1

Synchronization for Complex Networks with Multiple Weights Under Recoverable Attacks

1.1 Introduction

During the last decade, the dynamical behavior of complex networks (CNs) has aroused increasing attention because CNs prevalently exist in the real world. Particularly, synchronization has been an appealing research topic in CNs, and many meaningful results have been obtained [1–16]. By choosing appropriate adaptive state-feedback controllers and Lyapunov functional, Zhou et al. [1] discussed the global and local synchronization in a CN with uncertain coupling functions. In [4], the synchronization problem for a CN with switching disconnected topology was addressed, and some synchronization conditions were established for such a network model. Lv et al. [5] tackled the exponential synchronization problem for CNs with coupling delay based on the impulsive control and event-triggered control techniques. In [11], the synchronization problem for stochastic CNs was discussed via pinning control technique and graph theory, in which the topology structure may be unknown. Zhu et al. [14] used the adaptive control method to deal with the synchronization problem for a type of CNs with time-varying delay, in which the restriction that time delay is differentiable is removed.

For some practical networks, such as urban population flow networks, food webs, etc., may be better described by CNs with multiple weights (CNMWs). More recently, some authors have addressed the problem of synchronization for CNMWs [17–26]. Wang et al. [17] not only investigated the pinning synchronization in the CNMWs with undirected and directed topologies but also presented several feedback gains and coupling strengths adjustment schemes. In [18], a criterion of synchronization for output-strictly passive CNMWs was obtained, and the synchronization problem of CNMWs was further discussed based on the nodes- and edges-based pinning control approaches, and the output-strict passivity. Zhao et al. [23] introduced a multiple delayed CN model with uncertain inner coupling matrices and developed a criterion of synchronization through

Dynamical Behaviors of Multiweighted Complex Network Systems, First Edition.
Jin-Liang Wang, Shun-Yan Ren, Huai-Ning Wu, and Tingwen Huang.

the adaptive control scheme for such a network model. Dong et al. [24] took into account the exponential synchronization of multiple delayed CNs with switching and fixed topologies by employing the scramblingness property for adjacency matrix. Qin et al. [26] analyzed the robust synchronization of multiple delayed CNs, and a criterion for guaranteeing the robust synchronization was also developed by employing the adaptive state-feedback controller.

It is well known that the network topology may be destroyed owing to the various forms of attacks (e.g., power grids, military communication networks, and so on [27–29]), which might lead to undesirable dynamical behavior in the CNs. Consequently, it is very meaningful to study the dynamical behavior for CNs under attacks. Recently, some researchers have studied the synchronization problem of CNs suffering the attacks [30, 31]. Wang et al. [30] investigated the synchronization for multiple memristive neural networks with the communication links subject to attacks and developed several synchronization criteria based on inequality techniques, M-matrix properties, and event-triggered control method. Wang et al. [31] gave a global synchronization criterion for a network model suffering the successful but recoverable attacks by exploiting the switching system theory and derived the upper bounds of the average recovering time and the attack frequency. Regretfully, the network models with single coupling were discussed in these existing works about the synchronization for CNs under attacks [30, 31], and the synchronization for CNMWs subject to attacks has not yet been explored. Obviously, it is very valuable and significative to further address the synchronization problem of CNMWs suffering the attacks.

This chapter discusses the synchronization for CNs with multiple state couplings (CNMSCs) or CNs multiple delayed state couplings (CNMDSCs) under recoverable attacks, respectively. The main contributions of our work are summarized as follows. First, we not only give a sufficient condition for ensuring the synchronization of directed CNMSCs suffering the attacks but also further study the synchronization problem by selecting the suitable state-feedback controller. Second, the analysis and control for the synchronization problem of undirected CNMSCs subject to attacks are also discussed, and several synchronization criteria are presented based on some inequality techniques. Third, we not only develop several synchronization criteria for CNMDSCs under attacks by constructing appropriate Lyapunov functional but also devise the suitable state-feedback controller to ensure the network synchronization.

1.2 Preliminaries

1.2.1 Notations

Let $\mathbb{Z}^+ = \{0, 1, 2, \dots\}$; for any real square matrix K, $[K]^s = K + K^T$; $\lambda_{\min}(\cdot)$ and $\lambda_{\max}(\cdot)$ denote the smallest and the largest eigenvalues of real symmetric matrix.

1.2.2 Lemmas

Lemma 1.1 (*See* [32]) *Define*

$$D = \begin{bmatrix} 1 & -1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & -1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & -1 & 0 \\ 0 & \cdots & 0 & 0 & 1 & -1 \end{bmatrix} \in \mathbb{R}^{(N-1)\times N},$$

$$\Phi = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 1 & \cdots & 1 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 1 \\ 0 & \cdots & 0 & 0 & 1 \\ 0 & \cdots & 0 & 0 & 0 \end{bmatrix} \in \mathbb{R}^{N\times(N-1)},$$

and let the sum of each row in the matrix $P \in \mathbb{R}^{N\times N}$ *be equal. Then,* $\mathbb{R}^{(N-1)\times(N-1)} \ni M = DP\Phi$ *satisfying*

$$DP = MD.$$

Remark 1.2 The matrices D and Φ are very important for us to discuss the synchronization problem of CNMSCs and CNMDSCs, which will be utilized throughout this chapter.

Lemma 1.3 (*See* [33]) *The Kronecker product has the following properties:*

(*i*) $(A_1 \otimes A_2)^T = A_1^T \otimes A_2^T$,
(*ii*) $(\beta A_1) \otimes A_2 = A_1 \otimes (\beta A_2)$,
(*iii*) $(A_1 + A_2) \otimes A_3 = A_1 \otimes A_3 + A_2 \otimes A_3$,
(*iv*) $(A_1 \otimes A_2)(A_3 \otimes A_4) = (A_1A_3) \otimes (A_2A_4)$,

where $\beta \in \mathbb{R}, A_1, A_2, A_3$*, and* A_4 *are matrices with suitable dimensions.*

1.2.3 Network Models

In this chapter, two kinds of network models are considered as follows:

$$\dot{\alpha}_i(t) = z(\alpha_i(t)) + \sum_{m=1}^{\sigma}\sum_{j=1}^{N} a_m P_{ij}^m \Gamma^m \alpha_j(t), \tag{1.1}$$

$$\dot{\alpha}_i(t) = z(\alpha_i(t)) + \sum_{m=1}^{\sigma}\sum_{j=1}^{N} a_m P_{ij}^m \Gamma^m \alpha_j(t-\tau_m), \tag{1.2}$$

where $i = 1, 2, \dots, N$; $\alpha_i(t) = (\alpha_{i1}(t), \alpha_{i2}(t), \dots, \alpha_{in}(t))^T \in \mathbb{R}^n$ denotes the state vector of the ith node; $0 < a_m \in \mathbb{R}$ stands for the coupling strength; $z(\cdot) \in \mathbb{R}$ is a continuous function; $0 < \Gamma^m \in \mathbb{R}^{n\times n}$ denotes the inner coupling matrix; $0 < \tau_m \in \mathbb{R}$

represents the time delay; $P^m = (P^m_{ij})_{N\times N} \in \mathbb{R}^{N\times N}$ stands for the outer coupling matrix satisfying the following condition: if there is an edge from node i to node $j(i \neq j)$, then $P^m_{ij} > 0$; otherwise, $P^m_{ij} = 0$; and

$$P^m_{ii} = -\sum_{\substack{j=1\\ j\neq i}}^{N} P^m_{ij}, \quad i = 1, 2, \dots, N.$$

In this chapter, the function $z(\cdot)$ meets the following condition (see [34]):

$$\begin{aligned}&(\zeta_1 - \zeta_2)^T H[z(\zeta_1) - z(\zeta_2) - \Delta(\zeta_1 - \zeta_2)]\\ \leqslant &-\gamma(\zeta_1 - \zeta_2)^T(\zeta_1 - \zeta_2),\end{aligned} \tag{1.3}$$

for some constant matrices $\mathbb{R}^{n\times n} \ni H = \mathrm{diag}(h_1, h_2, \dots, h_n) > 0$ and $\Delta = \mathrm{diag}(\delta_1, \delta_2, \dots, \delta_n) \in \mathbb{R}^{n\times n}$, and a positive constant γ, where $\zeta_1, \zeta_2 \in \mathbb{R}^n$.

Remark 1.4 In the networks (1.1) and (1.2), the different coupling forms are required to have the same topology. In fact, this situation commonly exists in some real-life networks, such as inter-city population flow networks, urban public traffic networks, and so on. For instance, in the inter-city population flow networks, choosing each city as a node, the edge represents the population flow from any city to any other city. Obviously, the changes of the urban population depend on many factors, such as economic development, climate change, and education. Therefore, the intercity population flow networks should be modeled by CNMWs, in which each influencing factor corresponds to a coupling form. Apparently, the different coupling forms in the intercity population flow networks have the same topology.

Remark 1.5 In this chapter, the topology subject to the "successful" but recoverable attacks is discussed in CNMSCs (1.1) and CNMDSCs (1.2). Namely, the attacks happen at $t = t_{2k+1}$ and thus makes the topology to be broken, and the broken topology is recovered after $t = t_{2(k+1)}$, $k \in \mathbb{Z}^+$. In practice, this phenomenon exists in many real networks, such as military communications networks, and power grids [35, 36]. Therefore, some authors have studied the synchronization of CNs suffering the successful but recoverable attacks [30, 31]. However, the synchronization for CNMWs under the successful but recoverable attacks has not yet been discussed.

When $t = t_{2k+1}, k \in \mathbb{Z}^+$, the attacks happen and the topologies of the networks (1.1) and (1.2) are destroyed. After $t = t_{2(k+1)}$, the broken topology can be recovered. In this chapter, we assume that the networks (1.1) and (1.2) suffering the attacks have κ different types of topologies, and $\sup_{k\in\mathbb{Z}^+}\{t_{2(k+1)} - t_{2k+1}\} = T_M < +\infty$.

Therefore, one has

$$\begin{cases} \dot{\alpha}_i(t) = z(\alpha_i(t)) + \sum_{m=1}^{\sigma}\sum_{j=1}^{N} a_m P_{ij}^{m,\omega} \Gamma^m \alpha_j(t), & t \in [t_{2k+1}, t_{2(k+1)}), \\ \dot{\alpha}_i(t) = z(\alpha_i(t)) + \sum_{m=1}^{\sigma}\sum_{j=1}^{N} a_m P_{ij}^{m} \Gamma^m \alpha_j(t), & t \in [t_{2k}, t_{2k+1}), \end{cases}$$

$$\begin{cases} \dot{\alpha}_i(t) = z(\alpha_i(t)) + \sum_{m=1}^{\sigma}\sum_{j=1}^{N} a_m P_{ij}^{m,\omega} \Gamma P^m \alpha_j(t-\tau_m), & t \in [t_{2k+1}, t_{2(k+1)}), \\ \dot{\alpha}_i(t) = z(\alpha_i(t)) + \sum_{m=1}^{\sigma}\sum_{j=1}^{N} a_m P_{ij}^{m} \Gamma^m \alpha_j(t-\tau_m), & t \in [t_{2k}, t_{2k+1}), \end{cases}$$

where $i = 1, 2, \dots, N$, $\omega = 1, 2, \dots, \kappa$, $t_0 = 0$, $P^{m,\omega} = (P_{ij}^{m,\omega})_{N\times N} \in \mathbb{R}^{N\times N}$ represents the outer coupling matrix of the networks (1.1) and (1.2) subject to the attacks, in which $P_{ij}^{m,\omega}$ has the same definition as P_{ij}^{m}.

Denote

$$\begin{cases} P_{ij}^{m,\varphi(t)} = P_{ij}^{m}, & \text{if } t \in [t_{2k}, t_{2k+1}), \\ P_{ij}^{m,\varphi(t)} = P_{ij}^{m,\omega}, & \text{if } t \in [t_{2k+1}, t_{2(k+1)}), \end{cases}$$

where $\omega = 1, 2, \dots, \kappa$.

Then, we have

$$\dot{\alpha}_i(t) = z(\alpha_i(t)) + \sum_{m=1}^{\sigma}\sum_{j=1}^{N} a_m P_{ij}^{m,\varphi(t)} \Gamma^m \alpha_j(t), \tag{1.4}$$

$$\dot{\alpha}_i(t) = z(\alpha_i(t)) + \sum_{m=1}^{\sigma}\sum_{j=1}^{N} a_m P_{ij}^{m,\varphi(t)} \Gamma^m \alpha_j(t-\tau_m), \tag{1.5}$$

in which $i = 1, 2, \dots, N$.

Next, the synchronization definition for the network (1.4) [or (1.5)] is introduced as follows.

Definition 1.6 *The network* (1.4) *[or* (1.5)*] can achieve synchronization if*

$$\lim_{t\to+\infty} \|\alpha_i(t) - \alpha_j(t)\| = 0, \ i,j = 1, 2, \dots, N.$$

Denote

$$\begin{aligned} &\mathcal{Z}(t) = \left(z^T(\alpha_1(t)), z^T(\alpha_2(t)), \dots, z^T(\alpha_N(t))\right)^T, \\ &\alpha(t-\tau_m) = (\alpha_1^T(t-\tau_m), \alpha_2^T(t-\tau_m), \dots, \alpha_N^T(t-\tau_m))^T, \\ &\mathcal{Q} = 2I_{N-1} \otimes (H\Delta - \gamma I_n), \ \widetilde{H} = I_{N-1} \otimes H, \ \widetilde{D} = D \otimes I_n, \\ &\alpha(t) = (\alpha_1^T(t), \alpha_2^T(t), \dots, \alpha_N^T(t))^T, \ u(t) = (u_1^T(t), u_2^T(t), \dots, u_N^T(t))^T. \end{aligned}$$

1.3 Synchronization of CNMSCs Under Recoverable Attacks

1.3.1 Synchronization of CNMSCs with Directed Topology

(1) Synchronization analysis

Evidently, (1.4) can be rewritten as

$$\dot{\alpha}(t) = \mathcal{Z}(t) + \sum_{m=1}^{\sigma} a_m (P^{m,\varphi(t)} \otimes \Gamma^m)\alpha(t). \tag{1.6}$$

Theorem 1.7 *If there are two positive constants μ and ϖ satisfying*

(i) $\mu\widetilde{H} + \Xi \leqslant 0$;
(ii) $\varpi\widetilde{H} + \Xi_\omega \leqslant 0,\ \ \omega = 1, 2, \dots, \kappa$,

in which $\Xi = \mathcal{Q} + \sum_{m=1}^{\sigma} a_m[\widetilde{H}\widetilde{D}((P^m\Phi) \otimes \Gamma^m)]^s$, $\Xi_\omega = \mathcal{Q} + \sum_{m=1}^{\sigma} a_m[\widetilde{H}\widetilde{D}((P^{m,\omega}\Phi) \otimes \Gamma^m)]^s$, *the network* (1.4) *is synchronized.*

Proof: Let

$$\begin{aligned} e(t) &= (e_1^T(t), e_2^T(t), \dots, e_{N-1}^T(t))^T \\ &= \left((\alpha_1(t) - \alpha_2(t))^T, (\alpha_2(t) - \alpha_3(t))^T, \dots, (\alpha_{N-1}(t) - \alpha_N(t))^T\right)^T \\ &= \widetilde{D}\alpha(t). \end{aligned}$$

The Lyapunov function for the system (1.6) is given as follows:

$$\begin{aligned} V(t) &= \sum_{i=1}^{N-1} e_i^T(t) H e_i(t) \\ &= e^T(t)\widetilde{H}e(t) \\ &= \alpha^T(t)\widetilde{D}^T\widetilde{H}\widetilde{D}\alpha(t). \end{aligned}$$

Then, one gets

$$\begin{aligned} \dot{V}(t) + \mu V(t) &= 2\alpha^T(t)\widetilde{D}^T\widetilde{H}\widetilde{D}\dot{\alpha}(t) + \mu\alpha^T(t)\widetilde{D}^T\widetilde{H}\widetilde{D}\alpha(t) \\ &= \mu\alpha^T(t)\widetilde{D}^T\widetilde{H}\widetilde{D}\alpha(t) + 2\alpha^T(t)\widetilde{D}^T\widetilde{H}\widetilde{D}\mathcal{Z}(t) \\ &\quad + 2\alpha^T(t)\widetilde{D}^T\widetilde{H}\widetilde{D}\sum_{m=1}^{\sigma} a_m(P^m \otimes \Gamma^m)\alpha(t),\ \ t \in [t_{2k}, t_{2k+1}). \end{aligned}$$

By Lemma 1.1, one can derive

$$\begin{aligned} \widetilde{D}(P^m \otimes \Gamma^m) &= (D \otimes I_n)(P^m \otimes \Gamma^m) \\ &= (DP^m) \otimes \Gamma^m \end{aligned}$$

$$
\begin{aligned}
&= (DP^m \boldsymbol{\Phi} D) \otimes \Gamma^m \\
&= [(DA^m \boldsymbol{\Phi}) \otimes \Gamma^m](D \otimes I_n) \\
&= (D \otimes I_n)[(P^m \boldsymbol{\Phi}) \otimes \Gamma^m](D \otimes I_n) \\
&= \widetilde{D}[(P^m \boldsymbol{\Phi}) \otimes \Gamma^m]\widetilde{D},
\end{aligned}
$$

where $\boldsymbol{\Phi}$ has been defined in Lemma 1.1.

By (1.3), one has

$$
\begin{aligned}
\alpha^T(t)\widetilde{D}^T\widetilde{H}\widetilde{D}\mathcal{Z}(t) &= \sum_{i=1}^{N-1}(\alpha_i(t) - \alpha_{i+1}(t))^T H(z(\alpha_i(t)) - z(\alpha_{i+1}(t))) \\
&\leqslant \sum_{i=1}^{N-1}(\alpha_i(t) - \alpha_{i+1}(t))^T H\Delta(\alpha_i(t) - \alpha_{i+1}(t)) \\
&\quad - \gamma\sum_{p=1}^{N-1}(\alpha_i(t) - \alpha_{i+1}(t))^T(\alpha_i(t) - \alpha_{i+1}(t)) \\
&= \sum_{i=1}^{N-1} e_i^T(t)\left(H\Delta - \gamma I_n\right)e_i(t) \\
&= e^T(t)\left[I_{N-1} \otimes (H\Delta - \gamma I_n)\right]e(t).
\end{aligned}
$$

Therefore, one obtains

$$
\begin{aligned}
\dot{V}(t) + \mu V(t) &= \mu x^T(t)\widetilde{D}^T\widetilde{H}\widetilde{D}\alpha(t) + 2\alpha^T(t)\widetilde{D}^T\widetilde{H}\widetilde{D}\mathcal{Z}(t) \\
&\quad + 2\alpha^T(t)\widetilde{D}^T\widetilde{H}\widetilde{D}\sum_{m=1}^{\sigma}a_m[(P^m\boldsymbol{\Phi}) \otimes \Gamma^m]\widetilde{D}\alpha(t) \\
&\leqslant \mu e^T(t)\widetilde{H}e(t) + 2e^T(t)[I_{N-1} \otimes (H\Delta - \gamma I_n)]e(t) \\
&\quad + \sum_{m=1}^{\sigma}a_m e^T(t)\left[\widetilde{H}\widetilde{D}\left((P^m\boldsymbol{\Phi}) \otimes \Gamma^m\right)\right]^s e(t) \\
&= e^T(t)\left(\mu\widetilde{H} + \Xi\right)e(t),\ t \in [t_{2k}, t_{2k+1}).
\end{aligned}
$$

Then, we have

$$
\dot{V}(t) \leqslant -\mu V(t),\ t \in [t_{2k}, t_{2k+1}). \tag{1.7}
$$

Based on (1.7), one derives

$$
V(t) \leqslant e^{-\mu(t-t_{2k})}V(t_{2k}),\ t \in [t_{2k}, t_{2k+1}). \tag{1.8}
$$

Similarly, we can obtain

$$
\dot{V}(t) + \varpi V(t) \leqslant e^T(t)\left(\varpi\widetilde{H} + \Xi_\omega\right)e(t),\ t \in [t_{2k+1}, t_{2(k+1)}).
$$

Then, one has

$$
\dot{V}(t) \leqslant -\varpi V(t),\ t \in [t_{2k+1}, t_{2(k+1)}). \tag{1.9}
$$

From (1.9), we can derive

$$V(t) \leqslant e^{-\varpi(t-t_{2k+1})}V(t_{2k+1}),\ \ t \in [t_{2k+1}, t_{2(k+1)}). \tag{1.10}$$

By (1.8) and (1.10), we have

$$\begin{aligned}
V(t) &\leqslant e^{-\mu(t-t_{2k})}V(t_{2k}) \\
&\leqslant e^{-\mu(t-t_{2k})}e^{-\varpi(t_{2k}-t_{2k-1})}V(t_{2k-1}) \\
&\leqslant e^{-\mu(t-t_{2k})}e^{-\varpi(t_{2k}-t_{2k-1})}e^{-\mu(t_{2k-1}-t_{2k-2})}V(t_{2k-2}) \\
&= e^{(\mu-\varpi)(t_{2k}-t_{2k-1})}e^{-\mu(t-t_{2k-2})}V(t_{2k-2}) \\
&\ \ \vdots \\
&\leqslant e^{(\mu-\varpi)(t_{2k}-t_{2k-1}+t_{2k-2}-t_{2k-3}+\cdots+t_2-t_1)}e^{-\mu t}V(0), t \in [t_{2k}, t_{2k+1}),
\end{aligned} \tag{1.11}$$

and

$$\begin{aligned}
V(t) &\leqslant e^{-\varpi(t-t_{2k+1})}V(t_{2k+1}) \\
&\leqslant e^{(\mu-\varpi)(t_{2k}-t_{2k-1}+t_{2k-2}-t_{2k-3}+\cdots+t_2-t_1)}e^{-\varpi(t-t_{2k+1})}e^{-\mu t_{2k+1}}V(0) \\
&= e^{(\mu-\varpi)(t_{2k}-t_{2k-1}+t_{2k-2}-t_{2k-3}+\cdots+t_2-t_1)}e^{-\varpi t}e^{(\varpi-\mu)t_{2k+1}}V(0), \\
&\qquad\qquad\qquad\qquad t \in [t_{2k+1}, t_{2(k+1)}).
\end{aligned} \tag{1.12}$$

If $\mu > \varpi$, we can derive from (1.11) and (1.12) that

$$\begin{aligned}
V(t) &\leqslant e^{(\mu-\varpi)(t_{2k}-t_{2k-1}+t_{2k-2}-t_{2k-3}+\cdots+t_2-t_1)}e^{-(\mu-\varpi)t}e^{-\varpi t}V(0) \\
&= e^{(\mu-\varpi)(-t+t_{2k}-t_{2k-1}+t_{2k-2}-t_{2k-3}+\cdots+t_2-t_1+t_0)}e^{-\varpi t}V(0) \\
&\leqslant e^{-\varpi t}V(0), \quad t \in [t_{2k}, t_{2k+1}),
\end{aligned} \tag{1.13}$$

$$\begin{aligned}
V(t) &\leqslant e^{(\mu-\varpi)(-t_{2k+1}+t_{2k}-t_{2k-1}+t_{2k-2}-t_{2k-3}+\cdots+t_2-t_1+t_0)}e^{-\varpi t}V(0) \\
&\leqslant e^{-\varpi t}V(0),\ \ t \in [t_{2k+1}, t_{2(k+1)}).
\end{aligned} \tag{1.14}$$

If $\mu \leqslant \varpi$, we can get from (1.11) and (1.12) that

$$V(t) \leqslant e^{-\mu t}V(0),\ \ t \in [t_{2k}, t_{2k+1}), \tag{1.15}$$

$$\begin{aligned}
V(t) &\leqslant e^{(\mu-\varpi)(-t_{2k+1}+t_{2k}-t_{2k-1}+t_{2k-2}-t_{2k-3}+\cdots+t_2-t_1)}e^{(\mu-\varpi)t}e^{-\mu t}V(0) \\
&= e^{(\mu-\varpi)(t-t_{2k+1}+t_{2k}-t_{2k-1}+t_{2k-2}-t_{2k-3}+\cdots+t_2-t_1)}e^{-\mu t}V(0) \\
&\leqslant e^{-\mu t}V(0), \quad t \in [t_{2k+1}, t_{2(k+1)}).
\end{aligned} \tag{1.16}$$

From (1.13)–(1.16), one has

$$V(t) \leqslant e^{-\eta t}V(0),$$

where $\eta = \min\{\varpi, \mu\}$.

Based on the definition of $V(t)$, we can get

$$\lambda_{\min}(H)\|e(t)\|^2 \leqslant V(t) \leqslant e^{-\eta t}V(0) \leqslant e^{-\eta t}\lambda_{\max}(H)\|e(0)\|^2.$$

Then, one obtains

$$\|e(t)\| \leqslant \sqrt{\frac{\lambda_{\max}(H)}{\lambda_{\min}(H)}}\, \|e(0)\|\, e^{-\frac{\eta}{2}t}.$$

Therefore, the network (1.4) is synchronized. □

(2) State-feedback control for synchronization

In order to ensure the network (1.4) is synchronized, we devise the following state-feedback controller:

$$u_i(t) = -B\alpha_i(t), \tag{1.17}$$

in which $B \in \mathbb{R}^{n\times n}$.

Then, one has

$$\dot{\alpha}(t) = \mathcal{Z}(t) + \sum_{m=1}^{\sigma} a_m (P^{m,\varphi(t)} \otimes \Gamma^m)\alpha(t) - (I_N \otimes B)\alpha(t). \tag{1.18}$$

Theorem 1.8 *If there are two positive constants μ and ϖ, and matrix $B \in \mathbb{R}^{n\times n}$ satisfying*

(*i*) $\mu\widetilde{H} + \widetilde{\Xi} \leqslant 0$;

(*ii*) $\varpi\widetilde{H} + \widetilde{\Xi}_\omega \leqslant 0,\ \ \omega = 1, 2, \dots, \kappa$,

where $\widetilde{\Xi} = \mathcal{Q} + \left[\sum_{m=1}^{\sigma} a_m \widetilde{H}\widetilde{D}((P^m\Phi) \otimes \Gamma^m) - \widetilde{H}(I_{N-1} \otimes B)\right]^s$, $\widetilde{\Xi}_\omega = \mathcal{Q} + \left[\sum_{m=1}^{\sigma} a_m \widetilde{H}\widetilde{D}((P^{m,\omega}\Phi) \otimes \Gamma^m) - \widetilde{H}(I_{N-1} \otimes B)\right]^s$, *the network* (1.4) *under the state-feedback controller* (1.17) *is synchronized.*

Proof: Let

$$\begin{aligned} e(t) &= (e_1^T(t), e_2^T(t), \dots, e_{N-1}^T(t))^T \\ &= \widetilde{D}\alpha(t). \end{aligned}$$

The Lyapunov function for the system (1.18) is defined as follows:

$$\begin{aligned} V(t) &= \sum_{i=1}^{N-1} e_i^T(t) H e_i(t) \\ &= \alpha^T(t)\widetilde{D}^T\widetilde{H}\widetilde{D}\alpha(t). \end{aligned}$$

Then, one has

$$\begin{aligned} \dot{V}(t) + \mu V(t) = {} & \mu\alpha^T(t)\widetilde{D}^T\widetilde{H}\widetilde{D}\alpha(t) + 2\alpha^T(t)\widetilde{D}^T\widetilde{H}\widetilde{D}\mathcal{Z}(t) \\ & + 2\alpha^T(t)\widetilde{D}^T\widetilde{H}\widetilde{D}\sum_{m=1}^{\sigma} a_m (P^m \otimes \Gamma^m)\alpha(t) \\ & - 2\alpha^T(t)\widetilde{D}^T\widetilde{H}\widetilde{D}(I_N \otimes B)\alpha(t), t \in [t_{2k}, t_{2k+1}). \end{aligned}$$

Obviously, we can get

$$\begin{aligned}\widetilde{D}(I_N \otimes B) &= (D \otimes I_n)(I_N \otimes B)\\ &= D \otimes B\\ &= (I_{N-1}D) \otimes (BI_n)\\ &= (I_{N-1} \otimes B)(D \otimes I_n)\\ &= (I_{N-1} \otimes B)\widetilde{D}.\end{aligned} \tag{1.19}$$

From (1.19), one derives

$$\begin{aligned}\dot{V}(t) + \mu V(t) &= \mu\alpha^T(t)\widetilde{D}^T\widetilde{H}\widetilde{D}\alpha(t) + 2\alpha^T(t)\widetilde{D}^T\widetilde{H}\widetilde{D}\mathcal{Z}(t)\\ &\quad + 2\alpha^T(t)\widetilde{D}^T\widetilde{H}\widetilde{D}\sum_{m=1}^{\sigma} a_m[(P^m\Phi) \otimes \Gamma^m]\widetilde{D}\alpha(t)\\ &\quad -2\alpha^T(t)\widetilde{D}^T\widetilde{H}(I_{N-1} \otimes B)\widetilde{D}\alpha(t)\\ &\leqslant \mu e^T(t)\widetilde{H}e(t) + 2e^T(t)[I_{N-1} \otimes (H\Delta - \gamma I_n)]e(t)\\ &\quad + e^T(t)\left[\sum_{m=1}^{\sigma} a_m\widetilde{H}\widetilde{D}\left((P^m\Phi) \otimes \Gamma^m\right) - \widetilde{H}(I_{N-1} \otimes B)\right]^s e(t)\\ &= e^T(t)\left(\mu\widetilde{H} + \widetilde{\Xi}\right)e(t), \quad t \in [t_{2k}, t_{2k+1}).\end{aligned}$$

Then, we have

$$\dot{V}(t) \leqslant -\mu V(t), \ t \in [t_{2k}, t_{2k+1}).$$

Similarly, we can obtain

$$\dot{V}(t) + \varpi V(t) \leqslant e^T(t)\left(\varpi\widetilde{H} + \widetilde{\Xi}_\omega\right)e(t), \ t \in [t_{2k+1}, t_{2(k+1)}).$$

Then, one has

$$\dot{V}(t) \leqslant -\varpi V(t), \ t \in [t_{2k+1}, t_{2(k+1)}).$$

Utilizing the similar proof approach as in Theorem 1.7, we can easily get the following conclusion:

$$\|e(t)\| \leqslant \sqrt{\frac{\lambda_{\max}(H)}{\lambda_{\min}(H)}}\,\|e(0)\|\, e^{-\frac{\eta}{2}t},$$

where $\eta = \min\{\varpi, \mu\}$.

Therefore, the network (1.4) under the state-feedback controller (1.17) is synchronized. □

1.3.2 Synchronization of CNMSCs with Undirected Topology

A directed CNMSCs model (1.1) is discussed in Section 1.3.1. In other words, the outer coupling matrices $P^m, m = 1, 2, \dots, \sigma$ are nonsymmetric. However, in some cases, the topology of CNMSCs (1.1) may be undirected, that is, the outer coupling matrices $P^m, m = 1, 2, \dots, \sigma$ are symmetric. In this case, the outer coupling matrices $P^m = (P^m_{ij})_{N\times N} \in \mathbb{R}^{N\times N}, m = 1, 2, \dots, \sigma$, can be denoted as follows: if there is an edge between node i and node $j(i \neq j)$, then $P^m_{ij} = P^m_{ji} > 0$; otherwise, $P^m_{ij} = P^m_{ji} = 0$; and

$$P^m_{ii} = -\sum_{\substack{j=1\\ j\neq i}}^{N} P^m_{ij}, \quad i = 1, 2, \dots, N.$$

When $t = t_{2k+1}, k \in \mathbb{Z}^+$, the attacks happen and the topology of the undirected network (1.1) is destroyed. After $t = t_{2(k+1)}$, the broken topology can be recovered. Moreover, we suppose that the undirected network (1.1) subject to the attacks have κ different topologies in this section. Therefore, we have

$$\dot{\alpha}_i(t) = z(\alpha_i(t)) + \sum_{m=1}^{\sigma}\sum_{j=1}^{N} a_m P^{m,\varphi(t)}_{ij} \Gamma^m \alpha_j(t), \tag{1.20}$$

where $i = 1, 2, \dots, N$, and $P^{m,\varphi(t)}_{ij}$ satisfies

$$\begin{cases} P^{m,\varphi(t)}_{ij} = P^m_{ij}, & \text{if } t \in [t_{2k}, t_{2k+1}), \\ P^{m,\varphi(t)}_{ij} = P^{m,\omega}_{ij}, & \text{if } t \in [t_{2k+1}, t_{2(k+1)}), \end{cases}$$

in which $\omega = 1, 2, \dots, \kappa$, $A^{m,\omega} = (P^{m,\omega}_{ij})_{N\times N} \in \mathbb{R}^{N\times N}$ represents the outer coupling matrix of the undirected network (1.1) under the attacks, in which $P^{m,\omega}_{ij}$ has the same definition as P^m_{ij} in this section.

Remark 1.9 As we all know, plenty of real networks possess the directed topology, and meanwhile, the topology in many network models may be undirected. Therefore, it is very valuable and significative to, respectively, consider the directed and undirected topologies when we study the synchronization problem for CNMWs subject to recoverable attacks. Moreover, one of the main objectives of this chapter is to compare the difference between directed topology and undirected topology. Therefore, we not only analyze the synchronization of the undirected CNMSCs in this section but also design the appropriate state-feedback controller to guarantee the synchronization for such a network model.

(1) Synchronization analysis

By (1.20), we can derive

$$\dot{\alpha}(t) = \mathcal{Z}(t) + \sum_{m=1}^{\sigma} a_m (P^{m,\varphi(t)} \otimes \Gamma^m)\alpha(t). \tag{1.21}$$

Theorem 1.10 *If there are two positive constants* μ *and* ϖ *satisfying*

(*i*) $\mu\widetilde{H} + \hat{\Xi} \leqslant 0$;
(*ii*) $\varpi\widetilde{H} + \hat{\Xi}_{\omega} \leqslant 0,\ \ \omega = 1, 2, \dots, \kappa$,

where $\hat{\Xi} = \mathcal{Q} + \sum_{m=1}^{\sigma} a_m \left[\widetilde{H}\widetilde{D}\left((P^m\Phi) \otimes \Gamma^m\right) + \left((\Phi^T P^m) \otimes \Gamma^m\right)\widetilde{D}^T\widetilde{H}\right]$, $\hat{\Xi}_{\omega} = \mathcal{Q} + \sum_{m=1}^{\sigma} a_m \left[\widetilde{H}\widetilde{D}\left((P^{m,\omega}\Phi) \otimes \Gamma^m\right) + \left((\Phi^T P^{m,\omega}) \otimes \Gamma^m\right)\widetilde{D}^T\widetilde{H}\right]$, *the network* (1.20) *is synchronized.*

Proof: Utilizing the similar proof approach as in Theorem 1.7, the conclusion can be easily derived. □

(2) State-feedback control for synchronization

In order to ensure the network (1.20) is synchronized, we devise the following state-feedback controller:

$$u_i(t) = -B\alpha_i(t), \tag{1.22}$$

in which $B \in \mathbb{R}^{n\times n}$.

Then, one has

$$\dot{\alpha}(t) = \mathcal{Z}(t) + \sum_{m=1}^{\sigma} a_m (P^{m,\varphi(t)} \otimes \Gamma^m)\alpha(t) - (I_N \otimes B)\alpha(t). \tag{1.23}$$

Theorem 1.11 *If there are two positive constants* μ *and* ϖ*, and matrix* $B \in \mathbb{R}^{n\times n}$ *satisfying*

(*i*) $\mu\widetilde{H} + \hat{\hat{\Xi}} \leqslant 0$;
(*ii*) $\varpi\widetilde{H} + \hat{\hat{\Xi}}_{\omega} \leqslant 0,\ \ \omega = 1, 2, \dots, \kappa$,

where $\hat{\hat{\Xi}} = \mathcal{Q} + \sum_{m=1}^{\sigma} a_m \left[\widetilde{H}\widetilde{D}\left((P^m\Phi) \otimes \Gamma^m\right) + \left((\Phi^T P^m) \otimes \Gamma^m\right)\widetilde{D}^T\widetilde{H}\right] - \widetilde{H}(I_{N-1} \otimes B) - (I_{N-1} \otimes B^T)\widetilde{H}$, $\hat{\hat{\Xi}}_{\omega} = \mathcal{Q} + \sum_{m=1}^{\sigma} a_m \left[\widetilde{H}\widetilde{D}\left((P^{m,\omega}\Phi) \otimes \Gamma^m\right) + \left((\Phi^T P^{m,\omega}) \otimes \Gamma^m\right)\widetilde{D}^T\widetilde{H}\right] - \widetilde{H}(I_{N-1} \otimes B) - (I_{N-1} \otimes B^T)\widetilde{H}$, *the network* (1.20) *under the state-feedback controller* (1.22) *is synchronized.*

Proof: The following proof is similar to Theorem 1.8, and we omit its proof here. □

1.4 Synchronization of CNMDSCs Under Recoverable Attacks

1.4.1 Synchronization of CNMDSCs with Directed Topology

(1) Synchronization analysis

By (1.5), we can obtain

$$\dot{\alpha}(t) = \mathcal{Z}(t) + \sum_{m=1}^{\sigma} a_m (P^{m,\varphi(t)} \otimes \Gamma^m)\alpha(t-\tau_m). \tag{1.24}$$

Theorem 1.12 *If there are some matrices $0 < \Psi_m \in \mathbb{R}^{n(N-1)\times n(N-1)}, m = 1, 2, \ldots, \sigma$ satisfying*

(i) $\sum_{m=1}^{\sigma} a_m \Psi_m + \Upsilon < 0$,
(ii) $\sum_{m=1}^{\sigma} a_m \Psi_m + \Upsilon_\omega < 0, \quad \omega = 1, 2, \ldots, \kappa$,

where $\Upsilon = \mathcal{Q} + \sum_{m=1}^{\sigma} a_m \widetilde{H}\widetilde{D}[(P^m\Phi) \otimes \Gamma^m]\Psi_m^{-1}[(P^m\Phi)^T \otimes \Gamma^m]\widetilde{D}^T\widetilde{H}$, $\Upsilon_\omega = \mathcal{Q} + \sum_{m=1}^{\sigma} a_m \widetilde{H}\widetilde{D}[(P^{m,\omega}\Phi) \otimes \Gamma^m]\Psi_m^{-1}[(P^{m,\omega}\Phi)^T \otimes \Gamma^m]\widetilde{D}^T\widetilde{H}$, *the network* (1.5) *is synchronized.*

Proof: Let

$$\begin{aligned} e(t) &= (e_1^T(t), e_2^T(t), \ldots, e_{N-1}^T(t))^T \\ &= \widetilde{D}\alpha(t). \end{aligned}$$

The Lyapunov function for the system (1.24) is defined as follows:

$$\begin{aligned} \hat{V}(t) &= \sum_{i=1}^{N-1} e_i^T(t) H e_i(t) + \sum_{m=1}^{\sigma} a_m \int_{t-\tau_m}^{t} e^T(h)\Psi_m e(h)dh \\ &= \alpha^T(t)\widetilde{D}^T\widetilde{H}\widetilde{D}\alpha(t) + \sum_{m=1}^{\sigma} a_m \int_{t-\tau_m}^{t} \alpha^T(h)\widetilde{D}^T\Psi_m\widetilde{D}\alpha(h)dh. \end{aligned} \tag{1.25}$$

Then, one gets

$$\begin{aligned} \dot{\hat{V}}(t) &= 2\alpha^T(t)\widetilde{D}^T\widetilde{H}\widetilde{D}\mathcal{Z}(t) + 2\alpha^T(t)\widetilde{D}^T\widetilde{H}\widetilde{D}\sum_{m=1}^{\sigma} a_m (P^m \otimes \Gamma^m)\alpha(t-\tau_m) \\ &\quad - \sum_{m=1}^{\sigma} a_m \alpha^T(t-\tau_m)\widetilde{D}^T\Psi_m\widetilde{D}\alpha(t-\tau_m) + \sum_{m=1}^{\sigma} a_m \alpha^T(t)\widetilde{D}^T\Psi_m\widetilde{D}\alpha(t) \\ &\leqslant 2e^T(t)[I_{N-1} \otimes (H\Delta - \gamma I_n)]e(t) + \sum_{m=1}^{\sigma} a_m e^T(t)\Psi_m e(t) \\ &\quad + 2\sum_{m=1}^{\sigma} a_m e^T(t)\widetilde{H}\widetilde{D}\left[(P^m\Phi) \otimes \Gamma^m\right] e(t-\tau_m) \\ &\quad - \sum_{m=1}^{\sigma} a_m e^T(t-\tau_m)\Psi_m e(t-\tau_m) \\ &\leqslant 2e^T(t)[I_{N-1} \otimes (H\Delta - \gamma I_n)]e(t) + \sum_{m=1}^{\sigma} a_m e^T(t)\Psi_m e(t) \end{aligned}$$

$$
\begin{aligned}
&+\sum_{m=1}^{\sigma} a_m e^T(t)\widetilde{H}\widetilde{D}[(P^m\boldsymbol{\Phi})\otimes\Gamma^m]\Psi_m^{-1}[(P^m\boldsymbol{\Phi})^T\otimes\Gamma^m]\widetilde{D}^T\widetilde{H}e(t)\\
&=e^T(t)\left(\sum_{m=1}^{\sigma} a_m\Psi_m+\Upsilon\right)e(t)\\
&\leqslant \xi_1\|e(t)\|^2,\quad t\in[t_{2k},t_{2k+1}),
\end{aligned}
$$

where $\xi_1=\lambda_{\max}(\sum_{m=1}^{\sigma} a_m\Psi_m+\Upsilon)$.

Similarly, we have

$$
\begin{aligned}
\dot{\hat{V}}(t)&\leqslant e^T(t)\left(\sum_{m=1}^{\sigma} a_m\Psi_m+\Upsilon_\omega\right)e(t)\\
&\leqslant \xi_2\|e(t)\|^2,\quad t\in[t_{2k+1},t_{2(k+1)}),
\end{aligned}
$$

where $\xi_2=\max\limits_{\omega=1,2,\ldots,\kappa}\{\lambda_{\max}(\sum_{m=1}^{\sigma} a_m\Psi_m+\Upsilon_\omega)\}$.

Then, one has

$$
\dot{\hat{V}}(t)\leqslant \rho\|e(t)\|^2, \tag{1.26}
$$

where $\rho=\max\{\xi_1,\xi_2\}$.

From (1.25) and (1.26), we have

$$
\lim_{t\to+\infty}\hat{V}(t)=\hat{V}^*\geqslant 0, \tag{1.27}
$$

$$
\|e(t)\|^2\leqslant\frac{\dot{\hat{V}}(t)}{\rho}, \tag{1.28}
$$

Based on (1.27) and (1.28), one obtains

$$
\begin{aligned}
\lim_{t\to+\infty}\int_0^t\|e(h)\|^2dh&\leqslant\lim_{t\to+\infty}\int_0^t\frac{\dot{\hat{V}}(h)}{\rho}dh\\
&=\frac{\hat{V}^*-\hat{V}(0)}{\rho}.
\end{aligned}\tag{1.29}
$$

By means of (1.29), we can get

$$
\begin{aligned}
0&\leqslant\lim_{t\to+\infty}\int_{t-\tau_m}^t e^T(h)\Psi_m e(h)dh\\
&\leqslant\lambda_{\max}(\Psi_m)\lim_{t\to+\infty}\int_{t-\tau_m}^t\|e(h)\|^2dh\\
&=0,\quad m=1,2,\ldots,\sigma.
\end{aligned}\tag{1.30}
$$

In light of (1.25) and (1.30), one obtains

$$
\lim_{t\to+\infty}e^T(t)\widetilde{H}e(t)=\wp\geqslant 0.
$$

Suppose that $\wp > 0$. Then, there obviously exists $0 < \theta \in \mathbb{R}$ satisfying

$$e^T(t)\widetilde{H}e(t) > \frac{\wp}{2} \quad \text{for} \quad t \geqslant \theta.$$

Namely,

$$\|e(t)\|^2 > \frac{\wp}{2\lambda_{\max}\left(\widetilde{H}\right)}, \quad t \geqslant \theta. \tag{1.31}$$

By (1.26) and (1.31), one acquires

$$\dot{\hat{V}}(t) < \frac{\wp\rho}{2\lambda_{\max}\left(\widetilde{H}\right)}, \quad t \geqslant \theta.$$

Then, one gets

$$\begin{aligned}\hat{V}^* - \hat{V}(\theta) &= \int_\theta^{+\infty} \dot{\hat{V}}(t)dt \\ &< \int_\theta^{+\infty} \frac{\wp\rho}{2\lambda_{\max}\left(\widetilde{H}\right)}dt \\ &= -\infty,\end{aligned}$$

which is wrong. Therefore,

$$\lim_{t\to+\infty} e^T(t)\widetilde{H}e(t) = 0.$$

Then, we can derive

$$\lim_{t\to+\infty} \|e(t)\| = 0.$$

Therefore, the network (1.5) is synchronized. □

(2) State-feedback control for synchronization

In order to ensure the network (1.5) is synchronized, we devise the following state-feedback controller:

$$u_i(t) = -B\alpha_i(t), \tag{1.32}$$

in which $B \in \mathbb{R}^{n\times n}$.

Then, one has

$$\dot{\alpha}(t) = \mathcal{Z}(t) + \sum_{m=1}^{\sigma} a_m(P^{m,\varphi(t)} \otimes \Gamma^m)\alpha(t-\tau_m) - (I_N \otimes B)\alpha(t). \tag{1.33}$$

Theorem 1.13 *If there are some matrices* $B \in \mathbb{R}^{n\times n}$ *and* $0 < \Psi_m \in \mathbb{R}^{n(N-1)\times n(N-1)}$, $m = 1, 2, \ldots, \sigma$ *satisfying*

(i) $\sum_{m=1}^{\sigma} a_m\Psi_m + \hat{\Upsilon} < 0$,
(ii) $\sum_{m=1}^{\sigma} a_m\Psi_m + \hat{\Upsilon}_\omega < 0, \quad \omega = 1, 2, \ldots, \kappa$,

where $\hat{\Upsilon} = \mathcal{Q} + \sum_{m=1}^{\sigma} a_m \widetilde{H}\widetilde{D}[(P^m \Phi) \otimes \Gamma^m]\Psi_m^{-1}[(P^m \Phi)^T \otimes \Gamma^m]\widetilde{D}^T\widetilde{H} - [\widetilde{H}(I_{N-1} \otimes B)]^s$, $\hat{\Upsilon}_{\omega} = \mathcal{Q} + \sum_{m=1}^{\sigma} a_m \widetilde{H}\widetilde{D}[(P^{m,\omega}\Phi) \otimes \Gamma^m]\Psi_m^{-1}[(P^{m,\omega}\Phi)^T \otimes \Gamma^m]\widetilde{D}^T\widetilde{H} - [\widetilde{H}(I_{N-1} \otimes B)]^s$, *the network* (1.5) *under the state-feedback controller* (1.32) *is synchronized.*

Proof: Utilizing the similar proof approach as in Theorem 1.12, the conclusion can be easily obtained. □

1.4.2 Synchronization of CNMDSCs with Undirected Topology

In this section, the CNMDSCs (1.2) with undirected topology is considered. Then, the undirected network (1.2) under the successful but recoverable attacks can be described as follows:

$$\dot{\alpha}_i(t) = z(\alpha_i(t)) + \sum_{m=1}^{\sigma}\sum_{j=1}^{N} a_m P_{ij}^{m,\varphi(t)} \Gamma^m \alpha_j(t - \tau_m), \tag{1.34}$$

where $i = 1, 2, \dots, N$, and $P_{ij}^{m,\varphi(t)}$ satisfies

$$\begin{cases} P_{ij}^{m,\varphi(t)} = P_{ij}^{m}, & \text{if } t \in [t_{2k}, t_{2k+1}), \\ P_{ij}^{m,\varphi(t)} = P_{ij}^{m,\omega}, & \text{if } t \in [t_{2k+1}, t_{2(k+1)}), \end{cases}$$

in which $\omega = 1, 2, \dots, \kappa$, t_{2k+1}, $t_{2(k+1)}$, and κ have the same meanings as these in Section 1.4.1; $P^{m,\omega} = (P_{ij}^{m,\omega})_{N\times N} \in \mathbb{R}^{N\times N}$ and $P^m = (P_{ij}^m)_{N\times N} \in \mathbb{R}^{N\times N}$ have the identical definitions as these in (1.20).

Remark 1.14 In Section 1.3, two types of synchronization problems for CNMSCs are considered, that is, the cases with directed topology and with undirected topology. However, time delay always exists in CNs due to traffic congestions and finite transmission speeds. Furthermore, considering that different coupling forms may have nonidentical time delays in CNMSCs, it is necessary to further consider the CNMDSCs with directed and undirected topologies. Therefore, we study the synchronization and control problems for directed CNMDSCs in Section 1.4.1 and shall further consider the case that topology is undirected in this section.

(1) Synchronization analysis

By (1.34), we can derive

$$\dot{\alpha}(t) = \mathcal{Z}(t) + \sum_{m=1}^{\sigma} a_m (P^{m,\varphi(t)} \otimes \Gamma^m)\alpha(t - \tau_m). \tag{1.35}$$

Theorem 1.15 *If there are some matrices* $0 < \Psi_m \in \mathbb{R}^{n(N-1)\times n(N-1)}$, $m = 1, 2, \dots, \sigma$ *satisfying*

(*i*) $\sum_{m=1}^{\sigma} a_m \Psi_m + \widetilde{\Upsilon} < 0$,
(*ii*) $\sum_{m=1}^{\sigma} a_m \Psi_m + \widetilde{\Upsilon}_\omega < 0, \omega = 1, 2, \dots, \kappa$,

where $\widetilde{\Upsilon} = \mathcal{Q} + \sum_{m=1}^{\sigma} a_m \widetilde{H}\widetilde{D}[(P^m \Phi) \otimes \Gamma^m]\Psi_m^{-1}[(\Phi^T P^m) \otimes \Gamma^m]\widetilde{D}^T \widetilde{H}, \widetilde{\Upsilon}_\omega = \mathcal{Q} + \sum_{m=1}^{\sigma} a_m \widetilde{H}\widetilde{D}[(P^{m,\omega} \Phi) \otimes \Gamma^m]\Psi_m^{-1}[(\Phi^T P^{m,\omega}) \otimes \Gamma^m]\widetilde{D}^T \widetilde{H}$, *the network* (1.34) *is synchronized.*

Proof: Similar to the proof process of Theorem 1.12, the conclusion can be easily derived. □

(2) State-feedback control for synchronization

In order to ensure the network (1.34) is synchronized, we devise the following state-feedback controller:

$$u_i(t) = -B\alpha_i(t), \tag{1.36}$$

in which $B \in \mathbb{R}^{n\times n}$.

Then, one yields

$$\dot{\alpha}(t) = \mathcal{Z}(t) + \sum_{m=1}^{\sigma} a_m (P^{m,\varphi(t)} \otimes \Gamma^m)\alpha(t - \tau_m) - (I_N \otimes B)\alpha(t). \tag{1.37}$$

Theorem 1.16 *If there are some matrices* $B \in \mathbb{R}^{n\times n}$ *and* $0 < \Psi_m \in \mathbb{R}^{n(N-1)\times n(N-1)}$, $m = 1, 2, \dots, \sigma$ *satisfying*

(*i*) $\sum_{m=1}^{\sigma} a_m \Psi_m + \hat{\widetilde{\Upsilon}} < 0$,
(*ii*) $\sum_{m=1}^{\sigma} a_m \Psi_m + \hat{\widetilde{\Upsilon}}_\omega < 0, \quad \omega = 1, 2, \dots, \kappa$,

where $\hat{\widetilde{\Upsilon}} = \mathcal{Q} + \sum_{m=1}^{\sigma} a_m \widetilde{H}\widetilde{D}[(P^m \Phi) \otimes \Gamma^m]\Psi_m^{-1}[(\Phi^T P^m) \otimes \Gamma^m]\widetilde{D}^T \widetilde{H} - \widetilde{H}(I_{N-1} \otimes B) - (I_{N-1} \otimes B^T)\widetilde{H}$, $\hat{\widetilde{\Upsilon}}_\omega = \mathcal{Q} + \sum_{m=1}^{\sigma} a_m \widetilde{H}\widetilde{D}[(P^{m,\omega} \Phi) \otimes \Gamma^m]\Psi_m^{-1}[(\Phi^T P^{m,\omega}) \otimes \Gamma^m] \widetilde{D}^T \widetilde{H} - \widetilde{H}(I_{N-1} \otimes B) - (I_{N-1} \otimes B^T)\widetilde{H}$, *the network* (1.34) *under the state-feedback controller* (1.36) *is synchronized.*

Proof: The proof method is similar to Theorem 1.13, so we omit its proof here. □

Remark 1.17 The main difficulty for analysis and control of synchronization in CNs with multiple state or delayed state couplings comes from multiple weights and recoverable attacks. Based on inequality techniques, appropriate Lyapunov functionals, and the state-feedback controllers, some criteria of synchronization are developed in this chapter for CNMSCs and CNMDSCs with directed and undirected topologies.

1.5 Numerical Examples

In this section, two numerical examples are put forward to illustrate the correctness and effectiveness of the acquired results.

Example 1.18 Consider the following coupled Chua's circuits:

$$\dot{\alpha}_i(t) = z(\alpha_i(t)) + 0.3\sum_{j=1}^{6} P_{ij}^{1,\varphi(t)}\Gamma^1\alpha_j(t) + 0.4\sum_{j=1}^{6} P_{ij}^{2,\varphi(t)}\Gamma^2\alpha_j(t) + u_i(t), \tag{1.38}$$

where $i = 1, 2, \ldots, 6$, $\Gamma^1 = \text{diag}(0.2, 0.3, 0.3)$, $\Gamma^2 = \text{diag}(0.4, 0.3, 0.2)$,

$$P_{ij}^{1,\varphi(t)} = \begin{cases} P_{ij}^1, & \text{if } t \in [2k, 2k+1), \\ P_{ij}^{1,1}, & \text{if } t \in [4k+1, 4k+2), \\ P_{ij}^{1,2}, & \text{if } t \in [4k+3, 4(k+1)),\ k \in \mathbb{Z}^+, \end{cases}$$

$$P_{ij}^{2,\varphi(t)} = \begin{cases} P_{ij}^2, & \text{if } t \in [2k, 2k+1), \\ P_{ij}^{2,1}, & \text{if } t \in [4k+1, 4k+2), \\ P_{ij}^{2,2}, & \text{if } t \in [4k+3, 4(k+1)),\ k \in \mathbb{Z}^+, \end{cases}$$

$$z(\alpha_i(t)) = \begin{pmatrix} 10(-\alpha_{i1}(t) + \alpha_{i2}(t) - \ell(\alpha_{i1}(t))) \\ \alpha_{i1}(t) - \alpha_{i2}(t) + \alpha_{i3}(t) \\ -14.87\alpha_{i2}(t) \end{pmatrix},$$

in which $\ell(\alpha_{i1}(t)) = -0.68\alpha_{i1}(t) + 0.5(-1.27 + 0.68)(|\alpha_{i1}(t) + 1| - |\alpha_{i1}(t) - 1|)$,

$$P^1 = \begin{pmatrix} -0.7 & 0.3 & 0.2 & 0.2 & 0 & 0 \\ 0.2 & -0.5 & 0 & 0.3 & 0 & 0 \\ 0.2 & 0 & -0.4 & 0.2 & 0 & 0 \\ 0.1 & 0.2 & 0.3 & -0.8 & 0.2 & 0 \\ 0 & 0 & 0 & 0.3 & -0.5 & 0.2 \\ 0 & 0 & 0 & 0 & 0.3 & -0.3 \end{pmatrix},$$

$$P^2 = \begin{pmatrix} -0.4 & 0.1 & 0.2 & 0.1 & 0 & 0 \\ 0.2 & -0.3 & 0 & 0.1 & 0 & 0 \\ 0.3 & 0 & -0.5 & 0.2 & 0 & 0 \\ 0.2 & 0.1 & 0.4 & -0.9 & 0.2 & 0 \\ 0 & 0 & 0 & 0.2 & -0.4 & 0.2 \\ 0 & 0 & 0 & 0 & 0.5 & -0.5 \end{pmatrix},$$

$$P^{1,1} = \begin{pmatrix} -0.5 & 0.3 & 0.2 & 0 & 0 & 0 \\ 0.2 & -0.2 & 0 & 0 & 0 & 0 \\ 0.2 & 0 & -0.2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.2 & 0.2 & 0 \\ 0 & 0 & 0 & 0.3 & -0.5 & 0.2 \\ 0 & 0 & 0 & 0 & 0.3 & -0.3 \end{pmatrix},$$

$$P^{2,1} = \begin{pmatrix} -0.3 & 0.1 & 0.2 & 0 & 0 & 0 \\ 0.2 & -0.2 & 0 & 0 & 0 & 0 \\ 0.3 & 0 & -0.3 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.2 & 0.2 & 0 \\ 0 & 0 & 0 & 0.2 & -0.4 & 0.2 \\ 0 & 0 & 0 & 0 & 0.5 & -0.5 \end{pmatrix},$$

$$P^{1,2} = \begin{pmatrix} -0.3 & 0.3 & 0 & 0 & 0 & 0 \\ 0.2 & -0.2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.2 & 0.2 \\ 0 & 0 & 0 & 0 & 0.3 & -0.3 \end{pmatrix},$$

$$P^{2,2} = \begin{pmatrix} -0.1 & 0.1 & 0 & 0 & 0 & 0 \\ 0.2 & -0.2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.2 & 0.2 \\ 0 & 0 & 0 & 0 & 0.5 & -0.5 \end{pmatrix}.$$

Taking $H = I_3, \gamma = 1$, and $\Delta = \mathrm{diag}(10, 13, 8)$, one can ensure that (1.3) is satisfied.

Based on the MATLAB YALMIP Toolbox, the following parameters that satisfy the conditions of Theorem 1.8 can be acquired:

$$\mu = 0.5593, \ \ \varpi = 0.4407, \ \ B = \mathrm{diag}(9.9592, 12.9627, 7.9785).$$

From Theorem 1.8, the network (1.38) under the state-feedback controller (1.17) is synchronized. The changing curves of $e_{ij}(t), i = 1, 2, \dots, 5, j = 1, 2, 3$ and $\varphi(t)$ are, respectively, shown in Figures 1.1 and 1.2.

Example 1.19 Consider the following coupled Chua's circuits:

$$\begin{aligned} \dot{\alpha}_i(t) = z(\alpha_i(t)) + 0.6\sum_{j=1}^{5} P_{ij}^{1,\varphi(t)} \Gamma^1 \alpha_j(t-0.1) + 0.5\sum_{j=1}^{5} P_{ij}^{2,\varphi(t)} \Gamma^2 \alpha_j(t-0.2) \\ + u_p(t), \end{aligned} \tag{1.39}$$

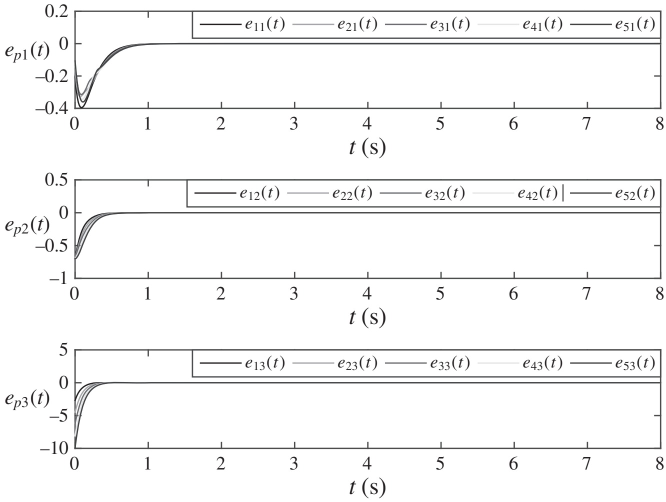

Figure 1.1 $e_{ij}(t)$, $i = 1, 2, \ldots, 5, j = 1, 2, 3$.

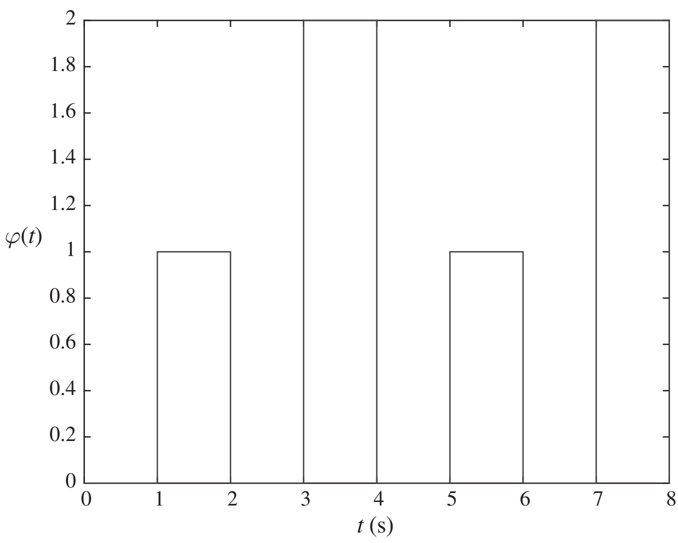

Figure 1.2 The CNMSCs (1.38) under the successful but recoverable attacks, where $\varphi(t) = 0$ represents the network topology has not been attacked or has been recovered.

where $i = 1, 2, \dots, 5$, $\Gamma^1 = \mathrm{diag}(0.4, 0.5, 0.5)$, $\Gamma^2 = \mathrm{diag}(0.3, 0.5, 0.4)$,

$$P_{ij}^{1,\varphi(t)} = \begin{cases} P_{ij}^{1}, & \text{if } t \in [2k, 2k+1), \\ P_{ij}^{1,1}, & \text{if } t \in [4k+1, 4k+2), \\ P_{ij}^{1,2}, & \text{if } t \in [4k+3, 4(k+1)),\ k \in \mathbb{Z}^+, \end{cases}$$

$$P_{ij}^{2,\varphi(t)} = \begin{cases} P_{ij}^{2}, & \text{if } t \in [2k, 2k+1), \\ P_{ij}^{2,1}, & \text{if } t \in [4k+1, 4k+2), \\ P_{ij}^{2,2}, & \text{if } t \in [4k+3, 4(k+1)),\ k \in \mathbb{Z}^+, \end{cases}$$

$$z(\alpha_i(t)) = \begin{pmatrix} 10(-\alpha_{i1}(t) + \alpha_{i2}(t) - \ell(\alpha_{i1}(t))) \\ \alpha_{i1}(t) - \alpha_{i2}(t) + \alpha_{i3}(t) \\ -14.87\alpha_{i2}(t) \end{pmatrix},$$

in which $\ell(\alpha_{i1}(t)) = -0.68\alpha_{i1}(t) + 0.5(-1.27 + 0.68)(|\alpha_{i1}(t) + 1| - |\alpha_{i1}(t) - 1|)$,

$$P^1 = \begin{pmatrix} -0.5 & 0.2 & 0.3 & 0 & 0 \\ 0.1 & -0.6 & 0.2 & 0.3 & 0 \\ 0.4 & 0.1 & -0.7 & 0 & 0.2 \\ 0 & 0.2 & 0 & -0.3 & 0.1 \\ 0 & 0 & 0.2 & 0.3 & -0.5 \end{pmatrix},$$

$$P^2 = \begin{pmatrix} -0.4 & 0.1 & 0.3 & 0 & 0 \\ 0.2 & -0.4 & 0.1 & 0.1 & 0 \\ 0.2 & 0.2 & -0.6 & 0 & 0.2 \\ 0 & 0.3 & 0 & -0.5 & 0.2 \\ 0 & 0 & 0.3 & 0.1 & -0.4 \end{pmatrix},$$

$$P^{1,1} = \begin{pmatrix} -0.3 & 0 & 0.3 & 0 & 0 \\ 0 & -0.3 & 0 & 0.3 & 0 \\ 0.4 & 0 & -0.4 & 0 & 0 \\ 0 & 0.2 & 0 & -0.3 & 0.1 \\ 0 & 0 & 0 & 0.3 & -0.3 \end{pmatrix},$$

$$P^{2,1} = \begin{pmatrix} -0.3 & 0 & 0.3 & 0 & 0 \\ 0 & -0.1 & 0 & 0.1 & 0 \\ 0.2 & 0 & -0.2 & 0 & 0 \\ 0 & 0.3 & 0 & -0.5 & 0.2 \\ 0 & 0 & 0 & 0.1 & -0.1 \end{pmatrix},$$

$$P^{1,2} = \begin{pmatrix} -0.2 & 0.2 & 0 & 0 & 0 \\ 0.1 & -0.1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.1 & 0.1 \\ 0 & 0 & 0 & 0.3 & -0.3 \end{pmatrix},$$

$$P^{2,2} = \begin{pmatrix} -0.1 & 0.1 & 0 & 0 & 0 \\ 0.2 & -0.2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.2 & 0.2 \\ 0 & 0 & 0 & 0.1 & -0.1 \end{pmatrix}.$$

Taking $H = I_3, \gamma = 1$, and $\Delta = \text{diag}(10, 13, 8)$, one can ensure that (1.3) is satisfied.

Based on the MATLAB YALMIP Toolbox, the following parameters that satisfy the conditions of Theorem 1.13 can be acquired:

$$B = \begin{pmatrix} 12.8475 & 0 & 0 \\ 0 & 12.9345 & 0 \\ 0 & 0 & 12.7880 \end{pmatrix},$$

$$\Psi_1 = \begin{pmatrix} 4.6461 & 0 & 0 \\ 0 & 1.2178 & 0 \\ 0 & 0 & 6.9206 \end{pmatrix},$$

$$\Psi_2 = \begin{pmatrix} 5.5266 & 0 & 0 \\ 0 & 1.6467 & 0 \\ 0 & 0 & 8.0851 \end{pmatrix}.$$

From Theorem 1.13, the network (1.39) under the state feedback controller (1.32) is synchronized. The changing curves of $e_{ij}(t), i = 1, 2, \dots, 4, j = 1, 2, 3$ and $\varphi(t)$ are, respectively, shown in Figures 1.3 and 1.4.

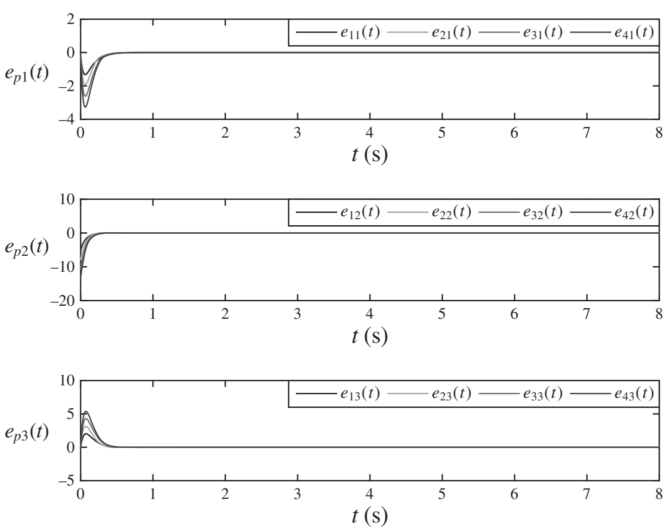

Figure 1.3 $e_{ij}(t), i = 1, 2, \dots, 4, j = 1, 2, 3$.

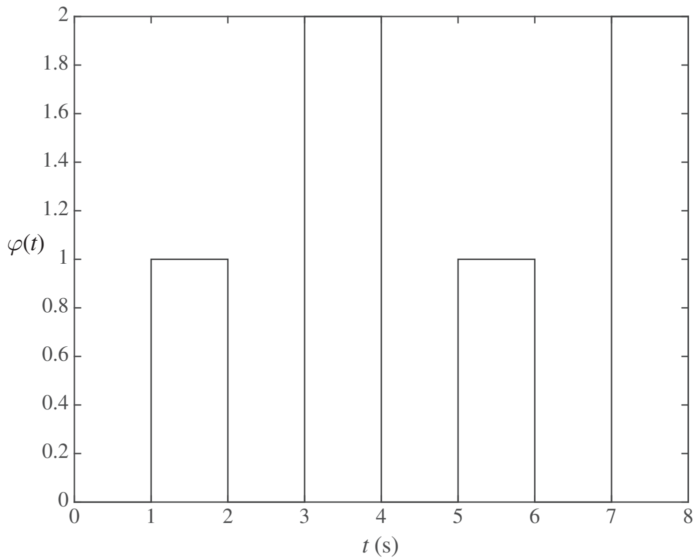

Figure 1.4 The CNMDSCs (1.39) under the successful but recoverable attacks, where $\varphi(t) = 0$ represents the network topology has not been attacked or has been recovered.

Remark 1.20 In recent years, some authors have taken the synchronization problem for CNs subject to recoverable attacks into consideration [30, 31]. But, CNs with single coupling were investigated in these existing results [30, 31]. In this chapter, we, respectively, establish several synchronization criteria for directed and undirected CNMWs subject to recoverable attacks, which cannot be obtained by those approaches used in [30, 31]. In addition, we can explicitly see from Figs. 1.1 and 1.3 that CNMSCs (1.38) and CNMDSCs (1.39) under the devised controllers are synchronized, which exhibit the correctness and effectiveness of our obtained criteria.

1.6 Conclusion

In this chapter, the synchronization problem for CNMSCs and CNMDSCs under recoverable attacks has been considered. On one side, several criteria to ensure the synchronization of CNMSCs and CNMDSCs under the case that the topology is directed or undirected have been developed by utilizing some inequality techniques. On the other side, the state-feedback control approach has been employed to study the synchronization of these network models. Finally, two numerical examples have been given to illustrate the validity of the obtained criteria. In the future, we will consider the synchronization of coupled reaction–diffusion neural networks with multiple state or spatial diffusion couplings under recoverable attacks.

References

1 J. Zhou, J. Lu, and J. Lv, "Adaptive synchronization of an uncertain complex network dynamical network," *IEEE Transactions on Automatic Control*, vol. 51, no. 4, pp. 652–656, 2006.

2 M. Chen, "Some simple synchronization criteria for complex dynamical networks," *IEEE Transactions on Circuits and Systems-II: Express Briefs*, vol. 53, no. 11, pp. 1185–1189, 2006.

3 M. Chen, "Chaos synchronization in complex networks," *IEEE Transactions on Circuits and Systems-I: Regular Papers*, vol. 55, no. 5. pp. 1335–1346, 2008.

4 Y. Chen, W. Yu, S. Tan, and H. Zhu, "Synchronizing nonlinear complex networks via switching disconnected topology," *Automatica*, vol. 70, pp. 189–194, 2016.

5 X. Lv, J. Cao, X. Li, M. Abdel-Aty, and U. A. Al-Juboori, "Synchronization analysis for complex dynamical networks with coupling delay via event-triggered delayed impulsive control," *IEEE Transactions on Cybernetics*, vol. 51, no. 11, pp. 5269–5278, 2021.

6 J. Qin, H. Gao, and W. X. Zheng, "Exponential synchronization of complex networks of linear systems and nonlinear oscillators: A unified analysis," *IEEE Transactions on Neural Networks and Learning Systems*, vol. 26, no. 3, pp. 510–521, 2015.

7 J.-L. Wang and H.-N. Wu, "Local and global exponential output synchronization of complex delayed dynamical networks," *Nonlinear Dynamics*, vol. 67, pp. 497–504, 2012.

8 G. Wen, W. Yu, G. Hu, J. Cao, and X. Yu, "Pinning synchronization of directed networks with switching topologies: A multiple Lyapunov functions approach," *IEEE Transactions on Neural Networks and Learning Systems*, vol. 26, no. 12, pp. 3239–3250, 2015.

9 W. Wu, W. Zhou, and T. Chen, "Cluster synchronization of linearly coupled complex networks under pinning control," *IEEE Transactions on Circuits and Systems-I: Regular Papers*, vol. 56, no. 4, pp. 829–839, 2009.

10 A. Wang, T. Dong, and X. Liao, "Event-triggered synchronization strategy for complex dynamical networks with the Markovian switching topologies," *Neural Networks*, vol. 74, pp. 52–57, 2016.

11 X.-J. Li and G.-H. Yang, "Graph theory-based pinning synchronization of stochastic complex dynamical networks," *IEEE Transactions on Neural Networks and Learning Systems*, vol. 28, no. 2, pp. 427–437, 2017.

12 X. Liu and T. Chen, "Synchronization of linearly coupled networks with delays via aperiodically intermittent pinning control," *IEEE Transactions on Neural Networks and Learning Systems*, vol. 26, no. 10, pp. 2396–2407, 2015.

13 Z. Xu, X. Li, and P. Duan, "Synchronization of complex networks with time-varying delay of unknown bound via delayed impulsive control," *Neural Networks*, vol. 125, pp. 224–232, 2020.

14 S. Zhu, J. Zhou, X. Yu, and J.-A. Lu, "Synchronization of complex networks with nondifferentiable time-varying delay," *IEEE Transactions on Cybernetics*, vol. 52, no. 5, pp. 3342–3348, 2022.

15 J. Zhuang, J. Cao, L. Tang, Y. Xia, and M. Perc, "Synchronization analysis for stochastic delayed multilayer network with additive couplings," *IEEE Transactions on Systems, Man, and Cybernetics: Systems*, vol. 50, no. 11, pp. 4807–4816, 2020.

16 Z. Yang, B. Luo, D. Liu, and Y. Li, "Adaptive synchronization of delayed memristive neural networks with unknown parameters," *IEEE Transactions on Systems, Man, and Cybernetics: Systems*, vol. 50, no. 2, pp. 539–549, 2020.

17 J.-L. Wang, P.-C. Wei, H.-N. Wu, T. Huang, and M. Xu, "Pinning synchronization of complex dynamical networks with multiweights," *IEEE Transactions on Systems, Man, and Cybernetics: Systems*, vol. 49, no. 7, pp. 1357–1370, 2019.

18 J.-L. Wang, M. Xu, H.-N. Wu, and T. Huang, "Passivity analysis and pinning control of multi-weighted complex dynamical networks," *IEEE Transactions on Network Science and Engineering*, vol. 6, no. 1, pp. 60–73, 2019.

19 X.-L. An, L. Zhang, Y.-Z. Li, and J.-G. Zhang, "Synchronization analysis of complex networks with multi-weights and its application in public traffic network," *Physica A*, vol. 412, pp. 149–156, 2014.

20 H.-A. Tang, J.-L. Wang, L. Wang, X. Hu, Y. Zhou, and S. Duan, "Impulsive control for passivity and exponential synchronization of coupled neural networks with multiple weights," *Journal of the Franklin Institute*, vol. 356, no. 10, pp. 5434–5463, 2019.

21 X.-L. An, L. Zhang, and J.-G. Zhang, "Research on urban public traffic network with multi-weights based on single bus transfer junction," *Physica A*, vol. 436, pp. 748–755, 2015.

22 X. Zhao, J. Zhou, and J.-A. Lu, "Pinning synchronization of multiplex delayed networks with stochastic perturbations," *IEEE Transactions on Cybernetics*, vol. 49, no. 12, pp. 4262–4270, 2019.

23 Y.-P. Zhao, P. He, H. S. Nik, and J. Ren, "Robust adaptive synchronization of uncertain complex networks with multiple time-varying coupled delays," *Complexity*, vol. 20, no. 6, pp. 62–73, 2015.

24 Y. Dong, J.-G. Xian, and D. Han, "New conditions for synchronization in complex networks with multiple time-varying delays," *Communications in Nonlinear Science and Numerical Simulation*, vol. 18, no. 9, pp. 2581–2588, 2013.

25 L. Chen, J. Cao, R. Wu, J. A. T. Machado, A. Lopes, and H. Yang, "Stability and synchronization of fractional-order memristive neural networks with multiple delays," *Neural Networks*, vol. 94, pp. 76–85, 2017.

26 Z. Qin, J.-L. Wang, Y.-L. Huang, and S.-Y. Ren, "Analysis and adaptive control for robust synchronization and $\mathcal{H}_\infty$ synchronization of complex dynamical networks with multiple time-delays," *Neurocomputing*, vol. 289, pp. 241–251, 2018.

27 Z. Ye, D. Zhang, Z.-G. Wu, and H. Yan, "A3C-based intelligent event-triggering control of networked nonlinear unmanned marine vehicles subject to hybrid attacks," *IEEE Transactions on Intelligent Transportation Systems*, vol. 23, no. 8, pp. 12921–12934, 2022.

28 Y. Xu, M. Fang, Y.-J. Pan, K. Shi, and Z.-G. Wu, "Event-triggered output synchronization for nonhomogeneous agent systems with periodic denial-of-service attacks," *International Journal of Robust and Nonlinear Control*, vol. 31, no. 6, pp. 1851–1865, 2021.

29 Y.-S. Ma, W.-W. Che, C. Deng, and Z.-G. Wu, "Distributed model-free adaptive control for learning nonlinear MASs under DoS attacks," *IEEE Transactions on Neural Networks and Learning Systems*, vol. 34, no. 3, pp. 1146–1155, 2023.

30 S. Wang, Y. Cao. T. Huang, Y. Chen, and S. Wen, "Event-triggered distributed control for synchronization of multiple memristive neural networks under cyber-physical attacks," *Information Sciences*, vol. 518, pp. 361–375, 2020.

31 Y.-W. Wang, H. O. Wang, J.-W. Xiao, and Z.-H. Guan, "Synchronization of complex dynamical networks under recoverable attacks," *Automatica*, vol. 46, no. 1, pp. 197–203, 2010.

32 C. W. Wu and L. O. Chua, "Synchronization in an array of linearly coupled dynamical systems," *IEEE Transactions on Circuits and Systems-I: Fundamental Theory and Applications*, vol. 42, no. 8, pp. 430–447, 1995.

33 J.-L. Wang, Z.-C. Yang, and H.-N. Wu, "Passivity analysis of complex dynamical networks with multiple time-varying delays," *Journal of Engineering Mathematics*, vol. 74, no. 1, pp. 175–188, 2012.

34 W. Lu and T. Chen, "New approach to synchronization analysis of linearly coupled ordinary differential systems," *Physica D*, vol. 213, no. 2, pp. 214–230, 2006.

35 W. He, Z. Mo, Q.-L. Han, and F. Qian, “Secure impulsive synchronization in Lipschitz-type multi-agent systems subject to deception attacks,” *IEEE/CAA Journal of Automatica Sinica*, vol. 7, no. 5, pp. 1326–1334, 2020.

36 W. He, F. Qian, Q.-L. Han, and G. Chen, “Almost sure stability of nonlinear systems under random and impulsive sequential attacks,” *IEEE Transactions on Automatic Control*, vol. 65, no. 9, pp. 3879–3886, 2020.