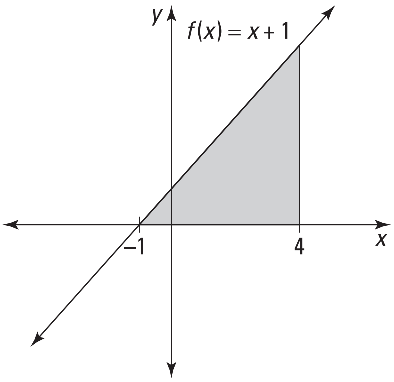
EXAMPLE

Q. Write a definite integral that defines the area of the shaded region in the figure.

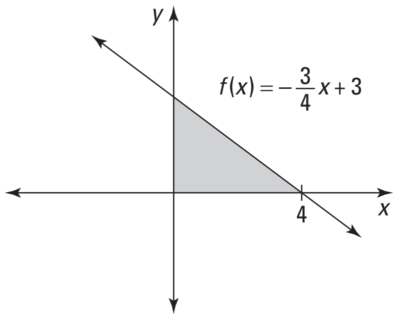
A. $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2\cos x\, dx$. The shaded region is vertically between $f(x) = 2\cos x$ and the x-axis, and horizontally between $x = -\frac{\pi}{2}$ and $x = \frac{\pi}{2}$.

In each of the following *xy*-graphs, write the definite integral that represents the shaded area.

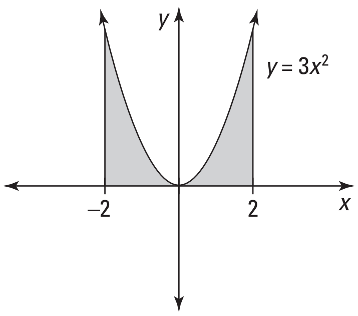
7



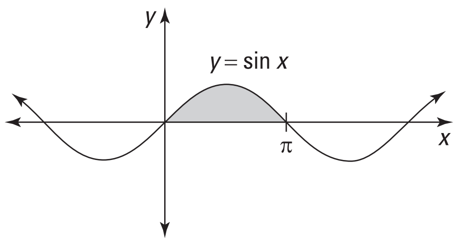
8



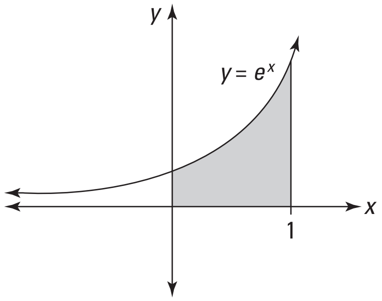
9



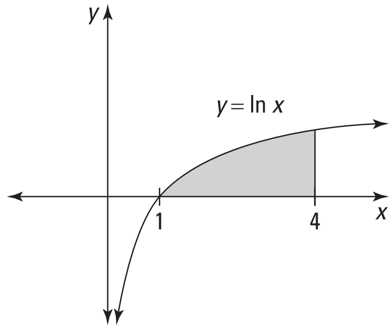
10



11



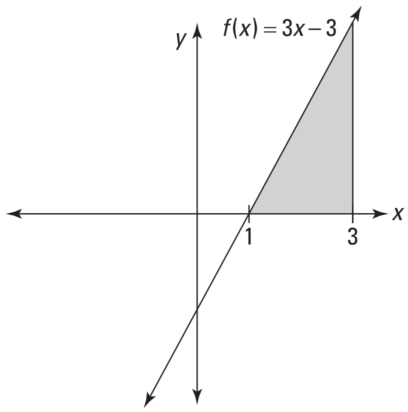
12



Calculating Area Defined by Functions and Curves on the *xy*-Graph

In the previous section, you discover how to use the definite integral to frame an area problem on the *xy*-graph. In Calculus II, you'll learn a ton of different ways to evaluate definite integrals to solve area problems.

But for many simple functions, you don't need calculus to evaluate a definite integral. For example:

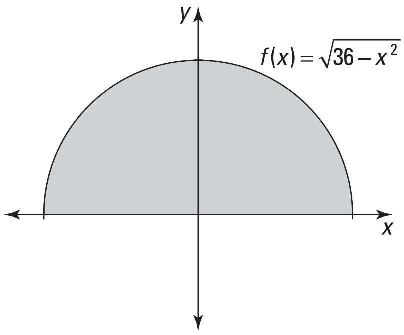


EXAMPLE

Q. Write a definite integral to describe the shaded area in the figure, and then use geometry to evaluate it.

A. $\boldsymbol{\int_1^3 3x - 3\ dx = 6}$**.** The shaded area is the triangle that resides vertically between the function $3x - 3$ and the x-axis and horizontally between $x = 1$ and $x = 3$. The base of this triangle is $3 - 1 = 2$. To find the height, plug in $f(3) = 3(3) - 3 = 6$. Now, to find the area, plug in 2 for the base and 6 for the height into the formula for a triangle:

$$\text{Area} = \frac{1}{2}bh = \frac{1}{2}(2)(6) = 6$$



EXAMPLE

Q. Use a definite integral to define the shaded region shown, and then find the area of that region without calculus.

A. $\boldsymbol{\int_{-6}^{6} \sqrt{36 - x^2}\ dx = 18\pi}$**.** The shaded area resides vertically between the function $\sqrt{36 - x^2}$ and the x-axis. To find the limits of integration, set this function to 0 and solve:

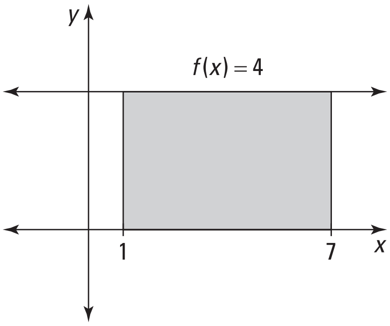
$$\begin{aligned} 0 &= \sqrt{36 - x^2} \\ 0 &= 36 - x^2 \\ x^2 &= 36 \\ x &= 6 \text{ and } -6 \end{aligned}$$

Thus, the limits of integration are $x = -6$ and $x = 6$. To find the area, plug in 6 for the radius into the formula for the area of a semi-circle:

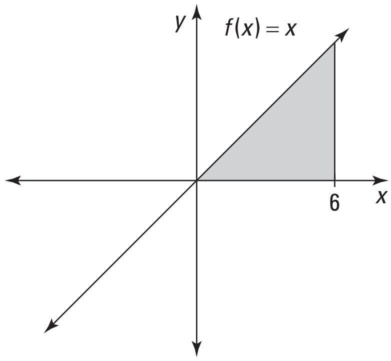
$$\frac{1}{2}\pi r^2 = \frac{1}{2}\pi(6)^2 = 18\pi$$

In each of the following xy-graphs, write and evaluate the definite integral that represents the shaded area.

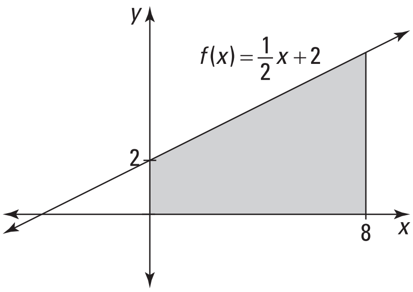
13



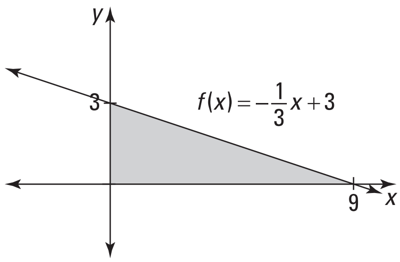
14



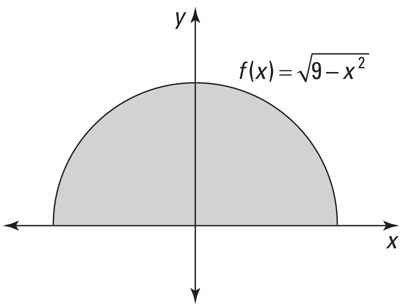
15



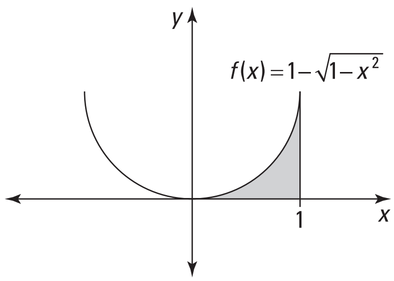
16



17



18



Answers and Explanations

1. **16.** $s^2 = 4^2 = 16$
2. **60.** $bh = (10)(6) = 60$
3. **16.** $\frac{1}{2}bh = \frac{1}{2}(8)(4) = 16$
4. **16.5.** $\frac{b_1 + b_2}{2}h = \frac{7+4}{2}(3) = 16.5$
5. **25π.** $\pi r^2 = \pi(5)^2 = 25\pi$
6. **32π.** $\frac{1}{2}\pi r^2 = \frac{1}{2}\pi(3)^2 = \frac{9}{2}\pi$
7. $\int_{-1}^{4} x + 1 \, dx$
8. $\int_{0}^{4} -\frac{3}{4}x + 3 \, dx$
9. $\int_{-2}^{2} 3x^2 \, dx$
10. $\int_{0}^{\pi} \sin x \, dx$
11. $\int_{0}^{1} e^x dx$
12. $\int_{1}^{4} \ln x \, dx$
13. $\int_{1}^{7} 4 \, dx = (6)(4) = 24$
14. $\int_{0}^{6} x \, dx = \frac{1}{2}(6)(6) = 18$
15. $\int_{0}^{8} \frac{1}{2}x + 2 \, dx = \frac{2+6}{2}(8) = 32$
16. $\int_{0}^{9} -\frac{1}{3}x + 3 \, dx = \frac{1}{2}(9)(3) = 13.5$
17. $\int_{-3}^{3} \sqrt{9 - x^2} \, dx = \frac{1}{2}\pi(3)^2 = 4.5\pi$
18. $\int_{0}^{1} 1 - \sqrt{1 - x^2} \, dx = 1 - \frac{\pi}{4}$