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Simple Harmonic Motion

In the physical world, there are many examples of things that vibrate or oscillate, i.e. perform periodic motion. Everyday examples include a swinging pendulum, a plucked guitar string and a car bouncing up and down on its springs. The most basic form of periodic motion is called simple harmonic motion (SHM). In this chapter, we develop quantitative descriptions of SHM. We obtain equations for the ways in which the displacement, velocity and acceleration of a simple harmonic oscillator vary with time and the ways in which the kinetic and potential energies of the oscillator vary. To do this, we discuss two particularly important examples of SHM: a mass oscillating at the end of a spring and a swinging pendulum. We then extend our discussion to electrical circuits and show that the equations that describe the movement of charge in an oscillating electrical circuit are identical in form to those that describe, for example, the motion of a mass at the end of a spring. Thus, if we understand one type of harmonic oscillator, then we can readily understand and analyse many other types. The universal importance of SHM is that, to a good approximation, many real oscillating systems behave like simple harmonic oscillators when they undergo oscillations of small amplitude. Consequently, the elegant mathematical description of the simple harmonic oscillator that we will develop can be applied to a wide range of physical systems.

1.1 Physical Characteristics of Simple Harmonic Oscillators

Observing the motion of a pendulum can tell us a great deal about the general characteristics of SHM. We could make such a pendulum by suspending an apple from the end of a length of string. When we draw the apple away from its *equilibrium position* and release it, we see that the apple swings back towards the equilibrium position. It starts off from rest but steadily picks up speed. We notice that it *overshoots* the equilibrium position and does not stop until it reaches the other extreme of its motion. It then swings back towards the equilibrium position and eventually arrives back at its initial position. This pattern then repeats with the apple swinging back and forth *periodically*. Gravity is the *restoring force* that attracts the apple back to its equilibrium position. It is the *inertia* of the mass that causes it to overshoot. The apple has kinetic energy because of its motion. We notice that its velocity is zero when its displacement from the equilibrium position is a maximum, and so its kinetic

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energy is also zero at that point. The apple also has potential energy. When it moves away from the equilibrium position, the apple's vertical height increases, and it gains potential energy. When the apple passes through the equilibrium position, its vertical displacement is zero, and so all of its energy must be kinetic. Thus, at the point of zero displacement, the velocity has its maximum value. As the apple swings back and forth, there is a continuous exchange between its potential and kinetic energies. These characteristics of the pendulum are common to all simple harmonic oscillators: (i) periodic motion, (ii) an equilibrium position, (iii) a restoring force that is directed towards this equilibrium position, (iv) inertia causing overshoot and (v) a continuous flow of energy between potential and kinetic. Of course, the oscillation of the apple steadily dies away due to the effects of dissipative forces such as air resistance, but we will delay the discussion of these effects until Chapter 2.

1.2 A Mass on a Spring

1.2.1 A Mass on a Horizontal Spring

Our first example of a simple harmonic oscillator is a mass on a horizontal spring, as shown in Figure 1.1. The mass is attached to one end of the spring, while the other end is held fixed. The equilibrium position corresponds to the unstretched length of the spring, and x is the displacement of the mass from the equilibrium position along the x -axis. We start with an idealised version of a real physical situation. It is idealised because the mass is assumed to move on a frictionless surface, and the spring is assumed to be weightless. Furthermore, because the motion is in the horizontal direction, no effects due to gravity are involved. In physics, it is quite usual to start with a simplified version or model because real physical situations are normally complicated and hard to handle. The simplification makes the problem tractable so that an initial, idealised solution can be obtained. The complications, e.g. the effects of friction on the motion of the oscillator, are then added in turn, and at each stage, a modified and improved solution is obtained. This process invariably provides a great deal of physical understanding about the real system and about the relative importance of the added complications.

Experience tells us that if we pull the mass so as to extend the spring and then release it, the mass will move back and forth in a periodic way. If we plot the displacement x of the mass with respect to time t , we obtain a curve like that shown in Figure 1.2. The *amplitude* of the oscillation is A , corresponding to the maximum excursion of the mass, and we note the *initial condition* that $x = A$ at time $t = 0$. The time for one complete cycle of oscillation is the period T . The frequency ν is the number of cycles of oscillation per unit time. The relationship between period and frequency is

$$\nu = \frac{1}{T}. \quad (1.1)$$

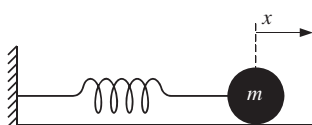
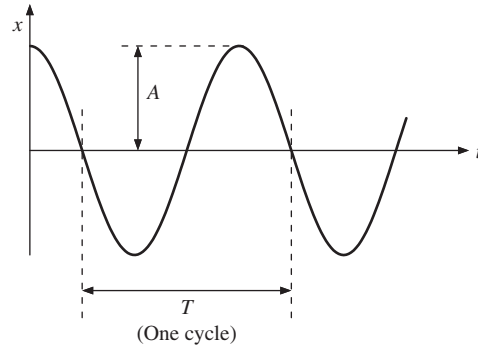


Figure 1.1 A simple harmonic oscillator consisting of a mass m on a horizontal spring.

Figure 1.2 Variation of displacement x with time t for a mass undergoing SHM.



The unit of frequency is hertz (Hz), where

$$1 \text{ Hz} \equiv 1 \text{ cycle per second} \equiv 1 \text{ s}^{-1}.$$

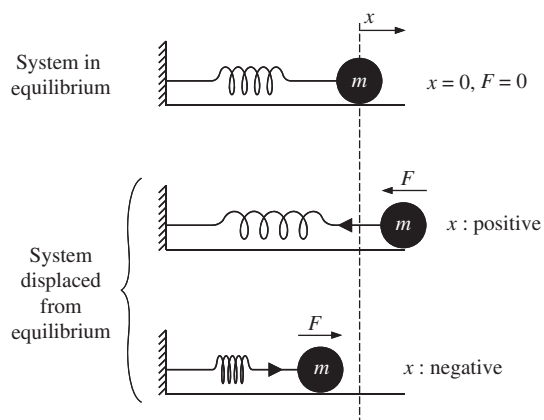
For small displacements, the force produced by the spring is described by Hooke's law, which says that the strength of the force is proportional to the extension (or compression) of the spring, i.e. $F \propto x$, where x is the displacement of the mass. The constant of proportionality is the spring constant k , which is defined as the force per unit displacement. When the spring is extended, i.e. x is positive, the force acts in the opposite direction to x to pull the mass back to the equilibrium position. Similarly, when the spring is compressed, i.e. x is negative, the force again acts in the opposite direction to x to push the mass back to the equilibrium position. This situation is illustrated in Figure 1.3, which shows the direction of the force at various points of the oscillation. We can therefore write

$$F = -kx, \quad (1.2)$$

where the minus sign indicates that the force always acts in the opposite direction to the displacement. All simple harmonic oscillators have forces that act in this way: (i) the magnitude of the force is directly proportional to the displacement and (ii) the force is always directed towards the equilibrium position.

The system must also obey Newton's second law of motion, which states that the force is equal to the mass m times the acceleration a , i.e. $F = ma$. We thus obtain the equation of

Figure 1.3 The direction of the force acting on the mass m at various values of displacement x .



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motion of the mass

$$F = ma = -kx. \tag{1.3}$$

Recalling that velocity v and acceleration a are, respectively, the first and second derivatives of displacement with respect to time, i.e.

$$a = \frac{dv}{dt} = \frac{d^2x}{dt^2}, \tag{1.4}$$

we can write Equation (1.3) in the form of the differential equation

$$m \frac{d^2x}{dt^2} = -kx \tag{1.5}$$

or

$$\frac{d^2x}{dt^2} = -\omega^2 x \tag{1.6}$$

where

$$\omega^2 = \frac{k}{m} \tag{1.7}$$

is a constant. Equation (1.6) is the equation of SHM, and *all* simple harmonic oscillators have an equation of this form. It is a linear second-order differential equation: linear because each term is proportional to x or one of its derivatives and second order because the highest derivative occurring in it is of second order. The reason for writing the constant as ω^2 will soon become apparent, but we note that ω^2 is equal to the restoring force per unit displacement per unit mass.

1.2.2 A Mass on a Vertical Spring

If we suspend a mass from a vertical spring, as shown in Figure 1.4, we have gravity also acting on the mass. When the mass is initially attached to the spring, the length of the spring increases by an amount Δl . Taking displacements in the downward direction as positive, the resultant force on the mass is equal to the gravitational force minus the force exerted upwards by the spring, i.e. the resultant force is given by $mg - k\Delta l$. The resultant force is equal to zero when the mass is at its equilibrium position. Hence,

$$k\Delta l = mg.$$

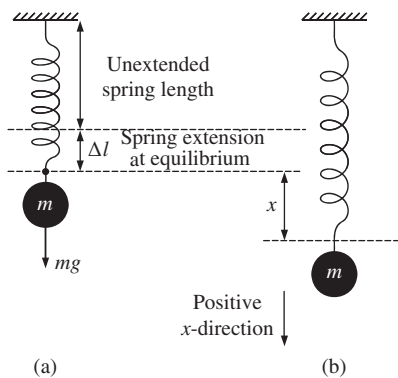


Figure 1.4 An oscillating mass on a vertical spring. (a) The mass at its equilibrium position. (b) The mass displaced by a distance x from its equilibrium position.

When the mass is displaced downwards by an amount x , the resultant force is given by

$$F = m \frac{d^2x}{dt^2} = mg - k(\Delta l + x) = mg - k\Delta l - kx,$$

i.e.

$$m \frac{d^2x}{dt^2} = -kx. \quad (1.8)$$

Perhaps not surprisingly, this result is identical to the equation of motion (1.5) of the horizontal spring: we simply need to measure displacements from the equilibrium position of the mass.

Worked Example The Young's modulus E of a wire is defined as

$$E = \frac{\text{tensile stress}}{\text{tensile strain}} = \frac{F/A}{\Delta l/l},$$

where F is the applied force, A is the cross-sectional area of the wire, l is its length and Δl is the resulting extension. A mass of 1.0 kg is attached to a piano wire of length 0.5 m and radius 0.25 mm. When the mass is displaced, a small distance from its equilibrium position and released, it oscillates in SHM with a frequency of 50 Hz. Determine the Young's modulus of the wire.

Solution

We have $F = \frac{AE\Delta l}{l}$, which compares with Hooke's law $F = kx$, where x is the extension produced by the force F . We can thus write $k = \frac{AE}{l}$ and the force exerted on the wire by the mass m is $mg = kx = \frac{AE}{l}x$. The force exerted on the mass is then $-\frac{AE}{l}x$.

This gives the equation of motion: $m \frac{d^2x}{dt^2} = -\frac{AE}{l}x$, for which $\omega^2 = \frac{AE}{ml}$.

Then, $v^2 = \frac{1}{4\pi^2} \frac{AE}{ml}$, or, $E = 4\pi^2 v^2 \frac{ml}{A}$.

Therefore,

$$E = 4\pi^2 50^2 \frac{1.0 \times 0.5}{\pi(0.25 \times 10^{-3})^2} = 2.5 \times 10^{11} \text{ Pa.}$$

1.2.3 Displacement, Velocity and Acceleration in SHM

To describe the harmonic oscillator, we need expressions for the displacement, velocity and acceleration as functions of time: $x(t)$, $v(t)$ and $a(t)$, respectively. These can be obtained by solving Equation (1.6) using standard mathematical methods. However, we will use our physical intuition to deduce them from the observed behaviour of a mass on a spring.

Observing the periodic motion shown in Figure 1.2, we look for a function $x(t)$ that also repeats periodically. Periodic functions that are familiar to us are $\sin \theta$ and $\cos \theta$. These are reproduced in Figure 1.5 over two complete cycles. Both functions repeat every time the angle θ changes by 2π . We can notice that the two functions are identical except for a shift of $\pi/2$ along the θ -axis. We also note the initial condition that the displacement x of the mass equals A at $t = 0$. Comparison of the actual motion with the mathematical functions in Figure 1.5 suggests the choice of a cosine function for $x(t)$. We write it as

$$x = A \cos\left(\frac{2\pi t}{T}\right), \quad (1.9)$$

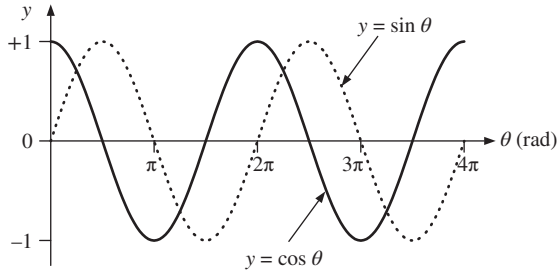


Figure 1.5 The functions $y = \cos \theta$ and $y = \sin \theta$ plotted over two complete cycles.

which has the correct form in that $(2\pi t/T)$ is an angle (in radians) that goes from 0 to 2π as t goes from 0 to T , and so repeats with the correct period. Moreover, x equals A at $t = 0$, which matches the initial condition. We also require that $x = \cos(2\pi t/T)$ is a solution to our differential Equation (1.6). We define

$$\omega = \frac{2\pi}{T}, \quad (1.10)$$

where ω is the *angular frequency* of the oscillator, with units of rad s^{-1} , to obtain

$$x = A \cos \omega t. \quad (1.11)$$

Then,

$$\frac{dx}{dt} = v = -\omega A \sin \omega t, \quad (1.12)$$

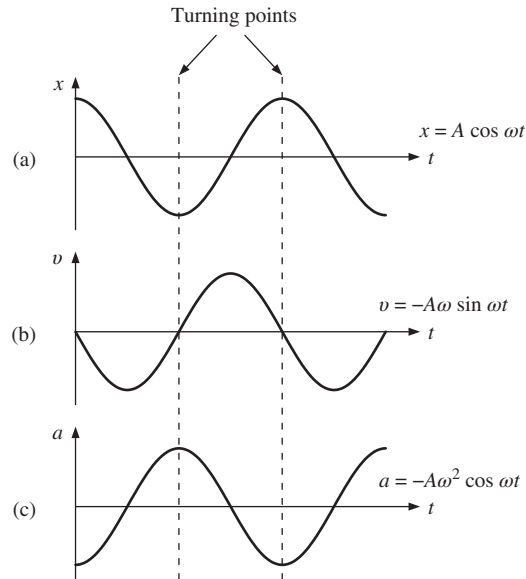
and

$$\frac{d^2x}{dt^2} = a = -\omega^2 A \cos \omega t = -\omega^2 x. \quad (1.13)$$

So, the function $x = A \cos \omega t$ is a solution of Equation (1.6) and correctly describes the physical situation. The reason for writing the constant as ω^2 in Equation (1.6) is now apparent: the constant is equal to the square of the angular frequency of oscillation. We have also obtained expressions for the velocity v and acceleration a of the mass as functions of time. All three functions are plotted in Figure 1.6. Since they relate to different physical quantities, namely displacement, velocity and acceleration, they are plotted on separate sets of axes, although the time axes are aligned with respect to each other.

Figure 1.6 shows that the behaviour of the three functions (1.11)–(1.13) agrees with our observations. For example, when the displacement of the mass is greatest, which occurs at the *turning points* of the motion ($x = \pm A$), the velocity is zero. However, the velocity is at a maximum when the mass passes through its equilibrium position, i.e. $x = 0$. Looked at in a different way, we can see that the maximum in the velocity curve occurs before the maximum in the displacement curve by one quarter of a period, which corresponds to an angle of $\pi/2$. We can understand at which points the maxima and minima of the acceleration occur by recalling that acceleration is directly proportional to the force. The force is maximum at the turning points of the motion but is of opposite sign to the displacement. The acceleration does indeed follow this same pattern, as is readily seen in Figure 1.6.

Figure 1.6 (a) The displacement x , (b) the velocity v and (c) the acceleration a of a mass undergoing SHM as a function of time t . The time axes of the three graphs are aligned.



1.2.4 General Solutions for SHM and the Phase Angle ϕ

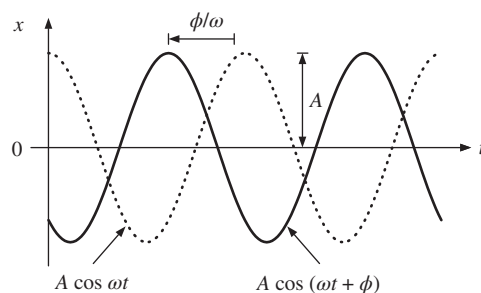
In the example above, we had the particular situation where the mass was released from rest with an initial displacement A , i.e. x equals A at $t = 0$. For the more general case, the motion of the oscillator will give rise to a displacement curve like that shown by the solid curve in Figure 1.7, where the displacement and velocity of the mass have arbitrary values at $t = 0$. This solid curve looks like the cosine function $x = A \cos \omega t$, which is shown by the dotted curve, but it is displaced horizontally to the left of it by a time interval $\phi/\omega = \phi T/2\pi$. The solid curve is described by

$$x = A \cos(\omega t + \phi), \quad (1.14)$$

where again A is the amplitude of the oscillation, and ϕ is called the *phase angle*, which has units of radians. [Note that changing ωt to $(\omega t - \phi)$ would shift the curve to the right in Figure 1.7.]

Equation (1.14) is also a solution of the equation of motion of the mass, Equation (1.6), as the reader can readily verify. In fact, Equation (1.14) is the *general solution* of Equation (1.6).

Figure 1.7 General solution for displacement x in SHM showing the phase angle ϕ , where $x = A \cos(\omega t + \phi)$.



We can state here a property of second-order differential equations that they always contain two arbitrary constants. In this case, A and ϕ are the two constants, which are determined from the initial conditions, i.e. from the position and velocity of the mass at time $t = 0$.

We can cast the general solution, Equation (1.14), in the alternative form

$$x = a \cos \omega t + b \sin \omega t, \quad (1.15)$$

where a and b are now the two constants. Equations (1.14) and (1.15) are entirely equivalent, as we can show in the following way. Since

$$A \cos(\omega t + \phi) = A \cos \omega t \cos \phi - A \sin \omega t \sin \phi \quad (1.16)$$

and $\cos \phi$ and $\sin \phi$ have constant values, we can rewrite the right-hand side of this equation as

$$a \cos \omega t + b \sin \omega t,$$

where

$$a = A \cos \phi \text{ and } b = -A \sin \phi. \quad (1.17)$$

We see that if we add sine and cosine curves of the *same* angular frequency ω , we obtain another cosine (or corresponding sine curve) of angular frequency ω . This is illustrated in Figure 1.8, where we plot $A \cos \omega t$ and $A \sin \omega t$, and also $(A \cos \omega t + A \sin \omega t)$, which is equal to $\sqrt{2}A \cos(\omega t - \pi/4)$. As the motion of a simple harmonic oscillator is described by sines and cosines, it is called 'harmonic', and because there is only a single frequency involved, it is called 'simple harmonic'.

There is an important difference between the constants A and ϕ in the general solution for SHM given in Equation (1.14) and the angular frequency ω . The constants are determined by the initial conditions of the motion. However, the angular frequency of oscillation ω is determined only by the properties of the oscillator: the oscillator has a *natural frequency of oscillation* that is independent of the way in which we start the motion. This is reflected in the fact that the SHM equation, Equation (1.6), already contains ω , which therefore has nothing to do with any particular solutions of the equation. This has important practical applications. It means, for example, that the period of a pendulum clock is independent of

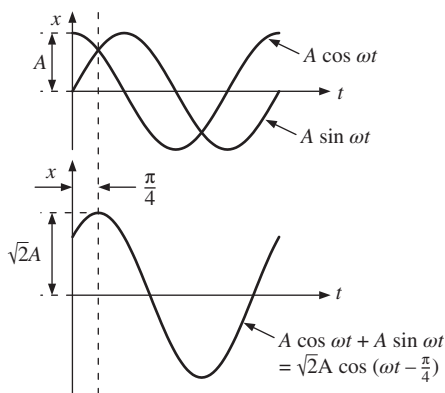


Figure 1.8 The addition of sine and cosine curves with the same angular frequency ω . The resultant curve also has angular frequency ω .

the amplitude of the pendulum so that it keeps time to a high degree of accuracy.¹ It means that the pitch of a note from a piano does not depend on how hard you strike the keys. For the example of a mass on a spring, $\omega = \sqrt{k/m}$. This expression tells us that the angular frequency becomes lower as the mass increases and becomes higher as the spring constant increases.

Worked Example In the example of a mass on a horizontal spring (cf. Figure 1.1), m has a value of 0.80 kg and the spring constant k is 180 N m⁻¹. At time $t = 0$, the mass is observed to be 0.04 m further from the wall than the equilibrium position and is moving away from the wall with a velocity of 0.50 m s⁻¹. Obtain an expression for the displacement of the mass in the form $x = A \cos(\omega t + \phi)$, obtaining numerical values for A , ω and ϕ .

Solution

The angular frequency ω depends only on the oscillator parameters k and m and not on the initial conditions. Substituting their values gives

$$\omega = \sqrt{k/m} = 15.0 \text{ rad s}^{-1}$$

To find the amplitude A : From $x = A \cos(\omega t + \phi)$, we obtain

$$v = -A\omega \sin(\omega t + \phi).$$

Substituting the initial values, i.e. at time $t = 0$, of x and v into these equations gives

$$0.04 = A \cos \phi, \quad 0.50 = -15A \sin \phi.$$

From $\cos^2 \phi + \sin^2 \phi = 1$, we obtain $A = 0.052$ m.

To find the phase angle ϕ : Substituting the value for A leads to two equations for ϕ :

$$\cos \phi = 0.04/0.052, \quad \text{giving } \phi = 39.8^\circ \text{ or } 320^\circ,$$

$$\sin \phi = -0.50/(15 \times 0.052), \quad \text{giving } \phi = -39.8^\circ \text{ or } 320^\circ.$$

Since ϕ must satisfy both equations, it must have the value of $\phi = 320^\circ$.

The angular frequency ω is given in rad s⁻¹. To convert ϕ to radians:

$$\phi = (\pi/180) \times 320 \text{ rad} = 5.59 \text{ rad. Hence, } x = 0.052 \cos(15t + 5.59) \text{ m.}$$

1.2.5 The Energy of a Simple Harmonic Oscillator

Consideration of the energy of a system is a powerful tool in solving physical problems. For one thing, scalar rather than vector quantities are involved, which usually simplifies the analysis. For the example of a mass on a spring (Figure 1.1), the mass has kinetic energy K and potential energy U . The kinetic energy is due to the motion and is given by $K = \frac{1}{2}mv^2$. The potential energy U is the energy stored in the spring and is equal to the work done in extending or compressing it, i.e. 'force times distance'. The work done on the spring, extending it from x' to $x' + dx'$ is $kx'dx'$. Hence, the work done in extending it

¹ This assumes that the pendulum is operating as an ideal harmonic oscillator, which is a good approximation for oscillations of small amplitude.

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from its unstretched length by an amount x , i.e. its potential energy when extended by this amount, is

$$U = \int_0^x kx' dx' = \frac{1}{2}kx^2. \quad (1.18)$$

Similarly, when the spring is compressed by an amount x , the stored energy is again equal to $\frac{1}{2}kx^2$.

Conservation of energy for the harmonic oscillator follows from Newton's second law, Equation (1.5). In terms of the velocity v , this becomes

$$m \frac{dv}{dt} = -kx.$$

Multiplying this equation by $dx = vdt$ gives

$$mvdv = -kxdx$$

and since $d(x^2) = 2xdx$ and $d(v^2) = 2vdv$, we obtain

$$d\left(\frac{1}{2}mv^2\right) = -d\left(\frac{1}{2}kx^2\right).$$

Integrating this equation gives

$$\frac{1}{2}mv^2 + \frac{1}{2}kx^2 = \text{constant},$$

where the right-hand term is a constant of integration. The two terms on the left-hand side of this equation are just the kinetic energy K and the potential energy U of the oscillator. It follows that the constant on the right-hand side is the total energy E of the oscillator, i.e. we have derived conservation of energy for this case:

$$E = K + U = \frac{1}{2}mv^2 + \frac{1}{2}kx^2 \quad (1.19)$$

Equation (1.19) enables us to calculate the energy E of the harmonic oscillator for any solution of the oscillator. If we take the general solution $x = A \cos(\omega t + \phi)$, we obtain the velocity

$$v = \frac{dx}{dt} = -\omega A \sin(\omega t + \phi) \quad (1.20)$$

and the potential and kinetic energies

$$U = \frac{1}{2}kx^2 = \frac{1}{2}kA^2 \cos^2(\omega t + \phi) \quad (1.21)$$

and

$$K = \frac{1}{2}mv^2 = \frac{1}{2}m\omega^2 A^2 \sin^2(\omega t + \phi) = \frac{1}{2}kA^2 \sin^2(\omega t + \phi), \quad (1.22)$$

where we substituted $\omega^2 = k/m$. Hence, the total energy E is given by

$$E = K + U = \frac{1}{2}kA^2 [\sin^2(\omega t + \phi) + \cos^2(\omega t + \phi)] = \frac{1}{2}kA^2. \quad (1.23)$$

Equation (1.23) shows that the energy of a harmonic oscillator is proportional to the square of the amplitude of the oscillation: the more we initially extend the spring, the more potential energy we store in it. The first line of Equation (1.23) also shows that the energy of

Figure 1.9 The variations of kinetic energy K and potential energy U with time t for a simple harmonic oscillator. The total energy of the oscillator E is the sum of the kinetic and potential energies and remains constant with time.

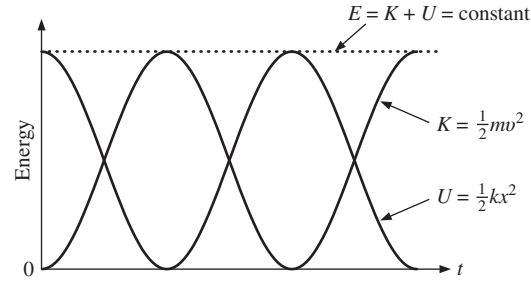
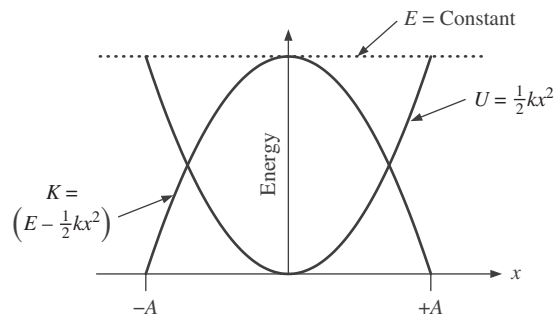


Figure 1.10 The variation of kinetic energy K and potential energy U with displacement x for a simple harmonic oscillator.



the system flows between kinetic and potential energies, although the total energy remains constant. This is illustrated in Figure 1.9, which shows the variation of the potential and kinetic energies with time. We have taken $\phi = 0$ in this figure. We can also plot the kinetic and potential energies as functions of the displacement x . The potential energy $U = \frac{1}{2} kx^2$ is a parabola in x , as shown in Figure 1.10. We do not need to work out the equivalent expression for the variation in kinetic energy since this must be equal to $(E - \frac{1}{2} kx^2)$ and is also shown in the figure.

1.2.6 The Physics of Small Vibrations

A mass on a spring is an example of a system in stable equilibrium. When the mass moves away from its equilibrium position, the restoring force pulls or pushes it back. We found that the potential energy of a mass on a spring is proportional to x^2 , so that the potential energy curve has the shape of a parabola given by $U(x) = \frac{1}{2} kx^2$ (cf. Figure 1.10). This curve has a minimum when $x = 0$, which corresponds to the unstretched length of the spring. The movement of the mass is constrained by the spring, and the mass is said to be confined in a potential well. The parabolic shape of this potential well gives rise to SHM. Any system that is in stable equilibrium will oscillate if it is displaced from its equilibrium state. We may think of a marble in a round-bottomed bowl. When the marble is pushed to one side, it rolls back and forth in the bowl. The universal importance of the harmonic oscillator is that nearly all the potential wells we encounter in physical situations have a shape that is parabolic when we are sufficiently close to the equilibrium position. Thus, *most oscillating systems will oscillate with SHM when the amplitude of oscillation is small*, as we shall prove

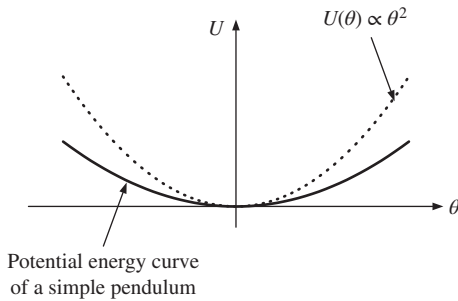


Figure 1.11 The solid curve represents the potential energy U of a simple pendulum as a function of its angular displacement θ . The dotted line represents the potential energy $U(\theta)$ of a simple harmonic oscillator for which the potential energy is proportional to θ^2 . For small angular amplitudes, where the two curves overlap, a simple pendulum behaves as a simple harmonic oscillator.

in a moment. This situation is illustrated in Figure 1.11, which shows as a solid line the potential energy of a simple pendulum as a function of the angular displacement θ . (We will discuss the example of the simple pendulum in detail in Section 1.3.) Superimposed on it as a dotted line is a parabolic-shaped potential well, i.e. proportional to θ^2 . Close to the equilibrium position ($\theta = 0$), the two curves lie on top of each other. So long as the amplitude of oscillation falls within the range where the two curves coincide, the pendulum will execute SHM.

We can see the above result mathematically using Taylor's theorem, which says that any function $f(x)$, which is continuous and possesses derivatives of all orders at $x = a$ can be expanded in a power series in $(x - a)$ in the neighbourhood of the point $x = a$, i.e.

$$f(x) = f(a) + \frac{(x - a)}{1!} \left(\frac{df}{dx} \right)_{x=a} + \frac{(x - a)^2}{2!} \left(\frac{d^2f}{dx^2} \right)_{x=a} + \dots, \quad (1.24)$$

where the derivatives df/dx , etc. are evaluated at $x = a$. (In practice, all the potential wells that we encounter in physical situations can be described by functions that can be expanded in this way.) We see that Taylor's theorem gives the value of a function $f(x)$ in terms of the value of the function at $x = a$ and the values of the first and higher derivatives of x evaluated at $x = a$. If we expand $f(x)$ about $x = 0$, we have

$$f(x) = f(0) + x \left(\frac{df}{dx} \right)_{x=0} + \frac{x^2}{2} \left(\frac{d^2f}{dx^2} \right)_{x=0} + \dots$$

In the case of a general potential well $U(x)$, we expand about the equilibrium position $x = 0$ to obtain

$$U(x) = U(0) + x \left(\frac{dU}{dx} \right)_{x=0} + \frac{x^2}{2} \left(\frac{d^2U}{dx^2} \right)_{x=0} + \dots \quad (1.25)$$

The first term $U(0)$ is a constant and has no physical significance in the sense that we can measure potential energy with respect to any position, and indeed we can choose it to be equal to zero. The first derivative of U with respect to x is zero because the curve is a minimum at $x = 0$. The second derivative of U with respect to x , evaluated at $x = 0$ will be a constant, as we will see. Thus, if we retain only the first non-zero term in the expansion, which is a good approximation so long as x is small, we have

$$U(x) = \frac{x^2}{2} \left(\frac{d^2U}{dx^2} \right)_{x=0} \quad (1.26)$$

This is indeed the form of the potential energy for the mass on a spring, with d^2U/dx^2 playing the role of the spring constant. Then the force close to the equilibrium position takes the general form

$$F = -\frac{dU}{dx} = -x \left(\frac{d^2U}{dx^2} \right)_{x=0} \quad (1.27)$$

The force is directly proportional to x and acts in the opposite direction, which is our familiar result for the simple harmonic oscillator.

The fact that a vibrating system will behave like a simple harmonic oscillator when its amplitude of vibration is small means that our physical world is filled with examples of SHM. To illustrate this diversity, Table 1.1 gives examples of a variety of physical systems that can oscillate and their associated periods of oscillation. These examples occur in both classical and quantum mechanics. Clearly the more massive the system, the greater the period of oscillation. For the case of a vibrating tuning fork, we can tell that the ends of the fork are oscillating at a single frequency because we hear a pure note that we can use to tune musical instruments. A plucked guitar string will also oscillate, and indeed, musical instruments provide a wealth of examples of SHM. These oscillations, however, will in general be more complicated than that of the tuning fork, but even here these complex oscillations are a superposition of SHMs, as discussed in Chapter 6. The balance wheel of a mechanical clock, the sloshing of water in a lake and the swaying of a skyscraper in the wind provide further examples of classical oscillators.

A good example of SHM in the microscopic world is provided by the vibrations of the atoms in a crystal. The forces between the atoms result in the regular lattice structure of the crystal. Furthermore, when an atom is slightly displaced from its equilibrium position, it is subject to a net restoring force. The shape of the resultant potential well approximates to a parabola for small amplitudes of vibration. Thus, when the atoms vibrate, they do so with SHM. Einstein used a simple harmonic oscillator model of a crystal to explain the observed variation of heat capacity with temperature (see also King,² Section 10.1). He assumed that the atoms were harmonic oscillators that vibrate independently of each other but with the same angular frequency, and he used a quantum mechanical description of these oscillators. As we have seen, in classical mechanics the energy of an oscillator

Table 1.1 Examples of systems that can oscillate and the associated periods of oscillation.

System	Period(s)
Sloshing of water in a lake	$\sim 10^2 - 10^4$
Large bridges and buildings	$\sim 1 - 10$
Clock pendulum or balance wheel	~ 1
String instruments	$\sim 10^{-3} - 10^{-2}$
Crystal in a quartz clock	$\sim 3 \times 10^{-5}$
Molecular vibrations	$\sim 10^{-15}$

² Physics of Matter, G.C. King, 2023, John Wiley & Sons, Ltd.

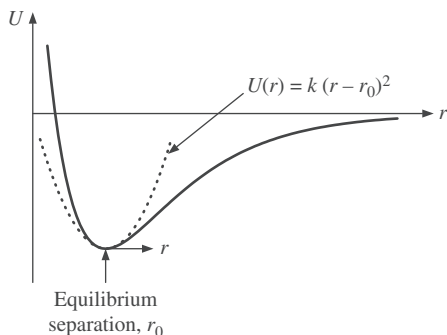


Figure 1.12 The solid curve represents the variation of potential energy of a hydrogen molecule as a function of the separation of the two hydrogen nuclei. The dotted curve represents the potential energy of a simple harmonic oscillator centred on the equilibrium separation r_0 of the two nuclei.

is proportional to the square of the amplitude and can take any value, i.e. the energy is continuous. A fundamental result of quantum mechanics is that the energy of a harmonic oscillator is quantised, i.e. only a discrete set of energies is possible. Einstein's quantum model predicted that the specific heat of a crystal, such as diamond, goes to zero as the temperature of the crystal decreases, unlike the classical result that the specific heat is independent of temperature. Experiments show that the specific heat of diamond does indeed go to zero at low temperatures.

Another example of SHM in quantum physics is provided by the vibrations of the two nuclei of a hydrogen molecule. The solid curve in Figure 1.12 represents the potential energy U of the hydrogen molecule as a function of the separation r between the nuclei, where we have taken the potential energy to be zero at infinite separation. This potential energy is due to the Coulomb interaction of the electrons and nuclei and the quantum behaviour of the electrons. The curve exhibits a minimum at $r_0 = 0.74 \times 10^{-10}$ m. At small separation ($r \rightarrow 0$) the potential energy tends to infinity, representing the strong repulsion between the nuclei. The nuclei perform oscillations about the equilibrium separation. The dotted line in Figure 1.12 shows the parabolic form of the potential energy of a harmonic oscillator, centred at the equilibrium separation r_0 . For small amplitudes of oscillation (i.e. when the nuclei are not too highly excited), the vibrations occur within the range where the two curves coincide. Again, according to quantum mechanics, only a discrete set of vibrational energies is possible. For a simple harmonic oscillator with angular frequency ω , the only allowed values of the energy are $\frac{1}{2}\hbar\omega$, $\frac{3}{2}\hbar\omega$, $\frac{5}{2}\hbar\omega$, ..., where \hbar is Planck's constant divided by 2π . The observed vibrational line spectra of molecules correspond to transitions between these energy levels with the emission of electromagnetic radiation that typically lies in the infrared part of the electromagnetic spectrum. These spectra provide valuable information about the properties of the molecule, such as the strength of the molecular bond.

Worked Example The H_2 molecule has a vibrational frequency ν of 1.32×10^{14} Hz. Calculate the strength of the molecular bond, i.e. the 'spring constant', assuming that the molecule can be modelled as a simple harmonic oscillator.

Solution

In previous cases, we considered a mass vibrating at one end of a spring while the other end of the spring was connected to a rigid wall. Now we have two nuclei vibrating against each

other, which we model as two equal masses connected by a spring. We can solve this new situation by realising that there is no translation of the molecule during the vibration, i.e. the centre of mass of the molecule does not move. Thus, as one hydrogen nucleus moves in one direction by a distance x , the other must move in the opposite direction by the same amount, and of course both vibrate at the same frequency. The total extension is $2x$ and the tension in the 'spring' is equal to $2kx$, where k represents the 'spring constant' or bond strength. The equation of motion of each nucleus of mass m is then given by

$$m \frac{d^2x}{dt^2} = -2kx$$

or

$$\frac{m}{2} \frac{d^2x}{dt^2} = -kx. \quad (1.28)$$

This equation is analogous to Equation (1.5), where m has been replaced by $m/2$, which is called the *reduced mass* of the system. The classical angular frequency of vibration ω of the molecule is then equal to $\sqrt{2k/m}$. The frequency of vibration $\nu = 1/T = \omega/2\pi$ and $m = 1.67 \times 10^{-27}$ kg. Therefore,

$$k = 4\pi^2\nu^2 \frac{m}{2} = \frac{4\pi^2(1.32 \times 10^{14})^2 1.67 \times 10^{-27}}{2} = 574 \text{ N m}^{-1}.$$

Worked Example The potential energy between two atoms can be represented by a plot of their potential energy $U(r)$ as a function of the distance r between their centres. A particular example that is useful for simple atoms is the *Lennard-Jones 6-12 potential*: $U(r) = \epsilon \left[\left(\frac{r_0}{r} \right)^{12} - 2 \left(\frac{r_0}{r} \right)^6 \right]$, where ϵ is the energy required to separate the two atoms to infinity and r_0 is the equilibrium separation. (i) Obtain an expression for the force $F(r)$ between the two atoms as a function of their separation r . (ii) Obtain an expression for $F(r_0 + \delta r)$, where $\delta r \ll r_0$. Hence, show that for small excursions from the equilibrium separation, the system vibrates in SHM. (iii) For the example of argon of atomic mass 40 u, $\epsilon = 1.70 \times 10^{-21}$ J and $r_0 = 3.4 \times 10^{-10}$ m. Use these data to determine the frequency of the vibrations.

Solution

$$\begin{aligned} \text{(i)} \quad \frac{dU(r)}{dr} &= \epsilon \left[-12r_0^{12} \left(\frac{1}{r} \right)^{13} + 2 \times 6r_0^6 \left(\frac{1}{r} \right)^7 \right], \\ F(r) &= -\frac{dU(r)}{dr} = \frac{12\epsilon}{r_0} \left[\left(\frac{r_0}{r} \right)^{13} - \left(\frac{r_0}{r} \right)^7 \right]. \\ \text{(ii)} \quad F(r_0 + \delta r) &= \frac{12\epsilon}{r_0} \left[\left(\frac{r_0}{r_0 + \delta r} \right)^{13} - \left(\frac{r_0}{r_0 + \delta r} \right)^7 \right], \\ &= \frac{12\epsilon}{r_0} \left[\left(\frac{1}{(1 + \delta r/r_0)} \right)^{13} - \left(\frac{1}{(1 + \delta r/r_0)} \right)^7 \right]. \end{aligned}$$

Then, using a Taylor expansion with $\delta r \ll r_0$:

$$F(r_0 + \delta r) = \frac{12\epsilon}{r_0} [1 - 13\delta r/r_0 - 1 + 7\delta r/r_0^7],$$

or

$$F(r_0 + \delta r) = -\frac{72\epsilon}{r_0^2}\delta r.$$

This is an equation of SHM, with frequency $\nu = \frac{1}{2\pi}\sqrt{\frac{72\epsilon}{\mu r_0^3}}$, where μ is the reduced mass of the system $= \frac{m_1 m_2}{m_1 + m_2}$. (iii) For argon, $\mu = 20$ u. Therefore,

$$\nu = \frac{1}{2\pi}\sqrt{\frac{72 \times 1.7 \times 10^{-21}}{20 \times 1.66 \times 10^{-27} (3.4 \times 10^{-10})^3}} = 9.0 \times 10^{11} \text{ Hz.}$$

1.3 The Pendulum

1.3.1 The Simple Pendulum

Timing the oscillations of a pendulum has been used for centuries to measure time accurately. The simple pendulum is the idealised form that consists of a point mass m suspended from a massless rigid rod of length l , as illustrated in Figure 1.13. For an angular displacement θ , the displacement of the mass *along* the arc of the circle of length l is $l\theta$. Hence, the angular velocity along the arc is $l d\theta/dt$, and the angular acceleration is $l d^2\theta/dt^2$. At a displacement θ , there is a tangential force on the mass acting along the arc that is equal to $-mg \sin \theta$: as usual, the minus sign indicates that it is a restoring force. Hence, by Newton's second law, we obtain

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l} \sin \theta. \quad (1.29)$$

This equation does not have the same form as the equation of SHM, Equation (1.6), as we have $\sin \theta$ on the right-hand side instead of θ . However, we can expand $\sin \theta$ in a power series in θ :

$$\sin \theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \dots \quad (1.30)$$

For small angular deflections, the second and higher terms are much smaller than the first term. For example, if θ is equal to 0.1 rad (5.7°), which is typical for a pendulum clock, then the second term is only 0.17% of the first term, and the higher terms are much smaller still. We can see this directly by plotting the functions $y = \sin \theta$ and $y = \theta$ on the same set of axes, as shown in Figure 1.14. The two curves are indistinguishable for values of θ below about $1/4$ rad ($\sim 15^\circ$). Thus, for small values of θ , we need to retain only the first term in the Equation (1.30) and replace $\sin \theta$ with θ (in radians) to give

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l} \theta. \quad (1.31)$$

This is the equation of SHM with $\omega = \sqrt{g/l}$ and $T = 2\pi \sqrt{l/g}$, and we can immediately write down an expression for the angular displacement θ of the pendulum:

$$\theta = \theta_0 \cos(\omega t + \phi), \quad (1.32)$$

where θ_0 is the angular amplitude of oscillation. The period is independent of amplitude for oscillations of small amplitude, and this is why the pendulum is so useful as an accurate

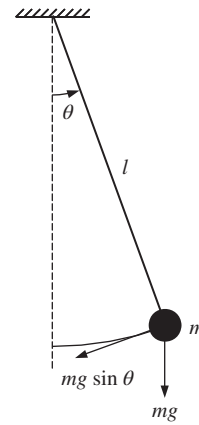
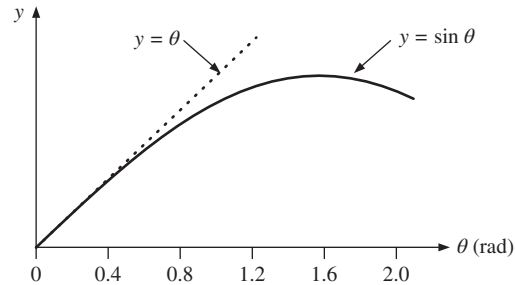


Figure 1.13 The simple pendulum of mass m and length l .

Figure 1.14 A comparison of the functions $y = \theta$ and $y = \sin \theta$ plotted against θ .



timekeeper. The period does, however, depend on the acceleration due to gravity, and so measuring the period of a pendulum provides a way of determining the value of g . (In practice, real pendulums do not have their mass concentrated at a point as in the simple pendulum as will be described in Section 1.3.3. So for an accurate determination of g , a more sophisticated pendulum has been developed called the *compound pendulum*.) We finally note that for $l = 1.00$ m and for a value of $g = 9.87$ m s⁻², the period of a simple pendulum is equal to $2\pi \sqrt{1.00/9.87} = 2.00$ s and indeed the second was originally defined as equal to one half the period of a 1 m simple pendulum.

Worked Example When the amplitude of oscillation of a pendulum is small, the approximation $\sin \theta = \theta$ can be used. However, when the amplitude is not sufficiently small, the second term in the expansion for $\sin \theta$ must be included:

$$\sin \theta = \theta - \frac{1}{6}\theta^3 + \dots$$

Then the equation of motion for the pendulum is

$$\frac{d^2\theta}{dt^2} + \omega_0^2\theta - \frac{\omega_0^2}{6}\theta^3 = 0.$$

Show that an approximate solution of this differential equation is

$$\theta = \theta_0 \sin \omega t + \varepsilon \theta_0 \sin 3\omega t,$$

where θ_0 is the amplitude of the oscillation and ε is a dimensionless constant $\ll 1$.

Solution

From the given solution for θ , we have:

$$\theta^3 = (\theta_0 \sin \omega t + \varepsilon \theta_0 \sin 3\omega t)^3,$$

giving

$$\theta^3 = \theta_0^3 \sin^3 \omega t + 3\varepsilon \theta_0^3 \sin^2 \omega t \sin 3\omega t,$$

where terms of order ε^2 and ε^3 have been discarded as $\varepsilon \ll 1$.

Using the identity $\sin^3 \alpha = \frac{3}{4} \sin \alpha - \frac{1}{4} \sin 3\alpha$ gives

$$-\frac{\omega_0^2}{6}\theta^3 = -\frac{3\omega_0^2}{24}\theta_0^3 \sin \omega t + \frac{\omega_0^2}{24}\theta_0^3 \sin 3\omega t - \frac{\omega_0^2}{2}\theta_0^3 \varepsilon \sin^2 \omega t \sin 3\omega t.$$

We have

$$\omega_0^2\theta = \omega_0^2\theta_0 \sin \omega t + \omega_0^2\varepsilon\theta_0 \sin 3\omega t$$

and

$$\frac{d^2\theta}{dt^2} = -\omega^2\theta_0 \sin \omega t - 9\omega^2\epsilon\theta_0 \sin 3\omega t.$$

The sum of the terms in the left-hand sides of these three equations is equal to zero. Then equating terms in $\sin \omega t$, we find

$$-\frac{3\omega_0^2}{24}\theta_0^3 + \omega_0^2\theta_0 - \omega^2\theta_0 = 0,$$

or,

$$\omega^2 = \omega_0^2 \left(1 - \frac{1}{8}\theta_0^2\right).$$

Then, $\omega = \omega_0 \left(1 - \frac{1}{8}\theta_0^2\right)^{1/2} \cong \left(1 - \frac{1}{16}\theta_0^2\right)$, for small θ .

This result shows that the frequency of the pendulum depends on its amplitude, which becomes apparent at large amplitudes. Note that the motion now contains two frequencies, ω and 3ω . These arise because the oscillator is a *non-linear* oscillator; the motion depends on the cubic term in θ as well as the linear term. Equating terms in $\sin 3\omega t$, we obtain

$$\frac{\omega_0^2}{24}\theta_0^2 - \frac{\omega_0^2}{2}\epsilon\theta_0^2 \sin^2\omega t + \omega_0^2\epsilon - 9\omega^2\epsilon = 0.$$

The second term on the left-hand side is much smaller than the other terms because it contains both ϵ and $\sin^2\omega t$ and can be neglected. Then setting $\omega^2 \cong \omega_0^2$, we obtain $\epsilon \cong \frac{\theta_0^2}{192}$. The value of ϵ gives the relative amplitudes of the fundamental frequency ω and the harmonic frequency 3ω . For example, if $\theta_0 = 30^\circ$, the relative amplitudes are in the ratio of $1 : 1.4 \times 10^{-3}$.

1.3.2 The Energy of a Simple Pendulum

We can also analyse the motion of a simple pendulum by considering its kinetic and potential energies. The geometry of the simple pendulum is shown in Figure 1.15. (The horizontal distance $x = l \sin \theta$ is not exactly the same as the distance along the arc, which is equal to $l\theta$. However, since $\sin \theta \cong \theta$ for small θ , the difference is negligible.) From the geometry, we have

$$l^2 = (l-y)^2 + x^2, \quad (1.33)$$

which gives

$$2ly = y^2 + x^2. \quad (1.34)$$

For small displacements of the pendulum, i.e. $x \ll l$, it follows that $y \ll x$, so that the term y^2 can be neglected and we can write,

$$y = \frac{x^2}{2l}. \quad (1.35)$$

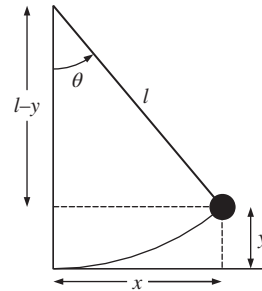


Figure 1.15 The geometry of the simple pendulum.

As the mass is displaced from its equilibrium position, its vertical height increases, and it gains potential energy. This gain in potential energy is equal to $mgy = mgx^2/2l$. The total energy of the system E is given by the sum of the kinetic and potential energies:

$$E = K + U = \frac{1}{2}mv^2 + \frac{1}{2}\frac{mgx^2}{l}. \quad (1.36)$$

At the turning point of the motion, when x equals A , the velocity v is zero giving

$$E = \frac{1}{2}\frac{mgA^2}{l}. \quad (1.37)$$

From conservation of energy, it follows that

$$\frac{mgA^2}{l} = mv^2 + \frac{mgx^2}{l} \quad (1.38)$$

is true for all times. We can use Equation (1.38) to obtain expressions for velocity v and displacement x :

$$v = \frac{dx}{dt} = \sqrt{\frac{g(A^2 - x^2)}{l}}. \quad (1.39)$$

This expression describes how the velocity changes with the displacement x in SHM in contrast to Equation (1.12), which describes how the velocity changes with time t . Since $v = dx/dt$, we have

$$\int \frac{dx}{\sqrt{A^2 - x^2}} = \sqrt{\frac{g}{l}} \int dt. \quad (1.40)$$

The integral on the left-hand side can be evaluated using the substitution $x = A \sin \theta$, giving

$$\sin^{-1}\left(\frac{x}{A}\right) = \sqrt{\frac{g}{l}} t + \phi, \quad (1.41)$$

where ϕ is the constant of integration, and

$$x = A \sin\left(\sqrt{\frac{g}{l}} t + \phi\right). \quad (1.42)$$

Equation (1.42) describes SHM with $\omega = \sqrt{g/l}$ and $T = 2\pi\sqrt{l/g}$ as before.

At this point we note the similarity in the expressions for the total energy of the two examples of SHM that we have considered.

$$\text{For the mass on a spring: } E = \frac{1}{2}mv^2 + \frac{1}{2}kx^2. \quad (1.43a)$$

$$\text{For the simple pendulum: } E = \frac{1}{2}mv^2 + \frac{1}{2}\frac{mg}{l}x^2. \quad (1.43b)$$

$$\text{Both expressions have the form: } E = \frac{1}{2}\alpha v^2 + \frac{1}{2}\beta x^2, \quad (1.43c)$$

where α and β are constants. It is a universal characteristic of simple harmonic oscillators that their total energy can be written as the sum of two parts, one involving the (velocity)² and the other the (displacement)². Just as $md^2x/dt^2 = -kx$, Equation (1.5) is the signature

of SHM in terms of forces, and Equation (1.43c) is the signature of SHM in terms of energies. If we obtain either of these equations in the analysis of a system, then we know we have SHM. We stress that the equations are the same for all simple harmonic oscillators: only the labels for the physical quantities change. We do not need to repeat the analysis again: we can simply take over the results already obtained. The constant α corresponds to the inertia of the system, through which it can store kinetic energy. The constant β corresponds to the restoring force per unit displacement through which the system can store potential energy. When we differentiate the conservation of energy equation for SHM, Equation (1.43c), with respect to time, we obtain

$$\frac{dE}{dt} = \alpha v \frac{dv}{dt} + \beta x \frac{dx}{dt} = 0,$$

giving

$$\frac{d^2x}{dt^2} = -\frac{\beta}{\alpha}x.$$

Comparing this with Equation (1.6), it follows that the angular frequency of oscillation ω is equal to $\sqrt{\beta/\alpha}$.

Worked Example A marble of radius r rolls back and forth without slipping in a spherical dish of radius R . Use energy considerations to show that the motion is simple harmonic for small displacements of the marble from its equilibrium position and deduce an expression for the period of the oscillations. The moment of inertia I of a solid sphere of mass m about an axis through its centre is equal to $2mr^2/5$.

Solution

The equilibrium and displaced positions of the marble are shown in Figure 1.16, where the arrows indicate the rotation of the marble when it rotates through an angle ϕ . If the marble were rotating through an angle ϕ on a *flat* surface, it would roll a distance $r\phi$. However, on a spherical surface, as in Figure 1.16, it rolls a distance l along the arc of radius R given by $l = r(\phi + \theta)$. As $l = R\theta$,

$$\phi = \frac{(R-r)}{r}\theta \quad \text{and} \quad \frac{d\phi}{dt} = \frac{(R-r)}{r} \left(\frac{d\theta}{dt} \right).$$

The total kinetic energy of the marble, as it moves along the surface of the dish, is equal to the kinetic energy of the translational motion of its centre of mass plus the kinetic energy

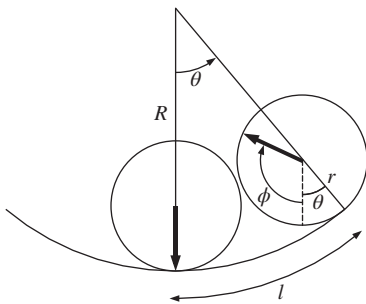


Figure 1.16 A marble of radius r that rolls back and forth without slipping in a spherical dish of radius R .

of its rotational motion about the centre of mass. Hence,

$$K = \frac{1}{2}mv^2 + \frac{1}{2}I\left(\frac{d\phi}{dt}\right)^2.$$

The translational kinetic energy is given by

$$\frac{1}{2}mv^2 = \frac{1}{2}m(R-r)^2\left(\frac{d\theta}{dt}\right)^2.$$

Therefore,

$$K = \frac{1}{2}m\left(\frac{7}{5}\right)(R-r)^2\left(\frac{d\theta}{dt}\right)^2,$$

where we have substituted for I . The potential energy is

$$U = mg(R-r)(1 - \cos \theta) = \frac{1}{2}mg(R-r)\theta^2$$

for small θ . Thus,

$$E = \frac{1}{2}m\left(\frac{7}{5}\right)(R-r)^2\left(\frac{d\theta}{dt}\right)^2 + \frac{1}{2}mg(R-r)\theta^2.$$

This has the general form of the energy Equation (1.43c) of a harmonic oscillator

$$E = \frac{1}{2}\alpha\left(\frac{d\theta}{dt}\right)^2 + \frac{1}{2}\beta\theta^2,$$

where now θ represents the displacement coordinate. Hence,

$$\omega = \sqrt{\frac{\beta}{\alpha}} = \sqrt{\frac{5g}{7(R-r)}} \quad \text{and} \quad T = 2\pi\sqrt{\frac{7(R-r)}{5g}}.$$

This example would be much more difficult to solve from force considerations.

1.3.3 The Physical Pendulum

In a physical pendulum, the mass is not concentrated at a point as in the simple pendulum but is distributed over the whole body. It is thus more representative of real pendulums. An example of a physical pendulum is shown in Figure 1.17. It consists of a uniform rod of length l that pivots about a horizontal axis at its upper end. This is a rotating system where the pendulum rotates about its point of suspension. For a rotating system, Newton's second law for linear systems, $m(d^2x/dt^2) = F$, becomes

$$I\frac{d^2\theta}{dt^2} = \tau, \quad (1.44)$$

where I is the moment of inertia of the body about its axis of rotation and τ is the applied torque. The centre of mass of a uniform rod is located at its midpoint, and its moment of inertia about its midpoint is $ml^2/12$, where m is its mass. The parallel-axis theorem says that if the moment of inertia of a body of mass m about an axis

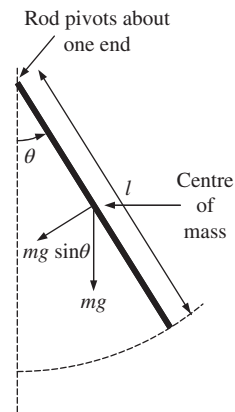


Figure 1.17 A rod that pivots about one of its ends, which is an example of a physical pendulum.

through its centre of mass is I_{com} , the moment of inertia I about any other axis that is parallel to the original one but displaced from it by distance d is $I = I_{\text{com}} + Md^2$. Hence, the moment of inertia of the rod about one end is

$$\frac{ml^2}{12} + m\left(\frac{l}{2}\right)^2 = \frac{ml^2}{3}.$$

The resultant torque τ on the rod when it is displaced through an angle θ is given by the product of the torque arm $l/2$ and the component of the force normal to the torque arm ($mg \sin \theta$), i.e.

$$\tau = \left(\frac{1}{2}l\right) \times (-mg \sin \theta).$$

Hence, we obtain

$$\frac{1}{3}ml^2 \frac{d^2\theta}{dt^2} = -\frac{1}{2}mgl \sin \theta, \quad (1.45)$$

giving

$$\frac{d^2\theta}{dt^2} = -\frac{3g}{2l} \sin \theta. \quad (1.46)$$

Again we can use the small-angle approximation to obtain

$$\frac{d^2\theta}{dt^2} = -\frac{3g}{2l} \theta. \quad (1.47)$$

This is SHM with $\omega = \sqrt{3g/2l}$ and $T = 2\pi\sqrt{2l/3g}$.

In a simple model, we can describe the walking pace of a person in terms of a physical pendulum. We model the human leg as a solid rod that pivots from the hip. Furthermore, when we walk, we do so at a comfortable pace that coincides with the natural period of oscillation of the leg. If we assume a value of 0.8 m for l , the length of the leg, then its natural period is ~ 1.5 s. One complete period of the swinging leg corresponds to two strides. Try this yourself. If the length of a stride is, say, 1 m, then we would walk at a speed of approximately $2/1.5 \text{ m s}^{-1}$, which corresponds to 4.8 km h^{-1} , or about 3 mph, which is in good agreement with reality.

Worked Example A uniform solid disc of mass M and radius R is suspended from a fixed support by a torsion wire. When the wire is twisted through angle θ , it exerts a restoring torque $c\theta$ on the disc, where c is the torsion constant. (a) When the disc is rotated through a small angle and released, it oscillates about its equilibrium position. Consider the total energy of the disc as it oscillates to show that it performs SHM. The mass M of the disc is 1.5 kg, and its radius R is 50 cm. If the period of the oscillation is 2.5 s, determine the value of the torsion constant. The moment of inertia of a disc about its centre is $MR^2/2$. (b) One form of the *equipartition of energy theorem* states that each quadratic term in the expression for the total energy of a body contributes $\frac{1}{2}kT$ to the total energy, where k is the Boltzmann constant and T is temperature. This means that even when it is not disturbed, the disc will jiggle about its equilibrium position. Use the theorem to obtain an expression for the root mean square value θ_{rms} of the angular displacement at $T = 20^\circ\text{C}$.

Solution

- (a) The rotational potential energy of the disc is $\int_0^\theta c\theta' d\theta' = \frac{1}{2}c\theta^2$, and its rotational kinetic energy is $\frac{1}{2}I\left(\frac{d\theta}{dt}\right)^2$. Hence, the total energy of the disc is $\frac{1}{2}I\left(\frac{d\theta}{dt}\right)^2 + \frac{1}{2}c\theta^2$. This has the form of the total energy of a simple harmonic oscillator:

$$E = \frac{1}{2}\alpha\dot{\psi}^2 + \frac{1}{2}\beta\psi^2, \text{ with } \omega = \sqrt{\beta/\alpha}.$$

Hence, $\omega = \sqrt{c/I}$. In terms of period τ ,

$$c = 4\pi^2 \frac{I}{\tau^2} = 4\pi^2 \frac{MR^2}{2\tau^2} = 4\pi^2 \frac{1.5 \times (50 \times 10^{-2})^2}{2 \times (2.5)^2} = 1.2 \text{ N rad}^{-1}.$$

- (b) From the equipartition of energy theorem, we have $\left\langle \frac{1}{2}c\theta^2 \right\rangle = \frac{1}{2}kT$, where the angular brackets indicate the average value. Hence,

$$\theta_{\text{rms}} = [\langle \theta^2 \rangle]^{1/2} = \left[\frac{kT}{c} \right]^{1/2} = \left[\frac{1.38 \times 10^{-23} \times 293}{1.2} \right]^{1/2} = 5.8 \times 10^{-11} \text{ rad}.$$

This small amount of jiggle can produce a limit to the sensitivity of measuring equipment that uses such a torsion balance, as, for example, in the Cavendish Boys experiment to measure the gravitational constant.

1.3.4 Numerical Solution of SHM³

When solving the equation of motion for an oscillating pendulum, we made use of the small-angle approximation, $\sin \theta \cong \theta$ when θ is small. This made the equation of motion much easier to solve. However, an alternative way, without resorting to the small-angle approximation, is to solve the equation numerically. The essential idea is that if we know the position and velocity of the mass at time t and we know the force acting on it, then we can use this knowledge to obtain good estimates of these parameters at time $(t + \delta t)$. We then repeat this process, step by step, over the full period of the oscillation to trace out the displacement of the mass with time. We can make these calculations as accurate as we like by making the time interval δt sufficiently small. To demonstrate this approach, we apply it to the simple pendulum. Figure 1.18 shows a simple pendulum and the angular position of the mass at three instants of time, each separated by δt , i.e. at t , $(t + \delta t)$ and $(t + 2\delta t)$. Using the notation $\dot{\theta}$ and $\ddot{\theta}$ for $d\theta/dt$ and $d^2\theta/dt^2$, respectively, we can write the equation of motion of the mass, Equation (1.29),

$$\ddot{\theta}(t) = -\frac{g}{l} \sin \theta(t). \quad (1.48)$$

If the angular position of the mass is $\theta(t)$ at time t , then its position at time $(t + \delta t)$ will be different by an amount equal to the angular velocity of the mass times the time interval δt (cf. the familiar expression $x = vt$ for linear motion). We might be tempted to use $\dot{\theta}$ for this angular velocity. However, as we know, the angular velocity varies during the time δt . A better estimate for the angular velocity is its *average* value between the times t and $(t + \delta t)$, i.e. $\dot{\theta}(t + \delta t/2)$. Thus, to a good approximation, we have

$$\theta(t + \delta t) = \theta(t) + \delta t \times \dot{\theta}(t + \delta t/2). \quad (1.49)$$

³ This section may be omitted as it is not required later in the book.

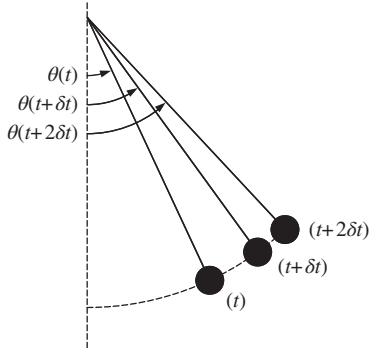


Figure 1.18 A simple pendulum showing the position of the mass at three instants of time separated by time interval δt .

In a similar way, we can relate the angular velocities of the mass at times separated by time δt , i.e. the new velocity will be different from the old value by an amount equal to $\delta t \times \ddot{\theta}(t)$, where $\ddot{\theta}(t)$ is the angular acceleration (cf. the familiar expression $v = u + at$ for linear motion). The acceleration also varies with time, and so again we will use its average value during the time interval δt . For the evaluation of $\dot{\theta}(t + \delta t/2)$, this translates to

$$\dot{\theta}(t + \delta t/2) = \dot{\theta}(t - \delta t/2) + \delta t \times \ddot{\theta}(t), \quad (1.50)$$

where $\ddot{\theta}(t)$ is the average value of the angular acceleration between the times $(t - \delta t/2)$ and $(t + \delta t/2)$, which we know from Equation (1.48). For the first step of this calculation, we need the value of the angular velocity at time $t = \delta t/2$. For this particular case, we use the expression

$$\dot{\theta}(\delta t/2) = (\delta t/2) \times \ddot{\theta}(0). \quad (1.51)$$

Armed with these expressions for angular position, velocity and acceleration we can trace the angular displacement of the mass step by step.

We proceed by building up a table of consecutive values of $\theta(t)$, $\dot{\theta}(t)$ and $\ddot{\theta}(t)$. As an example, we chose the length of the simple pendulum to give $T = 2.0$ s and $\omega = \pi$. We also chose a time interval δt of 0.02 s (equal to one hundredth of the period) and an angular amplitude θ_0 of 0.10 rad (5.7°). The values obtained in the first 10 steps of the calculation are shown in Table 1.2 and were obtained using a hand calculator. For comparison the final column of Table 1.2 shows the values obtained from the analytic solution $\theta(t) = \theta_0 \cos \omega t$. We see that the numerically calculated values of the displacement are in agreement with the analytic values up to the third significant figure. These two sets of values for a complete period of oscillation are plotted in Figure 1.19 and show the familiar variation of displacement with time. The solid curve corresponds to the values of displacement obtained from the analytic solution $\theta(t) = \theta_0 \cos \omega t$, while the dots (\bullet) correspond to the numerically computed values. The agreement is so good that the dots lie exactly on top of the analytic curve. These results demonstrate that the small-angle approximation is valid in this case and that the numerical method gives accurate results.

This numerical method allows us to explore what happens for large-amplitude oscillations where the small-angle approximation is no longer valid. Figure 1.20 shows the results for a very large angular amplitude of 1.0 rad (57°), which were obtained using a spreadsheet program. The solid curve corresponds to the values of displacement obtained from the solution $\theta(t) = \theta_0 \cos \omega t$, while the dotted curve is the one obtained from the numerically computed values. There is a significant difference between the two curves: the actual

Table 1.2 Computed values of angular displacement, velocity and acceleration of a simple pendulum. The last column on the right shows the values obtained from the analytic solution.

Time (s)	Angular displacement, $\theta(t)$ (rad)	Angular acceleration, $\ddot{\theta}(t)$ (rad s^{-2})	Angular velocity, $\dot{\theta}(t)$ (rad s^{-1})	$\theta(t) = 0.1 \cos \pi t$ (rad)
0.00	0.1000	-0.985	-0.0099	0.1000
0.02	0.0998	-0.983	-0.0295	0.0998
0.04	0.0992	-0.978	-0.0491	0.0992
0.06	0.0982	-0.968	-0.0685	0.0982
0.08	0.0968	-0.954	-0.0876	0.0969
0.10	0.0950	-0.937	-0.106	0.0951
0.12	0.0929	-0.915	-0.124	0.0930
0.14	0.0904	-0.891	-0.142	0.0905
0.16	0.0876	-0.863	-0.159	0.0876

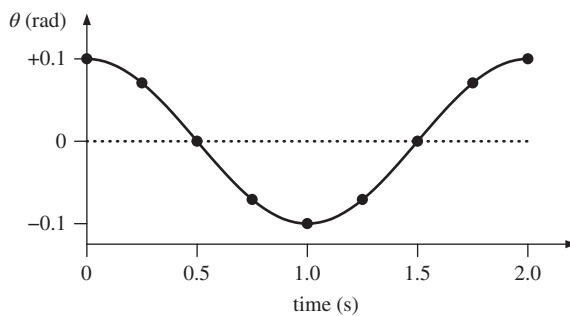


Figure 1.19 The angular displacement θ , plotted against time, for a simple pendulum with a small amplitude of oscillation; $\theta_0 = 0.1$ rad. The solid curve corresponds to the values of displacement obtained from the analytic solution $\theta(t) = \theta_0 \cos \omega t$, while the dots (\bullet) correspond to the numerically computed values. The agreement is so good that the computed values lie on top of the analytic curve.

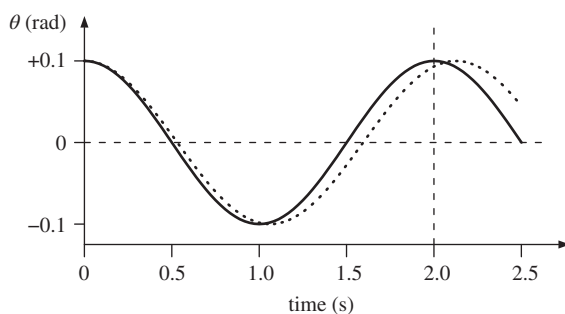


Figure 1.20 The angular displacement θ , plotted against time, of a simple pendulum for a large amplitude of oscillation; $\theta_0 = 1$ rad. The solid curve corresponds to the values of displacement obtained from the solution $\theta(t) = \theta_0 \cos \omega t$, while the dotted curve is obtained from the numerically computed results. For large-amplitude oscillations the period of the pendulum is no longer independent of amplitude and increases with amplitude.

angular displacement of the mass, which is given by the numerical values, no longer closely matches the analytic solution. In particular, the time period for the actual oscillations has increased to a value of 2.13 s: an increase of 6.5%. We see that for large-amplitude oscillations, the period of the pendulum is no longer independent of amplitude and that it increases with amplitude.

1.4 Oscillations in Electrical Circuits: Similarities in Physics

In this section, we consider oscillations in an electrical circuit. What we find is that these oscillations are described by a differential equation that is identical in form to Equation (1.6) and so has an identical solution: only the physical quantities associated with the differential equation are different. This illustrates that when we understand one physical situation, we can understand many others. It also means that we can simulate one system by another and, in this way, build analogue computers; i.e. we can build an electrical circuit consisting of resistors, capacitors and inductors that will exactly simulate the operation of a mechanical system.

1.4.1 The LC Circuit

The simplest example of an oscillating electrical circuit consists of an inductor L and capacitor C connected together in series with a switch, as shown in Figure 1.21. As usual, we start with an idealised situation where we assume that the resistance in the circuit is negligible. This is analogous to the assumption for mechanical systems that there are no frictional forces present. Initially, the switch is open and the capacitor is charged to voltage V_C . The charge q on the capacitor is given by $q = V_C C$, where C is the capacitance. When the switch is closed, the charge begins to flow through the inductor, and a current $I = dq/dt$ flows in the circuit. This is a time-varying current and produces a voltage across the inductor given by $V_L = LdI/dt$. We can analyse the LC circuit using *Kirchhoff's law*, which states that 'the sum of the voltages around the circuit is zero', i.e. $V_C + V_L = 0$. Therefore,

$$\frac{q}{C} + L \frac{dI}{dt} = 0, \quad (1.52)$$

giving

$$\frac{q}{C} + L \frac{d^2 q}{dt^2} = 0 \quad (1.53)$$

and

$$\frac{d^2 q}{dt^2} = -\frac{1}{LC} q. \quad (1.54)$$

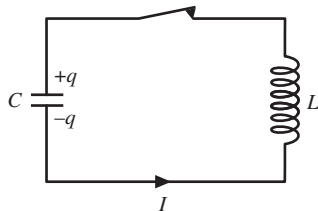
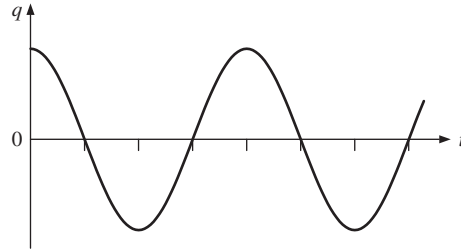


Figure 1.21 An electrical oscillator consisting of an inductor L and a capacitor C connected in series.



Figure 1.22 The variation of charge q with time on the capacitor in a series LC circuit. The charge oscillates in time in an analogous way to the displacement of a mass oscillating at the end of a spring.



This equation describes how the charge on a plate of the capacitor varies with time. It is of the same form as Equation (1.6) and represents SHM. The frequency of the oscillation is given directly by $\omega = \sqrt{1/LC}$. Since we have the initial condition that the charge on the capacitor has its maximum value at $t = 0$, then the solution to Equation (1.54) is $q = q_0 \cos \omega t$, where q_0 is the initial charge on the capacitor. The variation of charge q with respect to t is shown in Figure 1.22 and is analogous to the way the displacement of a mass on a spring varies with time.

We can also consider the energy of this electrical oscillator. The energy stored in a capacitor charged to voltage V_C is equal to $\frac{1}{2}CV_C^2$. This is electrostatic energy. The energy stored in an inductor is equal to $\frac{1}{2}LI^2$, and this is magnetic energy. Thus, the total energy in the circuit is given by

$$E = \frac{1}{2}LI^2 + \frac{1}{2}CV_C^2 \quad (1.55)$$

or

$$E = \frac{1}{2}LI^2 + \frac{1}{2}\frac{q^2}{C}. \quad (1.56)$$

For these electrical oscillations, the charge flows between the plates of the capacitor and through the inductor so that there is a continuous exchange between electrostatic and magnetic energy.

1.4.2 Similarities in Physics

We note the similarities between the equations for the mechanical and electrical cases

$$m\frac{d^2x}{dt^2} = -kx, \quad L\frac{d^2q}{dt^2} = -\frac{1}{C}q \quad (1.57a)$$

and

$$E = \frac{1}{2}m\left(\frac{dx}{dt}\right)^2 + \frac{1}{2}kx^2, \quad E = \frac{1}{2}L\left(\frac{dq}{dt}\right)^2 + \frac{1}{2}\frac{q^2}{C}, \quad (1.57b)$$

where we have written dx/dt for the velocity v and dq/dt for the current I , in order to bring out more sharply the similarity of the two cases. In both cases, we have the identical forms

$$\alpha\frac{d^2Z}{dt^2} = -\beta Z, \quad E = \frac{1}{2}\alpha\left(\frac{dZ}{dt}\right)^2 + \frac{1}{2}\beta Z^2, \quad (1.58)$$

where α and β are constants, and $Z = Z(t)$ is the oscillating quantity (see also Equations 1.43c). In the mechanical case, Z stands for the displacement x , and in



the electrical case for the charge q . Thus, all we have learnt about mechanical oscillators can be carried over to electrical oscillators. Moreover, we can see a direct correspondence between the two sets of physical quantities involved:

- q takes the place of x ;
- L takes the place of m ;
- $1/C$ takes the place of k .

For example, the inductance L is the electrical analogue of mechanical inertia m . These analogies enable us to build an electrical circuit that exactly mimics the operation of a mechanical system. This is useful because in the development of a mechanical system, it is much easier to change, for example, the value of a capacitor in the analogue circuit than to manufacture a new mechanical component.

Worked Example A plasma consists of an equal number of positive ions and electrons, as illustrated schematically in Figure 1.23(a). Suppose that an external electric field is applied so that all the electrons are moved a distance x , as in Figure 1.23(b); the much heavier positive ions can be considered to remain fixed in place. This produces two slabs of total charge of area A and width x . When the external field is removed, the electrons move towards the positive ions because of electrostatic attraction. However, the electrons overshoot their equilibrium positions and travel beyond the positive ions before turning back again towards the positive ions and so on. The electrons oscillate about their equilibrium positions just as the bob of a pendulum oscillates about its equilibrium position. The frequency at which they oscillate is called the *plasma frequency*. Obtain an expression for the plasma frequency, which involves n , the number of electrons per unit volume. Find the plasma frequency of a plasma that has $n = 1.2 \times 10^{15}$, which is a typical value for a laboratory plasma.

Solution

The total charge in the slab containing the electrons is $Q = -neAx$, and the total charge in the slab containing the ions is $Q = +neAx$. The electric field arising from two such slabs of charge is given by $E = Q/\epsilon_0 A$, giving $E = nex/\epsilon_0$, where ϵ_0 is the permittivity of free space. Therefore, the force on an electron is $-Ee = -ne^2x/\epsilon_0$, and its equation of motion is

$$m_e \frac{d^2x}{dt^2} = -\frac{ne^2}{\epsilon_0}x,$$

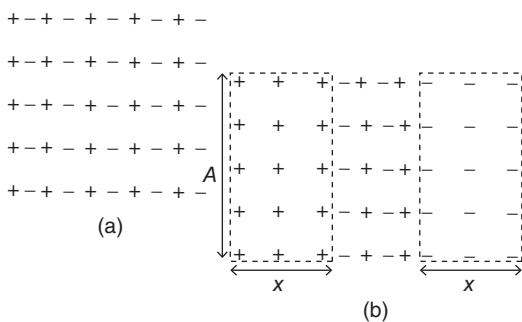


Figure 1.23 (a) A schematic diagram of a plasma consisting of an equal number of positive ions and electrons. (b) The plasma when an external electric field is applied so that all the electrons are moved a distance x . The much heavier positive ions can be considered to remain fixed in place. This produces two slabs of total charge of area A and width x .

where m_e is the mass of an electron. This is an equation of SHM with

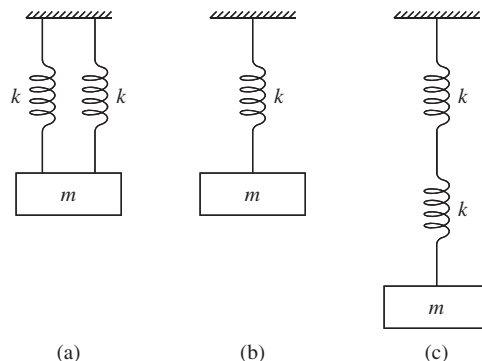
$$f_{\text{plasma}} = \frac{1}{2\pi} \sqrt{\frac{ne^2}{\epsilon_0 m_e}}$$

For $n = 1.2 \times 10^{15}$, we obtain

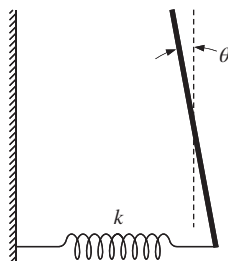
$$f_{\text{plasma}} = \frac{1}{2\pi} \sqrt{\frac{1.2 \times 10^{15} \times (1.6 \times 10^{-19})^2}{8.85 \times 10^{-12} \times 9.1 \times 10^{-31}}} = 310 \text{ MHz.}$$

Problems 1

- 1.1 A mass of 0.50 kg hangs from a light spring and executes SHM so that its position is given by $x = A \cos \omega t$. It is found that the mass completes 20 cycles of oscillation in 80 s. (a) Determine (i) the period of the oscillations, (ii) the angular frequency of the oscillations and (iii) the spring constant k . (b) Using a value of $A = 2.0$ mm, make sketches of the variations with time t of the displacement, velocity and acceleration of the mass.
- 1.2 The ends of a tuning fork oscillate at a frequency of 440 Hz with an amplitude of 0.50 mm. Determine (a) the maximum velocity and (b) the maximum acceleration of the ends.
- 1.3 A mass executes SHM at the end of a light spring. (a) What fraction of the total energy of the system is potential and what fraction is kinetic at the instant when the displacement of the mass is equal to half the amplitude? (b) If the maximum amplitude of the oscillations is doubled, what will be the change in (i) the total energy of the system, (ii) the maximum velocity of the mass and (iii) the maximum acceleration of the mass? Will the period of oscillation change?
- 1.4 A mass of 0.75 kg is attached to one end of a horizontal spring of spring constant 400 N m^{-1} . The other end of the spring is attached to a rigid wall. The mass is pushed so that at time $t = 0$ it is 4.0 cm closer to the wall than the equilibrium position and is travelling towards the wall with a velocity of 0.50 m s^{-1} . (a) Determine the total energy of the oscillating system. (b) Obtain an expression for the displacement of the mass in the form $x = A \cos(\omega t + \phi)$ m, giving numerical values for A , ω and ϕ .
- 1.5 A simple pendulum has a length of 0.75 m. The pendulum mass is displaced horizontally from its equilibrium position by a distance of 5.0 mm and then released. Calculate (a) the maximum speed of the mass and (b) the time it takes to reach this speed. (Assume $g = 9.8 \text{ m s}^{-2}$.)
- 1.6 The figure shows three systems of a mass m suspended by light springs that all have the same spring constant k . Show that the angular frequencies of oscillation for the three systems are in the ratio of $\omega_a : \omega_b : \omega_c = \sqrt{2} : 1 : \sqrt{1/2}$.

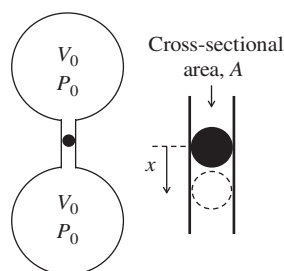


- 1.7** A simple pendulum is measured to have a period of 2.0 s on earth. The same pendulum is measured to have a period of 5.0 s on the moon. The radius of the earth and moon are known to have radii of 6.3×10^6 m and 1.7×10^6 m, respectively, and the average density of the earth to be 5.5×10^3 kg m⁻³. Use these data to determine the average density of the moon.
- 1.8** A simple pendulum consists of a mass m attached to a light rod of length l , with $m = 0.75$ kg and $l = 1.0$ m. The mass is pulled aside to give an angular displacement $\theta = 60^\circ$ and then released. Determine (i) the tangential velocity and (ii) the tangential acceleration of the mass as it passes through its equilibrium position, (iii) the minimum and (iv) the maximum tension in the string.
- 1.9** A platform oscillates in the vertical direction with SHM. Its amplitude of oscillation is 0.20 m. What is the maximum frequency of oscillation for a mass placed on the platform to remain in contact with the platform? (Assume $g = 9.8$ m s⁻².)
- 1.10** The figure shows a thin uniform rod of mass M and length $2L$ that is pivoted without friction about an axis through its midpoint. A horizontal light spring of spring constant k is attached to the lower end of the rod. The spring is at its equilibrium length when the angle θ with respect to the vertical is zero. Show that for oscillations of small amplitude, the rod will undergo SHM with a period of $2\pi\sqrt{M/3k}$. The moment of inertia of the rod about its midpoint is $ML^2/3$. (Assume the small-angle approximations: $\sin \theta \cong \theta$ and $\cos \theta \cong 1$.)



- 1.11** The period of a simple pendulum is found to be 1.0 s when the period is measured in the laboratory. The period of the pendulum is found to be 1.25 s when it is measured in a lift that is accelerating uniformly. Determine the acceleration of the lift.
- 1.12** A uniform disc of mass M and radius R pivots about a point a distance $R/2$ from its centre and oscillates in a vertical plane when disturbed from equilibrium. Obtain an expression for the frequency of small-amplitude oscillations of the disc. The moment of inertia of a uniform solid disc about a perpendicular axis that passes through its centre is $MR^2/2$.
- 1.13** A uniform disc of radius R and mass M is pivoted from a point that is a distance a from the centre of the disc. Find the value of a for which the period of oscillation of the disc is a minimum. The moment of inertia of a disc about an axis that passes through its centre and is perpendicular to its plane is $MR^2/2$.
- 1.14** A test tube is weighted by some lead shot and floats upright in a liquid of density ρ . When slightly displaced from its equilibrium position and released, the test tube oscillates with SHM. (a) Show that the angular frequency of the oscillations is equal to $\sqrt{A\rho g/m}$, where g is the acceleration due to gravity, A is the cross-sectional area of the test tube and m is its mass. (b) Obtain an expression for the total energy of the oscillating system in terms of the velocity of the test tube and its displacement from equilibrium.
- 1.15** We might assume that the period of a simple pendulum depends on the mass m , the length l of the string and g , the acceleration due to gravity, i.e. $T \propto m^\alpha l^\beta g^\gamma$, where α , β and γ are constants. Consider the dimensions of the quantities involved to deduce the values of α , β and γ .
- 1.16** If a spherical liquid drop is disturbed from equilibrium, it oscillates with frequency ν that depends on its radius r , density ρ and surface tension s . Use the method of dimensions to find the relationship between ν , r , ρ and s . The *oscillating drop technique* is widely used in the determination of the surface tension of liquids.
- 1.17** Imagine that a hole is drilled through the centre of planet Earth from the North Pole to the South Pole. Show that if an object was dropped into this hole, the object would perform SHM. Determine how long it would take for the object to travel between the two poles. The mean density of the Earth is $5.5 \times 10^3 \text{ kg m}^{-3}$, and the gravitational constant G is $6.67 \times 10^{-11} \text{ m}^3 \text{ kg}^{-1} \text{ s}^{-2}$.
- 1.18** The figure shows a schematic diagram of Ruchardt's method of determining γ , the ratio of the specific heats of a gas. The two spherical glass vessels are joined by a glass tube. The volume of each vessel is V_0 , and the pressure in both is P_0 . The spherical mass m is a close fit in the tube of cross-sectional area A . At equilibrium the mass sits in the middle of the tube. Show that when the mass is displaced slightly, it performs SHM with a period given by $T = 2\pi\sqrt{mV_0/2\gamma P_0 A^2}$. Assume that the changes

involved are adiabatic. It was found in one experiment with molecular nitrogen in the vessel that $T = 0.42$ s for $m = 3.0 \times 10^{-2}$ kg, $V_0 = 8.5 \times 10^{-3}$ m³, $A = 4.5 \times 10^{-4}$ m² and $P_0 = 1.0 \times 10^5$ Pa. Use these data to obtain the value of γ for nitrogen.



- 1.19** The potential energy $U(r)$ between two atoms in a diatomic molecule can be expressed approximately by $U(r) = -a/r^6 + b/r^{12}$, where r is the separation of the atoms and a and b are constants. (a) Obtain an expression for the force between the two atoms and hence show that the equilibrium separation r_0 of the atoms is equal to $(2b/a)^{1/6}$. (b) Show that the system will oscillate with SHM when slightly displaced from equilibrium with a frequency equal to $\sqrt{k/\mu}$, where μ is the reduced mass and $k = 36(a/2b)^{4/3}$.

- 1.20** The potential energy between the sodium and chlorine ions in a molecule of sodium chloride can be represented by $U(r) = A/r^9 - e^2/4\pi\epsilon_0 r^2$, where r is the separation of the two ions. The first term on the right-hand side is due to the repulsive part of the interaction, and the second term is due to the attractive part of the interaction. Show that for small excursions from the equilibrium separation r_0 , the molecule undergoes SHM and show that the restoring force per unit distance, i.e. force constant is $8e^2/4\pi\epsilon_0 r_0^3$. The value of r_0 for a sodium chloride molecule is 2.5×10^{-10} m. The atomic masses of sodium and chlorine are 23 u and 35.5 u, respectively. Determine the restoring force per unit distance for the sodium chloride molecule and the frequency of vibration.

- 1.21** A mass M oscillates at the end of a spring that has spring constant k and finite mass m . (a) Show that the total energy E of the system for oscillations of small amplitude is given by

$$E = \frac{1}{2} (M + m/3) v^2 + \frac{1}{2} kx^2,$$

where v and x are the velocity and displacement of the mass M , respectively. (Hint: To find the kinetic energy of the spring, consider it to be divided into infinitesimal elements of length dl and find the total kinetic energy of these elements, assuming that the mass of the spring is evenly distributed along its length.) (b) Hence, show that the frequency of the oscillations is equal to $\sqrt{k/(M + m/3)}$.

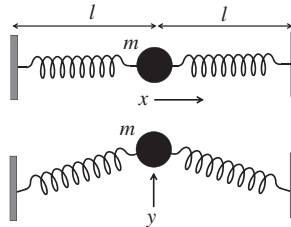
- 1.22** A particle oscillates with amplitude A in a one-dimensional potential $U(x)$ that is symmetric about $x = 0$, i.e. $U(x) = U(-x)$. (a) Show, from energy considerations, that the velocity v of the particle at displacement x from the equilibrium position ($x = 0$), is given by $v = \sqrt{2[U(A) - U(x)]/m}$. (b) Hence, show that the period of oscillation T is given by

$$T = 4\sqrt{\frac{m}{2U(A)}} \int_0^A \frac{dx}{\sqrt{1 - U(x)/U(A)}}$$

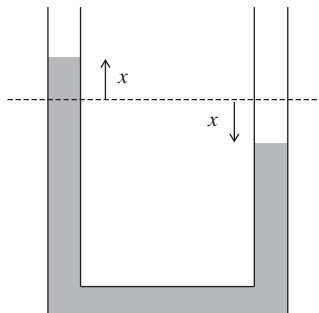
(c) If the potential $U(x)$ is given by $U(x) = \alpha x^n$, where α is a constant and $n = 2, 4, 6, \dots$ obtain the dependence of the period T on the amplitude A for different values of $n = 2, 4, 6, \dots$ (Hint: Introduce the new variable of integration $\xi = x/A$ in the above expression for the period T .)

- 1.23** Prove the identity: $\sin^3\theta = \frac{3}{4}\sin\theta - \frac{1}{4}\sin 3\theta$.

- 1.24** The figure shows a mass m held between two identical springs that are fixed to rigid supports. The springs have length l_0 when unstretched and length l when the mass is at the equilibrium position, and they have spring constant k . (i) Determine the frequency of longitudinal vibrations along the x -axis. (ii) Determine the frequency of transverse vibrations along the y -axis. Assume that $y \ll l$. (iii) Hence, show that the frequencies of the transverse and longitudinal vibrations, ω_T and ω_L , respectively, are related by $\frac{\omega_L}{\omega_T} = \frac{1}{\sqrt{(l-l_0)}}$.



- 1.25** The arrangement of a capacitor and an inductor like that shown in Figure 1.21 is used in the circuitry of the front section of an AM radio receiver. At what frequency would the circuit oscillate if $C = 100$ pF and $L = 300$ μ H? To what wavelength in the electromagnetic spectrum does this frequency correspond?
- 1.26** The figure shows a U-tube of cross-sectional area A that contains a length l of non-viscous fluid. When the liquid is slightly disturbed, it performs oscillations. (a) Show that the oscillations are simple harmonic by considering the force acting on the liquid when it oscillates. (b) Show that the oscillations are simple harmonic by considering the total energy of the liquid when it oscillates. If the length of the liquid is 25 cm, determine the period of the oscillations.



- 1.27** The displacement from equilibrium x of a particle undergoing SHM is given by $x = a \sin \omega t$, where a is the amplitude of oscillation. The particle spends time dt in moving from x to $x + dx$. Show that the probability of the particle being between x and $x + dx$ is $\frac{dx}{\omega(a^2 - x^2)^{1/2}}$.
- 1.28** The displacement of a harmonic oscillator of mass m is represented by $x = a \sin \omega t$. Obtain expressions for the kinetic energy of the particle and the potential energy of the particle. If the displacement x is plotted on the x -axis and the velocity \dot{x} is plotted on the y -axis, show that the locus of the points (x, \dot{x}) is an ellipse.
- 1.29** A particle moves on a smooth surface such that its displacement along the x and y axes are given by $x = a \cos \omega t$ and $y = b \sin \omega t$, respectively. Show that the particle follows an elliptical path on the surface and that this path represents a path of constant energy. Show that the quantity $(x\dot{y} + y\dot{x})$ is also constant. What does this quantity represent?
- 1.30** The figure shows the cross-section of a solid half-cylinder of mass M and radius R . The centre of mass of the half-cylinder is at distance $d = 4R/3\pi$ from point O , and the moment of inertia I_0 of the half-cylinder about an axis passing through point O and perpendicular to the page is $MR^2/2$. Show that when one end is pushed down slightly and released, the half-cylinder oscillates in SHM and that the period of the oscillations is $T = 0.78\sqrt{R/g}$. (Hint: You may assume that the half-cylinder pivots about point P , where the half-cylinder makes contact with the flat surface.)

