

1

Introductory Concepts

1.1 Introduction

This chapter is a review for most since the material is covered in undergraduate courses in the analysis of electromechanical devices [1]. The material is presented to start everyone with the same background. The chapter begins with coupled circuits (transformers) where the phasor equivalent circuit is established. Since phasors are not always taught the same, they are covered briefly in Appendix B to make sure everyone understands the concept of phasors as used in this text. Although we will give several approaches for the calculation of torque of electric machines; Section 1.1-3 sets forth a method of calculating force and torque that is generally taught at the undergraduate level.

Some instructors may choose to skip some material and/or select topics that were not covered in undergraduate courses at their school. As mentioned, the material will be a review for most and can be covered rather fast. On the other hand, Chapter 2 dives into machine analysis that contains new material and can be taught at a much slower pace.

1.2 Stationary Magnetically Coupled Circuits

Magnetically coupled electric circuits are central to the operation of transformers and electromechanical motion devices. In the case of transformers, stationary circuits are magnetically coupled for the purpose of changing the ac voltage and current levels. The two windings shown in Fig. 1.2-1 consist of turns N_1 and N_2 , and they are wound on a common core, which is a ferromagnetic material with a permeability large relative to that of air. The magnetic core is illustrated in two dimensions.

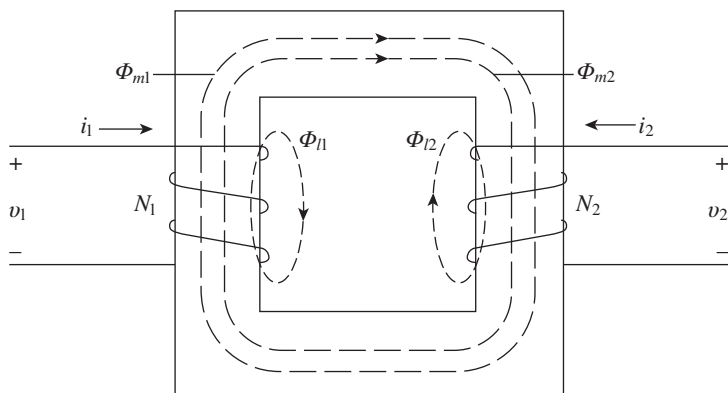


Figure 1.2-1 Magnetically coupled circuits.

The flux produced by each winding can be separated into two components: a leakage component denoted by the subscript l and a magnetizing component denoted by the subscript m . Each of these components is depicted by a single streamline with the positive direction determined by applying the right-hand rule to the directions of current flow in the winding. The leakage flux associated with a given winding links only that winding, whereas the magnetizing flux, whether it is due to current in winding 1 or winding 2, links both windings.

The flux linking of each winding may be expressed as

$$\Phi_1 = \Phi_{l1} + \Phi_{m1} + \Phi_{m2} \quad (1.2-1)$$

$$\Phi_2 = \Phi_{l2} + \Phi_{m2} + \Phi_{m1} \quad (1.2-2)$$

The leakage flux Φ_{l1} is produced by current flowing in winding 1, and it links only the turns of winding 1. Likewise, the leakage flux Φ_{l2} is produced by current flowing in winding 2, and it links only the turns of winding 2. The flux Φ_{m1} is produced by current flowing in winding 1, and it links all turns of windings 1 and 2. Similarly, the magnetizing flux Φ_{m2} is produced by current flowing in winding 2, and it also links all turns of windings 1 and 2. Both Φ_{m1} and Φ_{m2} are called *magnetizing fluxes*. With the selected positive directions of current flow and the manner in which the windings are wound, the magnetizing flux produced by positive current flowing in one winding can add to or subtract from the magnetizing flux produced by positive current flowing in the other winding. Thus, the mutual inductance can be positive or negative. In Fig. 1.2-1, it is positive.

It is appropriate to point out that this is an idealization of the actual magnetic system. It seems logical that all of the leakage flux will not link all the turns of the winding producing it; hence, Φ_{l1} and Φ_{l2} are “equivalent” leakage fluxes.

Similarly, all of the magnetizing fluxes of one winding may not link all of the turns of the other winding.

The voltage equations may be expressed as

$$v_1 = r_1 i_1 + \frac{d\lambda_1}{dt} \quad (1.2-3)$$

$$v_2 = r_2 i_2 + \frac{d\lambda_2}{dt} \quad (1.2-4)$$

In matrix form,

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} r_1 & 0 \\ 0 & r_2 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} + \frac{d}{dt} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} \quad (1.2-5)$$

The resistances r_1 and r_2 and the flux linkages λ_1 and λ_2 are related to windings 1 and 2, respectively. Since it is assumed that Φ_1 links the equivalent turns of winding 1 (N_1) and Φ_2 links the equivalent turns of winding 2 (N_2), the flux linkages may be written as

$$\lambda_1 = N_1 \Phi_1 \quad (1.2-6)$$

$$\lambda_2 = N_2 \Phi_2 \quad (1.2-7)$$

where Φ_1 and Φ_2 are given by (1.2-1) and (1.2-2), respectively.

If we assume that the magnetic system is magnetically linear (i.e., core losses and saturation are neglected), we may apply Ohm's law for magnetic circuits to express the fluxes. Thus, the fluxes may be written as

$$\Phi_{lk} = \frac{N_k i_k}{\mathfrak{R}_{lk}} \quad (1.2-8)$$

$$\Phi_{mk} = \frac{N_k i_k}{\mathfrak{R}_m} \quad (1.2-9)$$

where $k = 1$ or 2 and $\mathfrak{R}_{l1}$ and $\mathfrak{R}_{l2}$ are the reluctances of the leakage paths, and $\mathfrak{R}_m$ is the reluctance of the path of magnetizing fluxes. Typically, the reluctances associated with leakage paths are much larger than the reluctance of the magnetizing path. The reluctance associated with an individual leakage path is difficult to determine exactly, and it is usually approximated from test data or by using the computer to solve the field equations numerically. On the other hand, the reluctance of the magnetizing path of the core shown in Fig. 1.2-1 may be computed with sufficient accuracy.

For the iron

$$\mathfrak{R}_i = \frac{l_i}{\mu_r \mu_0 A_i} \quad (1.2-10)$$

where l_i is the length of the path in iron, μ_r is the relative permeability of iron, μ_0 is the permeability of free space, and A_i is the cross-sectional area of the flux

in the iron. In electromechanical devices, we will find that the magnetizing flux must transverse air gaps and

$$\mathfrak{R}_m = \mathfrak{R}_i + \mathfrak{R}_g \quad (1.2-11)$$

Substituting (1.2-8) and (1.2-9) into (1.2-1) and (1.2-2) yields

$$\Phi_1 = \frac{N_1 i_1}{\mathfrak{R}_{l1}} + \frac{N_1 i_1}{\mathfrak{R}_m} + \frac{N_2 i_2}{\mathfrak{R}_m} \quad (1.2-12)$$

$$\Phi_2 = \frac{N_2 i_2}{\mathfrak{R}_{l2}} + \frac{N_2 i_2}{\mathfrak{R}_m} + \frac{N_1 i_1}{\mathfrak{R}_m} \quad (1.2-13)$$

Substituting (1.2-12) and (1.2-13) into (1.2-6) and (1.2-7) yields

$$\lambda_1 = \frac{N_1^2}{\mathfrak{R}_{l1}} i_1 + \frac{N_1^2}{\mathfrak{R}_m} i_1 + \frac{N_1 N_2}{\mathfrak{R}_m} i_2 \quad (1.2-14)$$

$$\lambda_2 = \frac{N_2^2}{\mathfrak{R}_{l2}} i_2 + \frac{N_2^2}{\mathfrak{R}_m} i_2 + \frac{N_2 N_1}{\mathfrak{R}_m} i_1 \quad (1.2-15)$$

When the magnetic system is linear, the flux linkages are generally expressed in terms of inductances and currents. We see that the coefficients of the first two terms on the right-hand side of (1.2-14) depend on N_1 and the reluctance of the magnetic system, independent of the existence of winding 2. An analogous statement may be made regarding (1.2-15) with the roles of winding 1 and winding 2 reversed. Hence, the self-inductances are defined as

$$L_{11} = \frac{N_1^2}{\mathfrak{R}_{l1}} + \frac{N_1^2}{\mathfrak{R}_m} = L_{l1} + L_{m1} \quad (1.2-16)$$

$$L_{22} = \frac{N_2^2}{\mathfrak{R}_{l2}} + \frac{N_2^2}{\mathfrak{R}_m} = L_{l2} + L_{m2} \quad (1.2-17)$$

where L_{l1} and L_{l2} are the leakage inductances and L_{m1} and L_{m2} are the magnetizing inductances of windings 1 and 2, respectively. From (1.2-16) and (1.2-17), it follows that the magnetizing inductances may be related as

$$\frac{L_{m2}}{N_2^2} = \frac{L_{m1}}{N_1^2} \quad (1.2-18)$$

which is $1/\mathfrak{R}_m$.

The mutual inductances are defined as the coefficient of the third term on the right-hand side of (1.2-14) and (1.2-15). In particular,

$$L_{12} = \frac{N_1 N_2}{\mathfrak{R}_m} \quad (1.2-19)$$

$$L_{21} = \frac{N_2 N_1}{\mathfrak{R}_m} \quad (1.2-20)$$

We see that $L_{12} = L_{21}$ and, with the assumed positive direction of current flow and the manner in which the windings are wound as shown in Fig. 1.2-1, the mutual inductances are positive. If, however, the assumed positive directions of the current or the direction of the windings were such that Φ_{m1} opposed Φ_{m2} , then the mutual inductances would be negative.

The mutual inductances may be related to the magnetizing inductances. Comparing (1.2-16) and (1.2-17) with (1.2-19) and (1.2-20), we see that

$$L_{12} = \frac{N_2}{N_1} L_{m1} = \frac{N_1}{N_2} L_{m2} \quad (1.2-21)$$

The flux linkages may now be written as

$$\lambda_1 = L_{11}i_1 + L_{12}i_2 \quad (1.2-22)$$

$$\lambda_2 = L_{21}i_1 + L_{22}i_2 \quad (1.2-23)$$

where L_{11} and L_{22} are defined by (1.2-16) and (1.2-17), respectively, and L_{12} and L_{21} by (1.2-19) and (1.2-20), respectively. The self-inductances L_{11} and L_{22} are always positive; however, the mutual inductances $L_{12}(L_{21})$ may be positive or negative, as previously mentioned.

Although the voltage equations given by (1.2-3) and (1.2-4) may be used for purposes of analysis, it is customary to perform a change of variables that yields the well-known equivalent T circuit of two windings coupled by a linear magnetic circuit. To set the stage for this derivation, let us express the flux linkages from (1.2-22) and (1.2-23) as

$$\lambda_1 = L_{11}i_1 + L_{m1} \left(i_1 + \frac{N_2}{N_1} i_2 \right) \quad (1.2-24)$$

$$\lambda_2 = L_{12}i_2 + L_{m2} \left(\frac{N_1}{N_2} i_1 + i_2 \right) \quad (1.2-25)$$

With λ_1 in terms of L_{m1} and λ_2 in terms of L_{m2} , we see two logical candidates for substitute variables, in particular, $(N_2/N_1)i_2$ or $(N_1/N_2)i_1$. If we let

$$i'_2 = \frac{N_2}{N_1} i_2 \quad (1.2-26)$$

then we are using the substitute variable i'_2 , which, when flowing through winding 1, produces the same mmf as the actual i_2 flowing through winding 2; $N_1 i'_2 = N_2 i_2$. This is said to be referring the current in winding 2 to winding 1 or to a winding with N_1 turns, whereupon winding 1 becomes the reference or primary winding and winding 2 is the secondary winding and i'_2 is negative. On the other hand, if we let

$$i'_1 = \frac{N_1}{N_2} i_1 \quad (1.2-27)$$

then i'_1 is the substitute variable that produces the same mmf when flowing through winding 2 as i_1 does when flowing in winding 1; $N_2 i'_1 = N_1 i_1$. This change of variables is said to refer to the current of winding 1 to winding 2 or to a winding with N_2 turns, whereupon winding 2 becomes the reference or primary winding and winding 1 the secondary with i'_1 .

We will demonstrate the derivation of the equivalent T circuit by referring the current of winding 2 to a winding with N_1 turns; thus, i'_2 is expressed by (1.2-26). We want the instantaneous power to be unchanged by this substitution of variables. Therefore,

$$v'_2 i'_2 = v_2 i_2 \quad (1.2-28)$$

Hence,

$$v'_2 = \frac{N_1}{N_2} v_2 \quad (1.2-29)$$

Flux linkages, which have the units of $V \cdot s$, are related to the substitute flux linkages in the same way as voltages. In particular,

$$\lambda'_2 = \frac{N_1}{N_2} \lambda_2 \quad (1.2-30)$$

Now, replace $(N_2/N_1)i_2$ with i'_2 in the expression for λ_1 , given by (1.2-24). Next, solve (1.2-26) for i_2 and substitute it into λ_2 given by (1.2-25). Now, multiply this result by N_1/N_2 to obtain λ'_2 and then substitute $(N_2/N_1)^2 L_{m1}$ for L_{m2} in λ'_2 . If we do all this, we will obtain

$$\lambda_1 = L_{l1} i_1 + L_{m1} (i_1 + i'_2) \quad (1.2-31)$$

$$\lambda'_2 = L'_{l2} i'_2 + L_{m1} (i_1 + i'_2) \quad (1.2-32)$$

where

$$L'_{l2} = \left(\frac{N_1}{N_2} \right)^2 L_{l2} \quad (1.2-33)$$

The flux linkage equations given by (1.2-31) and (1.2-32) may also be written as

$$\lambda_1 = L_{11} i_1 + L_{m1} i'_2 \quad (1.2-34)$$

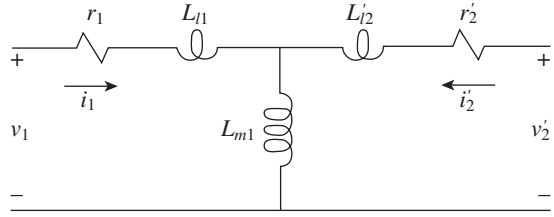
$$\lambda'_2 = L_{m1} i_1 + L'_{22} i'_2 \quad (1.2-35)$$

where

$$L'_{22} = \left(\frac{N_1}{N_2} \right)^2 L_{22} = L'_{l2} + L_{m1} \quad (1.2-36)$$

and L_{22} is defined by (1.2-17).

Figure 1.2-2 Equivalent T circuit with winding 1 selected as reference winding.



If we multiply (1.2-4) by N_1/N_2 to obtain v'_2 , the voltage equations become

$$\begin{bmatrix} v_1 \\ v'_2 \end{bmatrix} = \begin{bmatrix} r_1 & 0 \\ 0 & r'_2 \end{bmatrix} \begin{bmatrix} i_1 \\ i'_2 \end{bmatrix} + \frac{d}{dt} \begin{bmatrix} \lambda_1 \\ \lambda'_2 \end{bmatrix} \quad (1.2-37)$$

where

$$r'_2 = \left(\frac{N_1}{N_2} \right)^2 r_2 \quad (1.2-38)$$

The previous voltage equations, (1.2-37), together with the flux linkage equations, (1.2-34) and (1.2-35), suggest the equivalent T circuit shown in Fig. 1.2-2. This method may be extended to include any number of windings wound on the same core.

Example 1A The equivalent T circuit.

It is instructive to illustrate the method of deriving an equivalent T circuit from open- and short-circuit measurements. When winding 2 of the two-winding transformer shown in Fig. 1.2-2 is open circuited and a 60 Hz voltage of 110 V (rms) is applied to winding 1, the average power supplied to winding 1 is 6.66 W. The measured current in winding 1 is 1.05 A (rms). Next, with winding 2 short-circuited, the current flowing in winding 1 is 2 A (rms) when the applied voltage is 30 V at 60 Hz. The average input power is 44 W. If we assume $L_{11} = L'_{12}$, an approximate equivalent T circuit can be determined from these measurements with winding 1 selected as the reference winding.

With $\tilde{V}_1 = |\tilde{V}_1| / \theta_{ev}(0)$ and $\tilde{I}_1 = |\tilde{I}_1| / \theta_{ei}(0)$ then the average power supplied to winding 1 may be expressed as

$$P_1 = |\tilde{V}_1| |\tilde{I}_1| \cos \phi_{pf} \quad (1A-1)$$

where

$$\phi_{pf} = \theta_{ev}(0) - \theta_{ei}(0) \quad (1A-2)$$

Here, $\tilde{V}_1$ and $\tilde{I}_1$ are phasors with the positive direction of $\tilde{I}_1$ taken in the direction of the voltage drop, and $\theta_{ev}(0)$ and $\theta_{ei}(0)$ are the phase angles of $\tilde{V}_1$ and $\tilde{I}_1$,

respectively. Phasors are covered in Appendix B. Solving for ϕ_{pf} during the open-circuit test, we have

$$\phi_{pf} = \cos^{-1} \frac{P_1}{|\tilde{V}_1||\tilde{I}_1|} = \cos^{-1} \frac{6.66}{(110)(1.05)} = 86.7^\circ \quad (1A-3)$$

Although $\phi_{pf} = -86.7^\circ$ is also a legitimate solution of (1A-3), the positive value is taken since $\tilde{V}_1$ leads $\tilde{I}_1$ in an inductive circuit. With winding 2 open-circuited, the input impedance of winding 1 is

$$Z = \frac{\tilde{V}_1}{\tilde{I}_1} = r_1 + j(X_{l1} + X_{m1}) \quad (1A-4)$$

With $\tilde{V}_1$ as the reference phasor, $\tilde{V}_1 = 110/0^\circ$, $\tilde{I}_1 = 1.05/-86.7^\circ$. Thus,

$$r_1 + j(X_{l1} + X_{m1}) = \frac{110/0^\circ}{1.05/-86.7^\circ} = 6 + j104.6 \Omega \quad (1A-5)$$

If we neglect core losses, then, from (1A-5), $r_1 = 6 \Omega$. We also see from (1A-5) that $X_{l1} + X_{m1} = 104.6 \Omega$. For the short-circuit test, we will assume that $\tilde{I}_1 = -\tilde{I}'_2$ since transformers are designed so that at rated frequency $X_{m1} \gg |r'_2 + jX'_{l2}|$. Hence, using (1A-1) again,

$$\phi_{pf} = \cos^{-1} \frac{44}{(30)(2)} = 42.8^\circ \quad (1A-6)$$

In this case, the input impedance is $Z = (r_1 + r'_2) + j(X_{l1} + X'_{l2})$. This may be determined as

$$Z = \frac{30/0^\circ}{2/-42.8^\circ} = 11 + j10.2 \Omega \quad (1A-7)$$

Hence, $r'_2 = 11 - r_1 = 5 \Omega$ and, since it is assumed that $X_{l1} = X'_{l2}$, both are $10.2/2 = 5.1 \Omega$. Therefore, $X_{m1} = 104.6 - 5.1 = 99.5 \Omega$. In summary, $r_1 = 6 \Omega$, $L_{l1} = 13.5 \text{ mH}$, $L_{m1} = 263.9 \text{ mH}$, $r'_2 = 5 \Omega$, $L'_{l2} = 13.5 \text{ mH}$. Make sure we converted from X's to L's correctly.

1.2.1 Nonlinear Magnetic System

Although the analysis of transformers and electric machines is often performed assuming a magnetically linear system, economics and physics dictate that in the practical design of many of these devices, some saturation occurs and that heating of the magnetic material exists due to hysteresis loss [2]. The magnetization characteristics of transformer or machine materials are typically given in the form of

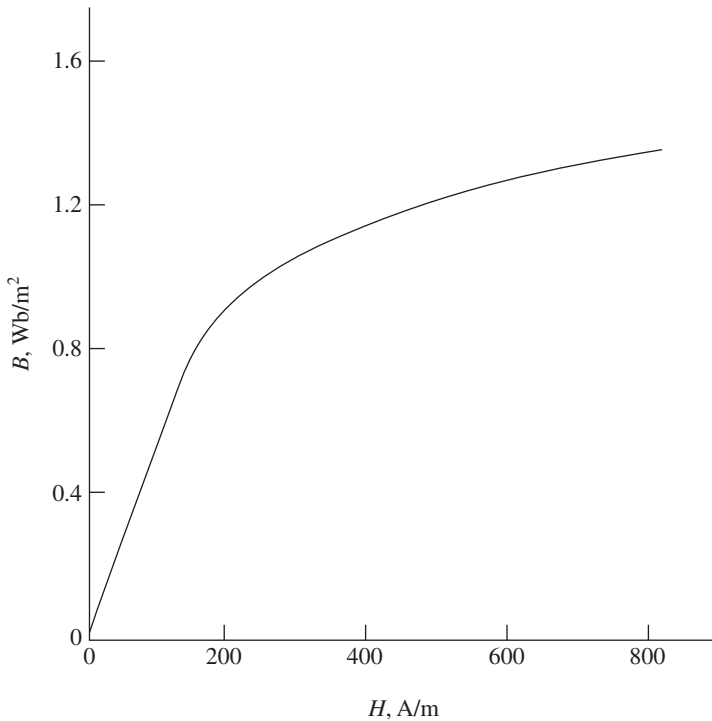


Figure 1.2-3 Typical B - H curve for silicon steel used in transformers.

the magnitude of flux density versus magnitude of field strength (B - H curve) as shown in Fig. 1.2-3.

If it is assumed that the magnetic flux is uniform through most of the core, then B is proportional to Φ and H is proportional to magnetomotive force (mmf). Hence, a plot of flux versus current is of the same shape as the B - H curve. A transformer is generally designed so that some saturation occurs during normal operation. During transients, saturation may occur resulting in large currents during startup transients. Electric machines are also designed similarly in that a machine generally operates slightly in the saturated region during normal, rated operating conditions. Since saturation causes coefficients of the differential equations describing the behavior of an electromagnetic device to be functions of the coil currents, transient analysis is difficult without the aid of a computer. Our purpose here is not to set forth methods of analyzing nonlinear magnetic systems. A method of incorporating the effects of saturation into a computer representation is of interest.

Formulating the voltage equations of stationary coupled windings appropriate for computer simulation is straightforward and yet this technique is fundamental

to the computer simulation of ac machines. Therefore, it is to our advantage to consider this method here. For this purpose, let us first write (1.2-31) and (1.2-32) as

$$\lambda_1 = L_{l1} i_1 + \lambda_m \quad (1.2-39)$$

$$\lambda'_2 = L'_{l2} i'_2 + \lambda_m \quad (1.2-40)$$

where

$$\lambda_m = L_{m1} (i_1 + i'_2) \quad (1.2-41)$$

Solving (1.2-39) and (1.2-40) for the currents yields

$$i_1 = \frac{1}{L_{l1}} (\lambda_1 - \lambda_m) \quad (1.2-42)$$

$$i'_2 = \frac{1}{L'_{l2}} (\lambda'_2 - \lambda_m) \quad (1.2-43)$$

If (1.2-42) and (1.2-43) are substituted into (1.2-37), and if we solve the resulting equations for flux linkages, the following equations are obtained:

$$\lambda_1 = \int \left[v_1 + \frac{r_1}{L_{l1}} (\lambda_m - \lambda_1) \right] dt \quad (1.2-44)$$

$$\lambda'_2 = \int \left[v'_2 + \frac{r'_2}{L'_{l2}} (\lambda_m - \lambda'_2) \right] dt \quad (1.2-45)$$

Substituting (1.2-42) and (1.2-43) into (1.2-41) yields

$$\lambda_m = L_a \left(\frac{\lambda_1}{L_{l1}} + \frac{\lambda'_2}{L'_{l2}} \right) \quad (1.2-46)$$

where

$$L_a = \left(\frac{1}{L_{m1}} + \frac{1}{L_{l1}} + \frac{1}{L'_{l2}} \right)^{-1} \quad (1.2-47)$$

We now have the equations expressed with λ_1 and λ'_2 as state variables. In the computer simulation, (1.2-44) and (1.2-45) are used to solve for λ_1 and λ'_2 and (1.2-46) is used to solve for λ_m . The currents can then be obtained from (1.2-42) and (1.2-43).

If the magnetization characteristics (magnetization curve) of the coupled winding are known, the effects of saturation of the mutual flux path may be incorporated into the computer simulation. Generally, the magnetization curve can be adequately determined from a test wherein one of the windings is open-circuited (winding 2, for example) and the input impedance of the other winding (winding 1) is determined from measurements as the applied

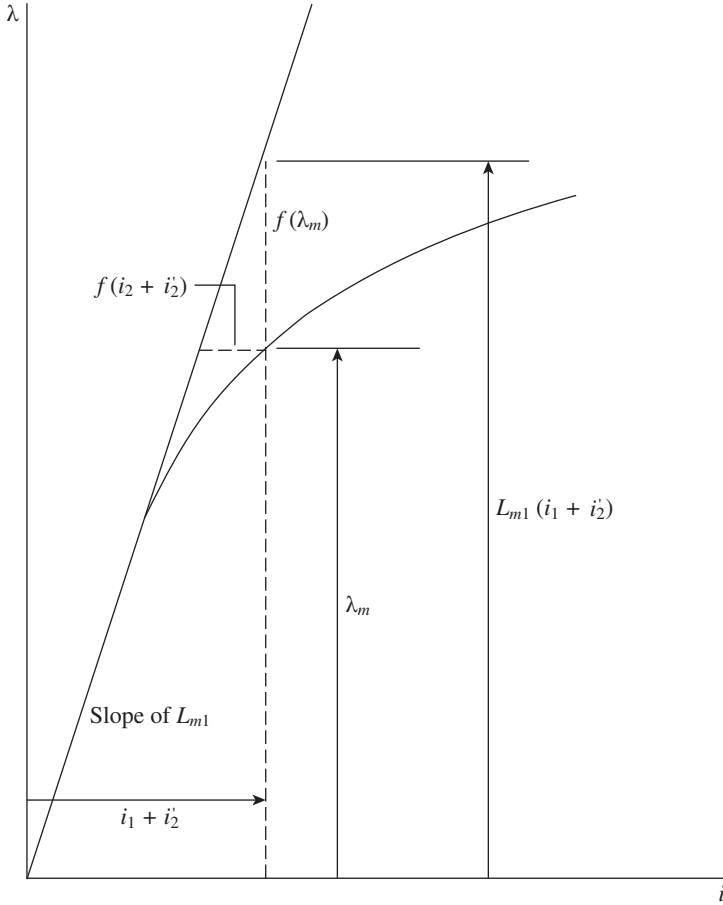


Figure 1.2-4 Magnetization curve.

voltage is increased in magnitude from zero to say 150% of the rated value. With information obtained from this type of test, we can plot λ_m versus $(i_1 + i'_2)$ as shown in Fig. 1.2-4 wherein the slope of the linear portion of the curve is L_{m1} . From Fig. 1.2-4, it is clear that in the region of saturation we have

$$\lambda_m = L_{m1}(i_1 + i'_2) - f(\lambda_m) \quad (1.2-48)$$

where $f(\lambda_m)$ may be determined from the magnetization curve for each value of λ_m . In particular, $f(\lambda_m)$ is a function of λ_m given by (1.2-48) and shown in Fig. 1.2-5. Therefore, the effects of saturation of the mutual flux path may be taken into account by replacing (1.2-41) with (1.2-48) for λ_m . Substituting (1.2-42) and (1.2-43) for i_1 and i'_2 , respectively, into (1.2-48) yields the following equation

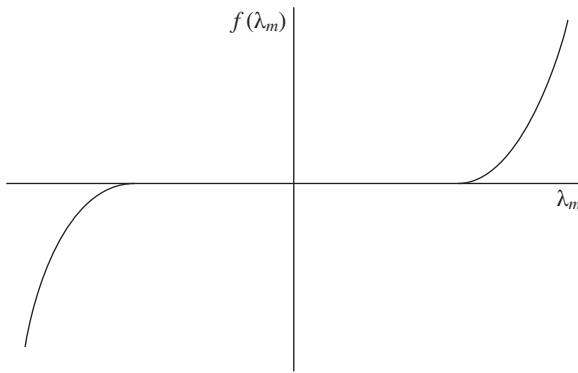


Figure 1.2-5 $f(\lambda_m)$ versus λ_m from Figure 1.2-4.

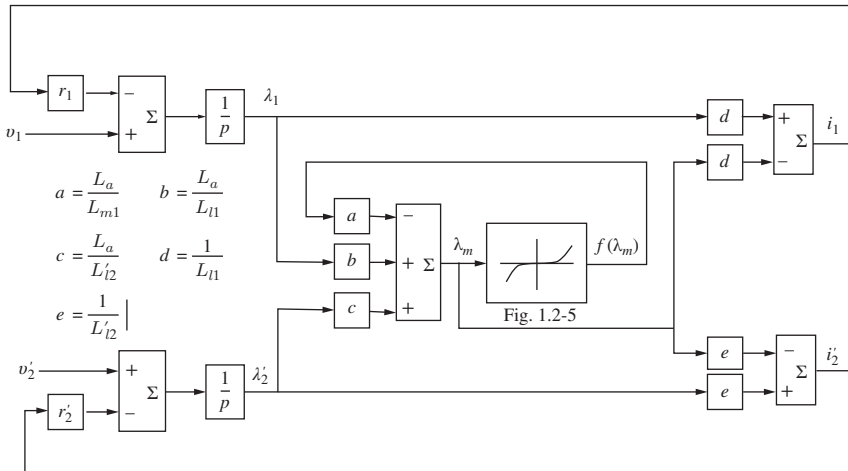


Figure 1.2-6 Time-domain block diagram of a two-winding transformer with saturation.

for λ_m

$$\lambda_m = L_a \left(\frac{\lambda_1}{L_{l1}} + \frac{\lambda_2'}{L'_{l2}} \right) - \frac{L_a}{L_{m1}} f(\lambda_m) \quad (1.2-49)$$

Hence, the computer simulation for including saturation involves replacing λ_m given by (1.2-46) with (1.2-49) where $f(\lambda_m)$ is a generated function of λ_m determined from the plot shown in Fig. 1.2-5. The time-domain block diagram of a two-winding transformer with saturation is shown in Fig. 1.2-6.

1.3 Energy Balance Relationships

Electromechanical systems consist of an electric system, a mechanical system, and a means whereby the electric and mechanical systems can interact. Interactions can take place through any and all electromagnetic and electrostatic fields that are common to both systems, and energy is transferred from one system to the other as a result of this interaction [3]. We will focus on the electromagnetic system, and the electrostatic system is treated in [2]. An electromechanical system with one electric subsystem, one mechanical subsystem, and one coupling field is depicted in Fig. 1.3-1. Electromagnetic radiation is neglected, and it is assumed that the electric system operates at a frequency sufficiently low so that the electric system may be considered a lumped-parameter system.

Heat loss will occur in the mechanical system due to friction, and the electric system will dissipate heat due to the resistance of the current-carrying conductors. Eddy current and hysteresis losses occur in the ferromagnetic materials. If W_E is the total energy supplied by the electric source and W_M the total energy supplied by the mechanical source, then the energy distribution could be expressed as

$$W_E = W_e + W_{eL} + W_{eS} \quad (1.3-1)$$

$$W_M = W_m + W_{mL} + W_{mS} \quad (1.3-2)$$

In (1.3-1), W_{eS} is the energy stored in the magnetic fields, which are not coupled with the mechanical system. The energy W_{eL} is the heat loss associated with the electric system excluding the coupling field losses. This loss occurs due to the resistance of the current-carrying conductors as well as the energy dissipated in the form of heat owing to hysteresis and eddy current losses external to the coupling field. W_e is the energy transferred to the coupling field by the electric system. The energies common to the mechanical system may be defined in a similar manner. In (1.3-2), W_{mS} is the energy stored in the moving member and compliances of the mechanical system, W_{mL} is the energy loss in the mechanical system in the form of heat, and W_m is the energy transferred to the coupling field. It is important to note that, with the convention adopted, the energy transferred to the coupling field by either source is considered positive. Also, W_E (W_M) is negative when energy is supplied to the electric source (mechanical source).



Figure 1.3-1 Block diagram of an elementary electromechanical system.

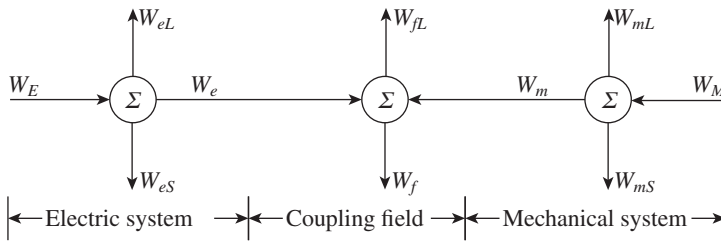


Figure 1.3-2 Energy balance.

If W_F is defined as the total energy transferred to the coupling field, then

$$W_F = W_f + W_{fL} \quad (1.3-3)$$

where W_f is energy stored in the coupling field and W_{fL} is the energy dissipated in the form of heat due to losses within the coupling field (eddy current or hysteresis losses). In order to comply with convention, we will use W_f to denote the energy stored in the coupling field rather than W_{fS} . The electromechanical system must obey the law of conservation of energy, thus,

$$W_f + W_{fL} = (W_E - W_{eL} - W_{eS}) + (W_M - W_{mL} - W_{mS}) \quad (1.3-4)$$

which may be written as

$$W_f + W_{fL} = W_e + W_m \quad (1.3-5)$$

This energy balance is shown schematically in Fig. 1.3-2.

The actual process of converting electric energy into mechanical energy (or vice versa) is independent of (1) the loss of energy in either the electric or the mechanical systems (W_{eL} and W_{mL}), (2) the energies stored in the electric or magnetic fields that are not common to both systems (W_{eS}), or (3) the energies stored in the mechanical system (W_{mS}). If the losses of the coupling field are neglected, then the field is conservative and (1.3-5) becomes

$$W_f = W_e + W_m \quad (1.3-6)$$

An example of an elementary electromechanical system is shown in Fig. 1.3-3. It has a magnetic coupling field with the space between the movable and stationary members exaggerated for clarity. The voltage of the electric source is denoted v , and f is an externally applied mechanical force. The electromagnetic force is denoted f_e . The resistance of the current-carrying conductor is denoted by r , with l denoting the inductance of a linear (conservative) electromagnetic system that does not couple with the mechanical system. In the mechanical system, M is the mass of the movable member, and the linear compliance and damper are represented by a spring constant K and a damping coefficient D . The displacement x_0

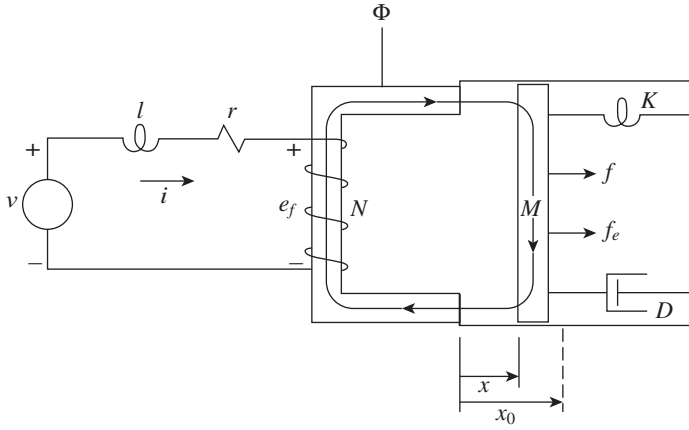


Figure 1.3-3 Electromechanical system with magnetic field.

is the zero force or equilibrium position of the mechanical system, which is the steady-state position of the mass with f_e and f equal to zero.

The voltage equation that describes the electric systems shown in Fig. 1.3-3 may be written as

$$v = ri + l \frac{di}{dt} + e_f \quad (1.3-7)$$

where e_f is the voltage drop due to the coupling field. The dynamic behavior of the translational mechanical systems may be expressed by employing Newton's law of motion. Thus,

$$f = M \frac{d^2x}{dt^2} + D \frac{dx}{dt} + K(x - x_0) - f_e \quad (1.3-8)$$

Since power is the time rate of energy transfer, the total energy supplied by the electric source is

$$W_E = \int v i dt \quad (1.3-9)$$

The total energy supplied by the mechanical source is

$$W_M = \int f dx \quad (1.3-10)$$

which may also be expressed as

$$W_M = \int f \frac{dx}{dt} dt \quad (1.3-11)$$

Substituting (1.3-7) into (1.3-9) yields

$$W_E = r \int i^2 dt + l \int i \frac{di}{dt} dt + \int e_f i dt \quad (1.3-12)$$

The first term on the right-hand side of (1.3-12) represents the energy loss due to the resistance of the conductors (W_{eL}). The second term represents the energy stored in the linear electromagnetic field external to the coupling field (W_{eS}). Therefore, the total energy transferred to the coupling field from the electric system is

$$W_e = \int e_f i dt \quad (1.3-13)$$

Similarly, for the mechanical system

$$W_M = M \int \frac{d^2x}{dt^2} dx + D \int \left(\frac{dx}{dt} \right)^2 dt + K \int (x - x_0) dx - \int f_e dx \quad (1.3-14)$$

Here, the first and third terms on the right-hand side of (1.3-14) represent the kinetic energy stored in the mass and the potential energy stored in the spring, respectively. The sum of these two stored energies is W_{mS} . You should take a moment to look at the first term on the right-hand side of (1.3-14) and recognize that it can be written as $\frac{1}{2}M(dx/dt)^2$. The second term is the heat loss due to friction (W_{mL}). Thus, the total energy transferred to the coupling field from the mechanical system is

$$W_m = - \int f_e dx \quad (1.3-15)$$

It is important to note from Fig. 1.3-3 that a positive force f_e is assumed to be in the same direction as a positive displacement dx . Substituting (1.3-13) and (1.3-15) into the energy balance relation, (1.3-6), yields

$$W_f = \int e_f i dt - \int f_e dx \quad (1.3-16)$$

The equations set forth may be readily extended to include an electromechanical system with any number of electric and mechanical inputs. Whereupon the field may be expressed as

$$W_f = \sum_{j=1}^J W_{ej} + \sum_{k=1}^K W_{mk} \quad (1.3-17)$$

wherein J electric and K mechanical inputs exist. The total energy supplied to the coupling field from the electric inputs is

$$\sum_{j=1}^J W_{ej} = \int \sum_{j=1}^J e_{fj} i_j dt \quad (1.3-18)$$

The total energy supplied to the coupling field from the mechanical inputs is

$$\sum_{k=1}^K W_{mk} = - \int \sum_{k=1}^K f_{ek} dx_k \quad (1.3-19)$$

In our analysis of electromechanical systems, we will consider devices with only one mechanical input, for example, the shaft of the electric machine or the moving arm of a magnetic solenoid. On the other hand, since machines may have more than one electric terminal, it is necessary to consider systems with multiple electric inputs. In all cases, however, the multiple electric inputs have a common coupling field. Therefore, we need not become too ambitious in the following derivations. More specifically, hereafter we will restrict our analysis to electromechanical devices with only one mechanical input. Thus, the k subscript will be dropped from f_e , x , and W_m . This reduces our work considerably without restricting the practical application of our results. With one mechanical input, the energy balance equation becomes

$$W_f = \int \sum_{j=1}^J e_{fj} i_j dt - \int f_e dx \quad (1.3-20)$$

In differential form, which will be the form we will use extensively,

$$dW_f = \sum_{j=1}^J e_{fj} i_j dt - f_e dx \quad (1.3-21)$$

Example 1B Calculation of inductance.

We will consider the electromechanical system shown in Fig. 1B-1. The system is at x_0 when v and f are both zero. The value of x_0 is 3 mm, which is very much exaggerated in Fig. 1B-1. The distance from c to d is 20 cm and 10 cm from a to b. The cross-sectional area of the iron and air gap is $A_i = A_g = 0.01 \text{ m}^2$. The relative permeability of the iron is 4000, and the permeability of air is $4\pi \times 10^{-7} \text{ H} \cdot \text{m}$.

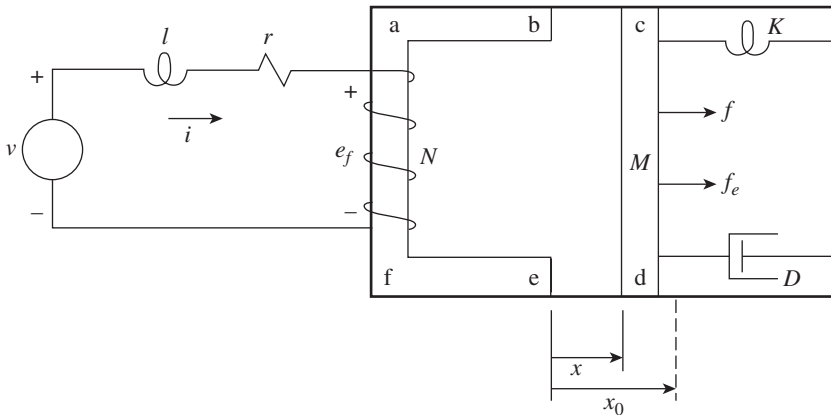


Figure 1B-1 Electromechanical system.

In Section 1.6, we let

$$L_m = \frac{k}{x} \quad (1B-1)$$

Determine N (turns) for $k = 6.283 \times 10^{-5} \text{ H} \cdot \text{m}$.

The reluctance to the magnetizing flux with $x = x_0$

$$\begin{aligned} \mathfrak{R}_m = \mathfrak{R}_i + \mathfrak{R}_g &= \frac{l_i}{\mu_r \mu_0 A_i} + \frac{2l_g}{\mu_0 A_g} \\ &= \frac{60 \times 10^{-2}}{4 \times 10^3 \times 4\pi \times 10^{-7} \times 1 \times 10^{-3}} + \frac{2 \times 3 \times 10^{-3}}{4\pi \times 10^{-7} \times 10^{-3}} \\ &= 1.19 \times 10^5 + 47.8 \times 10^5 \end{aligned} \quad (1B-2)$$

Clearly, the reluctance is dominated by the reluctance of the two air gaps.

Now, if we neglect the reluctance of the iron, then

$$L_m = \frac{N^2 \mu_0 A_g}{2x} \quad (1B-3)$$

Comparing (1B-1) and (1B-3) we see that

$$k = \frac{1}{2} N^2 \mu_0 A_g \quad (1B-4)$$

Solving for N yields

$$\begin{aligned} N &= \left(\frac{2k}{\mu_0 A_g} \right)^{\frac{1}{2}} = \left(\frac{2 \times 6.283 \times 10^{-5}}{4\pi \times 10^{-7} \times 1 \times 10^{-2}} \right)^{\frac{1}{2}} \\ &= 100 \text{ turns} \end{aligned} \quad (1B-5)$$

1.4 Energy in Coupling Field

Before using (1.3-21) to obtain an expression for the electromagnetic force f_e , it is necessary to derive an expression for the energy stored in the coupling field. Once we have an expression for W_f , we can take the total derivative to obtain dW_f , which can then be substituted into (1.3-21). When expressing the energy in the coupling field, it is convenient to neglect all losses associated with the magnetic coupling field, whereupon the field is assumed to be conservative and the energy stored therein is a function of the state of the electrical and mechanical variables. Although the effects of the core losses of the coupling field may be functionally accounted for by appropriately introducing resistance in the electric circuit, this refinement is generally not necessary since the ferromagnetic material is selected and arranged in laminations so as to minimize the hysteresis and eddy current losses. Moreover, most of the energy stored in the coupling field is stored in the

air gap of the electromechanical device. Since air is a conservative medium, all of the energy stored therein can be returned to the electric or mechanical systems. Therefore, the assumption of a lossless coupling field is not as restrictive as it might first appear.

The energy stored in a conservative field is a function of the state of the system variables and not the manner in which the variables reached that state. It is convenient to take advantage of this feature when developing a mathematical expression for the field energy. In particular, it is convenient to fix mathematically the position of the mechanical system associated with the coupling field and then excite the electric system with the displacement of the mechanical system held fixed. During the excitation of the electric inputs, $dx = 0$, hence, W_m is zero even though electromagnetic forces may occur. Therefore, with the displacement held fixed, the energy stored in the coupling field during the excitation of the electric inputs is equal to the energy supplied to the coupling field by the electric inputs. Thus, with $dx = 0$, the energy supplied from the electric system may be expressed from (1.3-20) as

$$W_f = \int \sum_{j=1}^J e_{fj} i_j dt \text{ with } dx = 0 \quad (1.4-1)$$

Let us consider a singly excited electromagnetic system similar to that shown in Fig. 1.3-3. In this case, $e_f = d\lambda/dt$, whereupon (1.4-1) becomes

$$W_f = \int i d\lambda \text{ with } dx = 0 \quad (1.4-2)$$

Here, $j = 1$; however, the subscript is omitted for the sake of brevity. The area to the left of the λi relationship, shown in Fig. 1.4-1 for a singly excited electromagnetic system, is the area described by (1.4-2). In Fig. 1.4-1, this area represents the energy stored in the field at the instant when $\lambda = \lambda_a$ and $i = i_a$. The λi relationship need not be linear; it need only be single-valued, a property that is characteristic of a conservative or lossless field. Moreover, since the coupling field is conservative, the energy stored in the field with $\lambda = \lambda_a$ and $i = i_a$ is independent of the excursion of the electrical and mechanical variables before reaching this state.

The area to the right of the λi curve is called the *coenergy* and can be expressed as

$$W_c = \int \lambda di \text{ with } dx = 0 \quad (1.4-3)$$

Although the coenergy has little or no physical significance, we will find it a convenient quantity for expressing the electromagnetic force. From Fig. 1.4-1, we see that the sum of W_f and W_c is λ times i , that is,

$$\lambda i = W_c + W_f \quad (1.4-4)$$

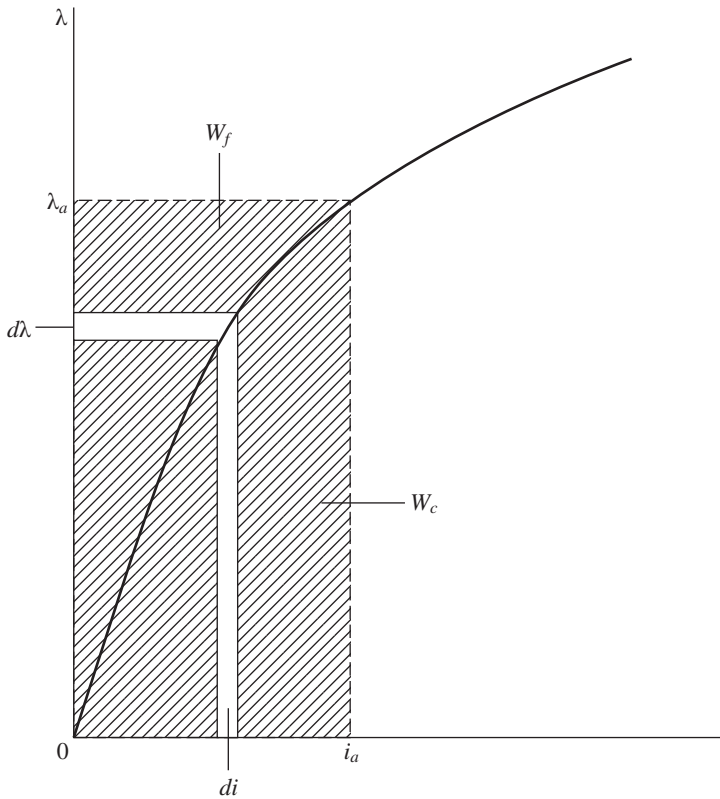


Figure 1.4-1 Stored energy and coenergy in a magnetic field of a singly excited electromagnetic device.

which is also valid for multiple electric inputs, where λi in (1.4-4) is replaced by $\sum_{j=1}^J \lambda_j i_j$. It should be clear that, for a linear magnetic system where the λi plots are straight-line relationships, $W_f = W_c = \frac{1}{2} \lambda i$.

The displacement x defines completely the influence of the mechanical system upon the coupling field; however, since λ and i are related, only one is needed in addition to x in order to describe the state of the electromechanical system. Therefore, we can select either λ and x as independent variables or i and x . If i and x are selected as the independent variables, it is convenient to express the field energy and the flux linkages as

$$W_f = W_f(i, x) \quad (1.4-5)$$

$$\lambda = \lambda(i, x) \quad (1.4-6)$$

With i and x as independent variables, we must express $d\lambda$ in terms of di before substituting into (1.4-2). Thus, from 1.4-6,

$$d\lambda = \frac{\partial \lambda(i, x)}{\partial i} di + \frac{\partial \lambda(i, x)}{\partial x} dx \quad (1.4-7)$$

In the derivation of an expression for the energy stored in the field, dx is set equal to zero. Hence, in the evaluation of field energy where $dx = 0$, $d\lambda$ is equal to the first term on the right-hand side of (1.4-7). Substituting into (1.4-2) yields

$$\begin{aligned} W_f(i, x) &= \int i \frac{\partial \lambda(i, x)}{\partial i} di \\ &= \int_0^i \xi \frac{\partial \lambda(\xi, x)}{\partial \xi} d\xi \text{ with } dx = 0 \end{aligned} \quad (1.4-8)$$

where ξ is a substitute variable of integration. Evaluation of (1.4-8) gives the energy stored in the field of the singly excited system. The coenergy in terms of i and x may be evaluated from (1.4-3) as

$$\begin{aligned} W_c(i, x) &= \int \lambda(i, x) di \\ &= \int_0^i \lambda(\xi, x) d\xi \end{aligned} \quad (1.4-9)$$

With λ and x as independent variables

$$W_f = W_f(\lambda, x) \quad (1.4-10)$$

$$i = i(\lambda, x) \quad (1.4-11)$$

The field energy may be evaluated from (1.4-2) as

$$\begin{aligned} W_f(\lambda, x) &= \int i(\lambda, x) d\lambda \\ &= \int_0^\lambda i(\xi, x) d\xi \end{aligned} \quad (1.4-12)$$

To evaluate the coenergy with λ and x as independent variables, we need to express di in terms of $d\lambda$. Thus, from (1.4-11),

$$di = \frac{\partial i(\lambda, x)}{\partial \lambda} d\lambda + \frac{\partial i(\lambda, x)}{\partial x} dx \quad (1.4-13)$$

Since $dx = 0$ in this evaluation, (1.4-13) becomes

$$\begin{aligned} W_c(\lambda, x) &= \int \lambda \frac{\partial i(\lambda, x)}{\partial \lambda} d\lambda \\ &= \int_0^\lambda \xi \frac{\partial i(\xi, x)}{\partial \xi} d\xi \text{ with } dx = 0 \end{aligned} \quad (1.4-14)$$

For a linear electromagnetic system, the λi plots are straight-line relationships. Thus, for the singly excited magnetically linear system,

$$\lambda(i, x) = L(x)i \quad (1.4-15)$$

or

$$i(\lambda, x) = \frac{\lambda}{L(x)} \quad (1.4-16)$$

where $L(x)$ is the inductance. Let us evaluate $W_f(i, x)$. With $dx = 0$ and, since $\frac{\partial \lambda(i, x)}{\partial i} = L(x)$, (1.4-7) becomes

$$d\lambda = L(x)di \quad (1.4-17)$$

Hence, from (1.4-8),

$$W_f(i, x) = \int_0^i \xi L(x) d\xi = \frac{1}{2} L(x) i^2 \quad (1.4-18)$$

It is left to the reader to show that by a similar procedure $W_f(\lambda, x)$, $W_c(i, x)$, and $W_c(\lambda, x)$ are equivalent to (1.4-18) for this magnetically linear system.

The field energy is a state function and the expression describing the field energy in terms of system variables is valid regardless of the variations in the system variables. For example, (1.4-18) expresses the field energy regardless of the variations in $L(x)$ and i . The fixing of the mechanical system so as to obtain an expression for the field energy is a mathematical convenience and not a restriction upon the result.

In the case of a multi-excited electromagnetic system, an expression for the field energy may be obtained by evaluating the following relation with $dx = 0$:

$$\int \sum_{j=1}^J i_j d\lambda_j \text{ with } dx = 0 \quad (1.4-19)$$

Since the coupling field is considered conservative, (1.4-19) may be evaluated independent of the order in which the flux linkages or currents are brought to their final values. To illustrate the evaluation of (1.4-19) for a multi-excited system, we will allow the currents to establish their final states one at a time while all other currents are mathematically fixed in their either unexcited or final states. This procedure may be illustrated by considering a doubly excited electric system with one mechanical input. An electromechanical system of this type could be constructed by placing a second winding, supplied from a second electric system, on either the stationary or movable member of the system shown in Fig. 1.3-3. In this evaluation, it is convenient to use currents and displacement as the independent variables. Hence, for a doubly excited electric system,

$$W_f(i_1, i_2, x) = \int [i_1 d\lambda_1(i_1, i_2, x) + i_2 d\lambda_2(i_1, i_2, x)] \text{ with } dx = 0 \quad (1.4-20)$$

In this determination of an expression for W_f , the mechanical displacement is held constant ($dx = 0$); thus, (1.4-20) becomes

$$W_f(i_1, i_2, x) = \int i_1 \left[\frac{\partial \lambda_1(i_1, i_2, x)}{\partial i_1} di_1 + \frac{\partial \lambda_1(i_1, i_2, x)}{\partial i_2} di_2 \right] \\ + i_2 \left[\frac{\partial \lambda_2(i_1, i_2, x)}{\partial i_1} di_1 + \frac{\partial \lambda_2(i_1, i_2, x)}{\partial i_2} di_2 \right] \text{ with } dx = 0 \quad (1.4-21)$$

We will evaluate the energy stored in the field by employing (1.4-21) twice. First, we will mathematically increase the current i_1 from zero to its desired final value while holding i_2 at zero. Thus, i_1 is the variable of integration and $di_2 = 0$. Energy is supplied to the coupling field from the source connected to winding 1. As the second evaluation of (1.4-21), i_2 is increased from zero to its desired final value while maintaining i_1 at the value attained in the preceding step. Hence, i_2 is the variable of integration and $di_1 = 0$. During this time, energy is supplied from both sources to the coupling field since $d\lambda_i$ is, in general, nonzero. The total energy stored in the coupling field is the sum of the two evaluations. Following this two-step procedure, the evaluation of (1.4-21) for the total field energy becomes

$$W_f(i_1, i_2, x) = \int i_1 \frac{\partial \lambda_1(i_1, 0, x)}{\partial i_1} di_1 \\ + \int \left[i_1 \frac{\partial \lambda_1(i_1, i_2, x)}{\partial i_2} di_2 + i_2 \frac{\partial \lambda_2(i_1, i_2, x)}{\partial i_2} di_2 \right] \quad (1.4-22)$$

which should be written

$$W_f(i_1, i_2, x) = \int_0^{i_1} \xi \frac{\partial \lambda_1(\xi, 0, x)}{\partial \xi} d\xi \\ + \int_0^{i_2} \left[i_1 \frac{\partial \lambda_1(i_1, \xi, x)}{\partial \xi} + \xi \frac{\partial \lambda_2(i_1, \xi, x)}{\partial \xi} d\xi \right] \quad (1.4-23)$$

The first integral on the right-hand side of (1.4-22) or (1.4-23) results from the first step of the evaluation with i_1 as the variable of integration and with $i_2 = 0$ and $di_2 = 0$. The second integral comes from the second step of the evaluation with i_1 equal to its final value ($di_1 = 0$) and i_2 as the variable of integration. The order of allowing the currents to reach their final state is irrelevant; that is, as our first step, we could have made i_2 the variable of integration while holding i_1 at zero ($di_1 = 0$) and then let i_1 become the variable of integration while holding i_2 at its final value. The results would be the same. For three electric inputs, the evaluation procedure would require three steps, one for each current to be brought mathematically to its final state.

Let us now evaluate the energy stored in a magnetically linear system with two electric inputs and one mechanical input. For this, let

$$\lambda_1(i_1, i_2, x) = L_{11}(x)i_1 + L_{12}(x)i_2 \quad (1.4-24)$$

$$\lambda_2(i_1, i_2, x) = L_{21}(x)i_1 + L_{22}(x)i_2 \quad (1.4-25)$$

where the self-inductances $L_{11}(x)$ and $L_{22}(x)$ include the leakage inductances. With the mechanical displacement held constant ($dx = 0$),

$$d\lambda_1(i_1, i_2, x) = L_{11}(x)di_1 + L_{12}(x)di_2 \quad (1.4-26)$$

$$d\lambda_2(i_1, i_2, x) = L_{21}(x)di_1 + L_{22}(x)di_2 \quad (1.4-27)$$

The coefficients on the right-hand side of (1.4-26) and (1.4-27) are the partial derivatives. For example, $L_{11}(x)$ is the partial derivative of $\lambda_1(i_1, i_2, x)$ with respect to i_1 . Appropriate substitution into (1.4-23) gives

$$W_f(i_1, i_2, x) = \int_0^{i_1} \xi L_{11}(x) d\xi + \int_0^{i_2} [i_1 L_{12}(x) + \xi L_{22}(x)] d\xi \quad (1.4-28)$$

which yields

$$W_f(i_1, i_2, x) = \frac{1}{2}L_{11}(x)i_1^2 + L_{12}(x)i_1i_2 + \frac{1}{2}L_{22}(x)i_2^2 \quad (1.4-29)$$

It follows that the total field energy of a linear electromagnetic system with J electric inputs may be expressed as

$$W_f(i_1, \dots, i_J, x) = \frac{1}{2} \sum_{p=1}^J \sum_{q=1}^J L_{pq} i_p i_q \quad (1.4-30)$$

1.5 Electromagnetic Forces

The stage is set for us to obtain expressions for the electromagnetic force in electromechanical devices. For this purpose, recall that e_{ff} in (1.3-21) may be expressed as

$$e_{ff} = \frac{d\lambda_j}{dt} \quad (1.5-1)$$

If we substitute (1.5-1) into (1.3-21) and if we solve for $f_e dx$, we obtain

$$f_e dx = \sum_{j=1}^J i_j d\lambda_j - dW_f \quad (1.5-2)$$

Although we will use (1.5-2), it is helpful to express it in an alternative form. For this purpose, let us first write (1.4-4) for multiple electrical inputs:

$$\sum_{j=1}^J \lambda_j i_j = W_c + W_f \quad (1.5-3)$$

If we take the total derivative of (1.5-3), we obtain

$$\sum_{j=1}^J \lambda_j di_j + \sum_{j=1}^J i_j d\lambda_j = dW_c + dW_f \quad (1.5-4)$$

We realize that when we evaluate the force f_e we must select the independent variables; that is, either the flux linkages and x or the currents and x . The flux linkages and the currents both cannot be considered independent variables when evaluating the force f_e . Nevertheless, (1.5-4), wherein both $d\lambda_j$ and di_j appear, is valid in general, before a selection of independent variables is made to evaluate f_e . If we solve (1.5-4) for the field energy dW_f and substitute the result into (1.5-2), we obtain

$$f_e dx = - \sum_{j=1}^J \lambda_j di_j + dW_c \quad (1.5-5)$$

Either (1.5-2) or (1.5-5) can be used to evaluate the electromagnetic force f_e . If flux linkages and x are selected as independent variables, (1.5-2) is the most direct, whereas (1.5-5) is the most direct if currents and x are selected.

With flux linkages and x as the independent variables, the currents are expressed functionally as

$$i_j(\lambda_1, \dots, \lambda_J, x) \quad (1.5-6)$$

For the purpose of compactness, we will denote $(\lambda_1, \dots, \lambda_J, x)$ as (λ, x) , where λ is an abbreviation for the complete set of flux linkages associated with the J windings. Let us write (1.5-2) with flux linkages and x as independent variables:

$$f_e(\lambda, x) dx = \sum_{j=1}^J i_j(\lambda, x) d\lambda_j - dW_f(\lambda, x) \quad (1.5-7)$$

If we take the total derivative of the field energy with respect to λ and x , and substitute that result into (1.5-7) we obtain

$$f_e(\lambda, x) dx = \sum_{j=1}^J i_j(\lambda, x) d\lambda_j - \sum_{j=1}^J \frac{\partial W_f(\lambda, x)}{\partial \lambda_j} d\lambda_j - \frac{\partial W_f(\lambda, x)}{\partial x} dx \quad (1.5-8)$$

Equating the coefficients of dx gives

$$f_e(\lambda, x) = - \frac{\partial W_f(\lambda, x)}{\partial x} \quad (1.5-9)$$

If we now select i and x as independent variables, where $(\mathbf{i}, x)$ is the abbreviated notation for $(i_1, \dots, i_J, x)$, then (1.5-5) can be written as

$$f_e(\mathbf{i}, x)dx = - \sum_{j=1}^J \lambda_j(\mathbf{i}, x)di_j + dW_c(\mathbf{i}, x) \quad (1.5-10)$$

If we take the total derivative of $W_c(i, x)$ with i and x as independent variables and substitute the result into (1.5-11), we obtain

$$f_e(\mathbf{i}, x)dx = - \sum_{j=1}^J \lambda_j(\mathbf{i}, x)di_j + \sum_{j=1}^J \frac{\partial W_c(\mathbf{i}, x)}{\partial i_j} di_j + \frac{\partial W_c(\mathbf{i}, x)}{\partial x} dx \quad (1.5-11)$$

Equating coefficients of dx yields

$$f_e(\mathbf{i}, x) = \frac{\partial W_c(\mathbf{i}, x)}{\partial x} \quad (1.5-12)$$

We will make extensive use of (1.5-12).

Although only translational mechanical systems have been considered, all force relationships developed herein may be modified for the purpose of evaluating the torque in rotational systems. In particular, when considering a rotational system, f_e is replaced with the electromagnetic torque T_e and x with the angular displacement θ_r of the rotating member. These substitutions are justified since the change of mechanical energy in a rotational system is expressed as

$$dW_m = -T_e d\theta_r \quad (1.5-13)$$

When dealing with machines, we perform a transformation that allows torque to be calculated more directly from the voltage equations. Nevertheless, the equations that we have developed for f_e given by (1.5-12) can be used to calculate T_e , that is

$$T_e = \frac{\partial W_c(i, \theta_r)}{\partial \theta_r} \quad (1.5-14)$$

Example 1C Detailed calculation of electromagnetic force.

One may prefer to determine the electromagnetic force or torque by starting with the relationship $dW_f = dW_e + dW_m$ rather than by selecting a formula from the text. To illustrate this procedure, let

$$\lambda = L(x)i \quad (1C-1)$$

First, we must evaluate the field energy. Since losses in the coupling field are neglected, W_f is a function of state. Hence, W_f may be evaluated by fixing the mechanical displacement. This is done by setting $dx = 0$, whereupon

$$dW_f = dW_e = id\lambda \text{ with } dx = 0 \quad (1C-2)$$

where dW_e is obtained from (1.3-13) with $e_f = d\lambda/dt$. From (1C-1) with $dx = 0$,

$$d\lambda = L(x)di \quad (1C-3)$$

Substituting (1C-3) into (1C-2) and solving for W_f yields

$$W_f = \int_0^i L(x)\xi d\xi = \frac{1}{2}L(x)i^2 \quad (1C-4)$$

To obtain an expression for f_e , we go back to the basic relationship that $dW_f = dW_e + dW_m$; however, now $dx \neq 0$. Thus, from (1C-4),

$$dW_f = \frac{1}{2}i^2 \frac{dL(x)}{dx} dx + L(x)idi \quad (1C-5)$$

Now,

$$dW_e = id\lambda = i^2 \frac{dL(x)}{dx} + L(x)idi \quad (1C-6)$$

and from (1.3-15),

$$dW_m = -f_e dx \quad (1C-7)$$

Substituting into $dW_f = dW_e + dW_m$ yields

$$\frac{1}{2}i^2 \frac{dL(x)}{dx} dx + L(x)idi = i^2 \frac{dL(x)}{dx} dx + L(x)idi - f_e dx \quad (1C-8)$$

Equating the coefficients of dx ,

$$f_e = \frac{1}{2}i^2 \frac{dL(x)}{dx} \quad (1C-9)$$

1.6 Steady-State and Dynamic Performance of an Electromechanical System

It is instructive to consider the steady-state and dynamic performance of the elementary electromagnetic system shown in Fig. 1.3-3. The differential equations that describe this system are given by (1.3-7) for the electrical system and (1.3-8) for the mechanical system. The electromagnetic force f_e is expressed by (1.5-12). If the applied voltage, v , and the applied mechanical force, f , are constant, all derivatives with respect to time are zero during steady-state operation and the behavior can be predicted by

$$v = ri \quad (1.6-1)$$

$$f = K(x - x_0) - f_e \quad (1.6-2)$$

Equation (1.6-2) may be written as

$$-f_e = f - K(x - x_0) \quad (1.6-3)$$

The magnetic core of the system in Fig. 1.3-3 is generally constructed of ferromagnetic material with a relative permeability in the order of 2000–4000 (see Example 1B). $L(x)$ can be adequately approximated by

$$L(x) = \frac{k}{x} \quad (1.6-4)$$

In the actual system, the inductance will be a large finite value rather than infinity, as predicted by (1.6-4), when $x = 0$. Nevertheless, (1.6-4) is quite sufficient to illustrate the action of the system for $x > 0$. Substituting (1.6-4) into (1.5-12) where $W_c(i, x) = \frac{1}{2}L(x)i^2$ yields

$$f_e(i, x) = -\frac{ki^2}{2x^2} \quad (1.6-5)$$

A plot of (1.6-3), with f_e replaced by (1.6-5), is shown in Fig. 1.6-1 for the following system parameters:

$$\begin{aligned} r &= 10 \, \Omega & x_0 &= 3 \, \text{mm} \\ K &= 2667 \, \text{N/m} & k &= 6.283 \times 10^{-5} \, \text{H} \cdot \text{m} \end{aligned}$$

In Fig. 1.6-1, the plot of the negative of the electromagnetic force is for an applied voltage of 5 V whereupon the steady-state current of 0.5 A. The straight lines represent the right-hand side of (1.6-3) with $f = 0$ (lower straight line) and $f = 4 \, \text{N}$ (upper straight line). Both lines intersect the $-f_e$ curve at two points. In particular, the upper line intersects the $-f_e$ curve at 1 and 1'; the lower line intersects at 2 and 2'. Stable operation occurs at only points 1 and 2. The system will not operate stably at points 1' and 2'. This can be explained by assuming the system is operating at one of these points (1' and 2') and then showing that any system disturbance will cause the system to move away from these points. If, for example, x increases slightly from its value corresponding to point 1', the restraining force $f - K(x - x_0)$ is larger in magnitude than $-f_e$, and x will continue to increase until the system reaches operating point 1. If x increases beyond its value corresponding to operating point 1, the restraining force is less than the electromagnetic force. Therefore, the system will establish steady-state operation at 1. If, on the other hand, x decreases from point 1' the electromagnetic force is larger than the restraining force and the movable member will move until it comes in contact with the stationary member ($x = 0$). It is clear that the restraining force that yields a straight line below the $-f_e$ curve will not permit stable operation with $x > 0$.

The dynamic behavior of the system during step changes in the source voltage is shown in Fig. 1.6-2, and in Figs. 1.6-3 and 1.6-4 for step changes in the applied

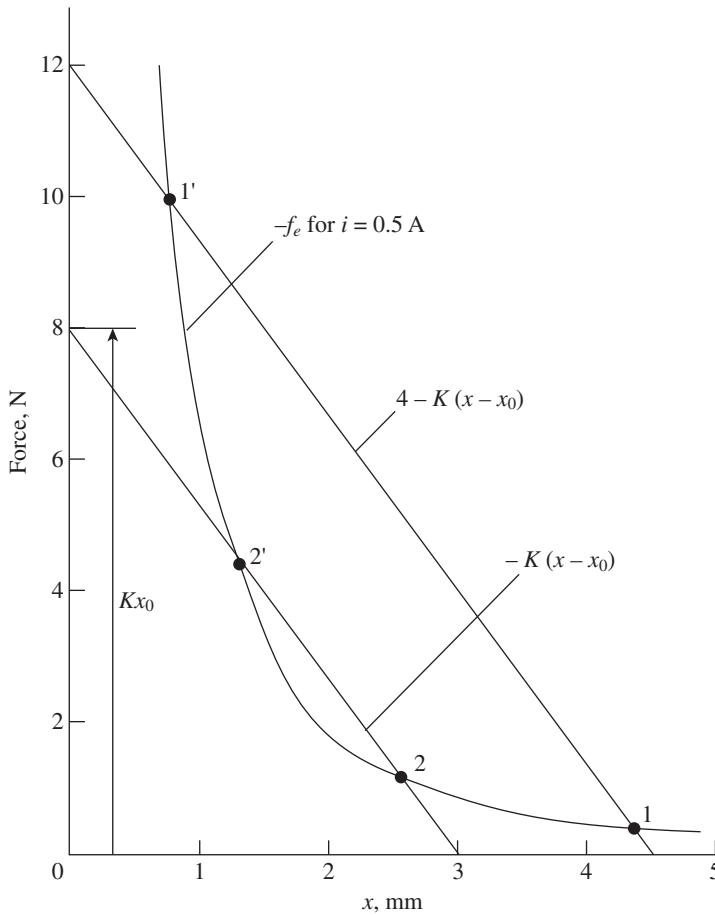


Figure 1.6-1 Steady-state operation of an electromechanical system in Fig. 1.3-1.

force f . The following system parameters were used in addition to those given previously:

$$l = 0 \quad M = 0.055 \text{ kg} \quad D = 4 \text{ N} \cdot \text{s/m}$$

The computer traces shown in Fig. 1.6-2 depict the dynamic performance of the example system when the applied voltage is stepped from zero to 5 V and then back to zero with the applied mechanical force held equal to zero. The following system variables are plotted: e_f , λ , i , f_e , x , W_e , W_f , and W_m . The energies are plotted in millijoules (mJ). Initially, the mechanical system is at rest with $x = x_0$ (3 mm). When the source voltage is applied, x decreases, and when steady-state operation is

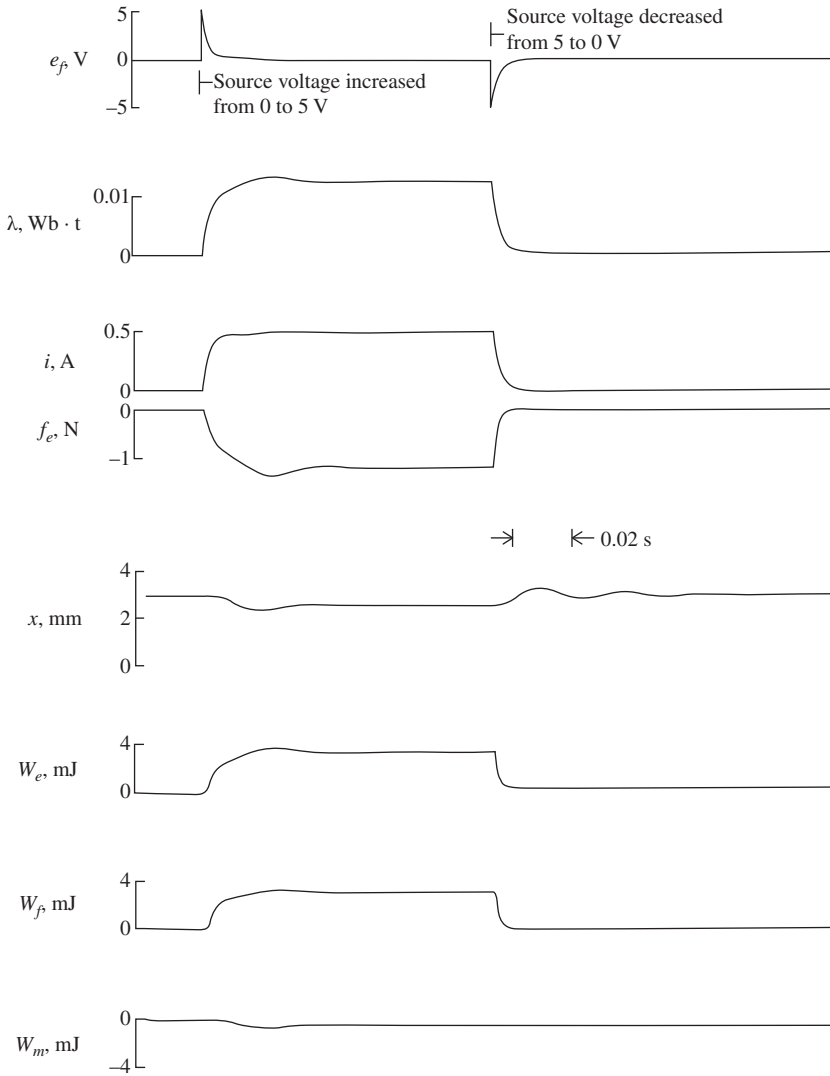


Figure 1.6-2 Dynamic performance of the electromechanical system shown in Fig. 1.3-3 during step changes in the source voltage.

reestablished, x is approximately 2.5 mm. Energy enters the coupling field via W_e . The bulk of this energy is stored in the field (W_f) with a smaller amount transferred to the mechanical system; some of that is dissipated in the damper during the transient period while the remainder is stored in the spring. When the applied

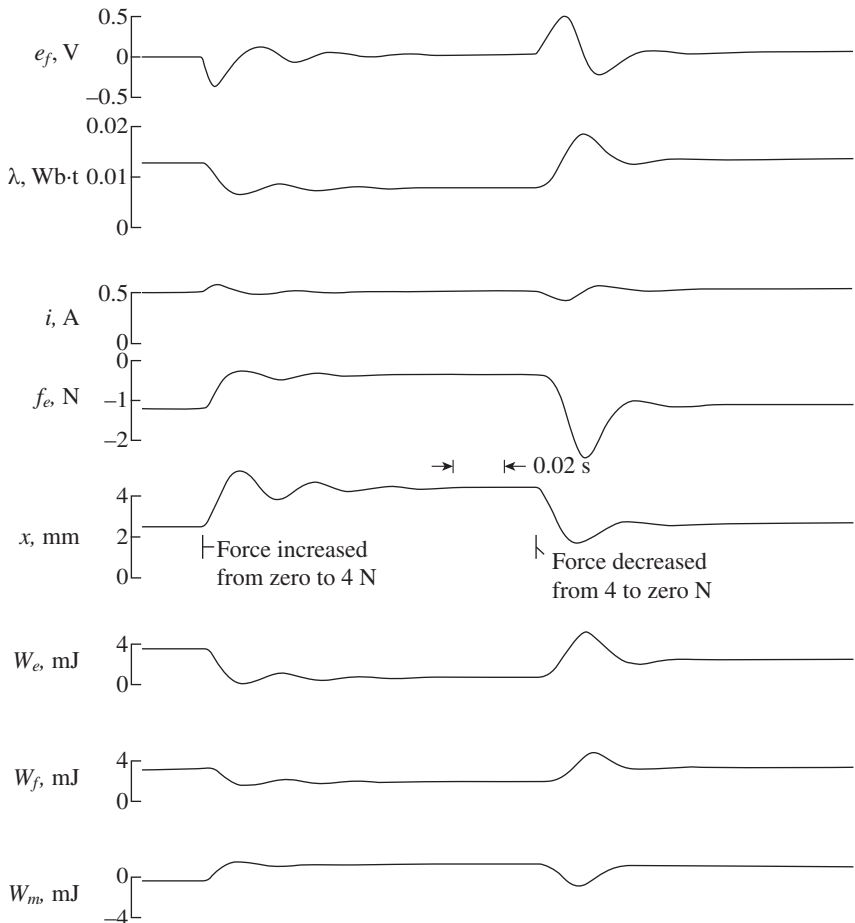


Figure 1.6-3 Dynamic performance of the electromechanical system shown in Fig. 1.3-3 during step changes in the applied force.

voltage is removed, the electrical and mechanical systems return to their original states. The change in W_m is small, increasing only slightly. Hence, during the transient period, there is an interchange of energy between the spring and mass that is finally dissipated in the damper. The net change in W_f during the application and removal of the applied voltage is zero; hence, the net change in W_e is positive and equal to the negative of the net change in W_m . The energy transferred to the mechanical system during this cycle is dissipated in the damper since f is fixed at zero, and the mechanical system returns to its initial rest position with zero energy stored in the spring.

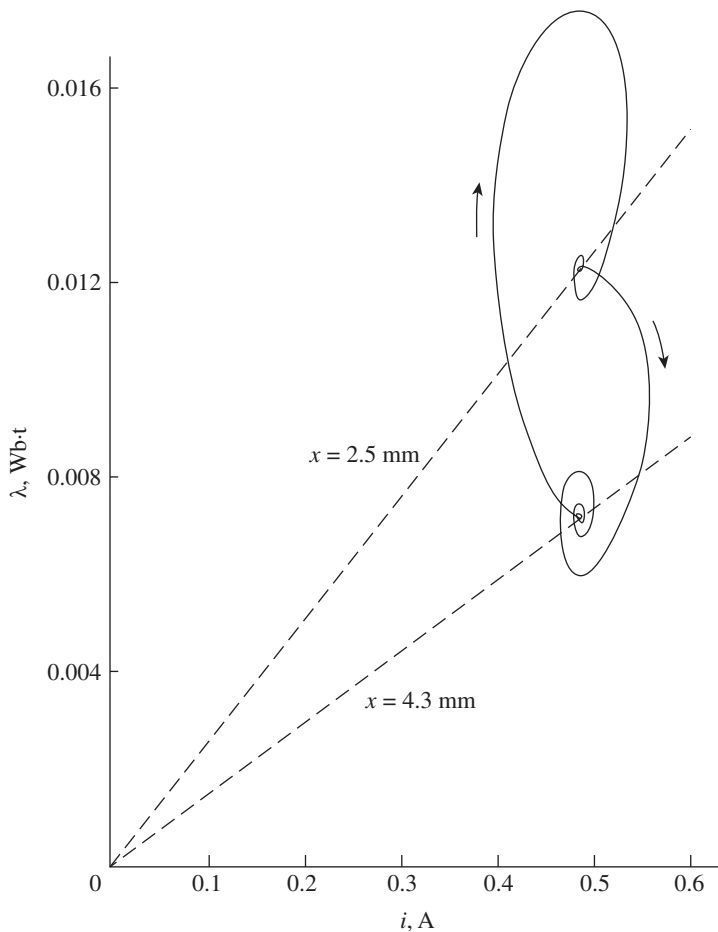


Figure 1.6-4 System response shown in Fig. 1.6-2.

In Fig. 1.6-3, the initial state is that shown in Fig. 1.6-2 with 5 V applied to the electrical system. The mechanical force f is increased from zero to 4 N, whereupon energy enters the coupling field from the mechanical system. Energy is transferred from the coupling field to the electrical system, some coming from the mechanical system and some from the energy originally stored in the magnetic field. Next, the force is stepped back to zero from 4 N. The electrical and mechanical systems return to their original states. During the cycle, a net energy has been transferred from the mechanical system to the electrical system that is dissipated in the resistance. This cycle is depicted on the λ versus i plot shown in Fig. 1.6-4.

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- 2 P. C. Krause, *Analysis of Electric Machinery*, McGraw-Hill, 1986.
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Problems

- 1.1 A two-winding, iron-core transformer is shown in Fig. 1P-1. $N_1 = 50$ turns, $N_2 = 100$ turns, and $\mu_r = 4000$. Calculate L_{m1} and L_{m2} .
- 1.2 Repeat Problem 1 if the iron core has an air gap of 0.2 cm in length and is cut through the complete cross section. Assume that fringing does not occur in the air gap, that is, the effective cross-sectional area of the air gap is 25 cm^2 .
- 1.3 Two coupled coils have the following parameters:

$$L_{11} = 100 \text{ mH} \quad r_1 = 10$$

$$L_{22} = 25 \text{ mH} \quad r_2 = 2.5$$

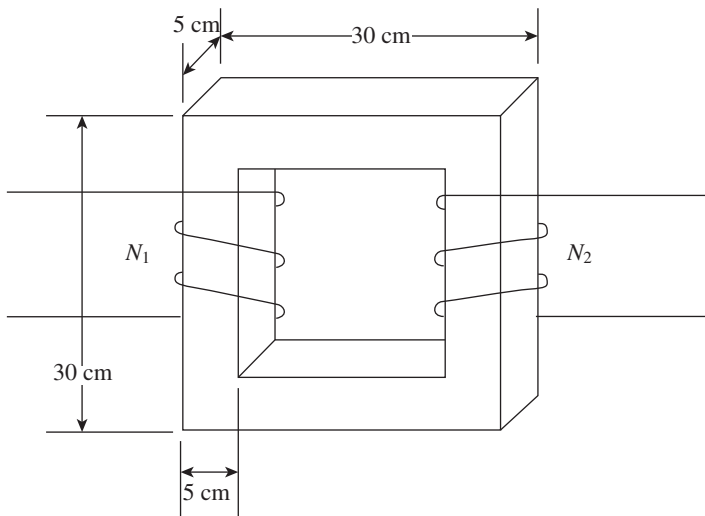


Figure 1P-1 Two-winding, iron-core transformer.

$$N_1 = 1000 \text{ turns} \quad N_2 = 500 \text{ turns}$$

$$L_{l1} = 0.1 L_{11} \quad L_{l2} = 0.1 L_{22}$$

Develop a T equivalent circuit with coil 1 as the reference coil. Repeat with coil 2 as the reference coil.

- 1.4** A system with two windings has a flux linkage versus a current profile of

$$\lambda_1 = \left(0.1 + \frac{0.03}{x}\right) i_1 - \frac{0.01}{x} i_2$$

$$\lambda_2 = \left(0.0111 + \frac{0.03}{9x}\right) i_2 - \frac{0.01}{x} i_1$$

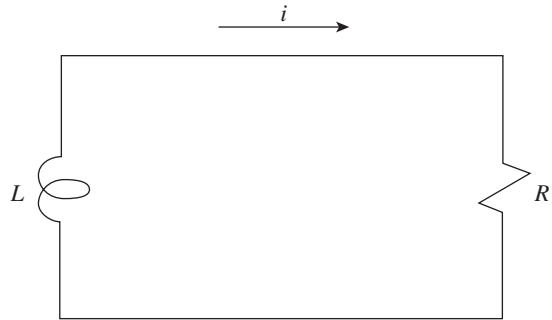
The resistance of the coils is $r_1 = 1 \, \Omega$ and $r_2 = 0.3 \, \Omega$. The winding voltage equations can be expressed in the form:

$$v_1 = r_1 i_1 + p \lambda_1$$

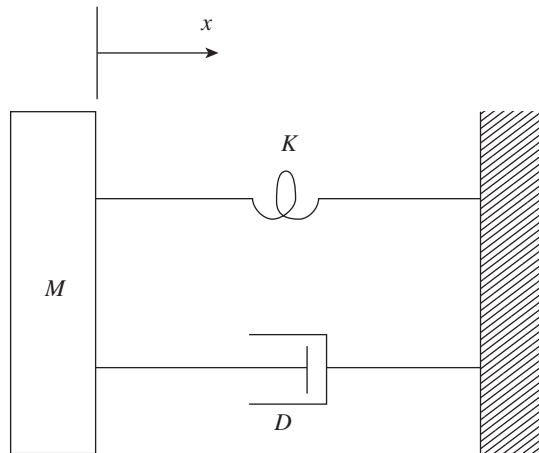
$$v_2 = r_2 i_2 + p \lambda_2$$

Derive the T equivalent circuit model for this system, assuming coil 1 as the reference. Show all component values. Label directions of all currents and voltages.

- 1.5** Obtain (1.2-47).
- 1.6** Determine the input impedance of the coupled circuits given in Problem 3 if the applied frequency to coil 1 is 60 Hz with coil 2 (a) open-circuited and (b) short-circuited. Repeat (b) with the current flowing in the magnetizing reactance neglected.
- 1.7** A third coil is wound on the ferromagnetic core shown in Fig. 1P-1. The resistance is r_3 and the leakage and magnetizing inductances are L_{l3} and L_{m3} , respectively. The coil is wound so that positive current (i_3) produces Φ_{m3} in the same direction as Φ_{m1} and Φ_{m2} . Derive the T equivalent circuit for this three-winding transformer. Actually, one should be able to develop the equivalent circuit without derivation.
- 1.8** A resistor and an inductor are connected as shown in Fig. 1P-2 with $R = 15 \, \Omega$ and $L = 250 \text{ mH}$. Determine the energy stored in the inductor W_{eS} and the energy dissipated by the resistor W_{eL} for $i > 0$ if $i(0) = 10 \text{ A}$.

Figure 1P-2 *R-L circuit.*

- 1.9** Consider the spring-mass-damper system shown in Fig. 1P-3. At $t = 0$, $x(0) = x_0$ (rest position) and $dx/dt = 1.5$ m/s. $M = 0.8$ kg, $D = 10$ N · s/m, and $K = 120$ N · m. For $t > 0$, determine the energy stored in the spring W_{ms1} , the kinetic energy of the mass W_{ms2} , and the energy dissipated by the damper W_{mL} .

Figure 1P-3
Spring-mass-damper system.

- 1.10** For the system shown in Fig. 1P-4, determine the winding inductance if the leakage inductance is one-tenth of the magnetizing inductance. If 10 V is applied to the winding at $t = 0$ s, determine W_f and the force of attraction that acts to attempt to reduce the gap at $t = 1$ s.
- 1.11** Express $W_f(i, x)$ and $W_c(i, x)$ for (a) $\lambda(i, x) = i^{2/3}x^2$ and (b) $\lambda(i, x) = ki \sin(x/a)\pi - xi$.

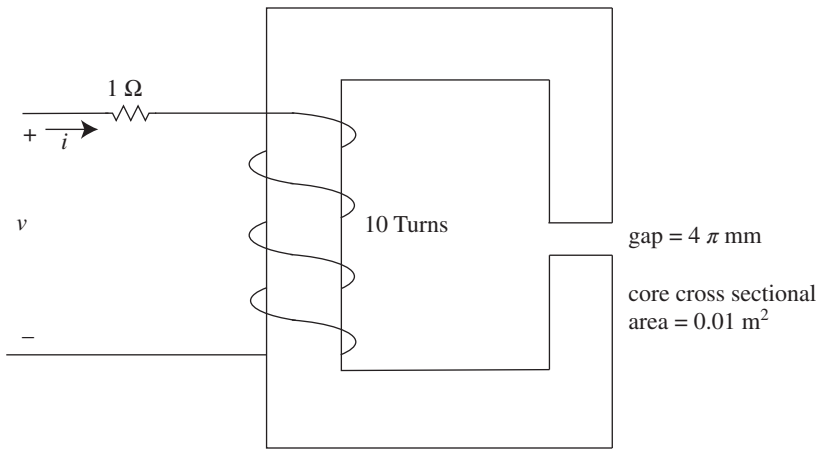


Figure 1P-4 Core configuration.

- 1.12** An electromechanical system has two electrical inputs. The flux linkages may be expressed as

$$\lambda_1(i_1, i_2, x) = x^2 i_1^2 + x i_2$$

$$\lambda_2(i_1, i_2, x) = x^2 i_2^2 + x i_1$$

Express $W_f(i_1, i_2, x)$ and $W_c(i_1, i_2, x)$.

- 1.13** Express $f_e(i, x)$ for the electromechanical systems described by the relations given in Problem 11.
- 1.14** Express $f_e(i_1, i_2, x)$ for the electromechanical system given in Problem 12.
- 1.15** Refer to Fig. 1.6-2. Following the system transients due to the application of the source voltage ($v = 5$ V), the system assumes steady-state operation. For this steady-state operation, calculate W_{eS} , W_f , and W_{mS} .
- 1.16** Refer to Fig. 1.6-3. Repeat Problem 15 for steady-state operation following the application of $f = 4$ N.
- 1.17** Refer to Fig. 1.6-4. Identify the area corresponding to ΔW_m when (a) x moves from 2.5 to 4.3 mm and (b) x moves from 4.3 to 2.5 mm.