Chapter 1

Introduction

In this chapter, we provide a conceptual description of the method of probability metrics and discuss direct and indirect applications in the field of finance, which are described in more detail throughout the book.

1.1 Probability Metrics

The development of the *theory of probability metrics* started with the investigation of problems related to limit theorems in probability theory. Limit theorems occupy a very important place in probability theory, statistics, and all their applications. A well-known example is the celebrated central limit theorem (CLT) but there are many other limit theorems, such as the generalized CLT, the max-stable CLT, functional limit theorems, etc. In general, the applicability of the limit theorems stems from the fact that the limit law can be regarded as an approximation to the stochastic model under consideration and, therefore, can be accepted as an approximate substitute. The central question arising is how large an error we make by adopting the approximate model and this question can be investigated by

A Probability Metrics Approach to Financial Risk Measures by Svetlozar T. Rachev, Stoyan V. Stoyanov and Frank J. Fabozzi

studying the distance between the limit law and the stochastic model. It turns out that this distance is not influenced by the particular problem. Rather, it can be studied by a theory based on some universal principles.

Generally, the theory of probability metrics studies the problem of measuring distances between random quantities. On one hand, it provides the fundamental principles for building probability metrics – the means of measuring such distances. On the other, it studies the relationships between various classes of probability metrics. Another realm of study concerns problems which require a particular metric while the basic results can be obtained in terms of other metrics. In such cases, the metrics relationship is of primary importance.

Certainly, the problem of measuring distances is not limited to random quantities only. In its basic form, it originated in different fields of mathematics. Nevertheless, the theory of probability metrics was developed due to the need of metrics with specific properties. Their choice is very often dictated by the stochastic model under consideration and to a large extent determines the success of the investigation. Rachev (1991) provides more details on the methods of the theory of probability metrics and its numerous applications in both theoretical and more practical problems.

1.2 Applications in Finance

There are no limitations in the theory of probability metrics concerning the nature of the random quantities. This makes its methods fundamental and appealing. Actually, in the general case, it is more appropriate to refer to the random quantities as random *elements*. They can be random variables, random vectors, random functions or random elements in general spaces. For instance, in the context of financial applications, we can study the distance between two random stocks prices, or between vectors of financial variables that are used to construct portfolios, or between yield curves which are much more complicated objects. The methods of the theory remain the same, irrespective of the nature of the random elements. This represents the most direct application of the theory of probability metrics in finance: that is, it provides a method for measuring how different two random elements are. We explain the axiomatic construction of probability metrics and provide financial interpretations in Chapter 2.

Financial economics, like any other science relying on statistical methods, considers statistical information about the objects it studies on several levels. In some theories in the area of finance, conclusions are drawn only on the basis of certain characteristics of the corresponding distributions. For example, an investor would oftentimes use a risk-reward ratio to rank investment opportunities. Essentially, this reduces to computing the measure of reward (e.g., the expected return) and the measure of risk (e.g., value-at-risk, conditional value-at-risk, standard deviation). Both the measure of reward and the measure of risk represent two characteristics of the corresponding distributions. In effect, the final decision is made on the basis of these two characteristics which, from the investor's perspective, aggregate the information available in the distribution functions.

The theory describing investor choice under uncertainty, the fundamentals of which we discuss in Chapter 3, uses a different approach. Various criteria were developed for first-, second-, and higher-order stochastic dominance based on the distributions themselves. As a consequence, investment opportunities are compared directly through their distribution functions, which is a superior approach from the standpoint of the utilized information.

As another example, consider the problem of building a diversified portfolio. The investor would be interested not only in the marginal distribution characteristics (i.e., the characteristics of the assets on a stand-alone basis), but also in how the assets depend on each another. This requires an additional piece of information which cannot be recovered from the distribution functions of the asset returns. The notion of stochastic dependence can be described by considering the joint behavior of assets returns.

The theory of probability metrics offers a systematic approach towards such a hierarchy of ways to utilize statistical information.

It distinguishes between *primary*, *simple*, and *compound* types of distances which are defined on the space of characteristics, the space of distribution functions, and the space of joint distributions, respectively. Therefore, depending on the particular problem, one can choose the appropriate distance type and this represents another direct application of the theory of probability metrics in the field of finance. This classification of probability distances is explained in Chapter 4.

Besides direct applications, there are also a number of indirect ones. For instance, one of the most important problems in risk estimation is formulating a realistic hypothesis for the asset return distributions. This is largely an empirical question because no arguments exist that can be used to derive a model from some general principles. Therefore, we have to hypothesize a model that best describes a number of empirically confirmed phenomena about asset returns: (1) volatility clustering, (2) autoregressive behavior, (3) short- and long-range dependence, and (4) fat-tailed behavior of the building blocks of the time-series model which varies depending on the frequency (e.g., intra-day, daily, monthly). The theory of probability metrics can be used to suggest a solution to (4). The fact that the degree of heavy-tailedness varies with the frequency may be related to the process of aggregation of higher-frequency returns to obtain lower frequency returns. Generally, the residuals from higher-frequency return models tend to have heavier tails and this observation together with a result known as a *pre-limit theorem* can be used to derive a suggestion for the overall shape of the return distribution. Furthermore, the probability distance used in the pre-limit theorem indicates that the derived shape is most relevant for the body of the distribution. As a result, through the theory of probability metrics we can obtain an approach to construct reasonable models for asset return distributions. We discuss in more detail limit and pre-limit theorems in Chapter 7.

Another central topic in finance is quantification of risk and uncertainty. The two notions are related but are not synonymous. Functionals quantifying risk are called *risk measures* and functionals quantifying uncertainty are called *deviation measures* or *dispersion* *measures*. Axiomatic constructions are suggested in the literature for all of them. It turns out that the axioms defining measures of uncertainty can be linked to the axioms defining probability distances, however, with one important modification. The axiom of symmetry, which every distance function should satisfy, appears unnecessarily restrictive. Therefore, we can derive the class of deviation measures from the axiomatic construction of asymmetric probability distances which are also called *probability quasi-distances*. The topic is discussed in detail in Chapter 5.

As far as risk measures are concerned, we consider in detail advantages and disadvantages of value-at-risk, average value-at-risk (AVaR), and spectral risk measures in Chapter 5 and Chapter 6. Since Monte Carlo-based techniques are quite common among practitioners, we discuss in Chapter 7 Monte Carlo-based estimation of AVaR and the problem of stochastic stability in particular. The discussion is practical, based on simulation studies, and is inspired by the classical application of the theory of probability metrics in estimating the stochastic stability of probabilistic models. We apply the CLT and the Generalized CLT to derive the asymptotic distribution of the AVaR estimator under different distributional hypotheses and we discuss approaches to improve its stochastic stability.

We mentioned that adopting stochastic dominance rules for prospect selection rather than rules based on certain characteristics leads to a more efficient use of the information contained in the corresponding distribution functions. Stochastic dominance rules, however, are of the type "X dominates Y" or "X does not dominate Y": that is, the conclusion is qualitative. As a consequence, computational problems are hard to solve in this setting. A way to overcome this difficulty is to transform the nature of the relationship from qualitative to quantitative. We describe how this can be achieved in Chapter 8, which is the last chapter in the book. Our approach is fundamental and is based on asymmetric probability semidistances, which are also called *probability quasi-semidistances*.

The link with probability metrics theory allows a classification of stochastic dominance relations in general. They can be primary, simple, or compound but also, depending on the underlying structure, they may or may not be generated by classes of investors, which is a typical characterization in the classical theory of choice under uncertainty. This is also a topic discussed in Chapter 8.

References

Rachev, S. T. (1991), *Probability Metrics and the Stability of Stochastic Models*, Wiley, New York.