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Monte Carlo and Numerical Integration Methods

Much of this book provides multiple examples that solve the following integral by Monte Carlo or numerical techniques rather than analytical methods, i.e.,

$$I = \int_a^b f(x)dx$$

In general, this integral has an integrand $f(x)$ with boundaries between $x = a$ and $x = b$. Many computations treat the lower case x as a variable X that is a random variable (r.v.) with a probability density function (pdf) $p_X(x)$.¹

Two broad classes of numerical integration include:

- Monte Carlo or random methods
- Deterministic or nonrandom methods

Examples of Monte Carlo methods include:

sample mean, Accept–Reject or rejection sampling, hit-or-miss, inverse transform, importance sampling, Metropolis–Hastings, Gibbs, Slice, Hamiltonian Monte Carlo, and Sequential Monte Carlo or particle filters.

Examples of deterministic methods include:

rectangular (midpoint), trapezoidal, Simpson, Gauss quadrature, and MATLAB integral.

Monte Carlo techniques use random sampling to perform numerical integration.² An important distinction between deterministic and Monte Carlo methods involves the selected sampling method.

1 Random variables are shown in capitals.

2 Robert, C.P. and Casella, G. (2004). *Monte Carlo Statistical Methods*, 2e. Springer.

In the methods discussed here, the continuous integral I is replaced by an estimate \hat{I} having a discrete representation in terms of the number of samples N , i.e.,

$$\hat{I} = \frac{b-a}{N} \sum_{i=1}^N f(x_i)$$

where x_i , $i = 1, \dots, N$ are independent samples (realizations) drawn from random variable X having a uniform pdf over the interval (a, b) .

By choosing N sufficiently large, the estimate converges to an accurate value of the integral due to the Law of Large Numbers (LLN). With many independent samples from the same distribution, the LLN indicates that the average of the samples approaches the mean of the actual distribution. A further consequence of the Central Limit Theorem (CLT) states that as the sample size approaches infinity, the distribution of the sample mean approaches a normal distribution independent of the original distribution.

Prior to delving directly into Monte Carlo methods, it is instructive to review historical numerical integration methods. An important aspect of this direction is the use of the MathWorks® tool in MATLAB® called “integral” based on adaptive quadrature and error tolerances.

The MathWorks® provides numerous examples using program `integral` where several of the possibilities include:

- Limits can be plus or minus infinity.
- Parameters can be inserted in the integrand.
- Tolerances are specified.
- Integration is performed over the complex plane.
- Double integrals can be evaluated using MATLAB “`integral2`.”

A more complete description of this tool is obtained by typing “`help integral`” in the MATLAB command window.