1

Introduction and Basic Concepts

Understanding how electrical circuits behave is fundamental to all of integrated circuit (IC) design. Whether we are designing amplifiers, filters, or data converters, every system can ultimately be reduced to a network of interconnected elements that obey a few simple physical laws. This chapter develops the methods used to analyze such networks systematically and efficiently. The material discussed here feeds into the rest of the chapters of this book in one way or another.

We begin by revisiting the essential circuit laws—Ohm's law, Kirchhoff's current law (KCL), and Kirchhoff's voltage law (KVL)—and use them to relate currents and voltages in electrical networks [1]. These laws form the basis for setting up equations that describe circuit behavior. We give the physical intuition behind the Tellegen's theorem, a result that seems very intriguing at first [2].

We then introduce the concept of nodal analysis, which provides a structured way of formulating circuit equations in linear networks. From there, we build up to modified nodal analysis (MNA) [3], a powerful and general technique that can be applied to any linear circuit. MNA forms the computational foundation of circuit simulators such as SPICE [4] and provides a unified framework for hand and computer-based analysis [5] alike.

As circuits become more complex, elements such as controlled sources must be incorporated. We will see how these components can be represented through compact "MNA stamps" that can be directly inserted into circuit equations. Using this approach, we can efficiently analyze circuits that include both passive and active devices without changing the overall formulation. We use the MNA approach to conclude that any transfer function in a linear network can be related to a component value in a bilinear form. This leads to the so-called extra-element theorem [6].

It turns out that the MNA formulation of network equations naturally leads to the concept of reciprocity and the adjoint (inter-reciprocal) network. The adjoint network is a *very* valuable computational and analytical tool that forms the basis for advanced topics such as noise and sensitivity computation [7]. The concepts of reciprocity and inter-reciprocity [8] are a continuous theme throughout this book.

1.1 Classification of Circuits

1.1.1 Linear and Nonlinear Circuits

An electric circuit is linear if its response (with all initial conditions being zero) is proportional to the excitation. For such a circuit, superposition is applicable, i.e., the overall response is the sum of the responses due to the individual sources. Mathematically, if the circuit's response to $x_1(t)$ and $x_2(t)$ are $y_1(t)$ and $y_2(t)$ respectively, the response to $\alpha x_1(t) + \beta x_2(t)$ is $\alpha y_1(t) + \beta y_2(t)$ for all constants α and β (Figure 1.1).

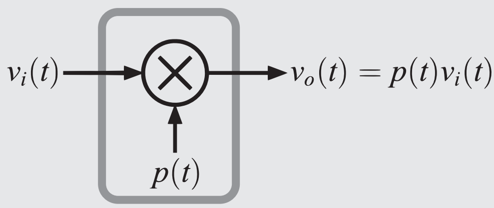


Figure 1.1 Is this a linear system?

The system above has an input/output relationship given by $v_o(t) = p(t)v_i(t)$. Is it a linear system?

Input	Output
$v_a(t)$	$p(t)v_a(t)$
$v_b(t)$	$p(t)v_b(t)$
$\alpha v_a(t) + \beta v_b(t)$	$p(t)(\alpha v_a(t) + \beta v_b(t))$
	$= \alpha p(t)v_a(t) + \beta p(t)v_b(t)$

From the input–output table above, we see that superposition is valid. The system, therefore, is linear.

Circuits where superposition does not hold are nonlinear. In practice, electronic circuits may behave approximately linearly, but only within a range of signals (Figure 1.2).

The system above has an output given by

$$v_o(t) = \left\{A(\alpha v_1^3(t) + v_2(t))\right\}^{1/3}\beta,$$

indicating that it is nonlinear in general. However, *in the special case that* $v_2 = 0$, $v_o(t) = (\alpha A)^{1/3}\beta v_1(t)$, leading us to conclude that the system behaves linearly in spite of signals being processed in a nonlinear fashion inside. Such systems are called externally linear, internally nonlinear systems.

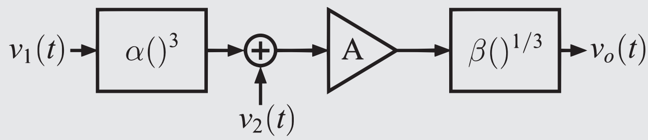


Figure 1.2 Is this system linear or nonlinear?

1.1.2 Time Invariant and Time-Varying Circuits

A circuit is time invariant if its response, when initially relaxed, to a given excitation is the same regardless of the time instant at which the source is connected. It is time-varying otherwise. Time (in)variance and (non)linearity are different aspects of a circuit that do not have anything to do with each other. The reader is encouraged to think of examples of linear and time-invariant, nonlinear and time-invariant, linear and time-varying, and nonlinear and time-varying systems.

1.1.3 Passive and Active Circuits

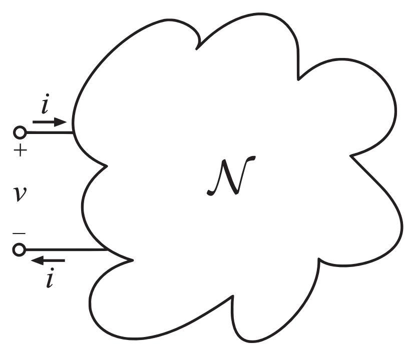


Figure 1.3 Associated directions of voltage and current at a network port. $\mathcal{N}$ is a symbol used to denote a network.

A network $\mathcal{N}$ is passive if it cannot provide more energy to the outside world than what was stored in it. Assume that $\mathcal{N}$ is connected to the outside world through two terminals (Figure 1.3). Note that the reference directions of the voltage and currents are such that the reference current flows from the positive to the negative terminal in the circuit. These are associated directions and will be assumed throughout this work. The power flowing into the network from the outside at time t is then $P(t) = v(t)i(t)$, and the energy delivered into it during the time period from 0 to t is

$$E(t) = \int_0^t v(t)i(t)\,dt. \tag{1.1}$$

For a passive circuit, $E(t)$ is nonnegative. For $E(t) < 0$, the circuit is active. Strictly speaking, a device can only supply net electric energy to the outside if it converts chemical or solar energy into electric form, like a battery or solar cell. However, the adjective “active” is also often applied to circuits that are capable of converting the power from a fixed voltage or current source into a magnified signal, like a voltage amplifier. For a circuit with several terminal pairs, the total energy delivered at all terminal pairs determines the active or passive property.

Terminal pairs where the current entering at one terminal is the same as the current leaving at the other are called ports. Thus, the structure in Figure 1.3 is a one-port circuit. Circuits may be *lossless* or *lossy*. The energy stored in a lossless circuit equals the energy supplied to it from the outside. For a lossy circuit, the stored energy is less than the energy input from outside. Thus, in a lossy network, the electromagnetic energy is transformed into some other form of energy, such as heat or light. A circuit is *memoryless* if its response at a given time depends only on its excitation at the same time. This will happen if there are no energy-storing elements in the circuit. Otherwise, the circuit is called *memoried*, or *dynamic*.

Energy Efficiency of Charging a Capacitor

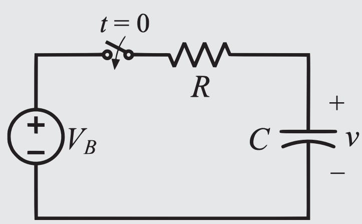


Figure 1.4 What is the energy supplied by the battery?

Figure 1.4 shows a simple circuit that illustrates some of the concepts discussed. A battery with a voltage V_B is switched at $t = 0$ to the series-connected R and C. (R may be the on-resistance of the switch.) The capacitor is initially uncharged. Let us determine the energy stored in the capacitor after the charging is completed, and the energy delivered by the battery. The final energy stored in C is $(1/2)CV_B^2$. The current flow during charging is $i = C dv/dt$, so the total energy E_B supplied by the battery at the end of the charging process is

$$E_B(\infty) = \int_0^\infty V_B C \frac{dv}{dt} dt = CV_B^2. \tag{1.2}$$

This is twice the energy stored in C. Thus, half of the energy provided by the battery is lost in the charging process, dissipated in the switch resistance R. This energy is given by

$$E_R(\infty) = \int_0^{\infty} Ri^2(t)dt = \frac{1}{2}CV_B^2, \tag{1.3}$$

as expected. It is interesting to note that the energy efficiency of the charging process is 50%, independent of the resistance of the switch. However, the speed of the charging process is increased when R is lowered.

1.2 Circuit Components

The circuit components discussed next include passive elements such as $R,\ L,\ C$, as well as active elements such as controlled sources and operational amplifiers. It will be assumed that all elements are ideal, i.e., linear and time-invariant (LTI). Then, their defining voltage–current relations are the following:

$$\text{Resistor } R : v = Ri \quad \text{and} \quad i = Gv, \text{ where } G = 1/R. \tag{1.4}$$

The equation above is Ohm's law. Here, R is the resistance and G the conductance of the resistor. Both have positive values. The power dissipated in the resistor is given by $P = vi = Ri^2$. P is clearly nonnegative, and therefore so is its time integral, the energy E. Hence, the resistor is a passive element.

$$\text{Capacitor } C : i = C\frac{dv}{dt} \quad \text{and} \quad v = \frac{1}{C}\int_0^t i(x)dx + v(0)\,. \tag{1.5}$$

Here, $C > 0$ is the capacitance of the element. Finding the energy entering a capacitor using the time integral of the power vi shows that its value at time t is $E(t) = (1/2)Cv^2(t) > 0$. Thus, the capacitor is also a passive element. From (1.5), we see that there cannot be a step change in the voltage across a capacitor unless an impulse current flows through it. Equivalently, if the voltage across a capacitor experiences a step change, one must conclude that an impulse current has flown through it.

$$\text{Inductor } L : v = L\frac{di}{dt} \quad \text{and} \quad i = \frac{1}{L}\int_0^t v(x)dx + i(0)\,. \tag{1.6}$$

Here, $L > 0$ is the inductance of the element. Integrating the power vi gives $E(t) = (1/2)Li^2(t) > 0$ for the energy that entered the inductor by time t. Thus, the inductor is also a passive element. From (1.6), we see that there cannot be a step-change in the current through an inductor unless an impulse voltage is applied across it. This also means that if the current through an inductor experiences a step change, an impulse voltage must have occurred across its terminals.

The energy entering the resistor leaves the circuit in the form of radiation of heat or an electromagnetic wave and is lost for the circuit. However, the energy entering the capacitor is stored in the electric field between the plates; the energy of the inductor is stored in the magnetic field within and around the device. The stored energies can be redeemed by the circuit.

The voltages and currents in (1.4)–(1.6) are physical quantities. They are generally functions of time. As the equations indicate, the relations describing the operation of the circuit in which they are present are integro-differential ones. They can be reduced to algebraic equations, and thus their solution greatly simplified, by using the Laplace transformation. Denoting the transformed variables by capital letters, this changes the voltage–current relation of the capacitor into $I = sCV$ and $V = I/sC$. Similarly, for an inductor, the relations become $V = sLI$ and $I = V/sL$. For the resistor, Ohm's law remains unchanged between V and I.

The ratio $Z = V/I$ between the Laplace-transformed quantities is called the impedance, and its reciprocal $Y = I/V$ is called the admittance. The Laplace-transformed variables are closely related to the phasors used in sinusoidal steady-state analysis. The equations used for Laplace-transformed variables in terms of the variable s remain valid for phasors if s is replaced by $j\omega$. The admittance of a capacitor is $j\omega C$, which is zero at DC. This means that the average current flowing through a capacitor in a circuit "that works" (in the sense that none of the voltages or currents blow up to ∞) is zero. Likewise, since the DC impedance of an inductor is zero, the average voltage across it must be zero.

The active components often encountered include the independent and dependent (controlled) sources. The voltage across an independent voltage source is defined by the designer, and it is independent of the current drawn from it. Similarly, the current

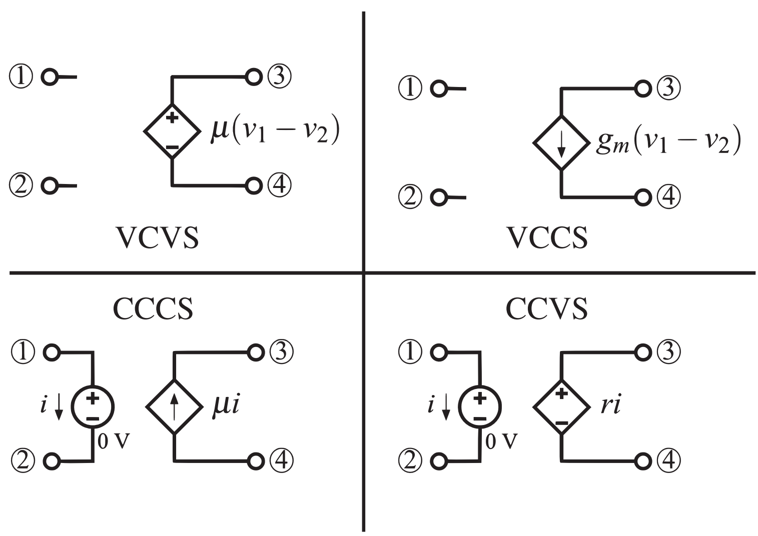


Figure 1.5 The four linear dependent (controlled) sources.

provided by an independent current source is a given quantity and is independent of the voltage across it.

A dependent voltage source provides a voltage v_j in branch j which is a function of another voltage v_k (VCVS) or current i_k (CCVS) in the circuit. Usually, the function is linear, so the voltage is given by $v_j = \mu v_k$ or by $v_j = ri_k$. Similarly, a dependent current source has a value determined by the voltage (VCCS) or current (CCCS) in another branch. Usually, it is a linear dependence, so that the current is given by $i_j = gv_k$ or $i_j = \mu i_k$. Figure 1.5 illustrates the symbols used for independent and dependent sources.

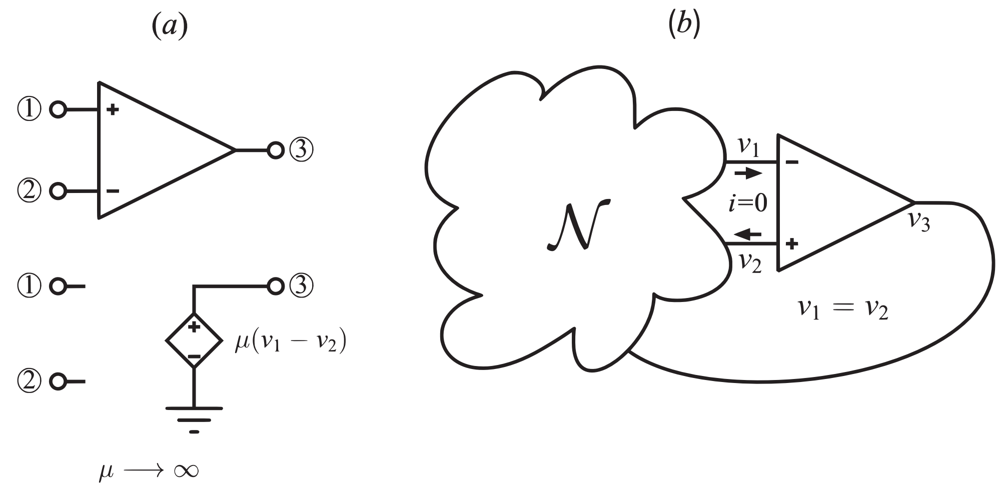


Figure 1.6 (a) Model for an ideal opamp. (b) The input terminals form a virtual short if the opamp has DC negative feedback around it.

An important concept in analog circuit design is that of the operational amplifier (Figure 1.6a), often called the workhorse of analog engineering. It can be thought of as a voltage-controlled voltage source with an output voltage $v_3 = \mu(v_1 - v_2)$, where μ is ideally infinite. When an opamp is embedded in a circuit as shown in Figure 1.6b, *and there is DC negative feedback around it*, v_1 and v_2 are equal. Since the opamp is a VCVS, $i = 0$. Its input terminals, therefore, are said to form a *virtual* short.[1]

1.3 Kirchhoff's Laws and the Incidence Matrix

The analysis of circuits is based on Kirchhoff's current law (KCL) and voltage law (KVL). The KCL states that the sum of all currents leaving any node at any time is zero. This is because charges ideally cannot accumulate at a node. (In a real circuit, parasitic capacitances are present at all nodes. Their currents must be included to make

[1]When there is a *real* short circuit between two nodes, the voltage between them is zero, and the current can be arbitrary. In a virtual short, the voltage between the nodes is zero, and the current flowing between the nodes *is also zero*.

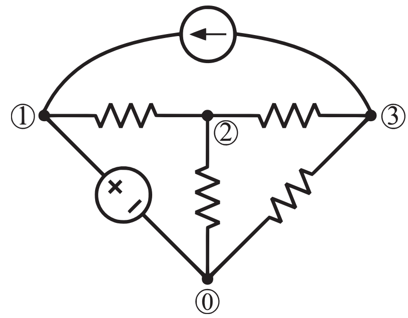


Figure 1.7 Example network with four nodes and six branches.

this statement valid.) The KVL states that the sum of all branch voltages in a loop is zero. Kirchhoff's laws are illustrated for a simple circuit in Figure 1.7. The "skeleton" of the circuit, called its *graph,* is shown in Figure 1.8. The junctions between elements are the nodes, and the elements themselves are abstracted out as branches. Our example circuit consists of four nodes and six branches. The nodes and branches are numbered. The voltage of the jth node is denoted by v_j. The voltage across and current through the kth branch are denoted by e_k and i_k, respectively. As discussed earlier (see Figure 1.3), the direction of current in a branch is chosen so that $e_k(t)i_k(t)$ represents the instantaneous power dissipated in the branch.

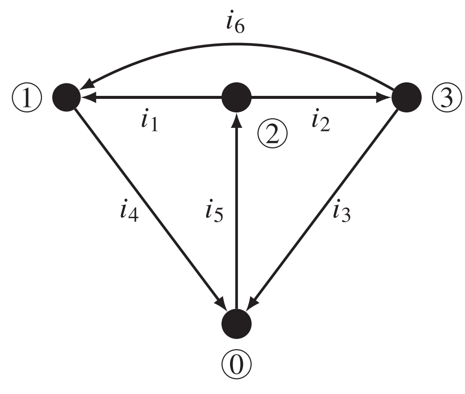


Figure 1.8 Graph of the network of Figure 1.7.

The aim of circuit analysis is to "solve" the network, which means that we need to determine the node voltages, branch voltages, and branch currents. Since currents only flow in response to potential differences, any one of the nodes can be chosen as the datum or reference with respect to which the potential of all other nodes is measured. This is numbered as node ⓪ and is often referred to as the *ground node.* In a network with n nodes, therefore, there are $(n-1)$ node voltages to be found. The network is solved by using Kirchhoff's current law (KCL), Kirchhoff's voltage law (KVL), and the branch relationships, i.e., the relationships between e_k and i_k.

KCL for our example circuit leads to

$$\begin{aligned} -i_1 + i_4 - i_6 &= 0 \\ i_1 + i_2 - i_5 &= 0 \\ -i_2 + i_3 + i_6 &= 0 \\ -i_3 - i_4 + i_5 &= 0. \end{aligned}$$

It is efficient at this point to introduce vector and matrix notations. We define the incidence matrix **A** of the graph of the circuit. Its element in row j and column k is

$$\begin{aligned} a_{jk} &= +1, && \text{if branch } k \text{ leaves node } j \\ a_{jk} &= -1, && \text{if branch } k \text{ enters node } j \\ a_{jk} &= 0, && \text{otherwise.} \end{aligned}$$

The incidence matrix for the graph of Figure 1.8 is shown below. Each row corresponds to a node, and there is a column for each branch. **A** contains all the information there is in the graph. For instance, examining the third column indicates that the branch b_3 originates at node 3 and terminates at node 0. The second row of **A** tells us that branches b_1 and b_2 leave node 2 while b_5 arrives into the node. Notice that there are precisely two nonzero entries in every column and that the sum of every column is zero. This means that one row of **A** can be removed without losing any information.

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 0 \end{array} \begin{array}{c} \begin{array}{cccccc} b_1 & b_2 & b_3 & b_4 & b_5 & b_6 \end{array} \\ \begin{bmatrix} -1 & 0 & 0 & 1 & 0 & -1 \\ 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \\ 0 & 0 & -1 & -1 & 1 & 0 \end{bmatrix} \end{array}.$$

Removing the (redundant) row corresponding to the ground node, we have the so-called *reduced* incidence matrix denoted by $\mathbf{A}_r$ given below.

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccccc} b_1 & b_2 & b_3 & b_4 & b_5 & b_6 \end{array} \\ \begin{bmatrix} -1 & 0 & 0 & 1 & 0 & -1 \\ 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \end{bmatrix} \end{array}.$$

$\mathbf{A}_r$ is an $(n-1) \times b$ matrix, where n and b denote the number of nodes and branches in the circuit, respectively. The branch currents can be collected into a column vector $\mathbf{i} = \begin{bmatrix} i_1 & i_2 & \dots & i_b \end{bmatrix}^T$. Then, all the KCL equations can be written in the simple form

$$\mathbf{A}_r \mathbf{i} = \mathbf{0}. \tag{1.7}$$

where $\mathbf{0}$ is the column vector of $(n-1)$ zeros. The (reduced) incidence matrices only depend on the network's graph, and say nothing of the components constituting the branches.

Kirchhoff's voltage law (KVL) can simply be stated as follows. The branch voltage is the difference between its node voltages. This is equivalent to saying that the sum of branch voltages in a loop is zero. Since the potentials of all other nodes are measured with respect to that of node 0, the latter's voltage is assumed to be 0. KVL for our circuit example gives the equations

$$\begin{aligned}
e_1 &= v_2 - v_1 \\
e_2 &= v_2 - v_3 \\
e_3 &= v_3 \\
e_4 &= v_1 \\
e_5 &= -v_2 \\
e_6 &= v_3 - v_1.
\end{aligned}$$

Similarly, collecting the branch and node voltages into column vectors $\mathbf{e}$ and $\mathbf{v}$, respectively, the KVL equations can be written in the concise form

$$\mathbf{e} = \mathbf{A}_r^T \mathbf{v}. \tag{1.8}$$

The equations (1.7) and (1.8) above hold for all networks—linear, nonlinear, time invariant or time varying.

1.4 Tellegen's Theorem

The sum total of the instantaneous power dissipated in all branches of the network is given by

$$P_{\text{tot}} = \sum_k e_k i_k = \mathbf{e}^T \mathbf{i}. \tag{1.9}$$

Using (1.8) and (1.7), we see that

$$P_{\text{tot}} = \mathbf{v}^T \underbrace{\mathbf{A}_r \mathbf{i}}_{= \mathbf{0}\,(\text{KCL})} = \mathbf{v}^T \mathbf{0} = 0. \tag{1.10}$$

This indicates that the sum of instantaneous powers in all branches of a network is zero; the power dissipation in a branch is due to power generated in other branches of the circuit. It is important to note that this is satisfied at *every instant* of time—not in some average sense.[2] This is one form of the famous *Tellegen's Theorem*. The reader might wonder what all the fuss is about; net power cannot be generated or dissipated

[2]Evidently, Nature (unlike our economists) takes Polonius' advice of neither being a borrower nor lender (of power) seriously.

in a network after all! However, there is more to this than meets the eye, as we see below.

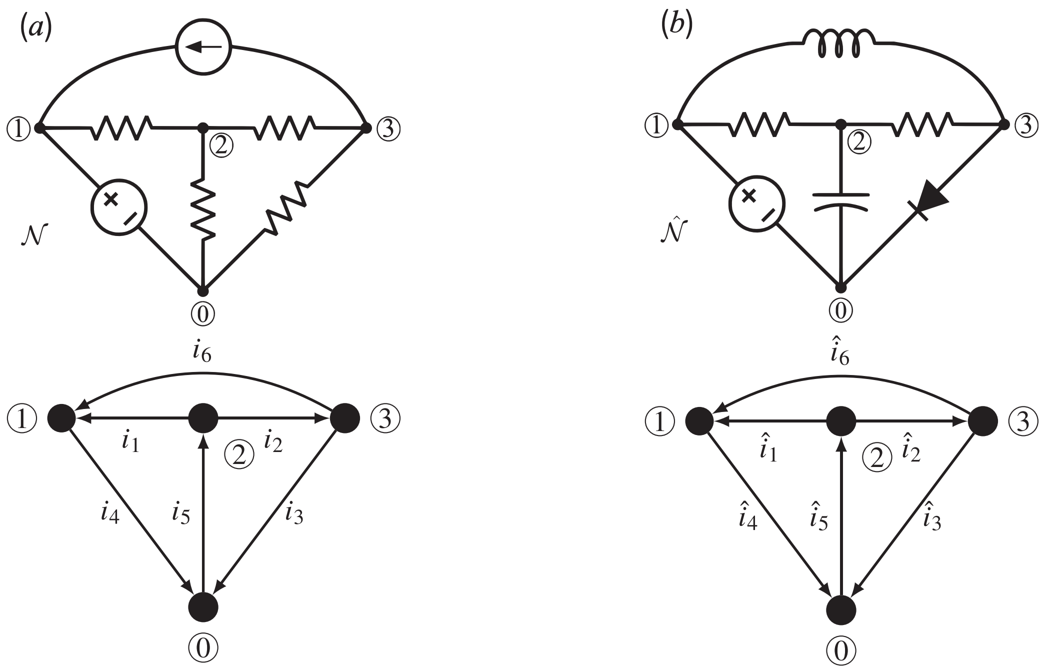


Figure 1.9 Two networks with the same graph. The branch voltages and currents of the networks are denoted by e_k, i_k and $\hat{e}_k$, $\hat{i}_k$ respectively.

Consider the networks $\mathcal{N}$ and $\hat{\mathcal{N}}$ shown in Figure 1.9a,b respectively. They are evidently different. Assuming the resistors obey Ohm's law, $\mathcal{N}$ is linear and memoryless, while $\hat{\mathcal{N}}$ is nonlinear and has memory. However, their graphs are identical, and the branches and nodes are numbered the same way. The branch voltages and currents of $\mathcal{N}$ are denoted by e_k, and i_k and those in $\hat{\mathcal{N}}$ are denoted by $\hat{e}_k$, and $\hat{i}_k$ respectively.

Let us compute the quantity $\sum_k e_k \hat{i}_k$, which has the dimensions of power. To be clear, what we are doing is computing the product of a branch voltage in $\mathcal{N}$ and the current in the corresponding branch in $\hat{\mathcal{N}}$ and summing this "power-like" quantity over all branches. In vector notation, this is $\mathbf{e}^T\hat{\mathbf{i}}$. By KVL, $\mathbf{e}^T = \mathbf{v}^T\mathbf{A}_r$. We thus have the surprising result

$$\sum_k e_k \hat{i}_k = \mathbf{e}^T\hat{\mathbf{i}} = \mathbf{v}^T\mathbf{A}_r\hat{\mathbf{i}} = \mathbf{v}^T\mathbf{0} = 0. \tag{1.11}$$

In a similar fashion, it is easy to see that

$$\sum_k \hat{e}_k \hat{i}_k = \sum_k e_k \hat{i}_k = \sum_k \hat{e}_k i_k = 0 \tag{1.12}$$

where the summation runs across all the branches in the network. What throws one off about this result is that it appears that $\mathcal{N}$ and $\hat{\mathcal{N}}$ must somehow know what is happening in the other network for the result to be true. Clearly, however, there is nothing in common between the networks other than that their graphs (and therefore their incidence matrices) are the same. It seems difficult to wrap our heads around the fact we can make any statement at all about $\sum_k e_k \hat{i}_k$ given that the networks are completely independent, let alone the sum being zero! Fortunately, as we see below, this result is not as intriguing as it seems at first.

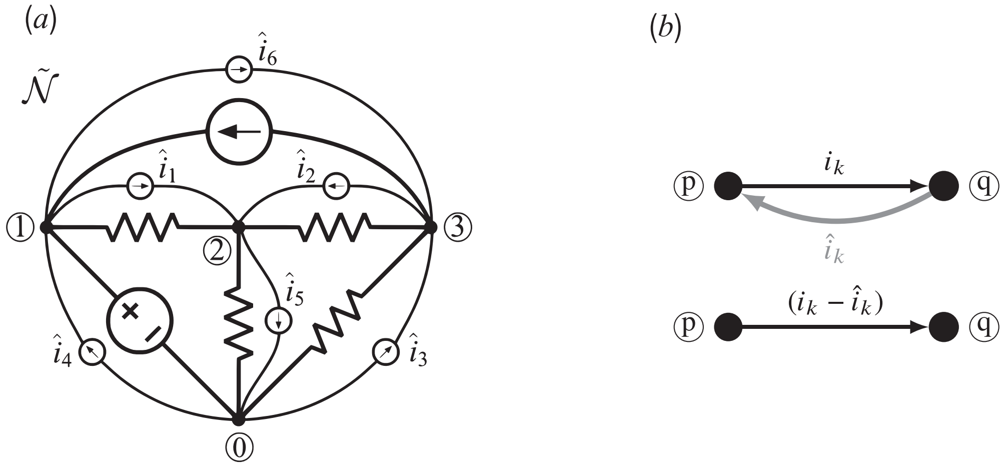


Figure 1.10 (a) $\tilde{\mathcal{N}}$ is formed by adding currents sources in parallel with the branches of $\mathcal{N}$. (b) Every branch in $\mathcal{N}$ has in (anti)parallel with it a current source whose value is the same as that of the corresponding branch in $\hat{\mathcal{N}}$. The composite branch carries a current $(i_k - \hat{i}_k)$.

To intuitively understand the result, consider the new network $\tilde{\mathcal{N}}$ shown in Figure 1.10a. It is formed from $\mathcal{N}$ using the following artifice. In parallel with every branch of $\mathcal{N}$, we add a current source whose value is identical to the current flowing in the corresponding branch of $\hat{\mathcal{N}}$, but in the opposite direction. Since the branch currents $\hat{i}_k$ satisfy KCL, it follows that the net current injected into each node is zero. In general, the kth branch has in parallel with it a current source $\hat{i}_k$ connected in the opposite direction as shown in Figure 1.10b. This composite branch can be thought of as a single branch with a current $(i_k - \hat{i}_k)$.

Since the currents injected into every node of $\mathcal{N}$ are zero, it follows that the node voltages $v_1, \ldots, v_3$ in $\tilde{\mathcal{N}}$ in Figure 1.10a are identical to those in $\mathcal{N}$ in Figure 1.9a.

The graph of $\tilde{\mathcal{N}}$ and a simplified version are given in Figure 1.11. Using Tellegen's theorem on the network of Figure 1.11b, we have

$$\sum_k e_k(i_k - \hat{i}_k) = \sum_k e_k i_k - \sum_k e_k \hat{i}_k = 0. \tag{1.13}$$

This is a just a statement of power conservation applied to $\tilde{\mathcal{N}}$. Since $\sum_k e_k i_k = 0$ (by power conservation in $\mathcal{N}$), it is evident that

$$\sum_k e_k \hat{i}_k = 0. \tag{1.14}$$

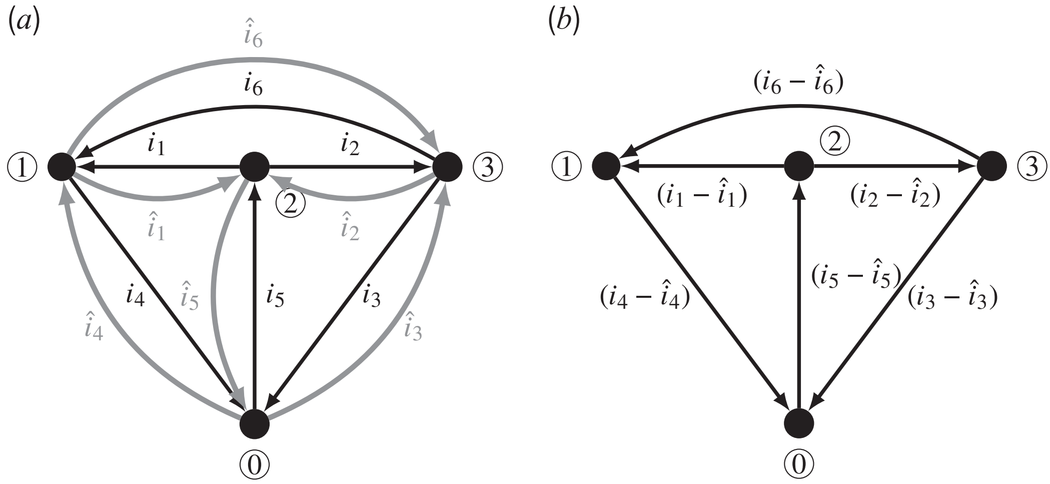


Figure 1.11 Graph of $\tilde{\mathcal{N}}$, and its simplified equivalent.

We started with $\mathcal{N}$ and added current sources to arrive at $\tilde{\mathcal{N}}$. Had we started with $\hat{\mathcal{N}}$ instead, we would conclude that

$$\sum_k \hat{e}_k i_k = 0. \tag{1.15}$$

To summarize, though Tellegen's theorem when applied to different networks looks intriguing at first, it can be intuitively understood.

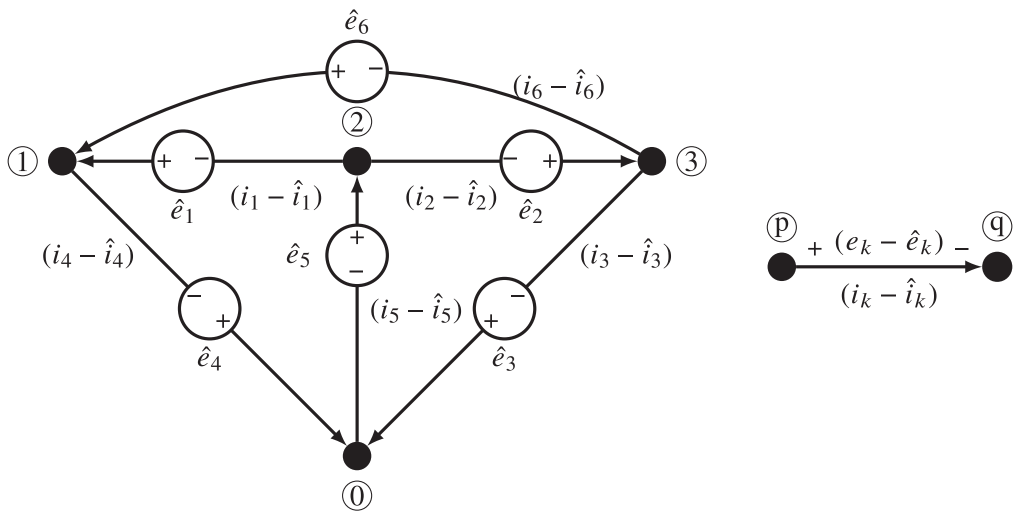


Figure 1.12 Inserting voltage sources in every branch of $\tilde{\mathcal{N}}$, in a direction as to oppose the branch voltage. The voltage source is chosen to be that of the corresponding branch in $\hat{\mathcal{N}}$.

We now describe a modification of the graph of Figure 1.11b that is useful in understanding the incremental behavior of nonlinear networks. As discussed earlier, the branch voltages in Figure 1.11b are identical to those in $\mathcal{N}$ since the net current we injected into every node when we formed $\tilde{\mathcal{N}}$ was zero. Taking this a step further, we insert in series with the kth branch of the graph of Figure 1.9b a voltage source identical to that in the corresponding branch of $\hat{\mathcal{N}}$, namely $\hat{e}_k$. The voltage source is inserted in the sense that it decreases the branch voltage as shown in Figure 1.12. Going around every loop, we see that the net voltage we have introduced into the loop is zero, since the $\hat{e}_k$'s satisfy KVL. As a consequence, the branch currents cannot change; they remain $(i_k - \hat{i}_k)$.

1.5 The Incremental Network and Small-Signal Analysis

What does all this mean in practice, and why is this construct useful? What we have concluded from the previous section is the following. Suppose that two networks, $\mathcal{N}$ and $\hat{\mathcal{N}}$, have the same graph and their branch voltages/currents are denoted by e_k/i_k and $\hat{e}_k/\hat{i}_k$, respectively. A new network with the same graph but with branch voltages/currents $(e_k - \hat{e}_k)/(i_k - \hat{i}_k)$ satisfy KVL and KCL, respectively. It turns out, as we see below, that this is the basis for small-signal (also known as incremental) circuit analysis.

Consider the two networks shown in Figure 1.13a,b. They are identical except that the sources in $\hat{\mathcal{N}}$ are increased by ΔV_a and ΔI_b. Note that the increments *need not be* small. $g_1(\cdot)$ and $g_2(\cdot)$ denote nonlinear conductances so that the branch current i_1 is related to the branch voltage according to $i_1 = g_1(e_1)$.

The "difference" network is shown in Figure 1.13c. The branch voltages and currents in this network are Δe_k and Δi_k, which are used to denote $(\hat{e}_k - e_k)$ and $(\hat{i}_k - i_k)$ respectively. We see that independent sources are the increments in the voltage and current excitations, namely ΔV_a and ΔI_b. The branch relations in the network of Figure 1.13c relate Δe_k to Δi_k. For instance, the first branch relation is derived as follows.

$$\begin{aligned} i_1 &= g_1(e_1) && \text{from } \mathcal{N} \\ i_1 + \Delta i_1 &= g_1(e_1 + \Delta e_1) && \text{from } \hat{\mathcal{N}} \\ \Delta i_1 &= g_1(e_1 + \Delta e_1) - g_1(e_1). \end{aligned} \tag{1.16}$$

Using the Taylor expansion for $g_1(\cdot)$ around e_1, we have

$$\Delta i_1 = \left.\frac{dg_1(x)}{dx}\right|_{x=e_1} \Delta e_1 + \frac{1}{2}\left.\frac{d^2 g_1(x)}{dx^2}\right|_{x=e_1} \Delta e_1^2 + \cdots \tag{1.17}$$

If the ΔV_a and ΔI_b qualify as small-signal excitations so that the higher-order terms in the Taylor series are negligible, the first branch can be replaced by a *linear* conductance whose value is given by $dg_1(x)/dx$ evaluated at $x = e_1$. e_1 is the so-called "operating point" of the nonlinear element $g_1(\cdot)$, and the difference network becomes linear. It is commonly referred to as the incremental circuit, where the component values depend on the operating point as shown in Figure 1.13d.

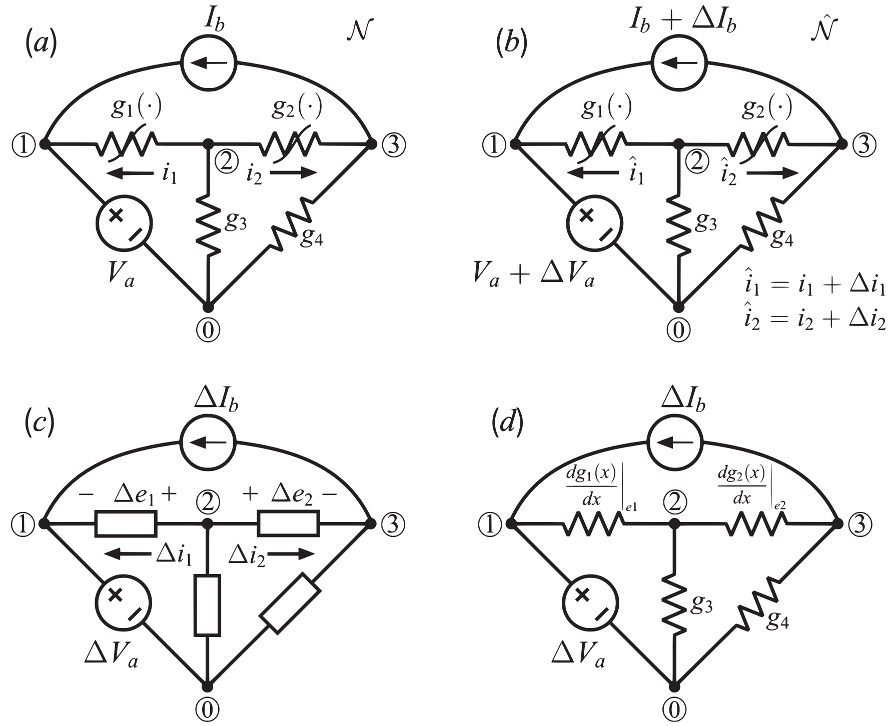


Figure 1.13 (a) Original network, consisting of nonlinear conductances $g_1(\cdot)$ and $g_2(\cdot)$. (b) Network in (a), but the exciting sources changed. The changes need not necessarily be small. (c) The changes in branch currents and voltages satisfy KCL/KVL. The elements in the network are nonlinear. (d) In the special case where the changes in branch voltages can be considered small, the nonlinear elements can be replaced by their incremental equivalents.

To summarize, to determine the changes in branch voltages and currents in a network $\mathcal{N}$ with nonlinear conductances due to a small change in the excitation sources, we proceed as follows.

a. Determine the branch voltages and currents in $\mathcal{N}$. Since the relations that govern the branch currents and voltages are nonlinear in general, it is often not possible to solve the set of KCL/KVL equations analytically. In such a case, the solution must be found numerically. The branch voltages/currents constitute the operating points of the corresponding conductances.

b. Create a linear network with the same graph as $\mathcal{N}$. This is the *incremental* network. The sources that excite this network are the *changes* in the exciting sources. The component values are linear conductances whose values are the slopes of the nonlinear conductances determined at the corresponding operating point. The branch voltages and currents in the incremental network correspond to changes in the branch voltages/currents in $\mathcal{N}$ due to a change in the exciting sources.

c. Solve the incremental network. This can be done analytically, since the network is linear. This will yield all the Δe_k and Δi_k. The branch voltages and currents in

$\mathcal{N}$ due to the changed excitations are simply given by $(e_k + \Delta e_k)$ and $(i_k + \Delta i_k)$ respectively.

1.6 Solving Linear Networks Using Modified Nodal Analysis

In this section, we develop the modified nodal analysis (MNA) method of writing down circuit equations. For simplicity, we consider memoryless networks. The same concepts hold for networks containing inductors and capacitors operating in the sinusoidal steady state. We start with the basic example of Figure 1.14, which consists of conductances and independent current sources. How do we "solve" the network? By "solve", we mean—determining all the node voltages and branch currents in the network.

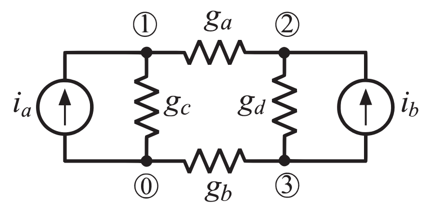


Figure 1.14 Example network to illustrate the nodal analysis.

a. Choose an arbitrary node as the ground node, with respect to which all other node potentials are measured. In the example of Figure 1.14, the node ⓪ is chosen as the ground node.
b. The unknowns to be evaluated are the node voltages. In our example network, they are v_1, v_2, and v_3.
c. Write Kirchhoff's current law (KCL) at every node other than the ground node. For our example network, we have the following.

$$
\begin{aligned}
g_a(v_1 - v_2) + g_c v_1 &= i_a \\
g_a(v_2 - v_1) + g_d(v_2 - v_3) &= i_b \\
g_b v_3 + g_d(v_3 - v_2) &= -i_b
\end{aligned}
$$

The set of equations above is expressed in matrix form as follows. Let us take a look at the anatomy of the matrices—this will perhaps help us write the matrices right away without having to go through the effort of writing KCL at each node, collecting terms, etc.

$$
\begin{array}{l}
\\
\text{KCL at ①} \\
\text{KCL at ②} \\
\text{KCL at ③}
\end{array}
\begin{array}{c}
\begin{array}{ccc} \text{①} & \text{②} & \text{③} \end{array} \\
\begin{bmatrix}
g_a + g_c & -g_a & 0 \\
-g_a & g_a + g_d & -g_d \\
0 & -g_d & g_b + g_d
\end{bmatrix}
\end{array}
\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}
=
\begin{bmatrix} i_a \\ i_b \\ -i_b \end{bmatrix}
\begin{array}{l}
\leftarrow\text{Indep. source} \\
\leftarrow\text{Indep. source} \\
\leftarrow\text{Indep. source}
\end{array}
\tag{1.18}
$$

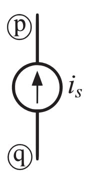
The unknowns are the node voltages $v_1,\ v_2$, and v_3. They form the column vector of unknowns on the left-hand side. In general, if the number of nodes is n, there will be $(n-1)$ unknowns. The square matrix on the left-hand side is 3×3: this makes sense, as there are three unknowns and three equations. Each row of the matrix arises from writing KCL at a node. Each column of the matrix corresponds to a node potential. For instance, the entry in the second row and first column, namely $-g_a$, indicates that $-g_a$ is the coefficient of v_1 in the KCL equation written for node 2. The right-hand side is a column vector with known entries, which are the independent currents *entering* the nodes.

d. Careful observation of the square matrix on the left-hand side of (1.18) reveals that each element in the network has a unique pattern of entries associated with it. For instance, the conductance g_a connected between nodes 1 and 2 appears in rows 1 and 2, and columns 1 and 2. This makes sense—after all, the current through g_a must appear in the KCL equations written at nodes 1 and 2. Furthermore, the current through g_a depends on v_1 and v_2; g_a must thereby appear in columns 1 and 2. This specific pattern is called the *stamp* of the conductance. The stamp of a conductance connected between nodes p and q is given in Figure 1.15. It is a square matrix whose dimension is equal to the number of unknowns. Its only nonzero entries are shown in the figure. In similar fashion, we see that a current source connected between nodes p and q is given in Figure 1.16. It is a column matrix with height equal to the number of unknowns. Its only nonzero entries are shown in the figure.

e. The unknown, namely the node-voltage vector, is determined by inverting the conductance matrix and multiplying it with the excitation vector.

$$\begin{array}{c c} & \begin{array}{cc} p & q \end{array} \\ \begin{array}{c} p \\ q \end{array} & \begin{bmatrix} g & -g \\ -g & g \end{bmatrix} \end{array}$$

Figure 1.15 Conductance connected between the pth and qth nodes, and its stamp.



$$\begin{array}{c c} \begin{array}{c} p \\ q \end{array} & \begin{bmatrix} i_s \\ -i_s \end{bmatrix} \end{array}$$

Figure 1.16 Current source connected between the pth and qth nodes, and its stamp.

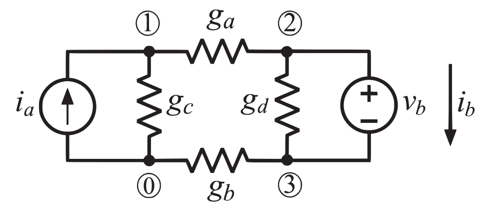


Figure 1.17 Example network containing a voltage-source excitation.

Consider now a network that contains a voltage-source excitation, such as the one shown in Figure 1.17. The voltage source poses a problem for nodal analysis, since the current that flows through is independent of the voltage. This is unlike a conductance, where the current depends on the voltage across it. Evidently, our procedure for solving the network must be modified. To address the problem, we denote the current through the voltage source v_b by i_b and treat it as an unknown. An additional unknown will need an additional equation. It is simply the constraint imposed by v_b, which is

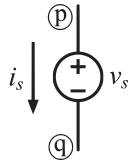
$$v_2 - v_3 = v_b. \tag{1.19}$$

In matrix form, the equations are given by

$$\underbrace{\begin{bmatrix} g_a + g_c & -g_a & 0 & 0 \\ -g_a & g_a + g_d & -g_d & 1 \\ 0 & -g_d & g_d + g_b & -1 \\ 0 & 1 & -1 & 0 \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ i_b \end{bmatrix}}_{\mathbf{X}} = \underbrace{\begin{bmatrix} i_a \\ 0 \\ 0 \\ v_b \end{bmatrix}}_{\mathbf{E}} \tag{1.20}$$

A few observations are in order. The shaded entries are due to the voltage source. The network is captured in the *augmented nodal matrix*, denoted by **A**. Its order has increased by 1 when compared to the network in Figure 1.14, since the number of unknowns is now 4. The 1 in the last column of row 2, and −1 in the last column of row 3 make explicit that i_b is being drawn from node ② and pushed into node ③. i_b is added to the column of unknowns, denoted by **X**. v_b is added as a new row to **E**, which is the vector of independent *excitation sources* on the right-hand side. **X** is seen to be given by $\mathbf{X} = \mathbf{A}^{-1}\mathbf{E}$. The way of writing network equations is nodal analysis, except that it has been *modified* to be able to support voltage sources. It is therefore called *modified nodal analysis* (MNA), and the stamps of the circuit elements are all referred to as MNA stamps. Figure 1.18 shows the MNA stamp of a voltage source connected between the p$^{\text{th}}$ and q$^{\text{th}}$ node.

To summarize, when a voltage source is encountered, a new row and column are added to the augmented nodal matrix. The extra column has 1 and −1 in the pth and qth positions, respectively. The new row is the transpose of the extra column. The



$$\begin{array}{c} \text{p} \\ \text{q} \\ \\ \\ \end{array}
\begin{array}{c}
\begin{bmatrix} & & 1 \\ & & -1 \\ 1 & -1 & \end{bmatrix} \\
\begin{matrix} \text{p} & \text{q} & \end{matrix}
\end{array}
\begin{array}{c}
\begin{bmatrix} \\ \\ i_s \end{bmatrix} \\ \\
\end{array}
\begin{array}{c} = \\ \\ \end{array}
\begin{array}{c}
\begin{bmatrix} \\ \\ v_s \end{bmatrix} \\ \\
\end{array}$$

Figure 1.18 Voltage source connected between the pth and qth nodes, and its MNA stamp.

current through the voltage source is appended to the column of unknowns. The value of the voltage source is appended to the vector of independent sources.

Let us now move to the MNA stamps of controlled sources. We first consider the voltage-controlled current source (VCCS), using the network of Figure 1.19 for illustration.

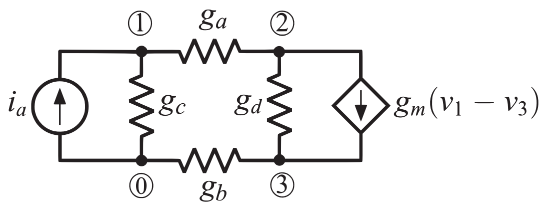
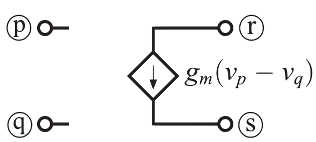


Figure 1.19 Example network containing a VCCS.

Applying KCL at each node yields three equations, which in matrix form are written as follows. The terms shaded in gray are due to the VCCS. Since the current source is connected between nodes 2 and 3, the g_m terms appear in the second and third rows. Furthermore, since the current is proportional to the potential difference between 1 and 3, g_m appears in the first and third columns. Since no new unknowns are introduced, the three node potentials need to be determined. Figure 1.20 shows a VCCS connected between arbitrary nodes and the corresponding MNA stamp.

$$\begin{bmatrix} g_a + g_c & -g_a & 0 \\ -g_a + g_m & g_a + g_d & -g_d - g_m \\ -g_m & -g_d & g_b + g_d + g_m \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} i_a \\ 0 \\ 0 \end{bmatrix}. \tag{1.21}$$



$$\begin{array}{c} \\ \text{r} \\ \text{s} \end{array}
\begin{array}{c}
\begin{matrix} \text{p} & \text{q} \end{matrix} \\
\begin{bmatrix} g_m & -g_m \\ -g_m & g_m \end{bmatrix}
\end{array}$$

Figure 1.20 A voltage-controlled current source and its MNA stamp.

Let us now derive the stamp of a voltage-controlled voltage source (VCVS). We use the network of Figure 1.21 for illustration. The four node voltages are unknowns, as is the current i_x through the voltage source.

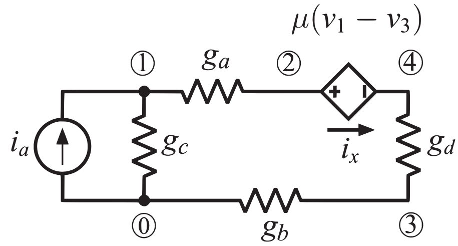


Figure 1.21 Example network containing a VCVS.

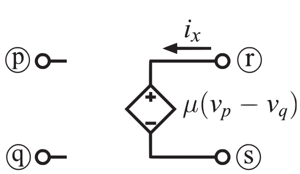
Applying KCL at each node yields four equations. In addition, the VCVS yields the additional equation

$$v_2 - v_4 - \mu v_1 + \mu v_3 = 0. \tag{1.22}$$

In matrix form, the five equations are expressed as follows. The terms shaded in gray are due to the VCVS. Since i_x flows from node 2 to 4, 1 and −1 appear in the last column of the second and fourth rows respectively.

$$\begin{bmatrix} g_a + g_c & -g_a & 0 & 0 & 0 \\ -g_a & g_a & 0 & 0 & 1 \\ 0 & 0 & g_b + g_d & -g_d & 0 \\ 0 & 0 & -g_d & g_d & -1 \\ -\mu & 1 & \mu & -1 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ i_x \end{bmatrix} = \begin{bmatrix} i_a \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{1.23}$$

Figure 1.22 shows a VCVS connected between arbitrary nodes, and the corresponding MNA stamp.



$$\begin{array}{c} \text{r} \\ \text{s} \\ \\ \end{array} \begin{bmatrix} & & & & 1 \\ & & & & -1 \\ -\mu & \mu & 1 & -1 & \end{bmatrix} \begin{bmatrix} \\ \\ i_x \end{bmatrix}$$
$$\begin{array}{cccc} \text{p} & \text{q} & \text{r} & \text{s} \end{array}$$

Figure 1.22 A VCVS and its MNA stamp.

Next, we derive the stamp of a current-controlled current source (CCCS). We use the network for Figure 1.23 for illustration. The four node voltages are unknowns, as is the current i_x through the zero-voltage source. Applying KCL at each node yields four equations. In addition, the zero-voltage source implies that

$$v_1 - v_4 = 0. \tag{1.24}$$

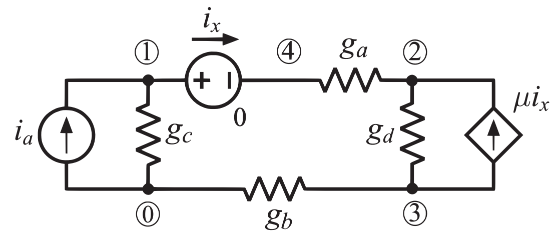


Figure 1.23 Example network containing a CCCS.

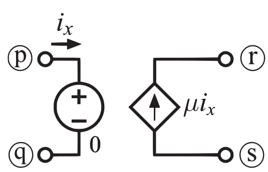
In matrix form, the five equations are expressed as follows. The terms shaded in gray are due to the CCCS. Since i_x flows from node ① to ④, 1 and −1 appear in the last column of the first and fourth rows, respectively.

$$\begin{bmatrix} g_c & 0 & 0 & 0 & 1 \\ 0 & g_a + g_d & -g_d & -g_a & -\mu \\ 0 & -g_d & g_b + g_d & 0 & \mu \\ 0 & -g_a & 0 & g_a & -1 \\ 1 & 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ i_x \end{bmatrix} = \begin{bmatrix} i_s \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{1.25}$$

Figure 1.24 shows a CCCS connected between arbitrary nodes, and the corresponding MNA stamp.

The last controlled source remaining is the current-controlled voltage source (CCVS). We derive its MNA stamp using the network of Figure 1.25 for illustration. The unknowns are the four node voltages and the currents i_x and i_y through the zero- and controlled-voltage sources, respectively. Applying KCL at each node yields four equations. There are *two* additional unknowns; two more equations are needed. They are the constraints imposed by the voltage sources and given by

$$v_1 - v_4 = 0 \tag{1.26}$$



$$\begin{array}{c} p \\ q \\ r \\ s \\ \\ \end{array} \begin{bmatrix} & & & & 1 \\ & & & & -1 \\ & & & & -\mu \\ & & & & \mu \\ 1 & -1 & & & \end{bmatrix} \begin{bmatrix} \\ \\ \\ \\ i_x \end{bmatrix}$$
$$\begin{array}{cc} p & q \end{array}$$

Figure 1.24 A CCCS, and its MNA stamp.

and

$$v_2 - v_3 = R i_x. \tag{1.27}$$

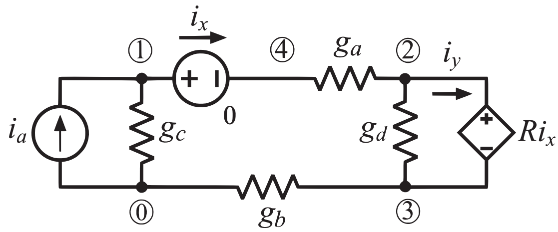
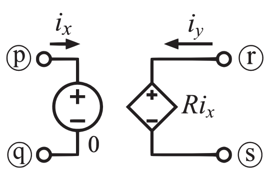


Figure 1.25 Example network containing a CCVS.

In matrix form, the six equations are expressed as shown below. The terms shaded in gray are due to the CCVS. Since i_x flows from node ① to ④, 1 and −1 appear in the penultimate column of the first and fourth rows, respectively. As i_y flows from node ② to ③, 1 and −1 appear in the last column of the second and third rows, respectively.

$$\begin{bmatrix} g_c & 0 & 0 & 0 & 1 & 0 \\ 0 & g_a+g_d & -g_d & -g_a & 0 & 1 \\ 0 & -g_d & g_b+g_d & 0 & 0 & -1 \\ 0 & -g_a & 0 & g_a & -1 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 & -R & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ i_x \\ i_y \end{bmatrix} = \begin{bmatrix} i_a \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{1.28}$$

Figure 1.26 shows a CCVS connected between arbitrary nodes and the corresponding MNA stamp.



$$\begin{array}{c} ⓟ \\ ⓠ \\ ⓡ \\ ⓢ \\ \\ \\ \\ \end{array} \begin{array}{c} \begin{bmatrix} & & & & 1 & \\ & & & & -1 & \\ & & & & & 1 \\ & & & & & -1 \\ 1 & -1 & & & 0 & 0 \\ & & 1 & -1 & -R & 0 \end{bmatrix} \\ \begin{array}{cccc} ⓟ & ⓠ & ⓡ & ⓢ \end{array} \end{array} \begin{bmatrix} \\ \\ \\ \\ i_x \\ i_y \end{bmatrix}$$

Figure 1.26 A CCVS and its MNA stamp.

The ideal opamp is a commonly used network element. *If an opamp has DC negative feedback around it,* its two input terminals form a virtual short. This means that they have the same potential, and no current flows into these terminals. Let us derive the MNA stamp of an opamp, using the network of Figure 1.27 for illustration. The unknowns are the five node voltages, the currents i_x and i_z flowing into the opamps, and i_y flowing into the input source v_s. Applying KCL at each node yields five equations. Three more equations are needed, as there are as many additional unknowns. The equations are

$$v_1 = v_s$$
$$v_1 = v_3$$
$$v_4 = 0.$$

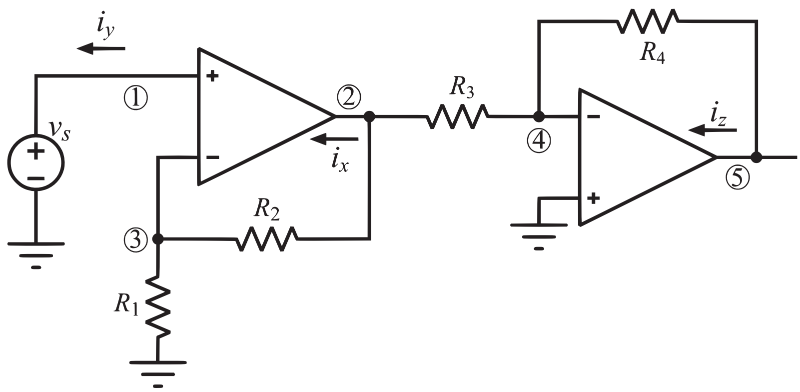
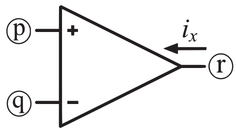


Figure 1.27 Example network containing ideal opamps.

The first of these is imposed by the voltage source, while the other two are the result of the opamps' input terminals forming a virtual short. In matrix form, the eight equations are expressed as shown below. The terms shaded in gray are due to the opamps. Since i_x flows out of ②, 1 appears in the penultimate column of the second row. Likewise, as i_z flows out of node ⑤, 1 appears in the last column of the fifth row.

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & g_2+g_3 & -g_2 & -g_3 & 0 & 0 & 1 & 0 \\ 0 & -g_2 & g_1+g_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & -g_3 & 0 & g_3+g_4 & -g_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & -g_4 & g_4 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ i_y \\ i_x \\ i_z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ v_s \\ 0 \\ 0 \end{bmatrix}. \tag{1.29}$$

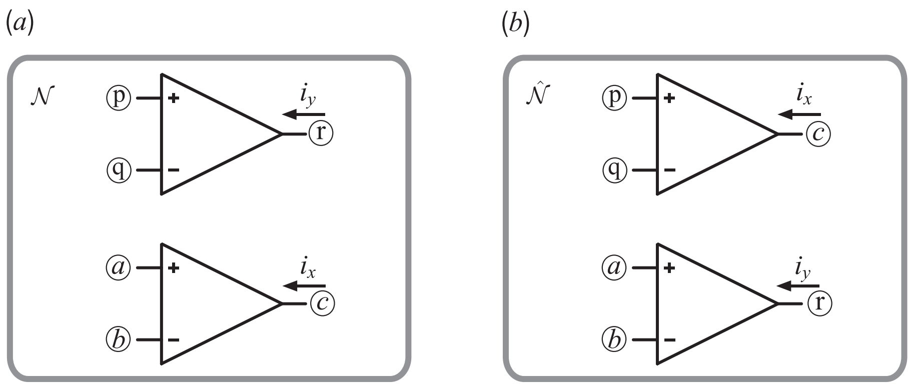
Figure 1.28 gives the MNA stamp of an opamp connected between arbitrary nodes. It is important to bear in mind that there must be DC negative feedback around the opamp for the virtual-short concept to hold.



$$\begin{array}{cc}
\begin{array}{c} \\ r \\ \\ \end{array} & \left[\begin{array}{cccc} & & & \\ & & & 1 \\ 1 & -1 & & \end{array}\right]\left[\begin{array}{c} \\ \\ i_x \end{array}\right] \\
& \begin{array}{cccc} p & q & & \end{array}
\end{array}$$

Figure 1.28 An ideal opamp and its MNA stamp.

1.6.1 Networks with Multiple Ideal Opamps



$$\begin{array}{cc}
\begin{array}{c} \\ r \\ c \\ \\ \\ \end{array} & \left[\begin{array}{cccccc} & & & & & \\ & & & & 1 & \\ & & & & & 1 \\ 1 & -1 & & & & \\ & & 1 & -1 & & \end{array}\right]\left[\begin{array}{c} \\ \\ \\ i_y \\ i_x \end{array}\right] \\
& \begin{array}{cccccc} p & q & a & b & & \end{array}
\end{array}
\qquad
\begin{array}{cc}
\begin{array}{c} \\ r \\ c \\ \\ \\ \end{array} & \left[\begin{array}{cccccc} & & & & & \\ & & & & 1 & \\ & & & & & 1 \\ 1 & -1 & & & & \\ & & 1 & -1 & & \end{array}\right]\left[\begin{array}{c} \\ \\ \\ i_y \\ i_x \end{array}\right] \\
& \begin{array}{cccccc} p & q & a & b & & \end{array}
\end{array}$$

Figure 1.29 (a) Network $\mathcal{N}$ containing two ideal opamps. (b) Network $\hat{\mathcal{N}}$ with the outputs of the opamps interchanged. All node voltages and branch currents remain unchanged since the MNA equations are identical.

We now make a useful observation in circuits that contain multiple ideal opamps. For starters, consider a network $\mathcal{N}$ with two ideal opamps connected to nodes p, q, r and a, b, c as shown in Figure 1.29a. The relevant portion of the MNA matrices, i.e., that pertaining to the two opamps is shown below $\mathcal{N}$. Let us now try the reverse—to determine the connections of the opamps from the MNA matrices. From the last two rows, it is evident that $v_p = v_q$ and $v_a = v_b$, indicating that p–q and a–b form virtual shorts. What it is not clear, however, is how rows r and c should be assigned to the opamps. Interchanging the roles of r and c, as seen in $\hat{\mathcal{N}}$ (Figure 1.29b) would result in *the same* MNA equations! The node voltages and branch currents in $\mathcal{N}$ and $\hat{\mathcal{N}}$ are thus seen to be identical, though the circuits might appear very different. In a circuit with m ideal opamps, there are therefore $m!$ seemingly different variations that basically are the same circuit.[3] The "trick" of interchanging opamp outputs, therefore, is useful to generate alternative circuits that attempt to realize the same functionality—some of which may be better than others when practical opamp nonidealities are considered. One should note that the signs of the opamps in each of the alternative circuits have to be reevaluated to ensure that there is indeed DC negative feedback around them.

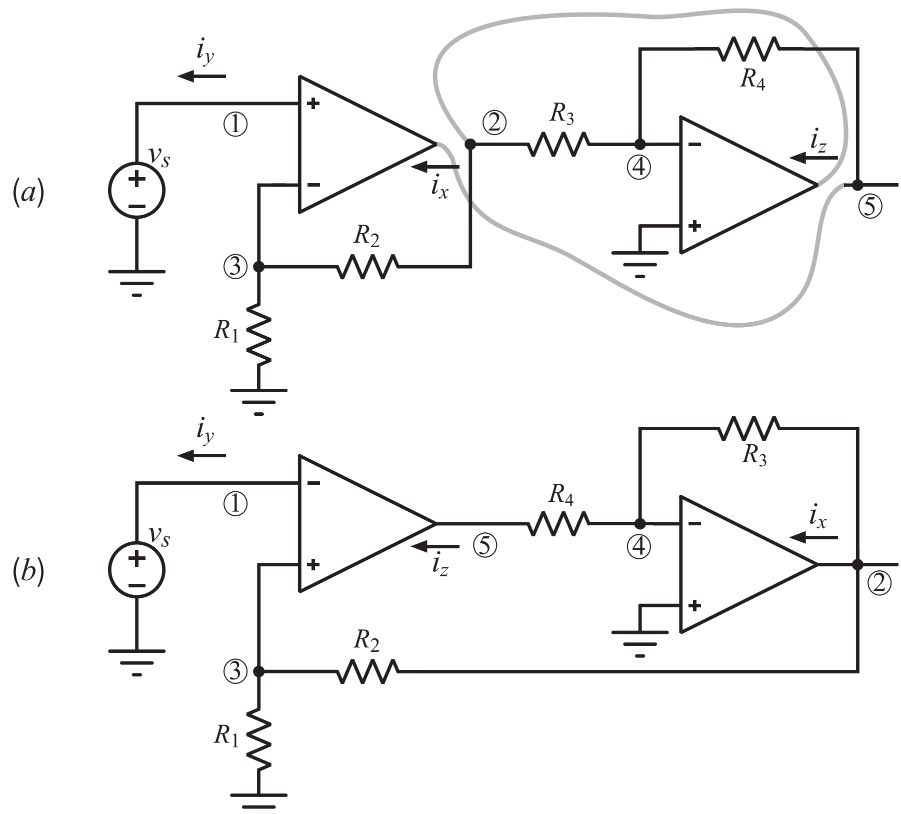


Figure 1.30 (a) Circuit of Figure 1.27, but with the output terminals of the opamps interchanged. The output is v_5. (b) Redrawing the circuit.

We illustrate with an example. Referring to Figure 1.27, let v_5 denote the output of the circuit. The voltage gain is given by

$$\frac{v_5}{v_s} = -\left(1 + \frac{R_2}{R_1}\right)\frac{R_4}{R_3}. \tag{1.30}$$

Figure 1.30a shows the circuit of Figure 1.27, but with the outputs of the opamps interchanged. Part (b) of the figure shows a redrawn version. The output is still v_5.

[3] The ! in $m!$ not only denotes the factorial but also excitement at this result!

Note that the signs of the first opamp have to be appropriately chosen so that both opamps have dc negative feedback around them. Furthermore, the output is the potential of node 5. Straightforward analysis shows that

$$\frac{v_2}{v_s} = \left(1 + \frac{R_2}{R_1}\right). \tag{1.31}$$

As a consequence,

$$\frac{v_5}{v_s} = -\left(1 + \frac{R_2}{R_1}\right)\frac{R_4}{R_3}. \tag{1.32}$$

The gain is indeed the same in both circuits, even though they look very different; the amplifier of Figure 1.27 is a cascade of two stages, while that of Figure 1.30b uses overall feedback. However, as our analysis shows, these two very different-looking circuits are actually the same *if the opamps are ideal*. The reader should verify that the MNA equations for the circuit of Figure 1.30b are identical to those for the network of Figure 1.27. It must therefore follow that the node voltages and branch currents in the circuits must therefore be the same. We reiterate that this only holds when the opamps are ideal—with real opamps, however, the performance of the two circuits will not be the same—one could perform better than the other.

To summarize, a systematic way of generating the MNA equations of a network is the following.

a. The augmented nodal matrix **A** is square with a dimension that equals the number of node voltages plus the number of currents through voltage sources. **A** is initialized with all zero entries.

b. Going element by element in the network, we add to **A** the MNA stamp of the corresponding element. It is thus seen that **A** is simply the sum of the MNA stamps of all the elements in the network.

c. The unknown vector **X** is determined using $\mathbf{X} = \mathbf{A}^{-1}\mathbf{E}$, where **E** is the source (or excitation) vector.

Figure 1.31a–e tabulates the MNA stamps of conductance and the four controlled sources for ready reference.

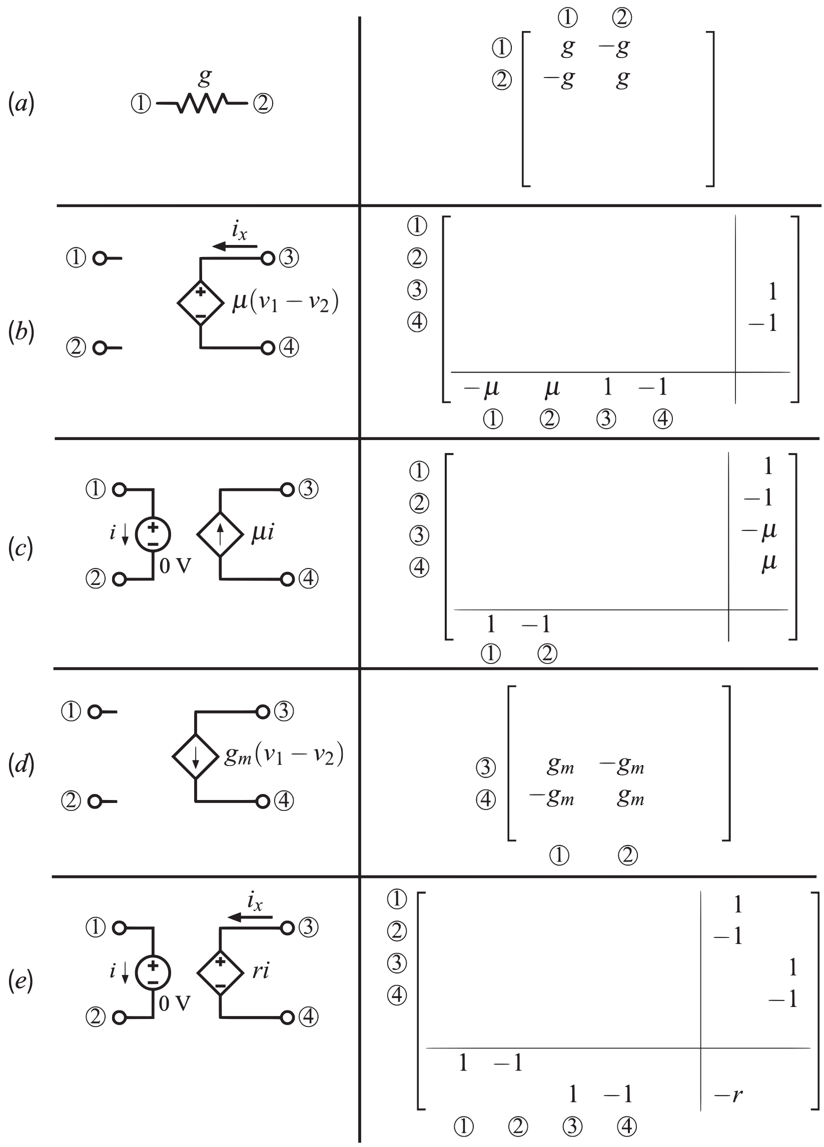


Figure 1.31 Summary of MNA stamps.

1.6.2 Sinusoidal Steady-State: Capacitors and Inductors

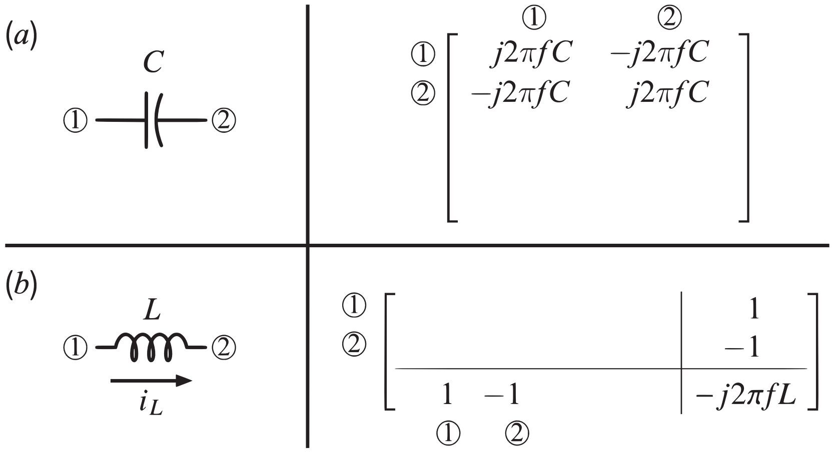


Figure 1.32 The MNA stamp of (a) a capacitor and (b) an inductor. Sinusoidal steady-state operation at a frequency f is assumed.

The discussion so far focused on memoryless networks. It is straightforward to see that the principles hold for networks operating in the sinusoidal steady state. The unknowns and excitations are now phasors. When capacitors and inductors are included in the network, their (frequency-dependent) MNA stamps are necessary. It is easy to see that the MNA stamp of the capacitor is a symmetric matrix as shown in Figure 1.32a. The stamp is seen to be similar to that of a conductance, except that the entries correspond to the frequency-dependent admittance of the capacitor. It is tempting to use the same approach for the inductor. This, however, is problematic since the entries in the stamp would then be of the form $\pm 1/(j2\pi fL)$ which go to infinity at DC. The solution to this problem is to introduce the inductor current as a new unknown I_L and add the equation $V_1 - V_2 - j2\pi fLI_L = 0$ to the set of nodal equations. In the nodal matrix, this corresponds to an additional row and column as shown in Figure 1.32b.

The same approach can be used to write the MNA equations when transfer functions need to be computed; $j2\pi f$ in the discussion above is replaced by the Laplace variable s.

The reader might wonder *why* all the discussion about writing MNA equations is relevant in the first place, when it seems more suited for a book devoted to describing the operation of circuit simulation programs. It turns out that understanding the MNA formulation is valuable in multiple ways, such as the following.

- As we will see going forward, the MNA formulation allows one to systematically derive reciprocal and inter-reciprocal networks, which are immensely useful in their own right.
- During the design of analog and mixed-signal subsystems and systems, it is often necessary to be able to simulate circuit blocks in the time domain. In many

cases, the level of detail needed is more than just the "top-level" transfer function. The MNA formulation allows a designer to create state-space models for circuit blocks, which can in turn be used for system-level simulations in tools like MATLAB.

Example: Descriptor State-Space Modeling of a Second-Order Filter

Suppose that the second-order filter structure shown in Figure 1.33 needs to be simulated in the time domain as part of a larger system. The opamp is modeled as a VCCS with a transconductance given by g_{ota}. Furthermore, assume that we are interested in the voltage waveform at *all* nodes. The reader is no doubt familiar with the state-space description of a linear system, namely

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \quad , \quad \mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}. \tag{1.33}$$

$\mathbf{x}$ is the vector of state variables, $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$ are the state matrices, and $\mathbf{y}$ and $\mathbf{u}$ denote the output and input vectors, respectively.

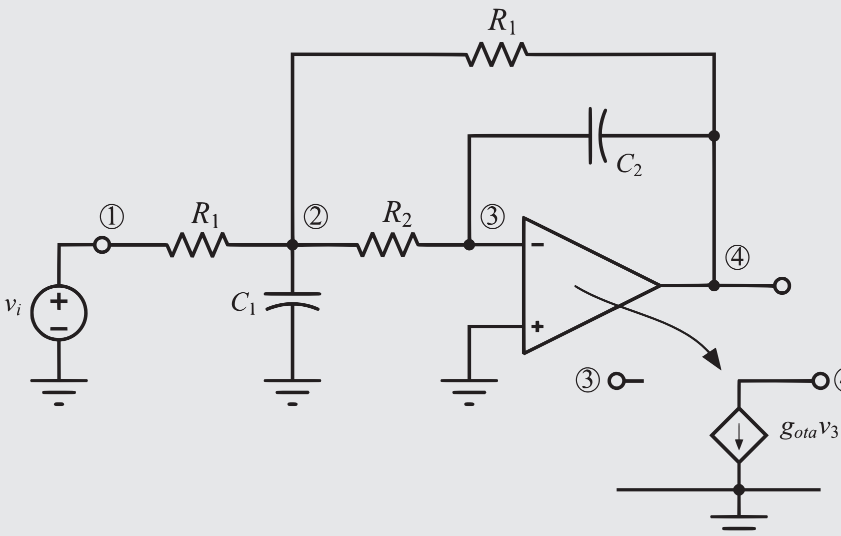


Figure 1.33 Modeling an analog sub-block for time-domain simulations.

When applied to the filter of Figure 1.33, the "standard" state-space description is problematic due to the following. The states are the capacitor voltages v_2 and $(v_3 - v_4)$. All other node voltages and voltage-source currents (a total of five in number) are linearly related to the state variables. Writing out the state matrices in terms of the node voltages is cumbersome.

(Continued)

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The problem above is avoided by the use of the descriptor-state space formulation, which allows us to use the MNA equations directly. The mechanics of the process is first described, following which we delve into practical details. The MNA equations, written in terms of the Laplace-transformed unknowns in the network (namely the node voltages and input-source current), are expressed as follows.

$$s\underbrace{\begin{bmatrix}0&0&0&0&0\\0&C_1&0&0&0\\0&0&C_2&-C_2&0\\0&0&-C_2&C_2&0\\0&0&0&0&0\end{bmatrix}}_{\mathbf{C}}\underbrace{\begin{bmatrix}V_1\\V_2\\V_3\\V_4\\I_x\end{bmatrix}}_{\mathbf{X}}+\underbrace{\begin{bmatrix}G_1&-G_1&0&0&1\\-G_1&2G_1+G_2&-G_2&-G_1&0\\0&-G_2&G_2&0&0\\0&-G_1&g_{ota}&G_1&0\\1&0&0&0&0\end{bmatrix}}_{\mathbf{G}}\underbrace{\begin{bmatrix}V_1\\V_2\\V_3\\V_4\\I_x\end{bmatrix}}_{\mathbf{X}}=\underbrace{\begin{bmatrix}0\\0\\0\\0\\V_i\end{bmatrix}}_{\mathbf{E}}$$

The "capacitor" terms in the augmented nodal matrix are separated from the conductance part, as shown above. Consequently, the MNA equations can be expressed as

$$s\mathbf{CX}+\mathbf{GX}=\mathbf{E}.$$

The capacitance matrix $\mathbf{C}$ is clearly singular, since two of its rows are zero. This makes sense since there are only two independent voltages (those across the capacitors), while there are five unknowns. In the time domain, the MNA equations reduce to

$$\mathbf{C\dot{x}}+\mathbf{Gx}=\mathbf{E}.$$

If $\mathbf{C}$ were nonsingular, both sides of the equation above could be multiplied by $\mathbf{C}^{-1}$, resulting in the state equations $\mathbf{\dot{x}}=-\mathbf{C}^{-1}\mathbf{GX}+\mathbf{C}^{-1}\mathbf{E}$. Since $\mathbf{C}$ is singular, its inverse does not exist. However, it is still possible to solve the set of equations above (these are called differential algebraic equations, or, DAEs). In MATLAB, the system above can be described using the `dss` command. The code fragment that can be used to plot all the unknowns due to a step change in v_i is given below.

Listing 1.1 Descriptor State-Space Systems

```
% MATLAB code
%
% C\dot{x} = -Gx + e;
% A is the matrix that multiplies x.
% B = e/v_i = [0 0 0 0 1]';
%
A = -G;
B = [0 0 0 0 1]';
```

```
D = 0;
sys_biquad = dss(A,B,eye(5),D,C);
% Defines biquad in descriptor state-space form
% The outputs are all the node voltages, and current
   through the input source.
[x, t] = step(sys_biquad,10); % computes step response
   for 10 seconds
```

1.7 Bilinear Form of a Transfer Function, and the Extra-Element Theorem

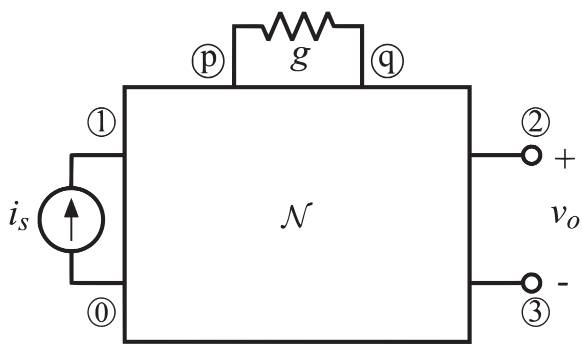

Figure 1.34 How does v_o/i_s depend on g?

Consider a network $\mathcal{N}$, as shown in Figure 1.34. How does the transimpedance v_o/i_s depend on a conductance g connected between nodes p and q? Let us use the "MNA way" to answer this question. Writing the nodal equations, we have

$$\begin{array}{c} \\ \\ p \\ q \\ \\ \end{array} \begin{array}{c} \begin{array}{cccc} \quad & p & q & \quad \end{array} \\ \begin{bmatrix} & & & \\ & g & -g & \\ & -g & g & \\ & & & \end{bmatrix} \end{array} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \end{bmatrix} = \begin{bmatrix} i_s \\ \\ \\ \\ \end{bmatrix}. \tag{1.34}$$

The empty spaces in the nodal matrix are entries that do not have g in them. Furthermore, the entries marked g and $-g$ could have additional terms in them. We express the equations as

$$\mathbf{A}(g)\mathbf{V} = \mathbf{E}, \tag{1.35}$$

to reflect the fact that the nodal matrix $\mathbf{A}$ depends on g. $\mathbf{V}$ is the vector of (unknown) node voltages, and $\mathbf{E}$ is the vector of excitations. The empty rows in $\mathbf{E}$ are zero. In the discussion that follows, we use $\mathbf{A}_k$ to denote a matrix where the k^{th} column of $\mathbf{A}$ is replaced by $\mathbf{E}/i_s$. As an example, $\mathbf{A}_1$ is given by

$$\mathbf{A}_1(g) = \begin{array}{cc} & \begin{matrix} 1 & p & q \end{matrix} \\ \begin{matrix} 1 \\ p \\ q \\ \end{matrix} & \begin{bmatrix} 1 & & \\ 0 & g & -g \\ 0 & -g & g \\ 0 & & \end{bmatrix} \end{array}. \tag{1.36}$$

From Figure 1.34, we see that $v_o = v_2 - v_3$. Using Cramer's rule, we have

$$\frac{v_o}{i_s} = \frac{\det(\mathbf{A}_2(g)) - \det(\mathbf{A}_3(g))}{\det(\mathbf{A}(g))}, \tag{1.37}$$

where $\det(\mathbf{A})$ denotes the determinant of $\mathbf{A}$. How does $\det(\mathbf{A}(g))$ depend on g? To answer this question, we perform a series of row and column operations on $\mathbf{A}(g)$ as follows.

$$\begin{array}{cc} & \begin{matrix} p & q \end{matrix} \\ \begin{matrix} p \\ q \end{matrix} & \begin{bmatrix} g & -g \\ -g & g \end{bmatrix} \end{array} \longrightarrow \begin{array}{cc} & \begin{matrix} p & q \end{matrix} \\ \begin{matrix} p \\ q+p \end{matrix} & \begin{bmatrix} g & -g \\ & \end{bmatrix} \end{array} \longrightarrow \begin{array}{cc} & \begin{matrix} p & q+p \end{matrix} \\ \begin{matrix} p \\ q \end{matrix} & \begin{bmatrix} g & \\ & \end{bmatrix} \end{array}$$

$$\longrightarrow \begin{array}{cc} & \begin{matrix} p & q \end{matrix} \\ \begin{matrix} p \\ q \end{matrix} & \begin{bmatrix} g & \\ & \end{bmatrix} \end{array}$$

We first add the pth row of $\mathbf{A}(g)$ to the qth row. As a consequence, the qth row will no longer have g in it. Next, we add the pth column of $\mathbf{A}(g)$ to the qth column. The resulting matrix will contain g only it its pth row and pth column. By inspection, it is clear that the determinant of this matrix must be of the form $k_1 g + k_2$. In other words, the highest power of g must be unity. Since this matrix is derived from $\mathbf{A}(g)$ through row and column additions, its determinant is identical to $\det(\mathbf{A}(g))$. We thus see that

$$\det(\mathbf{A}(g)) = k_1 g + k_2. \tag{1.38}$$

Using similar arguments, it is straightforward to see that

$$\det(\mathbf{A}_2(g)) - \det(\mathbf{A}_3(g)) = k_3 g + k_4. \tag{1.39}$$

From the equations above, we conclude that the transfer function v_o/i_s must be of the form

$$\boxed{\frac{v_o}{i_s} = \frac{k_3 g + k_4}{k_1 g + k_2}.} \tag{1.40}$$

It is thus seen that the transfer function is a *bilinear form*, i.e., a ratio of first-order polynomials in g. While we derived this for a current excitation and voltage output, it is straightforward to see that this is valid for *any* transfer function. It turns out, as we see below, that (1.40) lends itself to an alternative (and sometimes useful) interpretation. It can be recast as

$$\frac{v_o}{i_s} = \frac{k_4}{k_2}\frac{1+\frac{k_3}{k_4}g}{1+\frac{k_1}{k_2}g} = \left(\frac{v_o}{i_s}\right)_{oc}\frac{1+(z_n/R)}{1+(z_d/R)}, \tag{1.41}$$

where $R = 1/g$.

a. k_4/k_2 is the transfer function of $\mathcal{N}$ when $g = 0$, i.e., when the conductance is replaced by an open circuit. It is therefore denoted by $(v_o/i_s)_{oc}$.

b. z_n can be interpreted as follows. From (1.41), we see that choosing R to be $-z_n$ results in $v_o/i_s = 0$. In other words, if the conductance g is chosen so that the output v_o is zero (i.e., the output has been nulled) when $\mathcal{N}$ is excited by the current i_s, then that value of conductance is given by $-1/z_n$. Rather than vary g to obtain an output null, one could instead inject a current i_x between nodes ⓟ and ⓠ as shown in Figure 1.35, and vary it so that the output v_o is zero. Let the resulting voltage that is developed across the current source be v_x. It is clear that $v_x/i_x = z_n$.

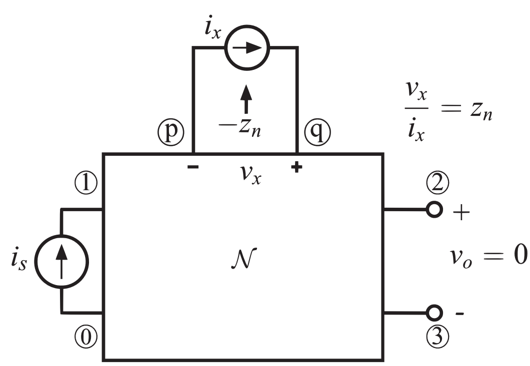

Figure 1.35 If R is replaced by $-z_n$, v_o will be zero. i_x is varied to obtain a null output, and $v_x/i_x = z_n$.

c. As seen from (1.40), z_d is given by k_1/k_2. Recall that

$$\det(\mathbf{A}(g)) = k_1 g + k_2. \tag{1.42}$$

This means that k_1 and k_2 can be obtained as follows.

$$k_1 = \frac{d}{dg}\det(\mathbf{A}(g))\bigg|_{g=0}, \qquad k_2 = \det(\mathbf{A}(0)). \tag{1.43}$$

Recall that the derivative of the determinant of a matrix with respect to a variable x can be expressed as

$$\frac{d}{dx}\det\begin{bmatrix} a(x) & b(x) & \cdots \\ c(x) & d(x) & \cdots \\ \cdots & \cdots & \cdots \end{bmatrix} = \det\begin{bmatrix} a'(x) & b(x) & \cdots \\ c'(x) & d(x) & \cdots \\ \cdots & \cdots & \cdots \end{bmatrix} + \det\begin{bmatrix} a(x) & b'(x) & \cdots \\ c(x) & d'(x) & \cdots \\ \cdots & \cdots & \cdots \end{bmatrix} + \cdots$$

where the prime denotes the derivative of the function with respect to its argument. Using this, we see that z_d can be expressed as follows:

$$z_d = \frac{k_1}{k_2} = \frac{1}{\det(\mathbf{A}(0))}\left.\frac{d(\det(\mathbf{A}(g)))}{dg}\right|_{g=0}$$

$$= \frac{\det\left\{\begin{array}{cc} & \begin{array}{cc} p & q \end{array} \\ \begin{array}{c} \\ p \\ q \\ \\ \end{array} & \begin{bmatrix} & 0 & & \\ & 1 & & \\ & -1 & & \\ & 0 & & \end{bmatrix} \end{array}\right\} - \det\left\{\begin{array}{cc} & \begin{array}{cc} p & q \end{array} \\ \begin{array}{c} \\ p \\ q \\ \\ \end{array} & \begin{bmatrix} & & 0 & \\ & & 1 & \\ & & -1 & \\ & & 0 & \end{bmatrix} \end{array}\right\}}{\det(\mathbf{A}(0))} = \frac{v_p - v_q}{1\,\mathrm{A}}.$$

In the expression above, the numerator is the difference between two determinants. These determinants are det($\mathbf{A}(0)$) with the pth and qth columns replaced by column vectors that contain 1 and -1 in the pth and qth rows, respectively. This makes sense, as only the pth and qth columns of $\mathbf{A}(g)$ contain g. It is clear that z_d can be interpreted as the difference between the potentials of nodes (p) and (q) when a 1A current source is connected between these nodes as shown in Figure 1.36. Note that the input excitation is disabled, i.e., $i_s = 0$.

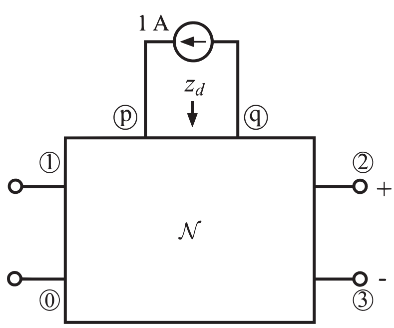

Figure 1.36 z_d is the impedance looking into nodes p–q when $i_s = 0$.

To summarize:

a. The transfer function of a network $\mathcal{N}$ is a bilinear function of its component values. Denoting the impedance of a component by z, any transfer function T can be expressed as

$$\boxed{T = T_\infty \left(\frac{1 + \frac{z_n}{z}}{1 + \frac{z_d}{z}} \right).} \tag{1.44}$$

T can be a voltage gain, current gain, (trans)impedance, or (trans)admittance. T_∞ denotes the value of T when z is infinite (1.44), which quantifies the effect of introducing z into $\mathcal{N}$ on T_∞ by multiplying with a correction factor, is called the **extra-element theorem**.

b. z_n is found by replacing z by a current source i_x whose value is such that the output is nulled and determining the ratio of the voltage developed across it to i_x. Since the network has two excitations and the output is zero, this is referred to as *null double injection*.

c. z_d is the impedance looking into $\mathcal{N}$ when the input excitation is set to 0; i.e., the input voltage (current) is replaced by a short (open) circuit as seen in Figure 1.37d.

Example: Using the Extra-Element Theorem

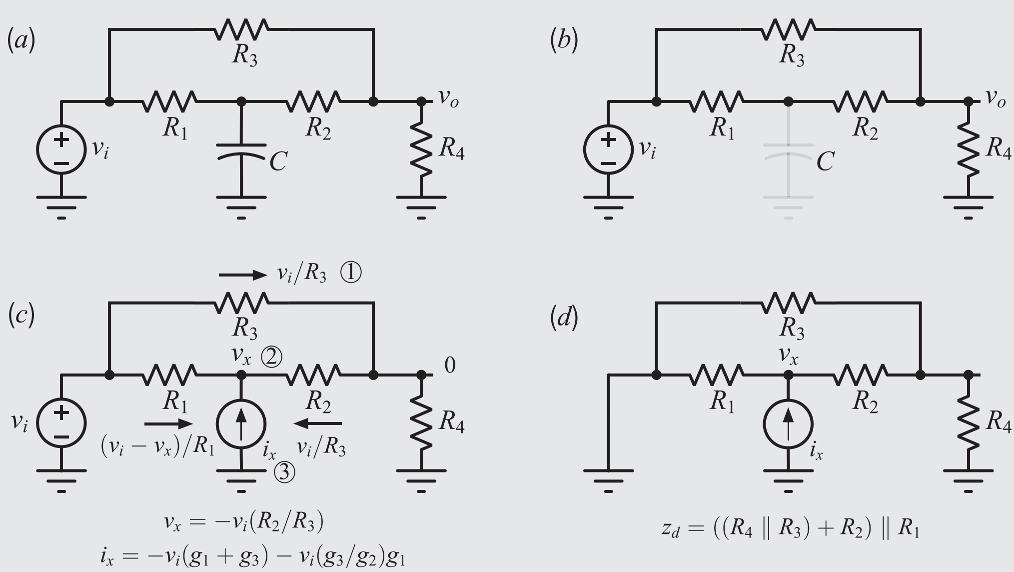

Figure 1.37 (a) $\mathcal{N}$: we treat C as the extra element. (b) Finding the transfer function without C. (c) Finding z_n. (d) Finding z_d.

(Continued)

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We illustrate the application of the extra-element theorem using the network of Figure 1.37a. Suppose we wish to find $V_o(s)/V_i(s)$. We treat the capacitor C as the "extra element", and proceed as follows.

a. We set $C = 0$ to find the transfer function $\left.\frac{V_o(s)}{V_i(s)}\right|_{oc}$. The resulting network is shown in Figure 1.37b. It is seen that

$$\left.\frac{V_o(s)}{V_i(s)}\right|_{oc} = \frac{R_4}{R_4 + (R_3 \parallel (R_1 + R_2))}. \tag{1.45}$$

b. To find z_n, we replace the capacitor with a current source i_x with the aim of nulling the output. The resulting circuit is shown in Figure 1.37c. We observe the following.

① Since $v_o = 0$, the current through R_3 is v_i/R_3.
② This current flows through R_2, resulting in $v_x = -v_i(R_2/R_3)$.
③ Applying KCL, we see that

$$-i_x = v_i g_3 + (v_i - v_x)g_1 = v_i(g_1 + g_3 + g_1 g_3/g_2). \tag{1.46}$$

z_n is thus given by

$$z_n = \frac{v_x}{i_x} = \frac{g_3/g_2}{g_1 + g_3 + g_1 g_3/g_2}. \tag{1.47}$$

c. To find z_d, we set $v_i = 0$ and find the looking-in impedance across the capacitor. By inspection, we have

$$z_d = ((R_4 \parallel R_3) + R_2) \parallel R_1. \tag{1.48}$$

d. By the extra-element theorem, the transfer function is

$$\frac{V_o(s)}{V_i(s)} = \frac{R_4}{R_4 + (R_3 \parallel (R_1 + R_2))} \frac{1 + sCz_n}{1 + sCz_d}. \tag{1.49}$$

1.8 Transposed MNA Stamps

We now draw the attention of the reader to an interesting aspect of MNA stamps. We ask the question—suppose we transpose the MNA stamp of an element. What circuit element would the transposed stamp correspond to? Again, the reader might wonder why this discussion is useful. First, this exercise might be seen as one that deepens our understanding of MNA stamps. More importantly, it turns out that it aids the seamless derivation of reciprocal and inter-reciprocal networks.

We begin with the simplest possible element, namely a conductance. Since the MNA stamp of a conductance is a symmetric matrix, the element corresponding to its transpose is the conductance itself. The MNA stamps of a capacitor and an inductor are also symmetric. It thus follows that the elements that correspond to the transpose of their MNA stamps are the elements themselves.

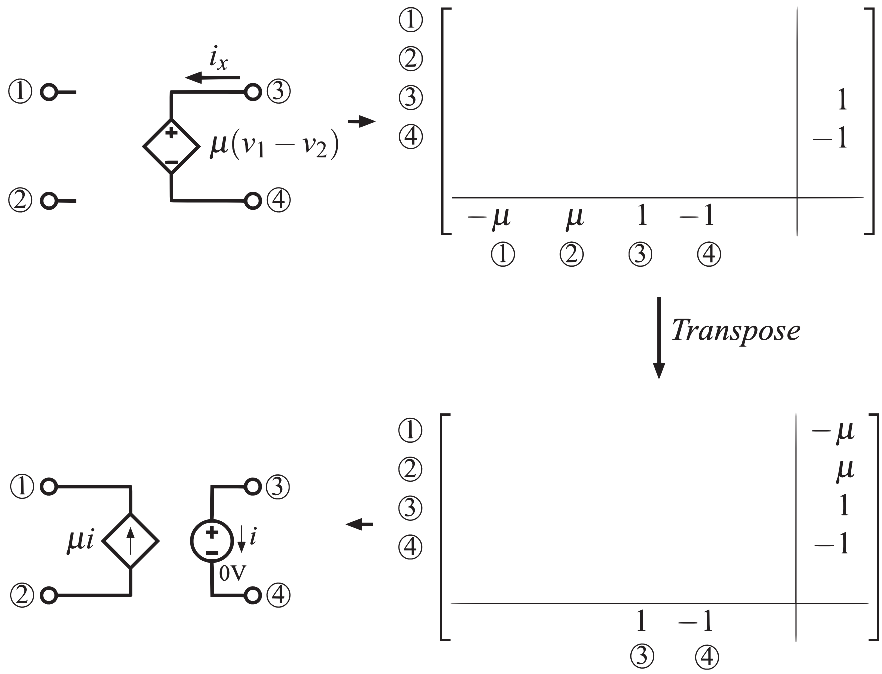


Figure 1.38 A VCVS and its MNA stamp. The transposed MNA stamp corresponds to a CCCS with its controlling and controlled ports interchanged.

The situation is different with controlled sources, as we explain with the help of an example (Figure 1.38). The figure shows a voltage-controlled voltage source and its MNA stamp. The transposed MNA stamp can be immediately recognized as that of a current-controlled current source. However, the controlling and controlled ports are reversed. A sanity check is the following. When $\mu = 0$, the VCVS is an open circuit (i.e., conductance of zero) between its input terminals and a short circuit (infinite conductance) between the output terminals. As we have seen earlier, a network of conductances remains unchanged when its MNA matrix is transposed. It must thus follow that with $\mu = 0$, the element corresponding to the transpose of the MNA stamp of the VCVS must also reduce to an open circuit between ① and ②, and a short circuit between nodes ③ and ④, which is indeed the case, as seen from Figure 1.38.

Practical Tip: Current Upwards or Downwards?

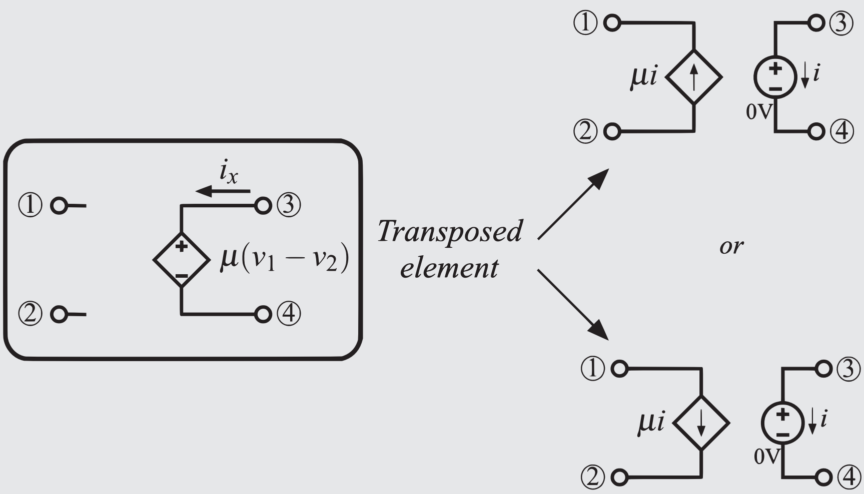


Figure 1.39 Does the μi in the transposed element go up or down?

A common source of confusion is the orientation of the CCCS. A simple way to mark the correct direction, without having to write out the MNA stamp, transpose it, and recognize the orientation, is the following. Assume that μ is positive. Applying a positive voltage at the input port of the VCVS yields a positive voltage at the output port (Figure 1.39). The same must happen in the CCCS. Attempting to apply a positive voltage to the controlling port of the CCCS causes current to flow downwards in Figure 1.38. That should cause an increase in the voltage of the controlled port, just like in the VCVS. This means that the current μi should be chosen to flow *upwards*.

Figure 1.40 shows the four controlled sources and the elements that correspond to the transpose of their MNA stamps (which are also referred to as the inter-reciprocal elements). In all cases, note that when the controlling parameter is set to zero, the port impedances in the original and the inter-reciprocal elements remain the same.

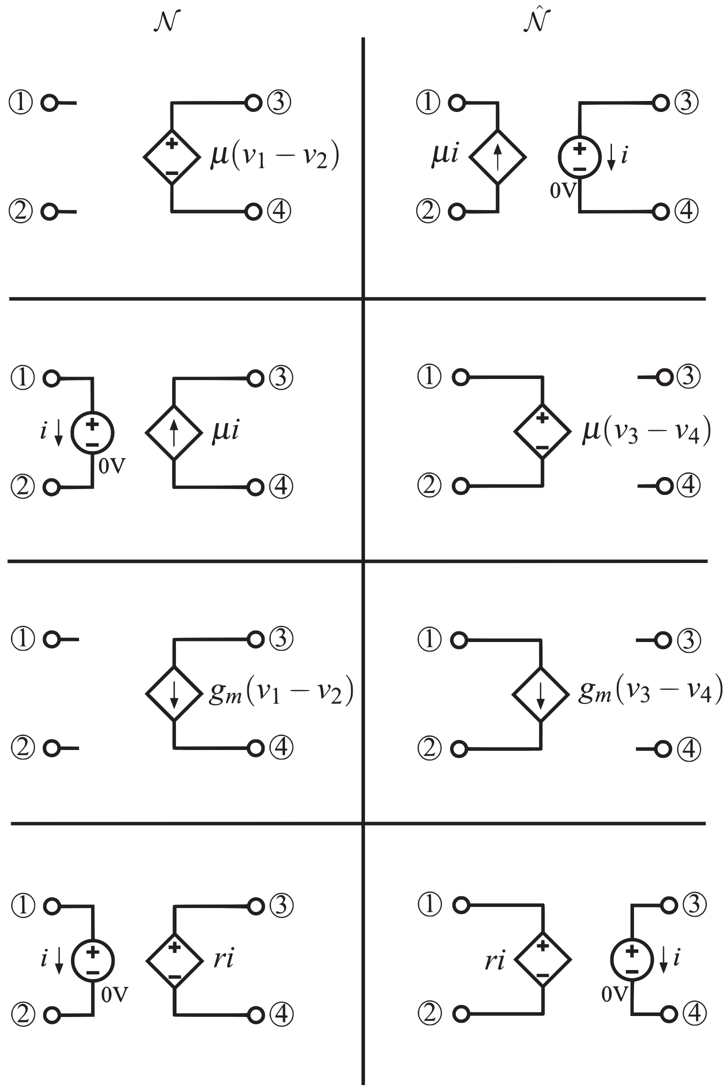


Figure 1.40 Inter-reciprocal elements corresponding to the four controlled sources.

1.9 Reciprocity and Inter-Reciprocity

In this section, we describe the property of inter-reciprocity using the background given in the previous section. We illustrate with an example. For simplicity, the discussions below use memoryless networks. Extension to networks with memory operating in the sinusoidal steady state is straightforward.

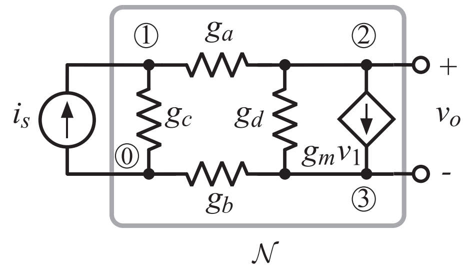


Figure 1.41 Example network used to illustrate reciprocity and inter-reciprocity.

Consider the network $\mathcal{N}$ of Figure 1.41. The network equations are seen to be

$$\underbrace{\begin{bmatrix} g_a + g_c & -g_a & 0 \\ -g_a + g_m & g_a + g_d & -g_d \\ -g_m & -g_d & g_b + g_d \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}}_{\mathbf{X}} = \underbrace{\begin{bmatrix} i_s \\ 0 \\ 0 \end{bmatrix}}_{\mathbf{E}}. \tag{1.50}$$

The output voltage is given by $v_o = (v_2 - v_3)$, and can be written in matrix form as

$$v_o = \begin{bmatrix} 0 & 1 - 1 \end{bmatrix} \mathbf{X}. \tag{1.51}$$

Using (1.50), we see that v_o can be expressed as

$$v_o = \underbrace{\begin{bmatrix} 0 & 1 & -1 \end{bmatrix}}_{\textit{Measurement}} \mathbf{A}^{-1} \underbrace{\begin{bmatrix} i_s \\ 0 \\ 0 \end{bmatrix}}_{\textit{Excitation}}. \tag{1.52}$$

The equation above can be interpreted as follows. The excitation vector $\mathbf{E}$ is processed by the network represented by the MNA matrix $\mathbf{A}$ to result in a node-voltage vector $\mathbf{X} = \mathbf{A}^{-1}\mathbf{E}$. The output voltage v_o is obtained by subjecting $\mathbf{X}$ to the *measurement* matrix, given by $\begin{bmatrix} 0 & 1 - 1 \end{bmatrix}$ in the example above.

Since v_o is a scalar, (1.52) can be rewritten by transposing its right-hand side, and moving the scalar i_s as shown below.

$$v_o = \underbrace{\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}}_{\text{Measurement}} (\mathbf{A}^T)^{-1} \underbrace{\begin{bmatrix} 0 \\ i_s \\ -i_s \end{bmatrix}}_{\text{Excitation}}. \tag{1.53}$$

The equation above can be *interpreted* as follows. The excitation vector $\begin{bmatrix} 0 & i_s - i_s \end{bmatrix}^T$ is processed by a network whose MNA matrix is $\mathbf{A}^T$. The resulting node-voltage vector is subjected to the measurement vector $\begin{bmatrix}1 & 0 & 0\end{bmatrix}$. The output that results is v_o. A few remarks are in order.

a. The exciting current is placed between nodes ② and ③. These nodes represented the output port in $\mathcal{N}$.
b. Recall that **A** was obtained by summing up the MNA stamps of the constituent network elements in $\mathcal{N}$. The network $\hat{\mathcal{N}}$ whose MNA matrix is $\mathbf{A}^T$ is thus seen to be comprised of elements whose MNA stamps are the transpose of the corresponding elements in $\mathcal{N}$. This suggests that if $\mathcal{N}$ is given, $\hat{\mathcal{N}}$ can be obtained by appropriate element-wise replacement. A conductance in $\mathcal{N}$ remains unchanged since the MNA stamp of a conductance is symmetric. Controlled sources are replaced according to the rules given in Figure 1.40. It is straightforward to see that the MNA stamp of the element on the right in Figure 1.40 is the transpose of that of the element on the left. $\hat{\mathcal{N}}$ is called the inter-reciprocal network. It is also referred to as the adjoint network.
c. The output voltage is measured between ① and ground. Recall that the exciting current source was connected between these nodes in $\mathcal{N}$.

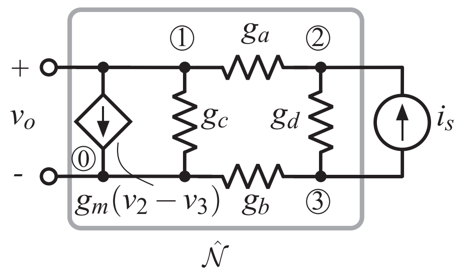


Figure 1.42 Exciting the inter-reciprocal network corresponding to the network of Figure 1.41 at the "output" port yields the same voltage v_o at the "input" port.

The interpretation of (1.53) as described above leads to the network shown in Figure 1.42. From the discussion above, we see that inter-reciprocity follows quite naturally from the way one interprets the relationship between v_o and i_s. In the special case of a network without any dependent sources, $\mathcal{N}$ and $\hat{\mathcal{N}}$ are identical, which leads to the more familiar concept of reciprocity.

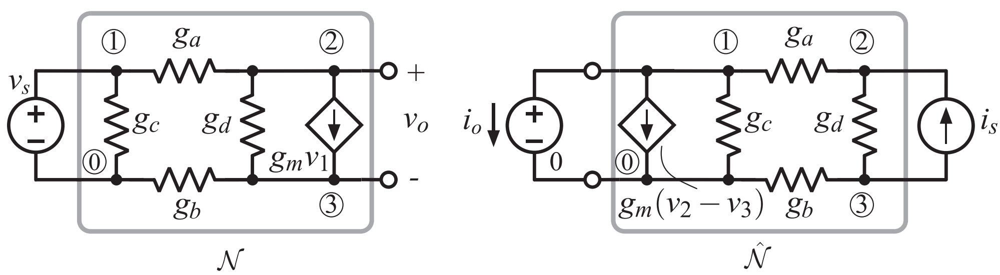


Figure 1.43 $\mathcal{N}$ is excited with a voltage, and the output is a voltage. The inter-reciprocal network is excited with a current, and the output is also a current.

Let us now try this out with a voltage-source excitation. Consider the network $\mathcal{N}$ of Figure 1.43. The MNA equations, written in matrix form, are given by

$$\underbrace{\begin{bmatrix} g_a+g_c & -g_a & 0 & 1 \\ -g_a+g_m & g_a+g_d & -g_d & 0 \\ -g_m & -g_d & g_b+g_d & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ i_s \end{bmatrix}}_{\mathbf{X}} = \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ v_s \end{bmatrix}}_{\mathbf{E}}. \tag{1.54}$$

The output voltage is given by $v_o = (v_2 - v_3)$, and can be written in matrix form as

$$v_o = \begin{bmatrix} 0 & 1-1 & 0 \end{bmatrix} \mathbf{X}. \tag{1.55}$$

Using (1.54), we see that v_o/v_s can be expressed as

$$\frac{v_o}{v_s} = \underbrace{\begin{bmatrix} 0 & 1 & -1 & 0 \end{bmatrix}}_{\text{Measurement}} \mathbf{A}^{-1} \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}}_{\text{Excitation}} = \underbrace{\begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}}_{\text{Measurement}} (\mathbf{A}^T)^{-1} \underbrace{\begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}}_{\text{Excitation}} = \frac{i_o}{i_s}. \tag{1.56}$$

The equation above can be *interpreted* as follows. The excitation vector $\begin{bmatrix} 0 & i_s - i_s & 0 \end{bmatrix}^T$ is processed by a network whose MNA matrix is $\mathbf{A}^T$. The resulting vector of unknowns is subjected to the measurement vector $\begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}$. Recall that the first three rows of the excitation vector represent currents. The first three columns of the measurement row matrix correspond to node voltages, while the last column corresponds to the current flowing through the voltage source connected between node 1 and ground. This interpretation is captured in the network $\hat{\mathcal{N}}$ in Figure 1.43. The network itself is derived from $\mathcal{N}$ using element-by-element replacement rules that we have already discussed. $\hat{\mathcal{N}}$ is excited by a current source between nodes 2 and 3, and by a zero-volt source between nodes 1 and 0. The "measurement" is the current flowing through the voltage source. By inter-reciprocity, v_o/v_s in $\mathcal{N}$ is identical to i_o/i_s in $\hat{\mathcal{N}}$.

Practical Point: Which of the Following Is Correct?

The figure above shows a common source of confusion with respect to reciprocal and inter-reciprocal networks. The network $\mathcal{N}$ with a voltage-source excitation is shown on the left. $\hat{\mathcal{N}}$ is the inter-reciprocal network. Which of the two pictures on the right (b or c) correctly demonstrates the concept of inter-reciprocity?

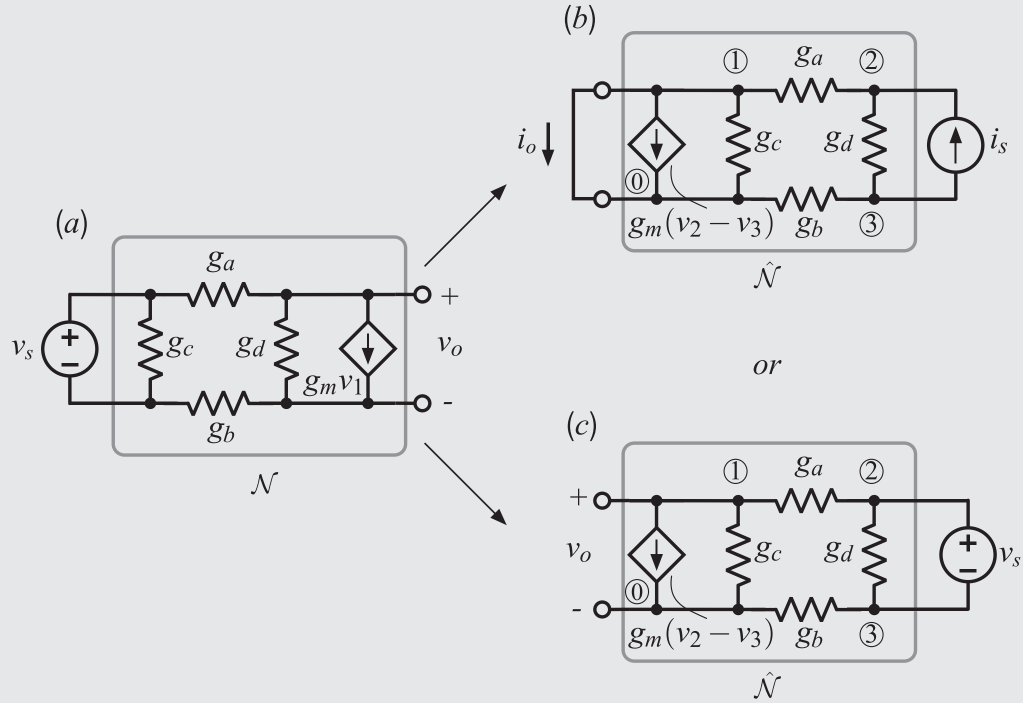


Figure 1.44 A common source of confusion.

The key point to note is that when the independent sources in $\mathcal{N}$ and $\hat{\mathcal{N}}$ are "de-energized", meaning that the voltage source is replaced by a short circuit and a current source by an open circuit, the port impedances of $\mathcal{N}$ and $\hat{\mathcal{N}}$ should be the same. For instance, the ports of the left- and right-hand sides of $\mathcal{N}$ are short and open circuits, respectively when $v_s = 0$. The same holds for the network of Figure 1.44b, but not for that of Figure 1.44c. Therefore, v_o/v_s for $\mathcal{N}$ and i_o/i_s for $\hat{\mathcal{N}}$ are identical. However, v_o/v_s in Figure 1.44c is *not* identical to v_o/v_s in Figure 1.44a.

The example of Figure 1.41 considered a network with an excitation that was a current, while the output was a voltage. It is straightforward to see (using the approach described above) that a voltage input and/or current output also behave in a similar manner, as shown in Figure 1.45. An observation is in order here. Notice that in Figure 1.45a–d, setting the excitation sources to zero makes the terminating impedances at ports 1–1′ and 2–2′ identical in $\mathcal{N}$ and $\hat{\mathcal{N}}$.

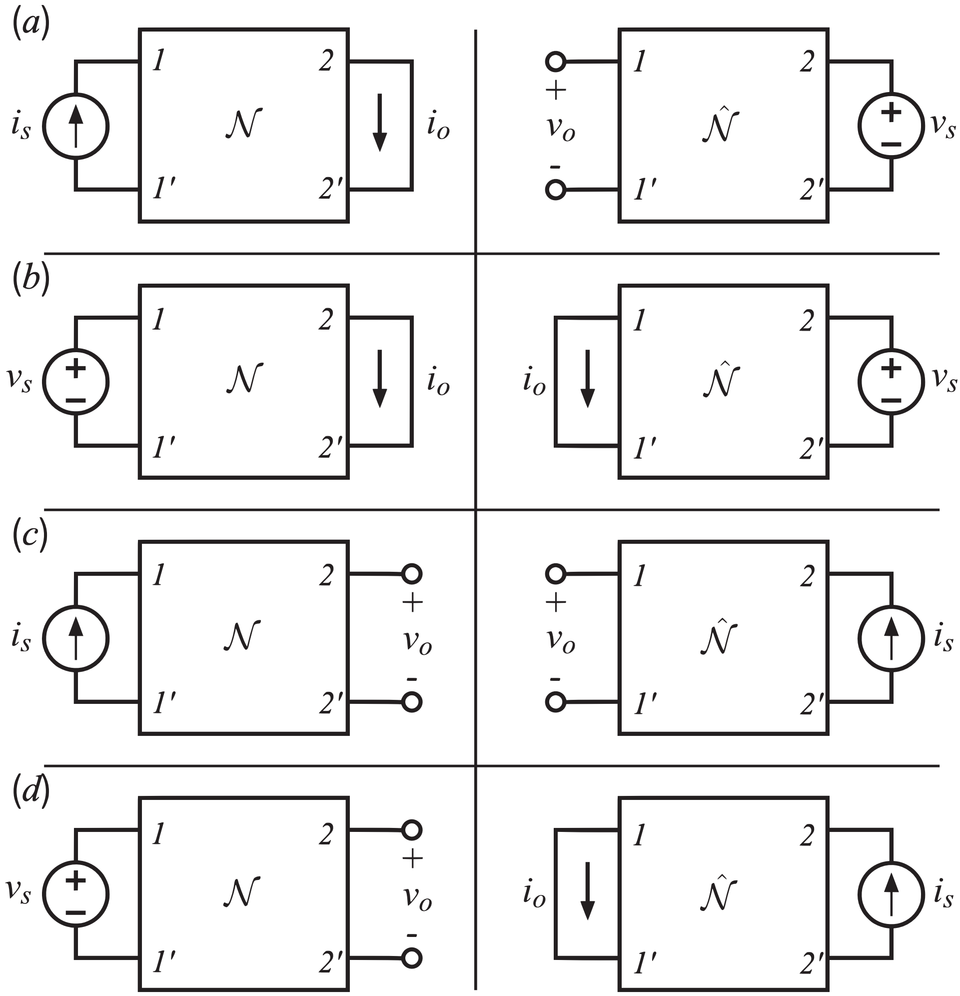


Figure 1.45 The ratio of the response to the excitation in $\mathcal{N}$ is identical to that in $\hat{\mathcal{N}}$. $\hat{\mathcal{N}}$ is derived by replacing every element in $\mathcal{N}$ with one whose MNA stamp is the transpose of that in $\mathcal{N}$.

1.10 Why are Reciprocity and Inter-Reciprocity Useful?

While the concepts described in the previous section are interesting and the end results intriguing, the immediate question that comes to mind is if all of this is anything more than a mathematical curiosity. It turns out, as we will see below, that reciprocity and inter-reciprocity are very relevant in practice. Consider the problem of determining the output of a network when excited by multiple inputs, as shown in Figure 1.46a. Such a situation occurs during noise analysis. For example, evaluating the output noise of a standard fully differential two-stage Miller-compensated opamp would need us to determine about a dozen transfer functions. One would naturally

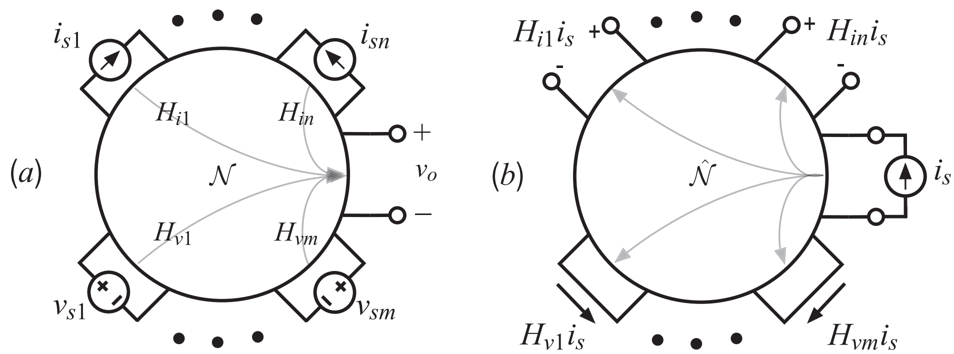


Figure 1.46 (a) The problem of determining the transfer functions from many inputs to a single output. Applying superposition would need the network to be evaluated $(m+n)$ times. (b) Using the inter-reciprocal (or adjoint) network yields all the $(m+n)$ transfer functions in one shot.

be tempted to solve this problem using superposition—i.e., by analyzing the network by considering one source at a time. The output is seen to be

$$v_o = \sum_{k=1}^{m} H_{vk} v_{sk} + \sum_{l=1}^{n} H_{il} i_{sl}. \tag{1.57}$$

For our example in Figure 1.46a, this would result in a total of $(m + n)$ evaluations of the network, with each evaluation yielding the transfer function from one source to the output. This is tedious and computationally inefficient.

The problem can be solved in a *much more* efficient manner by using the concept of inter-reciprocity, as shown in Figure 1.46b. We form the inter-reciprocal network using the rules described in Section 1.9. We excite the output port of $\hat{\mathcal{N}}$ with a current source and de-energize all the independent sources. Consequently, the voltage sources are replaced by short circuits, while the current sources are opened. We evaluate the network *only once*. By the properties of the inter-reciprocal network, the currents flowing in the short-circuited ports (corresponding to the exciting voltage sources in $\mathcal{N}$) are given by $H_{vk}i_s$. Similarly, the voltages across the open-circuited ports (which correspond to the exciting current sources in $\mathcal{N}$) are given by $H_{il}i_s$. Once the H_{vk} and H_{il} are determined, they can be used in (1.57) to determine v_o. The labor saved with respect to evaluating the desired $(m + n)$ transfer functions (naively) using superposition is evident. In a practical circuit simulator, the overhead in determining the adjoint is negligible, since all that one needs to do is to transpose the MNA matrix. This is easily accomplished in an algorithm that finds the matrix inverse, since all that is needed in a computer implementation is an interchange of the row and column addresses when fetching the corresponding matrix elements from memory. Not surprisingly, the adjoint (inter-reciprocity) network is routinely used in circuit simulators for the purposes of noise analysis.

Example: R-2R DAC

Figure 1.47a shows an R-2R DAC. The four current sources are identical and have strength I. Depending on the bits $b_0, \ldots, b_3$ the currents are steered either into the network of resistors or into ground. The opamp is ideal. To find the DAC output v_o, one could use superposition to find the transfer resistance from each current source to the output.

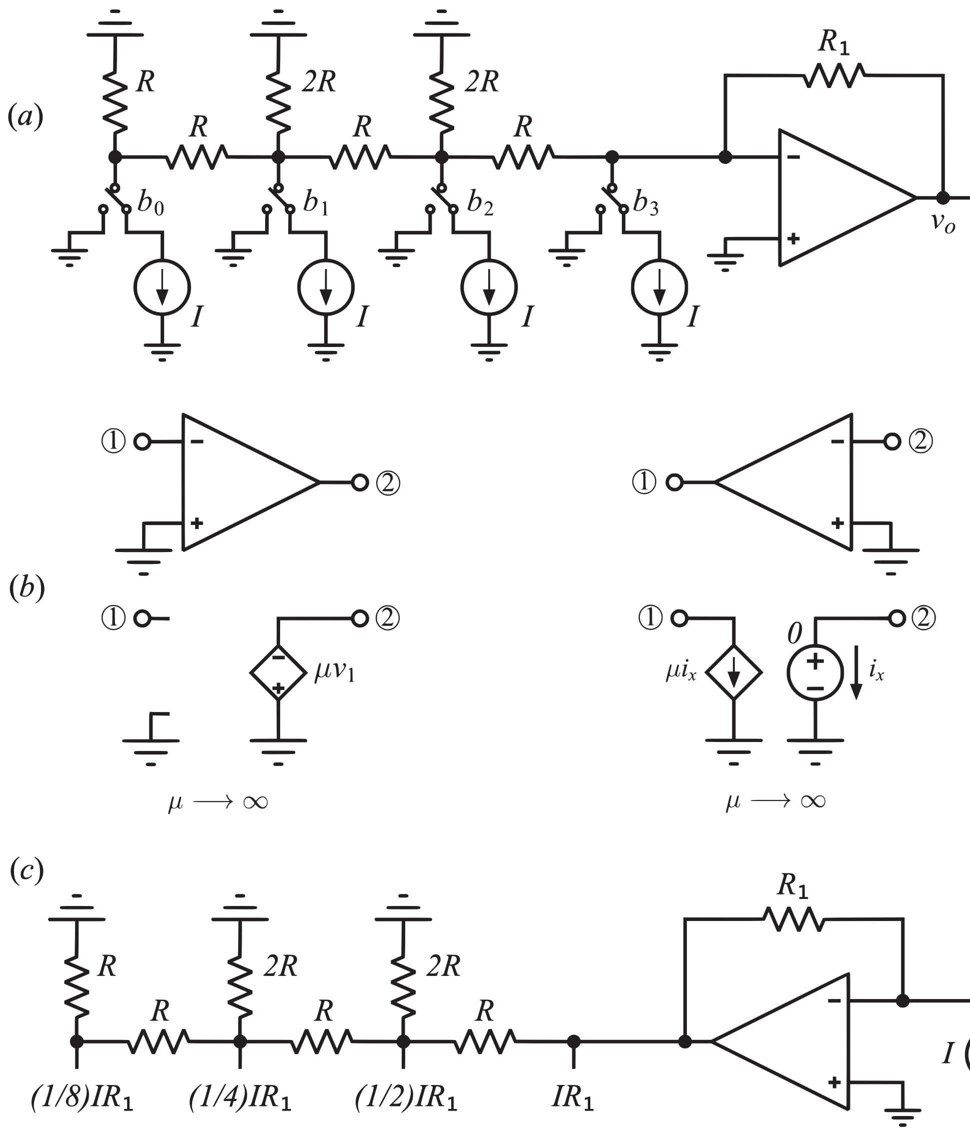


Figure 1.47 (a) An R-2R DAC. Transfer resistances from the four current sources to the output voltage v_o need to be found. (b) The opamp interpreted as a voltage-controlled voltage source. The transpose element corresponding to the opamp. $i_x \rightarrow 0$ as $\mu \rightarrow \infty$. (c) The inter-reciprocal (or adjoint) network can be used to obtain all the transfer functions in one shot.

A more efficient way is to use the inter-reciprocal network. To draw the adjoint, we need to replace every element with its inter-reciprocal cousin. The resistors, therefore, remain as they are in the original network. What do we do with the opamp? Remember that the opamp in the original network can be thought of as a VCVS with infinite gain, as shown in Figure 1.47b.[4] In the adjoint (Figure 1.47b), therefore, it must be replaced by a CCCS with infinite current gain, but with the controlling and controlled ports reversed. Again, thanks to negative feedback and infinite gain $i_x = 0$. We thus see that not only is the node potential 2 zero, but also $i_x = 0$. There is therefore a virtual short between 2 and ground. Consequently, the CCCS with infinite gain is also an ideal opamp. To draw the adjoint of the network of Figure 1.47a, therefore, the opamp needs to be "flipped" as shown in Figure 1.47c. By inter-reciprocity, the transfer resistances from the current sources to v_o in the original network can be obtained by exciting the "output" port of the inter-reciprocal network with a current I and determining the voltages developed across the current sources. It is seen (by inspection) that the transfer resistances are R_1, $(1/2)R_1, \dots, (1/8)R_1$. The DAC output is thus given by

$$v_o = IR_1\left(b_3 + \frac{1}{2}b_2 + \frac{1}{4}b_1 + \frac{1}{8}b_0\right). \tag{1.58}$$

1.10.1 Transfer Functions in a Biquadratic Section

Consider the network of Figure 1.48a. The opamps are ideal. The input source is v_s, and there are internal voltage sources $v_1, \dots, v_4$.[5] It is required to find the transfer functions from the internal sources to the output.

Straightforward analysis shows that

$$\frac{V_o(s)}{V_s(s)} = \frac{1}{D(s)} = \frac{1}{\frac{s^2}{\omega_o^2} + \frac{s}{Q\omega_o} + 1}$$

where $\omega_o^2 = 1/(R_1R_2C_1C_2)$ and $Q = \sqrt{C_2/C_1}R_3/\sqrt{R_1R_2}$. Using superposition to find the required transfer functions is too cumbersome. The inter-reciprocal network, shown in Figure 1.48b, comes in handy. The opamps are flipped, and the VCVS with a gain of −1 is replaced by a CCCS with the same gain but with the controlling and controlled ports flipped. The adjoint network is excited by a current source i_s. The transfer functions from i_s to $i_1, \dots, i_4$ in the adjoint are the same as those from $v_1, \dots, v_4$ to v_o in the original network. Furthermore, with respect to the adjoint, we note that:

a. The voltages across the resistors labeled R_1 are identical. Consequently, $i_1 = i_3$. Furthermore, $I_1(s)/I_s(s) = H(s)$.

[4] Negative feedback and infinite gain force the virtual short between nodes 1 and ground.
[5] As we will see in Chapter 7, these represent noise sources.

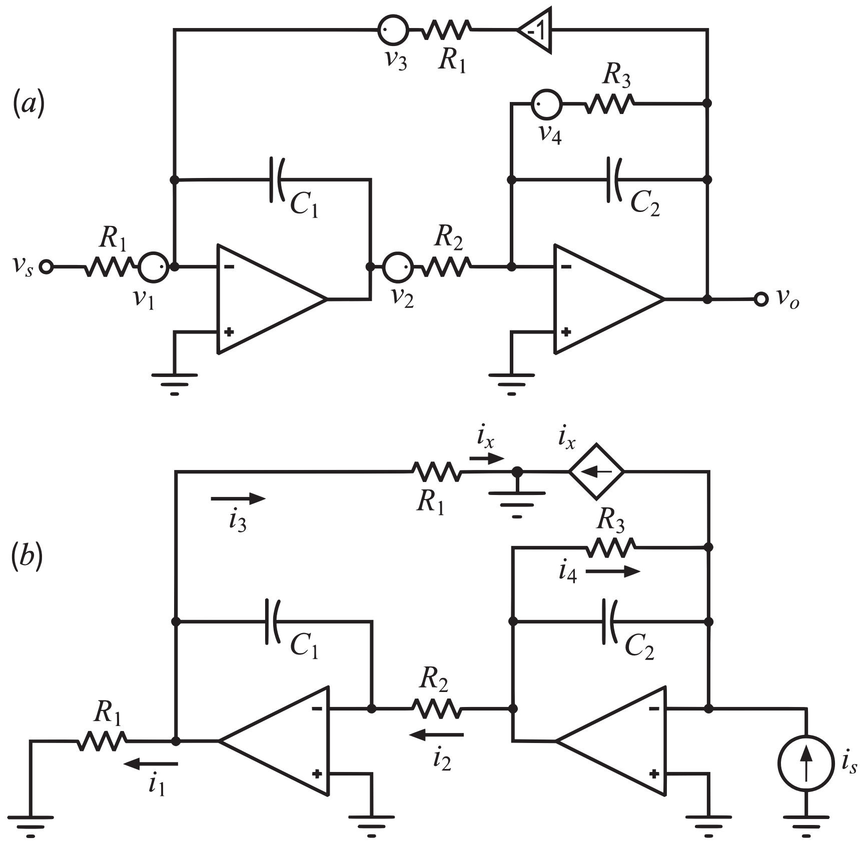


Figure 1.48 (a) A biquadratic section with several internal voltage sources. (b) The inter-reciprocal (or adjoint) network can be used to obtain all the transfer functions in one shot.

b. The voltages across the resistors labeled R_2 and R_3 are identical. As a result, $i_2R_2 = i_4R_3$.

c. $I_2(s) = -sC_1R_1I_1(s)$.

From the observations above, and using the properties of the inter-reciprocal network, we conclude that

$$\frac{V_o(s)}{V_1(s)} = \frac{I_1(s)}{I_s(s)} = H(s)$$
$$\frac{V_o(s)}{V_2(s)} = \frac{I_2(s)}{I_s(s)} = -sC_1R_1H(s)$$
$$\frac{V_o(s)}{V_3(s)} = \frac{I_3(s)}{I_s(s)} = H(s)$$
$$\frac{V_o(s)}{V_4(s)} = \frac{I_4(s)}{I_s(s)} = -sC_1R_1(R_2/R_3)H(s).$$

1.11 Transfer-Function Sensitivity

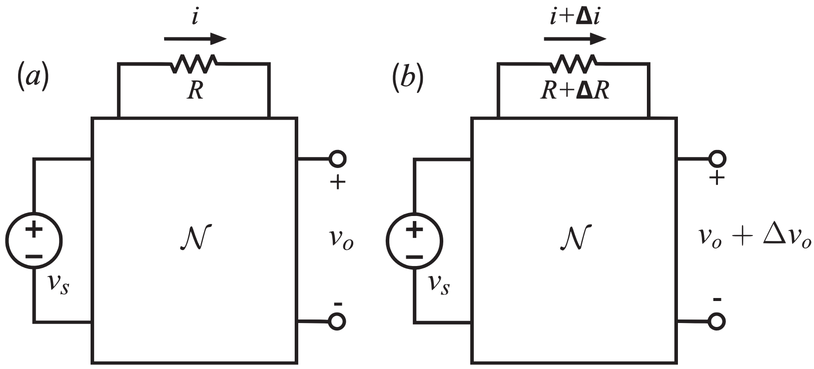


Figure 1.49 (a) Original network. (b) Network with a resistor changed from R to $R + \Delta R$.

It is often desired to find the change in the transfer function of a network due to small changes in the values of its elements. It turns out that the inter-reciprocal (or adjoint) network concept is very useful in such computations. To develop this, we first consider the change in the transfer function of a network $\mathcal{N}$ due to the change in a single resistor R. In the discussion that follows, we assume networks without memory, but the results also hold for Laplace-transformed quantities. The original network, shown in Figure 1.49a, is excited by v_s and results in an output $v_o = Hv_s$. Let i denote the current flowing through the resistor. Suppose R changes to $(R + \Delta R)$ as in Figure 1.49b. The resulting output is $(v_o + \Delta v_o)$ and the current is modified to $(i + \Delta i)$. The resistor $(R + \Delta R)$ can be split into a resistor R in series with ΔR.

Since the current flowing through ΔR is $(i + \Delta i)$, it can be replaced by a voltage source $\Delta v_1 = i\Delta R$ (neglecting the second-order term $\Delta i\Delta R$) as shown in Figure 1.50a.

Consider the network in Figure 1.50a, where a voltage source v_x is inserted in series with R. We denote the transfer function from v_s to v_1 by H_{s1}, while that from v_x to v_o is denoted by H_{1o}. Using this, we see that the output in Figure 1.50b can be expressed as

$$v_o + \Delta v_o = Hv_s + \underbrace{\frac{H_{s1}v_s}{R}}_{i} \Delta R H_{1o}. \tag{1.59}$$

The change in the transfer function H due to the change in the resistor R is thus seen to be

$$\Delta H = \frac{\Delta v_o}{v_s} = H_{s1}H_{1o}\frac{\Delta R}{R}. \tag{1.60}$$

The *sensitivity* of the transfer function H to the resistor R, formally defined as

$$S_R^H = \frac{\Delta H/H}{\Delta R/R}, \tag{1.61}$$

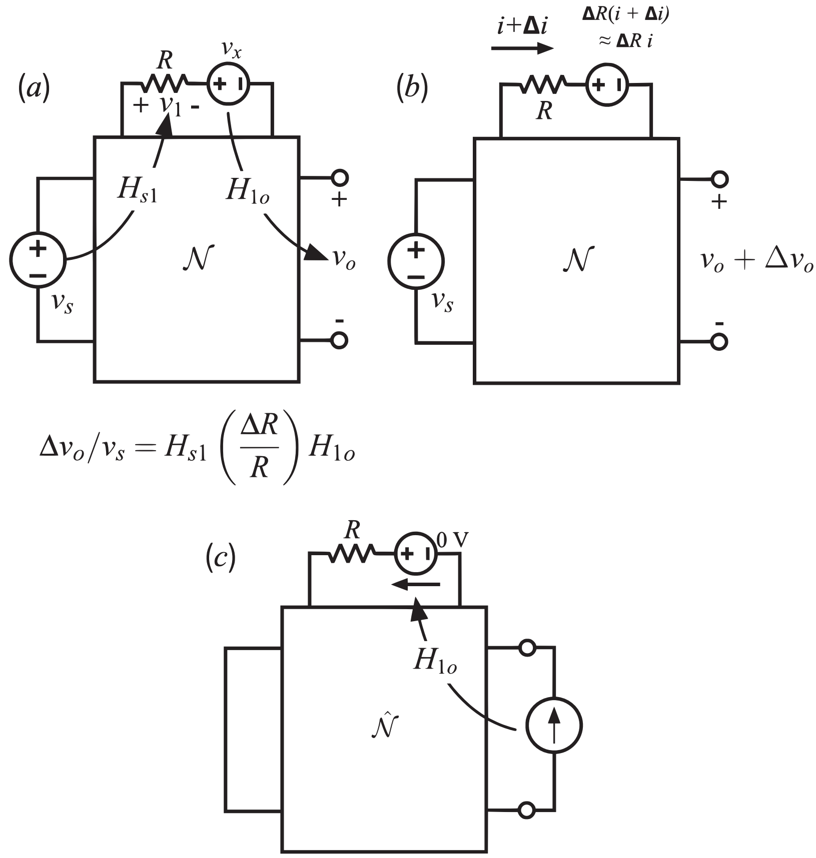


Figure 1.50 (a) Interpreting the effect of ΔR as a voltage source $I\Delta R$ in series with R. (b) Defining the transfer functions H_{s1} and H_{1o}. (c) Determining H_{1o} using the adjoint network.

is seen to be $S_R^H = H_{s1}H_{1o}/H$. How does one find H_{1o}? One way is to evaluate $\mathcal{N}$ again, now with v_x as the excitation. An alternative method is to form the adjoint network $\hat{\mathcal{N}}$, excite its output port with a current and determine the current flowing through v_x, as shown in Figure 1.50c. Regardless of the method chosen, we see that evaluation of the sensitivity to a single parameter needs the solution of *two* networks. This seems reasonable—to find the change in H due to a change in R, one would need to find H for the two different values of R and subtract the transfer functions. Furthermore, it seems that using the adjoint has no particular advantage in the sense that the analysis is no easier than the simple-minded method.

Now, what if one needed to determine the sensitivity to many (say m) parameters? The discussion above would have us believe that one would need to evaluate the network twice for each parameter change, thereby requiring a total of $2m$ analyses. Using the adjoint network concept, however, multi-parameter sensitivity can be computed by solving the network just *twice*, irrespective of the number of parameters. The idea is illustrated using Figure 1.51. Part (a) of the figure shows the network $\mathcal{N}$. Suppose

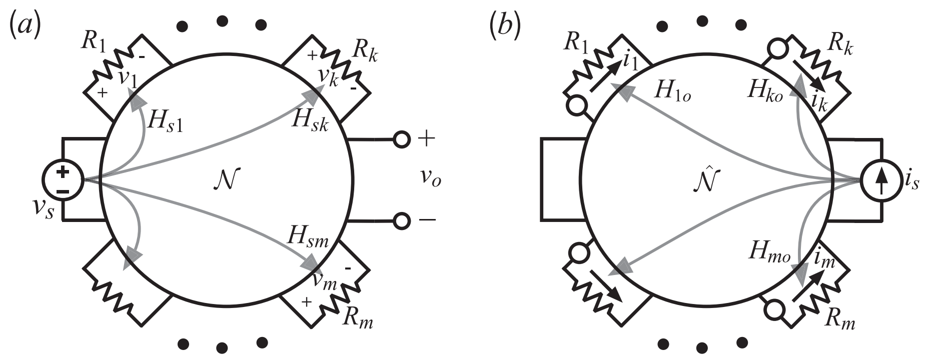


Figure 1.51 (a) Original network, the solution of which yields all the H_{sk}. (b) Solving the adjoint network yields all the H_{ko} in one shot.

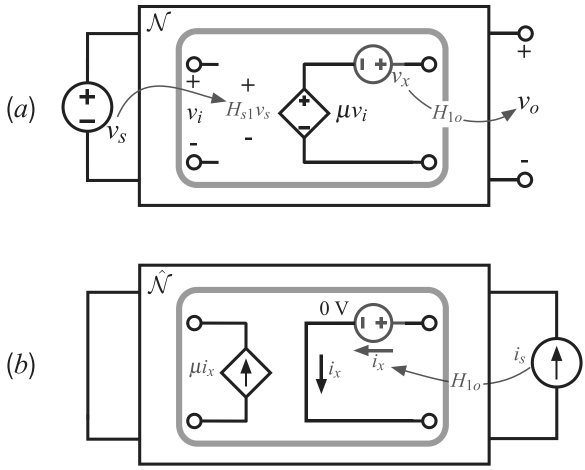
that we need to determine the change in the transfer function from v_s to v_o due to small changes in the values of the resistors $R_1, \ldots, R_m$. From the discussion in the single-parameter case, we see that

$$\Delta H = \frac{\Delta v_o}{v_s} = \sum_{k=1}^{m} H_{sk} H_{ko} \frac{\Delta R_k}{R_k}. \tag{1.62}$$

The H_{sk} represent the transfer functions from v_s to the voltages across the resistors R_k. Evaluating these transfer functions would need one evaluation of $\mathcal{N}$. The H_{ko} represent the transfer functions from voltage sources inserted in series with each of the resistors to the output. To evaluate these one at a time would need us to solve $\mathcal{N}$ m times. An efficient way of accomplishing this is to use the adjoint network $\hat{\mathcal{N}}$ as shown in Figure 1.51b. The input source is replaced by a short circuit. The output port is excited by a current i_s, and the currents through the voltage sources are determined to yield the H_{ko}. This needs $\hat{\mathcal{N}}$ to be evaluated only once. The H_{sk} and the H_{ko} found in the two steps above can then be used in (1.62) to yield ΔH.

The discussion above assumed that the resistors changed by small amounts; the extension of this analysis to controlled sources is straightforward and is illustrated for a VCVS in Figure 1.52. Part (a) of the figure shows a VCVS with gain μ. It is part of a larger network $\mathcal{N}$. To find the change in v_o due to a change in μ, the transfer function H_{s1} from v_s to the controlled voltage as well as H_{1o} (which is that from v_x to v_o) have to be found. The latter is determined from the adjoint network as shown in Figure 1.52b.

To summarize, therefore, the change in the transfer function due to m parameters can be determined by solving *two* networks once each—the original network $\mathcal{N}$ and its adjoint $\hat{\mathcal{N}}$.



$$\frac{\Delta v_o}{v_s} = H_{s1}H_{1o}\Delta\mu$$

Figure 1.52 (a) Original network. H_{s1} and H_{1o} need to be determined. (b) Determining H_{1o} using the adjoint network.

Example: Effect of Component Variations on a Series-Feedback Amplifier

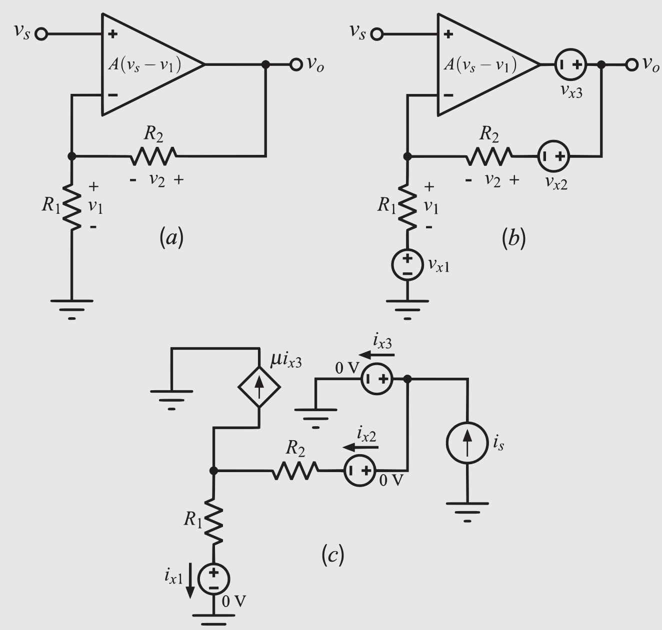


Figure 1.53 (a) Original network. (b) H_{s1} and H_{1o} need to be determined. (c) Determining H_{1o} using the adjoint network.

Consider the series-feedback amplifier shown in Figure 1.53a. It is desired to find the change in v_o due to changes in $A,\ R_1$, and R_2. The first step is to determine the H_{sk} from $v_o,\ v_1$, and v_2. Denoting

$$LG \equiv A\frac{R_1}{R_1 + R_2} \tag{1.63}$$

straightforward analysis yields

$$\begin{aligned} v_1 &= v_s\frac{LG}{(1+LG)} \\ v_2 &= v_s\frac{LG}{(1+LG)}\frac{R_2}{R_1} \\ v_o &= v_s\frac{LG}{(1+LG)}\left(1+\frac{R_2}{R_1}\right). \end{aligned} \tag{1.64}$$

The next step is to determine the H_{ko}. For this, voltage sources $v_{x1},\ v_{x2}$ and v_{x3} are inserted in series with $R_1,\ R_2$ and the output of the VCVS respectively, as shown in Figure 1.53b. To determine all the H_{ko} in one shot, the adjoint network is constructed. The result is shown in Figure 1.53c. The "output" port is excited by a current i_s, and the currents through the 0 V voltage sources are determined. Analysis of this network yields

$$\begin{aligned} i_{x1} &= -i_s\frac{LG}{(1+LG)}\frac{R_2}{R_1} \\ i_{x2} &= i_s\frac{LG}{(1+LG)} \\ i_{x3} &= i_s\frac{1}{(1+LG)}. \end{aligned} \tag{1.65}$$

Using the equations above along with (1.62), we obtain

$$\begin{aligned} \frac{\Delta v_o}{v_s} &= \frac{1}{(1+LG)^2}\Delta A - \frac{LG^2}{(1+LG)^2}\frac{R_2}{R_1^2}\Delta R_1 \\ &\quad + \frac{LG^2}{(1+LG)^2}\frac{\Delta R_2}{R_1}. \end{aligned} \tag{1.66}$$

The equation above is easily verified by differentiating the expression for the gain of the amplifier of Figure 1.53a. As $A \to \infty$, (1.66) reduces to

$$\frac{\Delta v_o}{v_s} \to -\frac{R_2}{R_1^2}\Delta R_1 + \frac{\Delta R_2}{R_1} \tag{1.67}$$

which makes intuitive sense.

Chapter Summary

- **Kirchhoff's Current and Voltage Laws, and Tellegen's Theorem:** Circuit theory rests on Ohm's law, Kirchhoff's Current Law (KCL), and Kirchhoff's Voltage Law (KVL), which together provide the framework for analyzing any linear electrical network. Tellegen's theorem is a consequence of KCL/KVL.
- **Modified Nodal Analysis (MNA) Formulation of Circuit Equations:** This is an extension of nodal analysis that accommodates components whose current cannot be determined from the branch voltage (e.g., voltage sources, inductors). Using MNA "stamps" of components, the circuit equations can be written by inspection. The MNA formulation is not only the cornerstone of circuit simulators but also immensely useful in understanding reciprocity and inter-reciprocity.
- **Bilinear Form of a Transfer Function and the Extra-Element Theorem:** Any transfer function in a linear network is a bilinear form with respect to a component value. This observation is not only useful in noise analysis (see Chapter 2) but also facilitates the derivation of the extra-element theorem.
- **Reciprocity, Inter-reciprocity, and Adjoint Networks:** These concepts are immensely useful when one is required to determine the transfer functions from multiple inputs to a single output. Such a scenario is routinely encountered in noise and sensitivity analysis. The MNA formulation of circuit equations is a way of understanding (inter)reciprocity from the ground up.

Problems

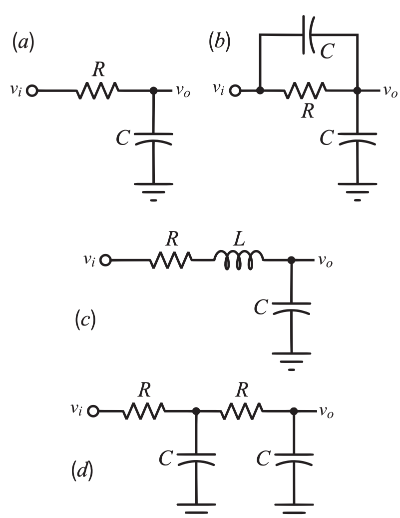


Figure 1.54

Problem 1. *Figure 1.54 shows four networks. v_i is a unit step.*

- *For each network, determine $v_o(0+)$, $v_o(\infty)$, and $\frac{dv_o(t)}{dt}$ at $t = 0+$.*
- *For the networks in Figure 1.54c,d determine $\frac{d^2v_o(t)}{dt^2}$ at $t = 0+$.*

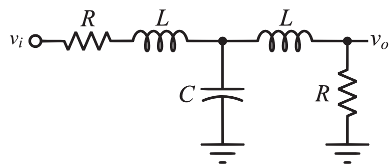


Figure 1.55

Problem 2. *In Figure 1.55, v_i is a unit step. Determine $\frac{d^3v_o(t)}{dt^3}$ at $t = 0+$.*

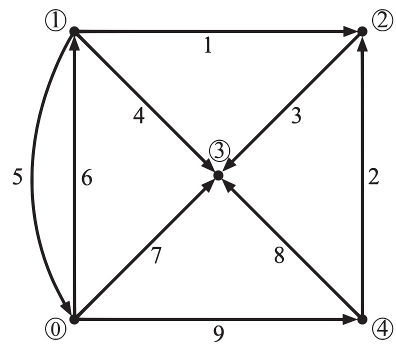


Figure 1.56

Problem 3. *Figure 1.56 shows the directed graph of a network, with the nodes and branches marked. Choose node 0 to be the reference (datum) and determine the incidence matrix of this graph.*

Problem 4. *Consider two topologically identical networks $\mathcal{N}$ and $\hat{\mathcal{N}}$. We denote the branch currents and branch voltages in these networks (as a function of time) by $v_k(t), i_k(t)$ and $\hat{v}_k(t), \hat{i}_k(t)$ respectively. Further, let t_1 and t_2 be two different time instants. Determine*

a. $\sum_k v_k(t_1)i_k(t_2)$

b. $\sum_k v_k(t_1)\hat{i}_k(t_2)$

c. $\sum_k \hat{v}_k(t_1)\left.\frac{di_k}{dt}\right|_{t=t_2}$

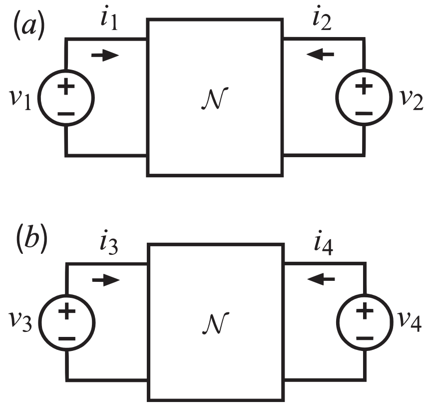


Figure 1.57

Problem 5. *$\mathcal{N}$ consists of resistors only. We perform the following two experiments indicated in Figure 1.57:*

Experiment 1. Drive $\mathcal{N}$ with two voltage sources v_1 and v_2 and measure the resulting port currents i_1 and i_2 respectively.

Experiment 2. Drive $\mathcal{N}$ with two voltage sources v_3 and v_4 and measure the resulting port currents i_3 and i_4 respectively.

Prove that the two sets of measurements are related as follows:

$$v_1 i_3 + v_2 i_4 = v_3 i_1 + v_4 i_2$$

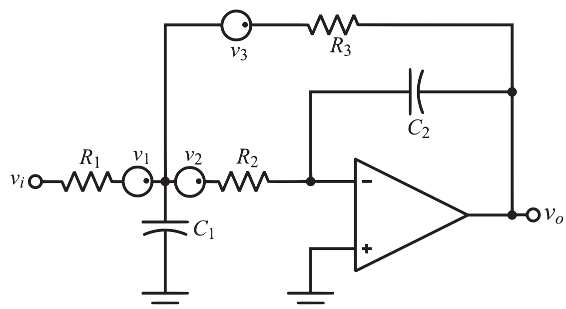


Figure 1.58

Problem 6. *In the network of Figure 1.58, the opamp is ideal. Determine the transfer functions from v_i, v_1, v_2, v_3 to v_o. The positive terminals of the voltage sources are marked with a dot.*

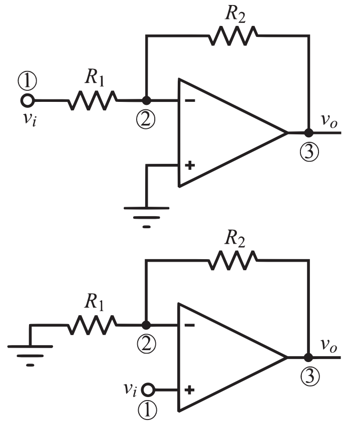


Figure 1.59

Problem 7. *In the networks of Figure 1.59, the opamps are ideal. Write the MNA equations for these networks. The nodes are numbered.*

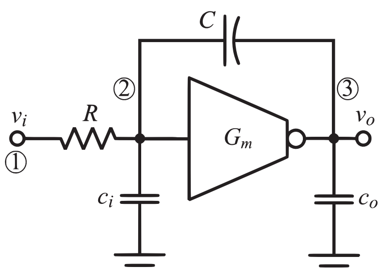


Figure 1.60

Problem 8. *Write the MNA equations for the active-RC integrator of Figure 1.60. The nodes are numbered. What is the input–output transfer function as $G_m \to \infty$?*

Problem 9. *Two integrators of the kind shown in Figure 1.60 are cascaded. Write the MNA equations for the cascade. You might want to use the results of the previous problem to reduce the effort involved.*

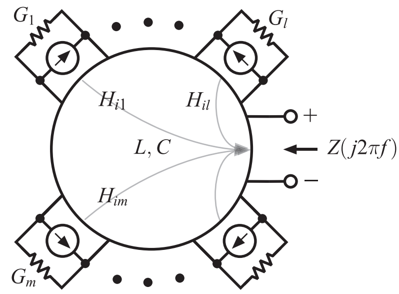


Figure 1.61

Problem 10. *Figure 1.61 shows a network that consists of only resistors, inductors and capacitors. All the resistors (m in total) are pulled out of the network as shown and current sources placed in parallel with them. Denote the transfer function from the lth current source to the output by $H_{il}(j2\pi f)$. If $Z(j2\pi f)$ denotes the impedance looking into the network (with all current sources turned off), show that*

$$\sum_{l=1}^{m} G_l |H_{il}(j2\pi f)|^2 = Re\{Z(j2\pi f)\},$$

where G_l denotes the lth conductance.

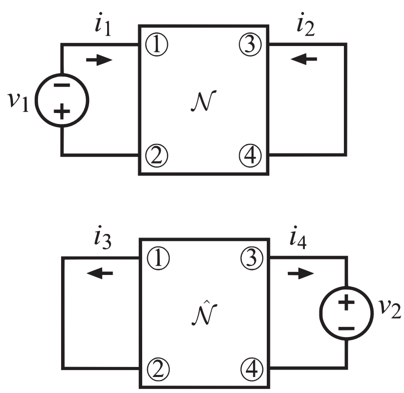


Figure 1.62

Problem 11. *In Figure 1.62, $\hat{\mathcal{N}}$ is the adjoint network corresponding to $\mathcal{N}$. Determine*

$$\frac{v_1 i_3}{v_2 i_2} + 2.$$

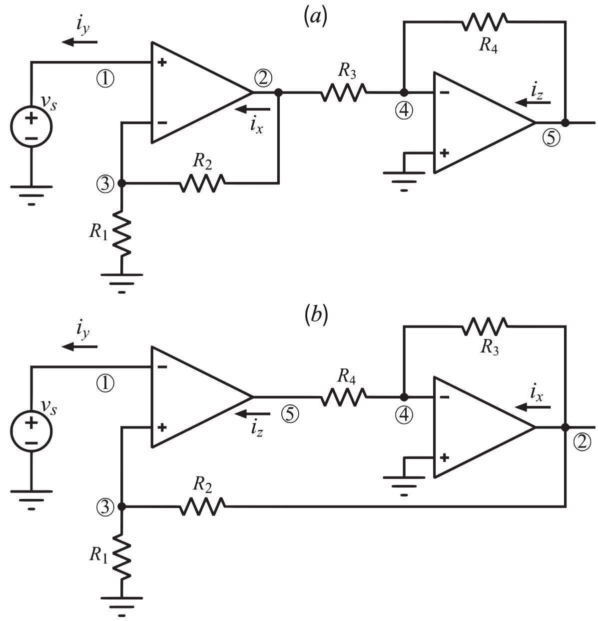


Figure 1.63

Problem 12. *As we have seen, the gain from v_s to v_5 is identical in both the circuits of Figure 1.63 if the opamps are ideal. Calculate the gain of the circuits if the gain of the opamps is not infinite, but a value $A \gg 1$. Repeat the exercise if the opamp's transfer function is modeled as $A(s) = \frac{2\pi f_u}{s}$.*

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