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Introduction to Sets

Concepts

- Definition of a set
- Elements of a set
- Cardinality of a set
- Notation (using braces, ellipsis, rosters, set builder notation)
- De Morgan's Laws

> ***Augustus De Morgan*** *(born June 27, 1806, Madura, India—died March 18, 1871, London, England) was an English mathematician and logician whose major contributions to the study of logic include the formulation of De Morgan's laws and work leading to the development of the theory of relations and the rise of modern symbolic, or mathematical, logic.*
>
> Encyclopedia Brittanica

1.1 Introduction

The landscape of history of mathematics, logic, and computer science is populated by gigantic figures: Sir Isaac Newton, Aristotle, Alan Turing, Galileo Galilei, Ludwig Wittgenstein, and countless others. As we embark on a journey to understand mathematics and its relationship to cybersecurity, we remark that not all of what we will discuss here will be straightforward (let alone simple!). Many of the results we present and use will be named after men and women of genius, and we will pause to honor them in our writing; it is to them that we all owe a measure of gratitude, awe, and respect.

We are, metaphorically speaking, walking in the footsteps of giants.

Be confident, however, that you are eminently capable of mastering the material we are about to present. You will emerge from the course with new ideas and techniques, new skills and insights, and a sense of accomplishment that will see you through the challenges that lie ahead.

Definition *Theorem: A theorem, in mathematics and logic, a proposition or statement that is demonstrated. Very loosely, the word can be interpreted as "a rule."*

Definition *Set: A well-defined collection of objects, typically gathered according to some unifying trait, denoted by a capital letter such as A, B, and C.*

Mathematics in Cybersecurity, First Edition. Alfred Basta, Stephan DeLong, and Stavros Basta.

Published 2026 by John Wiley & Sons, Inc.
Companion website: www.wiley.com/go/bastamic

Definition *Set Theory: A branch of mathematics that deals with the properties of well-defined collections of objects and serves as a basis for precise and adaptable terminology for the definition of complex and sophisticated mathematical concepts.*

1.1.1 The Definition of a Set

A set is a well-defined, unordered collection of objects containing no duplicate members. By "well-defined," we mean that membership in the collection is unambiguous, not open to opinion or interpretation, and strictly capable of determination through an objective investigation of the facts. By "unordered," we specify there is not necessarily an importance to the order in which the members of the set appear within the collection.

The set of "all wealthy countries in the world," for example, is not well-defined, while the set of "countries within the European Union" (EU) is well defined. The perception of what it means to be a wealthy country is open to debate and may not be interpreted the same way by different people. Consequently, whether Senegal is a member of the set of all countries within the European Union can be definitively established, while whether Senegal is a wealthy country cannot (unless we specifically define what we mean by the concept of "wealthy").

The set of numbers $\{1,2,3,4,5\}$ has an identical list of members as the set of numbers $\{3,1,4,2,5\}$. The order of the numbers is not the same, but this does not change the membership of the set.

In our text, sets will primarily be named using capital letters from the English alphabet. There are exceptions to this, as we will see, but this is the convention we will generally adopt. Thus, we can say $S = \{1,2,3,4,5\}$; any presentation of precisely this collection, however ordered, is an instantiation of the set S.

1.1.2 The Elements of a Set

We have mentioned a set is a well-defined, unordered collection of objects containing no duplicate members. The objects within or members of the set are called "elements."

It is common, we should say, for a single concept to be referred to in multiple ways. While occasionally creating a mild confusion, it is something with which we must deal. For example, the English word "dog" and the French word "chien" refer to the same concept, but are in no way similar.

Much of this variety in nomenclature arises from the long, slow climb of human knowledge. We take for granted that information today can be shared rapidly (even instantaneously), but this was not the case throughout history. Often, identical ideas were developed simultaneously in far-flung locations of the globe, with the developers using wildly different notation and terminology.

A particular symbol is used to represent the concept of membership in the set: $\in$. This is read as "is an element of," or some variant of this phrase. A certain amount of care must be taken in the use of this symbol, because subtleties of set theory can result in our misstatement of relationships.

For instance, suppose we consider the earlier mentioned set $S = \{1,2,3,4,5\}$. It would be true to state "1 is an element of the set S," which we represent by $1 \in S$. It would not be true, on the other hand, to state $\{1\} \in S$. This is a subtle but important distinction. The latter statement tells us: "the set consisting of the number 1 is an element of the (larger) set S." This is incorrect, though not apparent. If we had specified the membership of the set S was $\{\{1\},2,3,4,5\}$, THEN it would be correct to state $\{1\} \in S$, but incorrect to state $1 \in S$.

This is a fine point of "attention to detail," wherein we must pay very close attention to (and take great care in) our use of language. This, we point out, is one of the "reasons" all students are required to take classes in mathematics, as well (in an ideal world) as logic: the science is the ultimate proving ground for attention detail – a skill most of us believe we possess and yet in which we are not proficient.

In any of the precise sciences (mathematics, medicine, computer science, engineering, etc.), great care should be taken in the selection and use of notation and terminology. It is assumed we "know what we're talking about," and we should comport ourselves as such. A trivial example is the casual description of a particular shape as being a "parabola," when it merely *resembles* a parabola. Parabolic curves and shapes have properties unique to that particular categorization of objects and if we identify something as being a parabola, readers will assume they can exploit those properties. The results of such a misinterpretation could be disastrous.

There is a related notation about set membership with which we should be aware: the symbol indicating nonmembership. How are we to indicate a particular item is NOT an element of a given set? There is a convention widely used in logic and mathematics, placement of a "slash" through an already-established symbol, to indicate the concept of "not."

That is, since $\in$ represents "is an element of," the associated symbol $\notin$ represents "is not an element of." Therefore, since we have identified $S = \{1,2,3,4,5\}$, we can say $6 \notin S$.

1.1.3 Cardinality of a Set

Occasionally, it is important to refer to the quantity of elements lying within a set. We refer to this as the *cardinality* of the set.

The notation for cardinality varies; in this text, we will use two notations to indicate the concept: $|S|$ and $n(S)$. Each of these is read as **the cardinality of set *S***, or **the number of elements in the set *S***.

For our set $S = \{1,2,3,4,5\}$, we can say $|S| = 5$, or that $n(S) = 5$.

The cardinality of a set can reach extreme values. By this we mean the low end for the value of cardinality is zero (more will be said of this later), while the high end is infinity (∞). When the cardinality of a set is not infinite, we say it is finite. A specific value, however large, is considered finite. For instance, a set with one trillion elements is a large but finite set.

The cardinality of a finite set is a *discrete* concept. This means there is a definitive number of elements in a finite set. We cannot have, for instance, a set with cardinality π. Since we are determining a count of the members of the set, the result must be a whole number.

1.1.4 The Notation of Sets

The set $S = \{1,2,3,4,5\}$, which presents the membership as a list, incorporates two ideas we should make clear.

First is that the list of elements is surrounded by "curly parentheses," which we call **braces**. Note there is a distinction in mathematics between parentheses, (), brackets, [ ], and braces, {}. We also refer to left and right parentheses/brackets/braces.

Second is the list itself. We refer to a list of this sort as a ***roster*** of membership. It is permissible to informally describe this as a "list of elements," but our formal reference will be as a "roster."

It is, obviously, only convenient to utilize roster notation in the event the set consists of a (relatively) small group of elements. A lengthy roster would be unwieldy, and thus the choice not to use roster notation would be a matter of practicality. The reason we use the term "relatively" in this description is that "small" is a subjective word: what do we each mean by the idea of a "small" group of elements? This can vary from person to person and is often a judgment call. We will use caution when incorporating relative terms; it is typically too vague within the sciences to be loose with our phrasing, and we shall strive for precision.

We will presume, however, that for now that the idea of a "small group of elements" is reasonably agreeable to all of us, and that it is something about which we will not argue.

There are conventions we can adopt that circumvent the issue of "large" rosters. These are situationally driven and cannot always be used (for reasons we will make apparent through example).

Consider the set $T = \{1,2,3,4,5,6,7,8,9,10,11,12,13\}$. Is this a "large" set? Obviously, this could be debated; however, assume for the moment this set is "large."

In this case, the elements of the set follow a discernible pattern: they are the natural numbers from 1 through 13, inclusive. This might be less evident if the order in which the roster is given were "shuffled." Nevertheless, as presented the membership is rather clear. We can condense the list through the use of **ellipsis**, which is a series of three "dots," intended to indicate the observable pattern continues as seen, beyond the shown portion of the roster.

That is, the set T could be shown as $T = \{1,2,3,\ldots,13\}$.

We can see, based on the first few members displayed, the pattern existing among the elements. The ellipsis indicates the pattern continues through the number 13, where it terminates.

If no value follows the ellipsis, the list does not terminate; it is, instead, infinite. In other words, $V = \{1,2,3,\ldots\}$ is the entire list of natural numbers. Combining the notions of roster notation, ellipsis, and cardinality, we now state:

$S = \{1,2,3,4,5\}$, and $\mathrm{n}(S) = 5$
$T = \{1,2,3,\ldots,13\}$, and $\mathrm{n}(T) = 13$
$V = \{1,2,3,\ldots\}$, and $\mathrm{n}(V) = \infty$

There are other ways in which sets can be presented. A very commonly used (yet often misunderstood) method is **set builder notation**. This method incorporates braces, a separation bar delineating an identification of a symbol for set members and a **defining characteristic** for the elements of the set.

A defining characteristic is a quality or property possessed by the elements of the set. This can be established in written form or using an equation. The format we will adopt for set builder notation will be:

$$\{\textit{general symbol representing an element} \mid \textit{defining characteristic}\}$$

That notation may appear cumbersome; keep in mind it is merely a technical format and is presented in an extremely general form above.

Example 1.1 Describe the membership of the set, using words and a roster.

$$T = \{x \mid x \textit{ is a natural number less than or equal to } 13\}$$

Solution: The notation is read as "T is equal to the set of elements represented by the letter x, such that x is a natural number less than or equal to 13." We can make statement more concise by stating "T is the set of natural numbers less than or equal to 13," or "T is the set of natural numbers less than 14." An expression of set membership using words can always be modified somehow while still accurately describing the members.

A roster of the members of T is $\{1,2,3,4,5,6,7,8,9,10,11,12,13\}$, or $\{1,2,3,\ldots,13\}$.

Example 1.2 Provide a roster of the members of the given set E.

$$E = \{c|c \textit{ is a country in the European Union}\}$$

This would be read as, “E is equal to the set of elements represented by the letter c, such that c is a country in the European Union,” Or some close variant of this description. A roster of set E is:

$E =$ {Austria, Belgium, Bulgaria, Croatia, Republic of Cyprus, Czech Republic, Denmark, Estonia, Finland, France, Germany, Greece, Hungary, Ireland, Italy, Latvia, Lithuania, Luxembourg, Malta, Netherlands, Poland, Portugal, Romania, Slovakia, Slovenia, Spain, Sweden}.

One sees now the utility of set-builder notation: it is much more economical to produce the notation with the description of requirement for set membership rather than creating a roster.

An important observation to make is that the “vertical line” within the braces is always read as “such that.”

A final example, more mathematical, is:

$$P = \{(x, y)|\ 3x - 4y = 12\}$$

We read this as, “P is equal to the set of ordered pairs represented by (x,y), such that $3x - 4y = 12$.” Recall this criterion indicates the ordered pairs represent an infinitely large set of points all lying on the graph of the equation $3x - 4y = 12$ on the Cartesian Plane. It is worthwhile to remind ourselves that a straight line in the Plane is an example of a “curve.” Curves are drawings in the plane (having no width) that contain all the points lying on the graph of an equation. It seems odd, in a way, to refer to a straight line as a “curve,” but this is due to the definition offered a moment ago. There is a concept called “curvature” that describes “how curvy” a line is in the Plane, and a straight line is a curve with no curvature.

1.2 Types of Sets

- Empty set (null set)
- Singleton set
- Finite and infinite sets
- Special mathematical sets
- Subset and superset
- Power set

We now turn our attention to specific **types of sets**. When we refer to types of sets, we mean individual sets whose nature or properties elevate them to a standard where they are deserving of specific consideration and nomenclature.

1.2.1 The Empty Set (The Null Set)

In the first section of this chapter, we introduced the notion of cardinality. At that time, we mentioned there were extreme values of cardinality, the first extreme being a set having cardinality zero.

Such a set is called the **empty set**, or the **null set**, represented in one of several ways: first, a defining characteristic that is incapable of being met by any object whatsoever; second, a set of empty braces; third, the empty set symbol: $\emptyset$.

> *Note: There are organizations, such as the United States Military, where it is common to use a figure similar to the empty set symbol as a representation of the numeral “0.” While this is routinely used, it will create confusion for us here. Therefore, we shall adopt the convention the empty set symbol shall NOT be used synonymously with the numeral 0.*

An example of a defining characteristic impossible of fulfillment would be “the set of all odd numbers that are exactly divisible by 2.” No odd numbers are exactly divisible by 2 (this is a consequence of the definition of "odd numbers," which are "all numbers capable of representation in the form (2k + 1)," where k is an integer), and therefore the set of objects being described is empty. Another illustration is "the set of all human beings currently alive who were born in the 19th century." Since no such human beings exist, this set is empty.

However presented, the cardinality of the empty set is understood to be zero.

Recalling our earlier comment that there exist peculiarities and nuances within the theory of sets, we point out the following issues:

1) { } or ∅ are both used to represent the empty set, or null set
2) The set given by {∅} is not the same as the empty set, or null set

The first point highlights the different notations for the empty set; the second is a subtlety of detail. When we write $N = \{\emptyset\}$, we are referring to a set with one element … that element is the empty set. The cardinality of this set is 1; we will revisit this illustration later in this section, when we introduce the notion of a power set.

The second issue emphasizes a common error we must avoid: imprecise use (or lack of understanding) of notation. We have said ∅ is the empty set; the set {∅} is a set having as a member the empty set. While this may seem bewildering to you, consider the following illustration: suppose we use a thought experiment in which you have an empty cube of glass…the cube can be imagined to be the empty set. Next, suppose you have a cloth bag into which you place the cube. Though the cube remains the empty set, it is now contained within the bag; that is, the bag contains the empty set. Do you see how the two situations are different?

1.2.2 Singleton Sets

Sets having cardinality 1 are referred to as singleton sets. Examples are the sets $S = \{a\}$, $P = \{14\}$, $N = \{\emptyset\}$, $T = \{x|x$ is a natural number equal to its own square$\}$, and $V =$ the set of all states in the United States whose names begin with the letter R.

If we wish to maintain full generality, we could create a "generic" singleton set, $\{x\}$, with no specification as to the nature of the single element.

Though it may seem frivolous to concern ourselves with this terminology, singleton sets are fundamental components of set theory. The singleton sets are useful for expressing qualities of uniqueness and serve as building blocks for more elaborate sets.

1.2.3 Finite and Infinite Sets

The cardinality of a set may be finite (i.e., membership in the set is limited to a particular, possibly very large, quantity of elements) or infinite (there are an unlimited number of elements in the set). Essentially, any specific numerical value you can imagine is finite. At the moment of this writing, the National Debt of the United States of America is $34,687,005,429,763. Though extremely large, this number is finite.

It is worth noting the immensity of this value! Read aloud, this is "thirty-four trillion, six hundred eighty-seven billion, five million, four hundred twenty-nine thousand, seven hundred sixty-three dollars." Were we to count from 1 to this number, allowing ourselves no rest and taking just one second to read each number, it would take us over one million years to do so!

Care should be taken when referring to and using infinity within statements. We should not assert, for example, that space is infinite. This has not been proven, so we cannot make this claim. Moreover, intuition tends to break down when contemplating infinite quantities.

As an illustration, suppose we were told to find the sum of infinitely many positive numbers. The natural conclusion one would draw is that the sum is infinity. However, an elementary example proves this not to be the case! Consider the progression of numbers 1, 1/2, 1/4, 1/8, 1/16, …. We cannot dispute this list of numbers contains only positive values, and the pattern is clear (the denominator doubles from one number to the next). However, the values (when added) fail to reach 2.

The point of the last example serves to emphasize the bizarre nature of infinity, and the failure of intuition when considering the concept. This will not typically be of concern to us in this text, but suggests it is worthy of substantial investigation. The earliest significant work on the topic was performed by Georg Ferdinand Ludwig Philipp Cantor (1845–1918), a Russian-born mathematician who played a pivotal role in the creation of the field of mathematics we call Set Theory.

1.2.4 Special Mathematical Sets

There are collections of numbers that are used so frequently that it becomes convenient to identify them using special characters. Here we will list several, acknowledging that more do exist.

$\mathbb{N}$: the set of natural (counting) numbers

$\mathbb{W}$: the set of whole numbers

$\mathbb{Z}$: the set of integers

$\mathbb{Q}$: the set of rational numbers

$\mathbb{R}$: the set of real numbers

These sets are considered familiar to us, and we will not express detailed definitions here.

1.2.5 Subset and Superset

With the notion of a set now well in hand, we can begin to explore relationships between sets. The first such concept will be the "subset" and "superset" relationships.

Set A is a subset of set B if every member of set A is also a member of set B. The notation we use to indicate this is either $A \subset B$ or $A \subseteq B$, though those are not identical relationships. The first refers to a particular type of subset, called a PROPER subset. The second is a more inclusive and general subset notation.

Set A is a proper subset of set B if all the elements of set A are elements of set B, with the understanding that B contains additional elements not also contained in set A. For instance, the set $\mathbb{Q} \subset \mathbb{R}$. We could more generally say $\mathbb{Q} \subseteq \mathbb{R}$ and still be correct, but we will adopt the convention that we will use the proper subset notation if it is clear a particular subset is a proper subset of another set.

Note, once again, the peculiar nature of infinite cardinality. Though it is true $\mathbb{Q} \subset \mathbb{R}$, it turns out each has infinite cardinality. This appears to "fly in the face" of the assertion that $\mathbb{R}$ contains elements not in $\mathbb{Q}$, and suggests there may be "degrees of infinity," or some similar concept. As mentioned earlier, we will not explore that notion further, though it is definitely worthy of study.

The superset relationship is the opposing concept for subsets. That is, if set A is a subset of set B, then set B is considered a superset of set A. Therefore, $\mathbb{R}$ is a superset of $\mathbb{Q}$. We may express this by reversing the notation for subsets, using either $\supset$ or $\supseteq$. In other words, $\mathbb{R} \supset \mathbb{Q}$.

As an additional remark, we note the empty set is considered to be a subset of every other set, and that a given set is a subset of itself. The latter point seems odd to many who first encounter it, and thus we say set A is an improper subset of itself (yet, still a subset!).

1.2.6 Power Set

We have presented the concept of a subset of a given set and now may contemplate the complete list of subsets of that given set. The set of all subsets (including the empty set and the improper subset) of set A is referred to as the POWER SET of set A and is symbolized as $\mathcal{P}(A)$ or as 2^A.

Observe $\mathcal{P}(\emptyset) = \{\emptyset\}$, because the empty set is a subset of every set, including itself. Moreover, $\mathcal{P}(\{x\}) = \{\emptyset, \{x\}\}$, and $\mathcal{P}(\{x, y\}) = \{\emptyset, \{x\}, \{y\}, \{x, y\}\}$.

Though we could continue this list of illustrations, we make the following observation: if $n(A) = a$, then the cardinality of $\mathcal{P}(A) = n(\mathcal{P}(A)) = 2^a$. Adding an element to a set doubles the number of subsets that set possesses.

Listing full membership of the power set of a set can be extremely tedious. Owing to the rapidity of exponential growth, a set with (e.g.) 10 elements will have $2^{10} = 1024$ subsets!

Despite its unwieldiness, the power set is fundamental in set theory and has applications in various areas of mathematics and computer science, particularly in combinatorics (the branch of mathematics dealing with combinations of objects belonging to a finite set in accordance with particular constraints) and logic.

1.3 Operations on Sets

- Union of sets
- Intersection of sets
- Cardinality of unions and intersections
- Difference of sets

1.3.1 Union of Sets

When two or more sets are combined, we say that we are creating the "**union**" of the sets. We use a symbol resembling a capital "U" in the notation for union, though it is typically shorter and wider than the letter to which it bears a resemblance.

If the sets are A and B, the union of those sets is denoted $A \cup B$. By construction, both A and B are subsets of their union, and their union is hence a superset of both A and B. We refer to the separate sets A and B as the "component sets" of the union. If one set is a subset of the other, the union is equal to the superset.

Keep in mind the roster of a set does not contain duplication of elements, and so if there are elements common to both sets, they should be listed only once in the union.

Consider the set of students in your mathematics class and the set of members of your family; the union of the two sets would be the set of all the members of your math class, together with the members of your family.

We cannot say, at this time, the cardinality of the union is the sum of the cardinalities of the component sets. Why this is the case should, with some thought, be clear to you. We will address this matter later in the section, but it is worthwhile to see if you recognize the reason for this now.

Example 1.3 Let $A = \{1,3,5,7,9\}$ and $B = \{2,3,4,5,6\}$. What is $A \cup B$?

Solution: Combining the two sets, and noticing the elements 3 and 5 appear on both rosters, we find $A \cup B = \{1, 2, 3, 4, 5, 6, 7, 9\}$.

Referring to our remark from just prior to the example, observe $\mathrm{n}(A) = 5$ and $\mathrm{n}(B) = 5$, but $\mathrm{n}(A \cup B) \neq 5 + 5 = 10$. Now do you see why the cardinality of the union might not equal the sum of the component set cardinalities?

Example 1.4 Let $A = \{x \in \mathbb{R} \mid x^2 = x\}$ and $B = \{x \in \mathbb{R} \mid x^2 - x = 6\}$. What is $A \cup B$? Give your answer as a roster.

Solution: One way to give the union would be to employ the word "or" into our set-builder notation; that is, $A \cup B = \{x \in \mathbb{R} \mid x^2 = x \text{ or } x^2 - x = 6\}$. This, however, is not a roster (as was requested) and is uninformative. We know next to nothing about the membership of the set if we employ this technique.

We have several methods of attack open to us, and we shall explore both (we may discover something interesting). For set A, what is the roster of its members? We know the criteria for membership, so we solve the equation mentioned in the property area of set-builder notation.

$$x^2 = x$$
$$x^2 - x = 0$$
$$x(x-1) = 0$$
$$x = 0 \quad \text{or} \quad x - 1 = 0$$
$$x = 0 \quad \text{or} \quad x = 1$$

Therefore, A = {0, 1}.

For set B, we must solve:

$$x^2 - x = 6$$
$$x^2 - x - 6 = 0$$
$$(x-3)(x+2) = 0$$
$$x - 3 = 0 \quad \text{or} \quad x + 2 = 0$$
$$x = 3 \quad \text{or} \quad x = -2$$

Therefore, $B = \{3, -2\}$.

Combining our results, $A \cup B = \{0, 1, 3, -2\}$.

Now consider the following (flawed) reasoning: we know the members of the union must satisfy $x^2 = x$ or they must satisfy $x^2 - x = 6$. By substitution, since $x^2 = x$, we must have $x^2 - x^2 = 6$, or $0 = 6$. This is a contradiction, so the union of the two sets must be empty.

What we have done with this second approach, while appearing completely reasonable, was to change the nature of the problem. By using substitution in the manner described, we have actually resolved the issue regarding which values of x are such that $x^2 = x$ AND $x^2 - x = 6$. This subtle change in language has led us completely astray and points out to us that we must be very cautious in our choice of wording, and of our application of mathematical techniques!

1.3.2 Intersection of Sets

When the common elements of two or more sets are considered and gathered into a set, we say we are creating the "**intersection**" of the sets. We use the upside-down form of the union symbol to represent intersection.

If the sets are A and B, the intersection of those sets is denoted $A \cap B$. By construction, $A \cap B$ is a subset of both A and B, and both A and B are supersets of their intersection. We again refer to the separate sets A and B as the "component sets" of the intersection.

If A and B have the same set of elements, then the intersection is all elements in either of the component sets. If A and B have no common elements, the intersection is the empty set.

Example 1.5 Let A = the set of even integers between 0 and 20, inclusive, and B = the set of integers divisible by 3 between 0 and 20, inclusive. Find $A \cap B$.

Solution: Set $A = \{0,2,4,6,8,10,12,14,16,18,20\}$, set $B = \{0,3,6,9,12,15,18\}$. The members common to the two sets are 0,6,12, and 18, so $A \cap B = \{0,6,12,18\}$.

Example 1.6 Let $A = \{2,4,6,8,10\}$, $B = \{x|\ x^2 - 2x - 3 = 0\}$. Find $A \cap B$.

Solution: The membership of A is explicit, but the membership of B is not so clear. We will work to find the membership of B as a roster, and then the intersection should be apparent.

We begin by solving $x^2 - 2x - 3 = 0$. The expression on the left-hand side of the equation factors as $(x-3)(x+1) = 0$, and the zero-product rule of algebra tells us we must have either $x - 3 = 0$ or $x + 1 = 0$, which yields $x = 3$ or $x = -1$, so $B = \{3,-1\}$.

Next, observe there are no members common to A and B, and so we conclude the intersection is the empty set, and write $A \cap B = \emptyset$.

1.3.3 Cardinality of Unions and Intersections

Consider the following paraphrase of the definition of the union of two sets: "the union of two sets is created by combining the two sets together, without including duplications of common elements."

With this in mind, we can reason out the cardinality of the union. The cardinality of the union will be the sum of the cardinalities of the component sets, subtracting off the number of duplications of elements. The duplication of elements is precisely the intersection of the two sets, those elements common to both sets.

That is,

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

Rearrangement using algebra gives us a formula for the cardinality of the intersection:

$$n(A \cap B) = n(A) + n(B) - n(A \cup B)$$

Example 1.7 We saw that for A = the set of even integers between 0 and 20, inclusive, and B = the set of integers divisible by 3 between 0 and 20, inclusive, the intersection was $\{0,6,12,18\}$. Show the formula for the cardinality of the intersection and union holds for these sets.

Solution: We find $n(A) = 11$, $n(\mathrm{B}) = 7$, and $n(A \cap B) = 4$. Therefore, $n(A \cup B) = 11 + 7 - 4 = 14$. Further, we find $n(A \cap B) = 11 + 7 - 14 = 4$.

1.3.4 Difference of Sets

Recall that $A \subset B$ and $A \subseteq B$ indicate B is a superset of A. This means (in the case of a proper subset relationship) there exist elements in set B that are not in set A. This list of additional elements is called the "difference between A and B," denoted by $B - A$. Observe the superset is listed first.

The difference between sets could be infinite, if the superset is infinite. If both sets are finite, the difference is finite, and if the subset A is not a proper subset of B, the difference could be empty.

Note the rationale for using the term "empty set" becomes more evident as we proceed. In the case of the difference of sets, imagining the "additional elements" are in a theoretical "bag," if there is no difference between the sets then the bag is empty.

Note further this concept of "no difference" suggests, as we will soon discover, there exists a notion of set equality.

Example 1.8 For the sets $A = \{3,4,5,6,7,8\}$ and $B = \{1,2,3,4,5,6,7,8\}$, what is set $B - A$?

Solution: Set B contains elements 1 and 2 that are not in set A, so $B - A = \{1,2\}$.

Example 1.9 Let $A = \{x \mid x$ is an even non-negative integer less than 10$\}$, $B = \{x \mid x$ is a whole number less than 10$\}$. What is $B - A$?

Solution: The roster of set A is $\{0,2,4,6,8\}$, set B is $\{0,1,2,3,4,5,6,7,8,9\}$. We see that B is a superset of A, and that $B - A = \{1,3,5,7,9\}$.

Example 1.10 Let A = The set of integers whose absolute value is less than or equal to 10, and $B = \{0,1,2,3,4,5,6,7,8,9,10\}$. What is $B - A$?

Solution: The sets A and B coincide; that is, A and B have the same membership. Therefore, $B - A = \emptyset$.

1.4 Set Relations

- Equality of sets
- Equivalence of sets
- Disjoint sets
- Universal set
- Complement of a set
- Venn diagrams

Section 1.3 closed with the notion of the "difference between sets." If there is no difference (i.e., if the difference of sets is empty), the sets are equal.

Equal sets possess precisely the same elements as one another. The elements may be described differently, have their order shuffled, or presented in different manners, but the list of elements in the sets is precisely the same. This means equal sets have precisely the same cardinality, though sets with the same cardinality need not be equal.

Consider the following sets:

$A = \{x \mid x$ is an even natural number less than or equal to 100$\}$
$B = \{2,4,6,\ldots,100\}$
$C = \{x \mid x = 2n,\ n \in \mathbb{N},\ 1 \leq n \leq 50\}$

The sets A, B, and C are all equal. We have seen here and in the past that sets can be represented in multiple ways, and it is the membership (not the description) that establishes their equality.

The relationship of equality between sets possesses three fundamental properties we now point out explicitly:

1) Sets should be equal to themselves, that is, it should always be true that $A = A$. This requirement is referred to as *reflexivity*.
2) If $A = B$, then $B = A$. This requirement is referred to as *symmetry*.
3) If $A = B$ and $B = C$, then $A = C$. This is referred to *transitivity*.

These properties hold for many other relationships within other sets, including the real numbers. In particular, we shall see the properties hold for "directed graphs," which we shall encounter later in this work.

1.4.1 Equivalence of Sets

Informally, we can say that sets having precisely the same cardinality are said to be **equivalent** sets. It is evident that equal sets are also equivalent, but that the converse is not necessarily true.

That is, $A = \{1,3,5,7\}$ and $B = \{\alpha, \beta, \gamma, \delta\}$ both have cardinality 4, and so are equivalent sets. By inspection, we see the sets are not equal, as their elements are not identical.

The notation $A \sim B$ shall indicate the sets A and B are equivalent.

Definition Let be sets. We say that A is equivalent to B if and only if there exists a bijection $f : A \to B$. In the case of the example above, we can define such a bijection as:

$$f(1) = \alpha,\ f(2) = \beta,\ f(3) = \gamma,\ f(4) = \delta$$

Clearly, this function is both one-to-one and onto and serves as such a bijection (there could be many bijections between two equivalent sets; finding any particular one suffices).

In the matter of set equivalence, (as was the case in the concept of set equality) the reflexive, symmetric, and transitive relationships exist:

1) Things should be equivalent to themselves. That is, it should always be true that $A \sim A$.
2) If $A \sim B$, then $B \sim A$.
3) If $A \sim B$ and $B \sim C$, then you want $A \sim C$.

Note: These three qualities are, in general, referred to as the properties of an ***equivalence relation***.

1.4.2 Equivalence and Non-equivalence of Infinite Sets

We recall once more our admonition that one must be cautious when dealing with the concept of infinity. Are sets with infinite cardinality equivalent to one another? This is a challenging question, the answer for which can vary depending upon the situation: some infinite sets can be shown to be equivalent while others cannot.

The explanation for this once more falls upon our loss of intuition when it comes to infinity. Clearly, $\mathbb{N} \subset \mathbb{Q}$, because there are rational numbers that are not natural numbers. This situation presents us with a conundrum we will not explore in this course, though we do encourage you to pursue this issue as time permits.

Example 1.11 Show the sets $\mathbb{N}$ and $\mathbb{W}$ are equivalent.

Solution: The sets are each infinite, and we can readily produce a bijection between them. Consider the function $\mathrm{f}(x) = x - 1$ from $\mathbb{N}$ to $\mathbb{W}$. The function is clearly one-to-one and onto, and thus serves as a sample bijection. Therefore, the two sets are equivalent.

Example 1.12 Show the sets $\mathbb{N}$ and $\mathcal{P}(\mathbb{N})$ are not equivalent.

Solution: Since the natural numbers form an infinite set, the power set of the natural numbers is also an infinite set. However, we now show the sets are not equivalent.

There are many one-to-one functions from $\mathbb{N}$ to $\mathcal{P}(\mathbb{N})$, such as $f(n) = \{n\}$. The image, obviously, is an element of $\mathcal{P}(\mathbb{N})$. Notice the range of f, however, is a subset of $\mathcal{P}(\mathbb{N})$, but not equal to the entire set $\mathcal{P}(\mathbb{N})$.

Suppose the two sets ARE equivalent. Then there exists a bijection f that maps $\mathbb{N}$ to $\mathcal{P}(\mathbb{N})$. That is, every natural number is associated in a one-to-one, onto relationship with some set within the power set. Let us define a new set: $E = \{x \in \mathbb{N} \mid x \text{ is not an element of } f(x)\}$. The set E “collects” exactly those elements of S that are not in the subset they supposedly got paired with under the function f. Because E is a subset of $\mathbb{N}$, D is a member of $\mathcal{P}(\mathbb{N})$, which implies there is an element e in $\mathbb{N}$ such that $f(e) = E$. Consider the non-obvious question: is "e" in the set E, or not? If it is, then by the definition of set E, e is not an element of $f(e) = E$, which is a contradiction. On the other hand, if e is not an element of the set E, then by definition e is an element of $f(e) = E$, which is again a contradiction. Since we were supposing a bijection did exist and that led to contradictions, no such bijection exists.

1.4.3 Disjoint Sets

We have mentioned there is the possibility the intersection of two sets is empty. That is, two given component sets could have no members in common. When this happens, we say the sets are "disjoint."

The set of all even integers is disjoint from the set all odd integers, for instance. The set of human beings currently alive and the set of human beings who fought in The Revolutionary War are disjoint sets.

Note the consequence for the formulas, for cardinality of intersections and unions of sets, should the component sets be disjoint:

For unions:

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

$$n(A \cup B) = n(A) + n(B) - 0$$

$$n(A \cup B) = n(A) + n(B)$$

For intersections:

$$n(A \cap B) = n(A) + n(B) - n(A \cup B)$$

$$0 = n(A) + n(B) - n(A \cup B)$$

$$n(A \cup B) = n(A) + n(B)$$

As one would expect, the two formulas simplify to the same final expression. This is a good thing; had they not, there would be a fundamental problem with our notions of cardinality and our newly introduced quality of sets being disjoint.

This reminds us of an important and fundamental fact: mathematical ideas should not conflict with one another. If it appears they do, then there is either a problem with calculations or with definitions. Whatever the case, such conflicts must be investigated and resolved.

1.4.4 Universal Sets

When considering sets, an early step is to establish a context within which the set exists. It is customary to define this context by means of an all-encompassing set called the universal set, which we shall symbolize by U.

In each situation, all sets considered within a particular problem must take their members from that universal set. An example of a universal set could be a particular set of numbers, or a particular group of individuals. If no universal set is specified, it may be that its composition is irrelevant for the purposes of the problem, and so we can take it to be some larger set within which the members of our sets reside.

For a specific set, the universal set could vary from example to example. Suppose that our set was the set of all persons currently enrolled at your college. The universal set could (unless specifically stated) be the set of all persons within the town in which the college lies, the set of all persons enrolled at any college at all within the United States of America, or the individuals currently located within the United States of America. When the universal set is not made explicit by context, and when the identity of the universal set is of importance, we will state it specifically.

Once the universal set has been identified, we will only consider members from that set for the duration of the problem under consideration. If, for instance, we define U = {3,4,5,6,7,8}, then for the remainder of the problem, no other objects exist. Thus, using that universal set, if we then wanted to create a set consisting of all even numbers from the universal set, and call that set E, we would say E = {4,6,8}. While there are other even numbers from the broader context of all possible numbers, from the perspective of our universal set, these are the only even numbers in existence.

Consider now the letters q and Q. If the universal set were the 26 letters of the English alphabet, then we could rightly claim that both symbols represent the same thing, and thus are the same. If, on the other hand, the universal set is the 52-member collection of upper- and lower-case letters of the English alphabet, then the elements q and Q are distinct, and should not be treated as being one and the same.

The moral to that last example is that we must be very careful when we consider the members of sets because it is possible that the specification of the universal set could affect our observations and analysis.

1.4.5 The Complement of a Set

Given a universal set U, and a set A within that universal set, it is meaningful to speak of the "complement" of the set. Observe this should not be mistaken for the homonym "compliment." The "compliment" of a set might be, "You're a very attractive set." This is not the same as the "complement" of a set!

The complement of set A, represented by A^c or A', refers to the set of elements in the universal set not contained within set A. Without knowledge of the nature of the universal set, it is impossible to accurately describe the complement of set A.

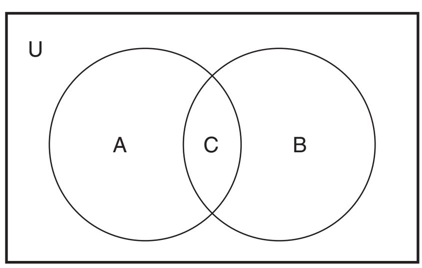
If we let $U = \{a,b,c,d,e,f,g\}$, then identify $V = \{a,e\}$, we find V^c = {b,c,d,f,g}.

The sets V and V^c are, by definition, disjoint, and their union is equal to the universal set.

1.4.6 Venn Diagrams

A Venn Diagram (named after its inventor, John Venn [1834–1923]) is a useful visual tool to represent sets and their relationships to one another.

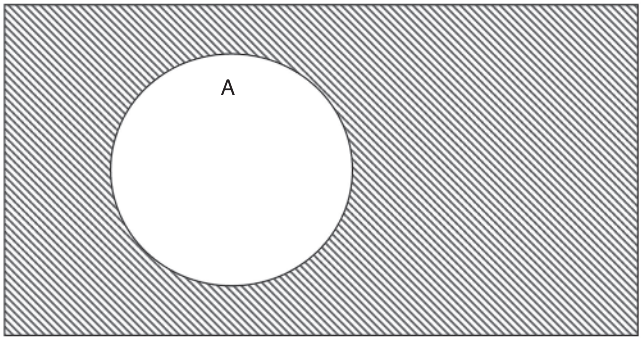
A typical Venn Diagram would look something like this:



In this situation, the two overlapping circles A and B each represent sets, and the area that is common to both circles is labeled C, which is the overlap, or intersection of sets A and B.

The sets are surrounded by a rectangle that encloses them; the rectangle represents what we have identified as the universal set.

When we consider a set drawn in a Venn Diagram, such as A above, we can envision the circle representing A breaking the universal set into two pieces: those things that are in set A, and those things that are not. Thus, the region within the rectangle but outside the circle represents A^c.



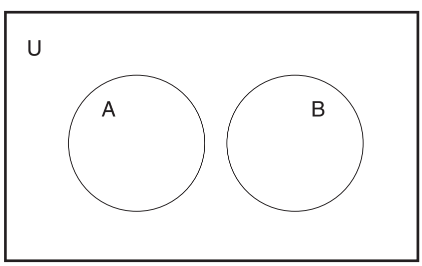
In the image, the shaded area of the rectangular representation of the universal set is A^c.

Consider the following example. Suppose that the universe consists of all the states in the United States. Set A represents the set of states whose names begin with the letter "N." Set $A = \{$Nebraska, Nevada, New Hampshire, New Jersey, New Mexico, New York, North Carolina, North Dakota$\}$. Those are the states sitting inside circle A. The other 42 states are OUTSIDE of the circle A and comprise A^c.

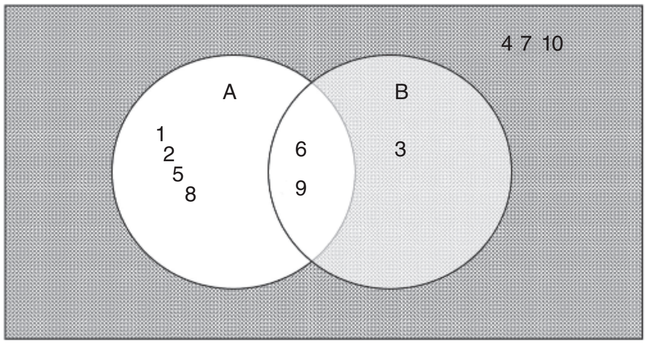
Suppose that set B represents the states in the "contiguous United States." That is, the states that are in the "main part" of the United States, all the connected states.

The area labeled C on diagram would be the "overlap," or "intersection" of the two sets. Since the only two states not within the contiguous United States are Alaska and Hawaii, the region represented by C is empty.

Though, in generality and if it is made clear which elements are within sets A and B, it is permissible to represent this situation with overlapping circles, the disjoint relationship of sets A and B is typically made clear by showing the sets A and B as non-intersecting circles.



Example 1.13 Construct a Venn diagram representation of the universal set $U = \{x \in \mathbb{Z} \mid 0 < x \leq 10\}$, with sets $A = \{1,2,5,6,8,9\}$ and $B = \{3,6,9\}$.



Note the universal set is not labeled with "U." This can be done, but we understand in this text, the all-encompassing rectangle will be the universal set and will suppress labeling of such.