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Chapter 1
Introducing Geometry and Geometry Proofs

In this chapter, you get started with some basics about geometry and shapes, a couple points about deductive logic, and a few introductory comments about the structure of geometry proofs. Time to get started!

What Is Geometry?

What is geometry?! C'mon, everyone knows what geometry is, right? *Geometry* is the study of shapes: circles, triangles, rectangles, pyramids, and so on. Shapes are all around you. The desk or table where you're reading this book has a shape. You can probably see a window from where you are, and it's probably a rectangle. The pages of this book are also rectangles. Your pen or pencil is roughly a cylinder (or maybe a right hexagonal prism — see Part 5 for more on solid figures). Your shirt may have circular buttons. The bricks of a brick house are right rectangular prisms. Shapes are ubiquitous — in our world, anyway.

For the philosophically inclined, here's an exercise that goes *way* beyond the scope of this book: Try to imagine a world — some sort of different universe — where there aren't various objects with different shapes. (If you're into this sort of thing, check out *Philosophy For Dummies*.)

Making the Right Assumptions

Okay, so geometry is the study of shapes. And how can you tell one shape from another? From the way it looks, of course. But — this may seem a bit bizarre — when you're studying geometry, you're sort of *not* supposed to rely on the way shapes look. The point of this strange treatment of geometric figures is to prohibit you from claiming that something is true about a figure merely because it looks true, and to force you, instead, to *prove* that it's true by airtight, mathematical logic.

When you're working with shapes in any other area of math, or in science, or in, say, architecture or design, paying attention to the way shapes look is very important: their proportions, their angles, their orientation, how steep their sides are, and so on. Only in a geometry course are you supposed to ignore to some degree the appearance of the shapes you study. (I say "to some degree" because, in reality, even in a geometry course — or when using this book — it's still quite useful most of the time to pay attention to the appearance of shapes.)

WARNING

When you look at a diagram in this or any geometry book, **you *cannot* assume any of the following just from the appearance of the figure**.

- **Right angles:** Just because an angle looks like an exact $90°$ angle, that doesn't necessarily mean it is one.
- **Congruent angles:** Just because two angles look the same size, that doesn't mean they really are. (As you probably know, *congruent* [symbolized by $\cong$] is a fancy word for "equal" or "same size.")
- **Congruent segments:** Just like with angles, you can't assume segments are the same length just because they appear to be.
- **Relative sizes of segments and angles:** Just because, say, one segment is drawn to look longer than another in some diagram, it doesn't follow that the segment really is longer.

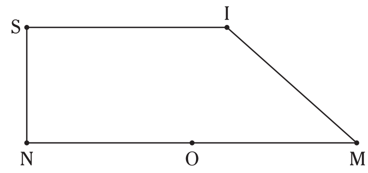
Sometimes size relationships are marked on the diagram. For instance, a small L-shaped mark in a corner means that you have a right angle. Tick marks can indicate congruent parts. Basically, if the tick marks match, you know the segments or angles are the same size.

You can assume pretty much anything not on this list; for example, if a line looks straight, it really is straight.

Before doing the following problems, you may want to peek ahead to Chapters 4 and 6 if you've forgotten or don't know the names of various triangles and quadrilaterals.

EXAMPLE

Q. What can you assume and what can't you assume about *SIMON*?



A. You *can* assume that

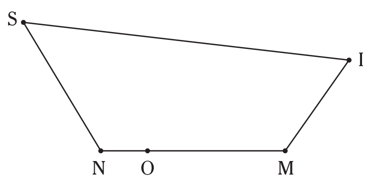
- $\overline{MN}$ (line segment *MN*) is straight; in other words, there's no bend at point *O*.

  Another way of saying the same thing is that $\angle MON$ is a *straight angle* or a 180° angle.
- $\overline{NS}$, $\overline{SI}$, and $\overline{IM}$ are also straight as opposed to curvy.
- Therefore, *SIMON* is a quadrilateral because it has four straight sides.

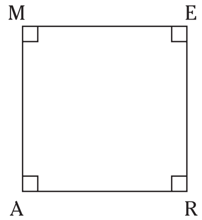
  (If you couldn't assume that $\overline{MN}$ is straight, there could actually be a bend at point *O* and then *SIMON* would be a pentagon, but that's not possible.)

  That's about it for what you can assume. If this figure were anywhere else other than a geometry book, you could safely assume all sorts of other things — including that *SIMON* is a trapezoid. But this *is* a geometry book, so you *can't* assume that. You also *can't* assume that
- $\angle S$ and $\angle N$ are right angles.
- $\angle I$ is an obtuse angle (an angle greater than 90°).
- $\angle M$ is an acute angle (an angle less than 90°).
- $\angle I$ is greater than $\angle M$ or $\angle S$ or $\angle N$, and ditto for the relative sizes of other angles.
- $\overline{NS}$ is shorter than $\overline{SI}$ or $\overline{MN}$, and ditto for the relative lengths of the other segments.
- *O* is the midpoint of $\overline{MN}$.
- $\overline{SI}$ is parallel to $\overline{MN}$.

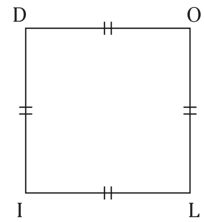
The "real" *SIMON* — weird as it seems — could actually look like this:



1 What type of quadrilateral is *AMER*? ***Note:*** See Chapter 6 for types of quadrilaterals.



2 What type of quadrilateral is *IDOL*?



3 Use the figure to answer the following questions (Chapter 4 can fill you in on triangles):

a. Can you assume that the triangles are congruent?

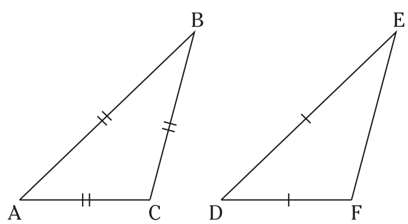
b. Can you conclude that ΔABC is acute? Obtuse? Right? Isosceles (with at least two equal sides)? Equilateral (with three equal sides)?

c. Can you conclude that ΔDEF is acute? Obtuse? Right? Isosceles? Equilateral?

d. What can you conclude about the length of $\overline{EF}$?

e. Might $\angle D$ be a right angle?

f. Might $\angle F$ be a right angle?



4 Can you assume or conclude

a. $\Delta ABC \cong \Delta WXY$?

b. $\Delta ABD \cong \Delta CBD$?

c. $\Delta ABD \cong \Delta WXZ$?

d. ΔABC is isosceles?

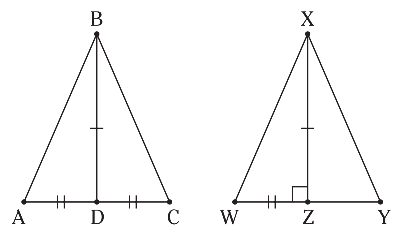
e. D is the midpoint of $\overline{AC}$?

f. Z is the midpoint of $\overline{WY}$?

g. $\overline{BD}$ is an altitude (height) of ΔABC?

h. $\angle ADB$ is supplementary to $\angle CDB$ (that is: $\angle ADB + \angle CDB = 180°$)?

i. ΔXYZ is a right triangle?



If-Then Logic: If You Bought This Book, Then You Must Love Geometry!

Geometry *theorems* (and their cousins, *postulates*) are basically statements of geometrical truth, like "All radii of a circle are congruent." As you can see in this section and in the rest of the book, theorems (and postulates) are the building blocks of proofs. (I may get hauled over by the geometry police for saying this, but if I were you, I'd just glom theorems and postulates together into a single group because, for the purposes of doing proofs, they work the same way. Whenever I refer to theorems, you can safely read it as "theorems and postulates.")

Geometry theorems can all be expressed in the form, "*If* blah blah blah, *then* blah blah blah," like "If two angles are right angles, then they are congruent" (although theorems are often written in some shorter way, like "All right angles are congruent"). You may want to flip through the book looking for theorem icons to get a feel for what theorems look like.

WARNING

An important thing to note here is that **the reverse of a theorem is not necessarily true**. For example, the statement, "If two angles are congruent, then they are right angles," is false. When a theorem does work in both directions, you get two separate theorems, one the reverse of the other.

The fact that theorems are not generally reversible should come as no surprise. Many ordinary statements in *if-then* form are, like theorems, not reversible: "If something's a ship, then it's a boat" is true, but "If something's a boat, then it's a ship" is false, right? (It might be a canoe.)

Geometry definitions (like all definitions), however, are reversible. Consider the definition of *perpendicular:* perpendicular lines are lines that intersect at right angles. Both if-then statements are true: 1) "If lines are perpendicular, then they intersect at right angles," and 2) "If lines intersect at right angles, then they are perpendicular." When doing proofs, you'll have the occasion to use both forms of many definitions.

EXAMPLE

Q. Read through some theorems.

a. Give an example of a theorem that's not reversible and explain why the reverse is false.

b. Give an example of a theorem whose reverse is another true theorem.

A. A number of responses work, but here's how you could answer:

a. "If two angles are vertical angles, then they are congruent." The reverse of this theorem is obviously false. Just because two angles are the same size, it does not follow that they must be vertical angles. (When two lines intersect and form an X, vertical angles are the angles straight across from each other — turn to Chapter 2 for more info.)

b. Two of the most important geometry theorems are a reversible pair: "If two sides of a triangle are congruent, then the angles opposite those sides are congruent" and "If two angles of a triangle are congruent, then the sides opposite those angles are congruent." (For more on these isosceles triangle theorems, check out Chapter 5.)

5 Give two examples of theorems that are not reversible and explain why the reverse of each is false. ***Hint:*** Flip through this book or your geometry textbook and look at various theorems. Try reversing them and ask yourself whether they still work.

6 Give two examples of theorems that work in both directions. ***Hint:*** See the hint for question 5.

What's a Geometry Proof?

Many students find two-column geometry proofs difficult, but they're really no big deal once you get the hang of them. Basically, they're just arguments like the following, in which you brilliantly establish that your Labradoodle, Fifi, will not lay any eggs on the Fourth of July:

1. Fifi is a Labradoodle.
2. Therefore, Fifi is a dog, because all Labradoodles are dogs.
3. Therefore, Fifi is a mammal, because all dogs are mammals.
4. Therefore, Fifi will never lay any eggs, because mammals don't lay eggs (okay, okay . . . except for platypuses and spiny anteaters, for you monotreme-loving nitpickers out there).
5. Therefore, Fifi will not lay any eggs on the Fourth of July, because if she will never lay any eggs, she can't lay eggs on the Fourth of July.

In a nutshell: Labradoodle $\rightarrow$ dog $\rightarrow$ mammal $\rightarrow$ no eggs $\rightarrow$ no eggs on July 4. It's sort of a domino effect. Each statement knocks over the next till you get to your final conclusion.

EXAMPLE

Check out Figure 1-1 to see what this argument or proof looks like in the standard two-column geometry proof format.

Given: Fifi is a Labradoodle.

Prove: Fifi will not lay eggs on the Fourth of July.

Statements (or Conclusions) These are the specific claims you make.	**Reasons (or Justifications)** These are the general rules you use to justify your claims. If after each claim you made, I said, "How do you know?" your response to me goes in this column.
I claim…	**How do I know?**
1) Fifi is a Labradoodle.	1) Because it was given as a fact.
2) Fifi is a dog.	2) Because all Labradoodles are dogs.
3) Fifi is a mammal.	3) Because all dogs are mammals.
4) Fifi doesn't lay eggs.	4) Because mammals don't lay eggs.
5) Fifi will not lay eggs on the Fourth of July.	5) Because something that doesn't lay eggs can't lay eggs on the Fourth of July.

FIGURE 1-1: A standard two-column proof listing statements and reasons.

TIP

Note that the left-hand column contains *specific* facts (about one particular dog, Fifi), **while the right-hand column contains *general* principles** (about dogs in general or mammals in general). This format is true of all geometry proofs.

Now look at the very same proof in Figure 1-2; this time, the reasons appear in *if-then* form. When reasons are written this way, you can see how the chain of logic flows.

REMEMBER

In a two-column proof, the idea or ideas in the *if* part of each reason must come from the statement column somewhere *above* the reason; and the single idea in the *then* part of the reason must match the idea in the statement on *the same line* as the reason. This incredibly important flow-of-logic structure is shown with arrows in the following proof.

Statements (or Conclusions)	**Reasons (or Justifications)**
1) Fifi is a Labradoodle.	1) Given.
2) Fifi is a dog.	2) If something is a Labradoodle, then it is a dog.
3) Fifi is a mammal.	3) If something is a dog, then it is a mammal.
4) Fifi doesn't lay eggs.	4) If something is a mammal, then it doesn't lay eggs.
5) Fifi will not lay eggs on the Fourth of July.	5) If something doesn't lay eggs, then it won't lay eggs on the Fourth of July.

FIGURE 1-2: A proof with the reasons written in if-then form.

In the preceding proof, each *if* clause uses only a single idea from the statement column. However, as you can see in the following practice problem, you often have to use more than one idea from the statement column in an *if* clause.

7 In the following facetious and somewhat fishy proof, fill in the missing reasons in *if-then* form and show the flow of logic as I illustrate in Figure 1-2.

Given: You forgot to set your alarm last night.

You've already been late for school twice this term.

Prove: You will get a detention at school today.

Note: To complete this "proof," you need to know the school's late policy: A student who is late for school three times in one term will be given a detention.

Statements (or Conclusions)	**Reasons (or Justifications)**
1) I forgot to set my alarm last night.	1) Given.
2) I will wake up late.	2)
3) I will miss the bus.	3)
4) I will be late for school.	4)
5) I've already been late for school twice this term.	5) Given.
6) This will be the third time this term I'll have been late.	6)
7) I'll get a detention at school today.	7)

Solutions

1. *AMER* looks like a square, but you can't conclude that because you can't assume the sides are equal. You do know, however, that the figure is a rectangle because it has four sides and four right angles.

2. *IDOL* also looks like a square, and again, like with question 1, you can't conclude that, but this time you can't conclude that because you can't assume that the angles are right angles. But because you do know that *IDOL* has four equal sides, you know that it's a rhombus.

3. Here are the answers (flip to Chapter 4 if you need to go over triangle classification):

   a. No. The triangles look congruent, but you're not allowed to assume that.

   b. The tick marks tell you that $\triangle ABC$ is equilateral. It is, therefore, an acute triangle and an isosceles triangle. It is neither a right triangle nor an obtuse triangle.

   c. The tick marks tell you that $\triangle DEF$ is isosceles and that, therefore, it is not scalene. That's all you can conclude. It may or may not be any of the other types of triangles.

   d. Nothing. $\overline{EF}$ could be the longest side of the triangle, or the shortest, or equal to the other two sides. And it may or may not have the same length as $\overline{BC}$.

   e. Yes. $\angle D$ might be a right angle, though you can't assume that it is.

   f. No. (If you got this question right, give yourself a pat on the back.) If $\angle F$ were a right angle, $\triangle DEF$ would be a right triangle with $\overline{DE}$ its hypotenuse. But $\overline{DE}$ is the same length as $\overline{DF}$, and the hypotenuse of a right triangle has to be the triangle's longest side.

4. Here are the answers:

   a. No. The triangles might not be congruent in any number of ways. For example, you know nothing about the length of $\overline{ZY}$, and if $\overline{ZY}$ were, say, a mile long, the triangles would obviously not be congruent.

   b. No. The triangles would be congruent only if $\angle ADB$ and $\angle CDB$ were right angles, but you don't know that. Point B is free to move left or right, changing the measures of $\angle ADB$ and $\angle CDB$.

   c. No. You don't know that $\angle ADB$ is a right angle.

   d. No. The figure *looks* isosceles, but you're not allowed to assume that $\overline{AB} \cong \overline{CB}$.

   e. Yes. The tick marks show it.

   f. No. Like with part *a*, you know nothing about the length of $\overline{ZY}$.

   g. No. You can't assume that $\overline{BD} \perp \overline{AC}$ (the upside-down *T* means "is perpendicular to").

   h. Yes. You *can* assume that $\overline{AC}$ is straight and that $\angle ADC$ is $180°$; therefore, $\angle ADB$ and $\angle CDB$ must add up to $180°$.

   i. Yes. $\angle WZY$ is $180°$ and $\angle WZX$ is $90°$, so $\angle YZX$ must also be $90°$.

5. Answers vary. One example is "If angles are complementary to the same angle, then they're congruent." The reverse of this is false because many angles, like obtuse angles, do not have complements (obtuse angles are already bigger than $90°$, so you can't add another angle to them to get a right angle).

6 Answers vary. Any of the parallel line theorems in Chapter 2 makes a good answer. For example, "If two parallel lines are cut by a transversal, then alternate interior angles are congruent." Both this theorem and its reverse are true. To wit (in abbreviated form): "If lines are parallel, then alternate interior angles are congruent," and "If alternate interior angles are congruent, then lines are parallel."

7

Statements (or Conclusions)	**Reasons (or Justifications)**
1) I forgot to set my alarm last night.	1) Given.
2) I will wake up late.	2) If someone forgets to set his alarm, then he will wake up late.
3) I will miss the bus.	3) If someone wakes up late, then he will miss the bus.
4) I will be late for school.	4) If someone misses the bus, then he will be late for school.
5) I've already been late for school twice this term.	5) Given.
6) This will be the third time this term I'll have been late.	6) If someone has already been late for school twice in one term and then he is late another time in the same term, then he will have been late three times in one term. ***Note:*** This reason basically amounts to saying that 2 + 1 = 3. Many steps in geometry proofs, like this step, are about incredibly obvious, *well-duh* things.
7) I'll get a detention at school today.	7) If someone is late for school three times in one term, then he will get a detention.

I hope it goes without saying that this is *not* an airtight, mathematical proof.