1 Basic Concepts and Measures

1.1 Meaning of "Population"

To a statistician, the term "population" refers to a collection of items, for example, balls in an urn. Demographers use the term in a similar way to denote the collection of persons alive at a specified point in time who meet certain criteria. Thus, they may refer to the "population of India on April 1, 1995," or to the "population of American black females in the Northeast on June 1, 1900." In both cases the criteria for inclusion in the population need further elaboration: do we count "legal residents" or simply those who can be found within the borders on that date? What do we mean by "black," or by "Northeast"? Do we refer to midnight or noon on the specified date? It is clear that "the population of India on April 1, 1995" is a shorthand description of what may be a rather long set of operational choices designed to minimize blurriness at the boundaries.

But demographers also use the term "population" to refer to a different kind of collectivity, one that persists through time even though its members are continuously changing through attrition and accession. Thus, "the population of India" may refer to the aggregate of persons who have ever been alive in the area we define as India and possibly even to those yet to be born there. The collectivity persists even though a virtually complete turnover of its members occurs at least once each century.

Demographic analysis focuses on this enduring collectivity. It is particularly addressed to studying changes in its size, its growth rates, and its composition. But while the emphasis is on understanding aggregate processes, demography is also attentive to the implications of those processes for individuals. Many of the indexes in common use in demography, such as life expectancy at birth and the total fertility rate, translate aggregate-level processes into statements about the demographic circumstances faced by an average or randomly-chosen individual. In turn, a frequent concern in demography is to trace out the consequences of changes in individual-level behavior for aggregate processes. Demography is one of the social science disciplines where micro- and macro-level analyses find perhaps their most complete and satisfactory articulation.

1.2 The Balancing Equation of Population Change

No matter how a population is defined, there are only two ways of entering it: being born into it; or migrating into it. If the definition of the population includes a social element in addition to the customary geographic/temporal elements, then "migration" can include a change in the social label, a process often referred to as "social mobility." For example, the population of American high school graduates can be entered by achieving a high school diploma, a form of social migration or mobility. Note in this example that the population cannot be entered at birth since the acquisition of the label of high school graduate requires the investment of years of life. Populations defined by marital status or occupation are other examples of populations that cannot normally be entered by birth (except for the default options, unmarried and no occupation). On the other hand, populations defined by characteristics fixed at birth, such as sex, race, or nativity, cannot be entered through migration but only through birth. So there are *at most* two ways of entering a population, birth and in-migration (= immigration).

Likewise, there are at most two ways of leaving a population, death and out-migration (= emigration). All populations can be left through death, but only those defined by characteristics not fixed at birth can be exited through migration. If one is born in the United States, one cannot leave the population of persons born in the United States by migration, but one can obviously leave the population resident in the United States by migration.

Because there are only four possible ways of entering or leaving a population, we can be sure that changes in the size of the population must be attributable to the magnitude of these flows. In particular,

$$N(T) = N(0) + B[0, T] - D[0, T] + I[0, T] - O[0, T], \qquad \textbf{(1.1)}$$

where

$N(T)$ = number of persons alive in the population at time T,
$N(0)$ = number of persons alive in the population at time 0,
$B[0, T]$ = number of births in the population between time 0 and time T,
$D[0, T]$ = number of deaths in the population between time 0 and time T,
$I[0, T]$ = number of in-migrations between time 0 and time T,
$O[0, T]$ = number of out-migrations from the population between time 0 and time T.

The unit of time in this equation, and throughout the book unless otherwise noted, is number of years. Thus, the time period in which births, deaths, and migrations are occurring is T years in length. T may be fractional and need not be an integer number.

Kenneth Boulding has called this equation the most fundamental in the social sciences. It is clearly an *identity* rather than an approximation or a hypothesized relation. However, when data are used to estimate the elements of this equation, it is no longer the case that both sides must be equal. Error in measuring any element will cause an imbalance in the equation, unless two or more errors happen to be exactly offsetting. An imbalance in the equation is sometimes

Box 1.1 *The Balancing Equation of Population Change*

$N(T) = N(0) + B[0, T] - D[0, T] + I[0, T] - O[0, T]$

Example: Sweden, 1988

Ending population Jan. 1, 1989		Starting population Jan. 1, 1988	Births between Jan. 1, 1988 and Jan. 1, 1989	Deaths between Jan. 1, 1988 and Jan. 1, 1989	In-migrations between Jan. 1, 1988 and Jan. 1, 1989	Out-migrations between Jan. 1, 1988 and Jan. 1, 1989
$N(1989.0)$	$=$	$N(1988.0)$	$+ B[1988.0, 1989.0]$	$- D[1988.0, 1989.0]$	$+ I[1988.0, 1989.0]$	$- O[1988.0, 1989.0]$
8,461,554	=	8,416,599	+ 112,080	− 96,756	+ 51,092	− 21,461

Data source: United Nations, *Demographic Yearbook* (various years).

referred to as an "error of closure." Box 1.1 demonstrates the application of the equation to data from Sweden, which are among the world's most reliable.

1.3 The Structure of Demographic Rates

The balancing equation of population change breaks down the changes in the size of the population into four flows. Each flow is the sum of events or transitions occurring to individuals. Three of the four types of events can be related to an individual present in the population prior to the event. While death and out-migration can be related to one individual, birth can be related to two individual parents, assuming that both belong to the population of interest. Analytical insight can be gained by relating the size of these flows (number of occurrences) to the size of the population producing them. This task is normally accomplished by constructing a demographic "rate."

The term "rate" is used in many fields and its meaning is not consistent. An unemployment rate, for example, is simply a *ratio* of the unemployed to the total labor force at a moment in time. In demography, rates are normally (but not invariably) what are known in statistical parlance as "occurrence/exposure rates." The typical form of demographic rates reflects the fact that the frequency of occurrences can be expected to be higher in a larger population, and that the total number of occurrences can also be expected to be higher the longer the members of the population are exposed to the "risk" of the occurrence. The amount of exposure in the denominator of an occurrence/exposure rate combines these two features – the number of persons in the population and the length of the time frame in which exposure is counted. The most conventional occurrence/exposure rate in demography takes the form of:

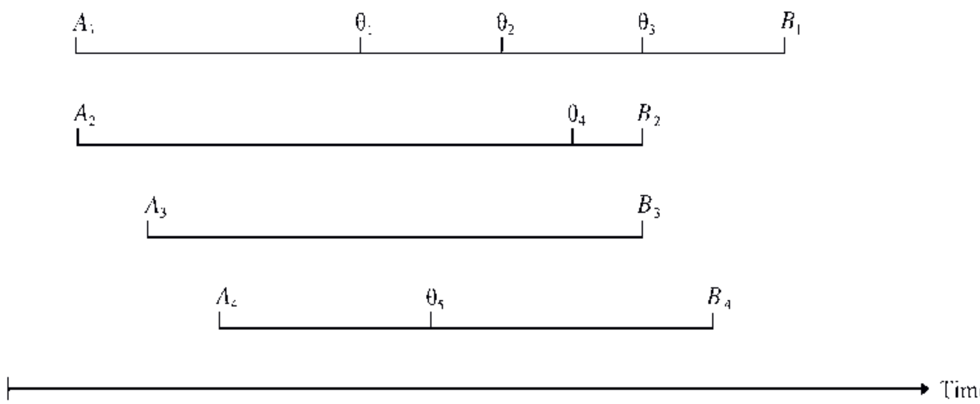
$$Rate = \frac{Number\ of\ Occurrences}{Person\text{-}years\ of\ Exposure\ to\ the\ Risk\ of\ Occurrence}$$

Demographic rates thus contain in the numerator a count of the number of events occurring within some defined time period, and in the denominator an estimate of the number of "person-years" lived in the population during that time period. The number of person-years functions in part as an indicator of the population's amount of exposure to the risk of the event, hence the term *occurrence/exposure rate*. When person-years are used in the denominator, a rate is referred to as an "annualized" rate.

Unlike occurrences, the number of person-years lived is rarely directly observed or counted. Nevertheless, the concept is central in demography. To deal with the concept in a population that is continuously changing its membership, it is useful to represent individual exposures as "life-lines." A life-line extends from an individual's birth (A) to the point where he or she experiences some terminal event (B), usually death. Occurrences of interest, θ_i, can be added to the life-line, as illustrated below:

In order to better connect events and exposure to the risk of experiencing the event, a life-line is sometimes restricted: if we are interested in the risk of giving birth, for instance, we may restrict analysis of life-lines to a certain age range. In our exposition, event A and B are simply birth and death respectively, but the concept can readily be extended to other types of bounding events.

For a group of individuals, however the group might be defined, the concept of the occurrence/exposure rate can be illustrated by a set of life-lines for each member of the group G:



where θ_j are the event occurrences in group G and A_i and B_i represent the birth and death of individual i in the group. The rate for the group defined over their entire lifetimes is

$$Rate^G = \frac{\sum_{i \in G} N_i}{\sum_{i \in G} T_i}$$

where N_i is the total number of occurrences in the lifetime of individual i, T_i is the length of time between A_i and B_i, and $\sum_{i \in G}$ is an instruction to take the sum across all individuals (i) who are a member of group G.

1.4 Period Rates and Person-years

A period rate for a population is constructed by limiting the count of occurrences and exposure times to those pertaining to members of the population during a specified period of time:

$$Rate\,[0, T] = \frac{\textit{Number of Occurrences between Time } 0 \textit{ and } \mathrm{T}}{\textit{Person-years Lived in the Population between Time } 0 \textit{ and } \mathrm{T}}$$

If a person lives one year between time 0 and time T, he or she has contributed one person-year to the denominator of the period rate. If a person lives 24 hours between 0 and T, he or she has contributed 1/365th of a person-year. The contributions from all individuals who were alive in the population at any time between 0 and T are simply added together in order to produce the denominator for our rates.

The idea is easily grasped by referring again to life-lines. If we are interested in period 0 to T, all life-lines can be truncated to the "window" 0 to T, since we will not count any occurrences outside that interval. Figure 1.1 shows the life-lines of 7 individuals in a small hypothetical population during the period from 12:00 A.M., January 1, 1981 to 12:00 A.M. on January 1, 1982.

Person 1, for example, is a member of the population for the entire year, whereas person 6 is born on April 1 and dies on October 1, thereby contributing only 6 months or one-half of a person-year to the sum of person-years. Adding exposure across individuals would be a convenient way to estimate person-years lived in country that had a population register which recorded exact dates of birth, death, and migration for each individual.

An alternative method of computing period person-years is to ignore individual histories, such as those provided by a population register, and simply record the number of persons alive

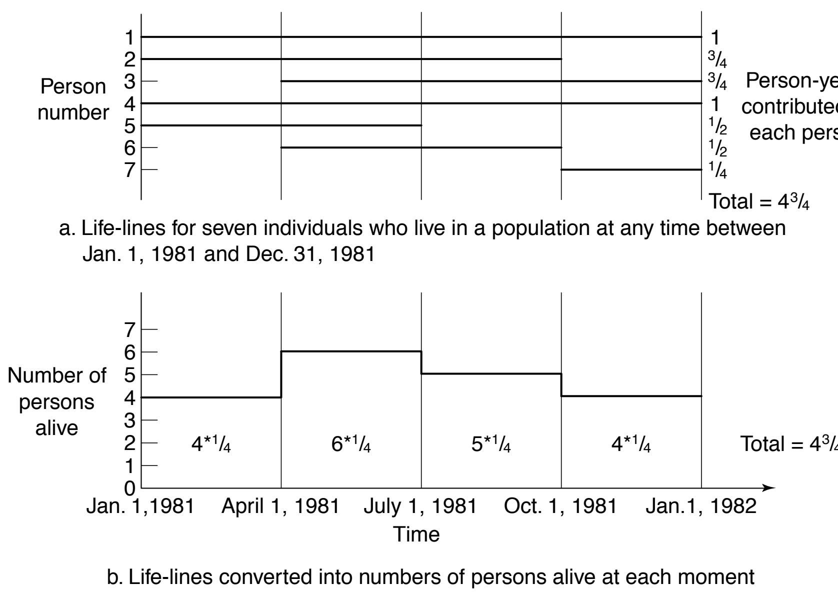


Figure 1.1 Demonstration of the equivalence of the two methods for recording person-years

at various points in time during the year. In our example, there were 4 persons alive from January 1, 1981 to April 1, 1981, so that this quarter-year contributed $4(\frac{1}{4}) = 1$ person year. The next quarter contributed $6(\frac{1}{4}) = 1.5$ person-years, and so on to a total of 4.75 person-years contributed during all of 1981. This value is of course the same number derived by following personal histories, as demonstrated in figure 1.1.

In this alternative approach, what we have done is to estimate the area under the $N(t)$ curve between January 1, 1981 and January 1, 1982. $N(t)$ is defined as the number of persons alive at time t. An area is found by taking the height of a figure times its width. In our case, $N(t)$ is the height and the proportion of the year that corresponds to our measurement of $N(t)$ is the width. Since height represents persons and width represents fractions of a year, it is natural to measure the product in units of person-years.

In our example the number of person-years was:

$$PY[1981.00, 1982.00] = 4(.25) + 6(.25) + 5(.25) + 4(.25) = 4.75$$

This sum can be written in conventional notation as:

$$PY[1981.00, 1982.00] = \sum_{i=1}^{4} N_i \cdot \Delta_i$$

where N_i is the number of persons alive in the ith quarter and Δ_i is the fraction of a year represented by that quarter. Had we measured the size of the population each *day* instead of each quarter, the sum would be represented as:

$$\begin{aligned} PY[1981.00, 1982.00] &= N(Jan.\ 1, 1981) \cdot \frac{1}{365} \\ &\quad + N(Jan.\ 2, 1981) \cdot \frac{1}{365} \\ &\quad + \cdots \\ &\quad + N(Dec.\ 31, 1981) \cdot \frac{1}{365} \\ &= \sum_{i=1}^{365} N_i \cdot \Delta_i \end{aligned}$$

If we were able to measure the height, $N(t)$, in tiny intervals of time *dt*, where *dt* represents the width of the interval, the area under the curve could be represented more accurately as:

$$PY[1981.00, 1982.00] = \int_{1981.00}^{1982.00} N(t) \cdot dt$$

Here an integral sign has replaced the summation sign and for the fraction of a year represented by the time interval, *dt* has replaced Δ_i.

We have seen that areas under a curve can be represented in two ways, using either algebraic or calculus notation. In demography, algebraic notation satisfies a practical need that arises when measurement occurs in discrete intervals. But calculus is often preferred for its compact notation and for its far more extensive body of theorems having direct applicability to population processes. We will use algebra and calculus interchangeably in this volume. One of the most frequent uses of calculus will occur in the issue we have already encountered, representing the area under a curve.

1.5 Principal Period Rates in Demography

We can now apply the concept of period rate to demographic events of interest, in particular the four components of the balancing equation of population change. When the elements of equation (1.1), the balancing equation of population growth, are each divided by the number of person-years lived between 0 and T, we define four rates:

The Crude Birth Rate between times 0 and T:

$$CBR[0, T] = \frac{\text{Number of births in the population between times 0 and } T}{\text{Number of person-years lived in the population between times 0 and } T}$$

The Crude Death Rate between times 0 and T:

$$CDR[0, T] = \frac{\text{Number of deaths in the population between times 0 and } T}{\text{Number of person-years lived in the population between times 0 and } T}$$

The Crude Rate of In-migration between times 0 and T:

$$CRIM[0, T] = \frac{\text{Number of in-migrations into the population between times 0 and } T}{\text{Number of person-years lived in the population between times 0 and } T}$$

The Crude Rate of Out-migration between times 0 and T:

$$CROM[0, T] = \frac{\text{Number of out-migrations from the population between times 0 and } T}{\text{Number of person-years lived in the population between times 0 and } T}$$

We could label the crude birth rate as we have defined it as the “true” crude birth rate, since it includes the actual births and actual person-years in the numerator and denominator, respectively. Throughout the book, the term “rates” will refer to the true or actual rates prevailing in a population. These should be distinguished from the “recorded” or “estimated” rates that are produced when data are used to estimate the value of the true rate.

A person is normally counted as having migrated during the period 0 to T if he or she has changed his or her principal place of residence during the period in a way that crosses the administrative boundaries defining “the population” of a region.

As is especially clear from our definition of the crude rate of in-migration, the connection between exposure and event is not always very precise in demography. No member of a population is literally exposed to the risk of in-migrating into that same population; those at risk are all outside of the population. Like any definitions, these contain an element of arbitrariness, and we could have chosen to put another element in the denominator. What the crude rate of in-migration expresses is the rate at which the population is growing as a result of in-migration. The other rates also indicate the rate at which the population is changing as a result of births, deaths, or out-migration. Using person-years as the denominator for all the major rates in demography provides a firm basis for developing and integrating many different functions and formulas involving population growth. This advantage should become evident in the course of this volume.

It is important to keep in mind the distinction between the reference period to which a rate pertains (i.e., the period for which the values are calculated) and the unit in which exposure time is measured. As noted, the conventional practice is to count exposure in the form of person-years lived, thus creating "annualized" rates. They express the number of events occurring *per year* of exposure. But a period rate need not refer to a single year of the population's experience. For example, we can readily define a crude death rate for 1990–1. Here the number of events in the numerator would include all deaths for calendar years 1990 and 1991, and the denominator would include all person-years lived in 1990 as well as those lived in 1991. Since both the numerator and denominator are, in size, approximately double what they would be if they referred to only a single calendar year, defining the rate over a 2-year period does not affect the scale of the rate. It is still an annualized rate, expressing the number of events per person-year. Likewise, we could define a crude death rate for May 1992, in which both numerator and denominator would be approximately one-twelfth of their value for all of 1992. The scale of the rate, and its annualized nature, is preserved.

Although a period rate in demography apparently can accommodate any length of reference period, it is important to recognize that it must have *some* reference period. The phrase, "the crude birth rate of the United States," has no meaning and there is no way to calculate its value. We must know in what period to count births for the numerator and person-years for the denominator.

1.6 Growth Rates in Demography

1.6.1 Crude growth rate

Let us rearrange the balancing equation of population change (1.1), by subtracting $N(0)$ from both sides and then dividing both sides by the total of person-years lived between 0 and T, $PY[0, T]$:

$$
\begin{aligned}
\frac{N(T) - N(0)}{PY[0,T]} &= \frac{B[0,T]}{PY[0,T]} - \frac{D[0,T]}{PY[0,T]} + \frac{I[0,T]}{PY[0,T]} - \frac{O[0,T]}{PY[0,T]} \\
CGR[0,T] &= CBR[0,T] - CDR[0,T] + CRIM[0,T] - CROM[0,T] \\
&= CRNI[0,T] + CRNM[0,T]
\end{aligned}
\tag{1.2}
$$

Here we have defined the crude growth rate between 0 and T, $CGR[0, T]$, as the change in the size of population divided by person-years lived between 0 and T. If $N(T)$ exceeds $N(0)$, then the growth rate will be positive; if $N(0)$ exceeds $N(T)$, it will be negative. Clearly, the crude growth rate as we have defined it is simply equal to the crude birth rate minus the crude death rate plus the crude rate of in-migration minus the crude rate of out-migration.

The difference between the crude birth rate and the crude death rate is usually termed the crude rate of natural increase (*CRNI*); also, the difference between the crude rate of in-migration and the crude rate of out-migration is usually termed the crude rate of net migration (*CRNM*). So the crude growth rate will equal the crude rate of natural increase plus the crude rate of net migration. Box 1.2 illustrates the calculation of crude demographic rates, again using the Swedish data in box 1.1 and estimating the person-years lived in 1988 by the population size on July 1, 1988. Table 1.1 presents the estimated value of demographic rates for major regions of the world.

Box 1.2 *Principal Period Rates in Demography*

$$\frac{N(T)-N(0)}{PY[0,T]} = \frac{B[0,T]}{PY[0,T]} - \frac{D[0,T]}{PY[0,T]} + \frac{I[0,T]}{PY[0,T]} - \frac{O[0,T]}{PY[0,T]}$$

$$CGR[0,T] = CBR[0,T] - CDR[0,T] + CRIM[0,T] - CROM[0,T]$$
$$= CRNI[0,T] + CRNM[0,T]$$

Example: Sweden, 1988

Person-years lived in Sweden between January 1, 1988 and January 1, 1989 = 8,438,477 (mid-year population)

$$\frac{N(1989.0)-N(1988.0)}{PY[1988.0,1989.0]} = \frac{B[1988.0,1989.0]}{PY[1988.0,1989.0]} - \frac{D[1988.0,1989.0]}{PY[1988.0,1989.0]} + \frac{I[1988.0,1989.0]}{PY[1988.0,1989.0]} - \frac{O[1988.0,1989.0]}{PY[1988.0,1989.0]}$$

$$CGR[1988.0,1989.0] = CBR[1988.0,1989.0] - CDR[1988.0,1989.0] + CRIM[1988.0,1989.0] - CROM[1988.0,1989.0]$$

$$\frac{8{,}461{,}554 - 8{,}416{,}599}{8{,}438{,}477} = \frac{112{,}080}{8{,}438{,}477} - \frac{96{,}756}{8{,}438{,}477} + \frac{51{,}092}{8{,}438{,}477} - \frac{21{,}461}{8{,}438{,}477}$$

$$0.00533 = 0.01328 - 0.01147 + 0.00605 - 0.00254$$

$$CGR[1988.0,1989.0] = CRNI[1988.0,1989.0] + CRNM[1988.0,1989.0]$$

$$0.00533 = 0.00182 + 0.00351$$

Data source: United Nations, *Demographic Yearbook* (various years).

The crude growth rate is only one of several types of growth rate encountered in demography. The term "growth rate" is used to refer to other measures as well, and it is important to distinguish the various forms.

1.6.2 *Instantaneous growth rate*

As any rate, the crude growth rate can be computed for any period of time. What happens when we compute the growth rate during a very short period of time, between time t and $t+\Delta t$, as Δt approaches 0? Denote the population change, $N(t+\Delta t)-N(t)$, as $\Delta N(t)$ and the growth rate as $r(t)$. Since the person-years lived over the period $[t, t+\Delta t]$ is now $N(t)\Delta t$, the crude growth rate for the period is $r(t) = \Delta N(t)/N(t)\Delta t$. But the limit of $\Delta N(t)/\Delta t$ when Δt approaches 0 is simply the derivative of the $N(t)$ function, designated $dN(t)/dt$. Therefore:

$$r(t) = \lim_{\Delta t \to 0} \frac{\Delta N(t)}{N(t)\,\Delta t} = \frac{\frac{dN(t)}{dt}}{N(t)} = \frac{d\ln[N(t)]}{dt} \tag{1.3}$$

where "ln" refers to the natural logarithm. The time interval is very short, dt years, so that $r(t)$ pertains to the tiny interval of time between t and $t+dt$. Because it is measured in time units of years, $r(t)$ continues to be an annualized rate. It is referred as "the growth rate at time t" or "the instantaneous growth rate at time t." It is, of course, also the crude growth rate in the tiny interval of time from t to $t+dt$.

Table 1.1: *Population size and components of change in major areas of the world, 1995–2000*

Major area	*Population size (thousands)*		*Births (thousands)*	*Deaths (thousands)*	*Net international migrants (thousands)*	*Crude growth rate (percentage)*	*Crude birth rate (per 1000)*	*Crude death rate (per 1000)*	*Crude rate of natural increase (per 1000)*	*Crude rate of net migration (per 1000)*
	1995	2000	1995–2000	1995–2000	1995–2000	1995–2000	1995–2000	1995–2000	1995–2000	1995–2000
World	5,666,360	6,055,049	649,050	260,360	0	1.33	22.1	8.9	13.2	0.0
Africa	696,963	784,445	140,575	51,655	−1,435	2.37	38.0	13.9	24.1	−0.4
Asia	3,436,281	3,682,550	389,765	137,460	−6,035	1.38	21.9	7.7	14.2	−0.3
Europe	727,912	728,887	37,465	41,240	4,750	0.03	10.3	11.3	−1.0	1.3
Latin America and the Caribbean	479,954	519,143	57,770	16,225	−2,355	1.57	23.1	6.5	16.6	−0.9
Northern America	296,762	309,631	20,860	12,640	4,650	0.85	13.8	8.3	5.5	3.1
Oceania	28,488	30,393	2,635	1,135	405	1.30	17.9	7.7	10.2	2.8

Source: United Nations, 1999.

The concept of the instantaneous growth rate enables us to develop a new expression for population change over a longer time interval. Integrating formula (1.3) between exact times 0 and T (also measured in years), gives:

$$\int_0^T r(t)\,dt = \int_0^T \frac{d \ln N(t)}{dt} dt = \ln N(t) \Big]_0^T$$

So:

$$\int_0^T r(t)\,dt = \ln\left(\frac{N(T)}{N(0)}\right) \tag{1.4}$$

Taking exponentials on both sides we have:

$$e^{\int_0^T r(t)\,dt} = \frac{N(T)}{N(0)}$$

or

$$N(T) = N(0)e^{\int_0^T r(t)\,dt} \tag{1.5}$$

Formula (1.5) is extremely important in demography. It appears in many guises in many different applications. It expresses the change in population size during a particular discrete time period (in this case from 0 to T) as a simple function of the set of instantaneous growth rates that prevailed during that period. Note that the proportionate growth in population over the period, $N(T)/N(0)$, is a simple function of the sum of growth rates. The order in which those growth rates are applied is immaterial; all that matters is their sum.

Viewing $r(t)$ as a continuously varying function raises questions about the commonly encountered term, "exponential growth." Any growth that occurs, including zero growth or negative growth, must obey equation (1.5). An exponential appears in that formula because we have *defined* our measure of growth – the growth rate – in proportionate terms. In this sense the term "exponential growth" is a redundancy; all growth is exponential by our measure of growth as the proportionate rate of change in population size. When people use the term "exponential growth" they are often (but not invariably) referring to an $N(t)$ sequence produced by a *constant positive* growth rate within some time interval. Such a sequence is probably more precisely characterized by the term Malthus chose for it, "geometric growth," or by "constant growth rate." If the instantaneous growth rate is in fact constant between time 0 and time T at some value r^*, then equation (1.5) simplifies to:

$$N(T) = N(0)e^{r^* \cdot T} \tag{1.6}$$

This formula follows from the fact that:

$$\int_0^T r^*\,dt = r^* \Big]_0^T = r^* \cdot T - r^* \cdot 0 = r^* \cdot T$$

Rearranging equation (1.6) and taking natural logarithms gives:

$$r^* = \frac{\ln\left(\frac{N(T)}{N(0)}\right)}{T} \tag{1.7}$$

Equation (1.7) shows that, if the instantaneous growth rate is constant during the interval 0 to T, one can solve for its value by observing the population size at the beginning and end of the interval.

1.6.3 Mean annualized growth rate

If we divide both sides of equation (1.4) by T, the length of the time interval over which growth is occurring, we have:

$$\frac{\int_0^T r(t)\,dt}{T} = \frac{\ln\left[\frac{N(T)}{N(0)}\right]}{T}$$

The left-hand side of this equation is simply the mean value of the instantaneous growth rate over the period 0 to T, which we will designate as $\bar{r}[0, T]$. It is the area under the $r(t)$ function between 0 and T, divided by the length of the time interval. Thus:

$$\bar{r}[0, T] = \frac{\ln\left[\frac{N(T)}{N(0)}\right]}{T} \tag{1.8}$$

Note that the right-hand side of equation (1.8) is identical to that of (1.7); if the growth rate is constant between 0 and T, equation (1.8) provides a way of estimating its value. But (1.8) is clearly a more general expression since it requires no assumption of constancy. Performing the simple operation given by the right-hand side of equation (1.8) provides the "mean annualized growth rate between 0 and T."

1.6.4 Doubling time

If population size doubles between time 0 and time T, then $N(T)/N(0) = 2$ and:

$$\ln[N(T)/N(0)] = \ln[2] = .693$$

A population thus doubles in size beyond some initial date whenever the sum of its annualized growth rates beyond that date equals 0.693. If the growth rate is constant at r^*, the population will double whenever the product of r^* and T, the length of time (in years) over which it is applied, is 0.693.

So with constant growth rate r^*,

$$\textit{Doubling time} = \frac{.693}{r^*}$$

Under a constant annual growth rate of 0.03, the population will double in $.693/.03 = 23.1$ years. With a constant growth rate of 0.01, it will double in $.693/.01 = 69.3$ years. Since $e^{-.693} = 1/e^{.693} = 0.5$, a population will be reduced to half of its initial size whenever the sum of annual growth rates equals $-.693$.

1.6.5 Comparison of crude growth rate and mean annualized growth rate

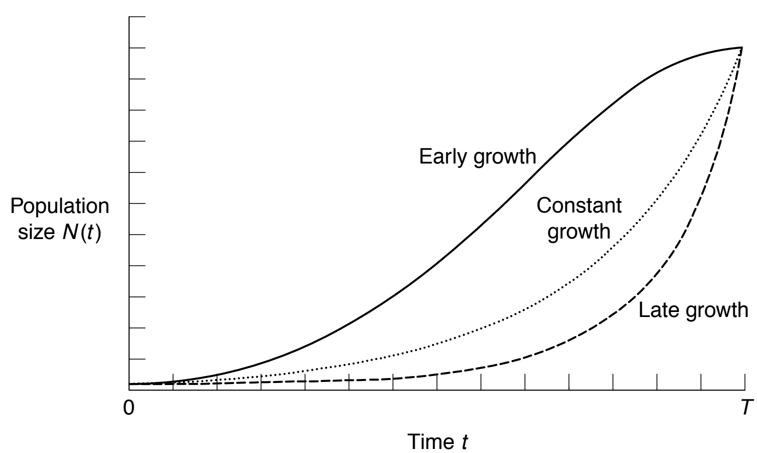
We have now developed two formulas for period growth rates over the discrete interval between 0 and T: the crude growth rate and the mean annualized growth rate. This section, which is included for completeness and can be skipped by many readers, compares the two rates. The basic lesson is that the two growth rates will be the same when the instantaneous growth rate is constant during the period 0 to T. Otherwise, the two rates will not, in general, have the same value. However, differences between them can usually be ignored for practical purposes unless the period of measurement is very long (say, longer than 10 years) *and* the growth rate function, $r(t)$, is very irregular.

From (1.2), the crude growth rate between 0 and T can be written as:

$$CGR[0,T] = \frac{B[0,T] - D[0,T] + I[0,T] - O[0,T]}{\int_0^T N(t)\,dt} = \frac{N(T) - N(0)}{\int_0^T N(t)\,dt} \quad \textbf{(1.9)}$$

As is clear in (1.8), $\bar{r}[0,T]$ does not depend on the order in which growth rates occur between 0 and T. The numerator of $CGR[0,T]$ in (1.9) is also independent of the order in which growth rates occur. But the denominator of $CGR[0,T]$ in (1.9), person-years lived between 0 and T, *does* depend on the order in which growth rates occur. A distribution of positive growth rates that is heavily skewed toward the beginning of the period will raise person-years lived relative to a distribution that is skewed toward the end of the period. This tendency is illustrated in figure 1.2.

So it is clear that, in general, there can be no equality between CGR and $\bar{r}$. An "early" distribution of growth rates will lower CGR relative to $\bar{r}$, and a "late" distribution will



The sum of growth rates, $\int_0^T r(t)dt$, is the same in the three cases, since $N(0)$ and $N(T)$ are the same.

Person-years lived – the area under the $N(t)$ curve – are different, however.

Figure 1.2 Population growth sequences between times 0 and T under three different assumptions about the time sequence of growth rates raise CGR relative to $\bar{r}$. There is, however, one circumstance in which CGR will equal $\bar{r}$. This occurs when the growth rates are constant between 0 and T. Suppose that $r(t) = r^*$ for $0 \le t \le T$. Then:

$$\begin{aligned}\int_0^T N(t)\,dt &= \int_0^T N(0)e^{r^*t}\,dt = N(0)\int_0^T e^{r^*t}\,dt \\ &= N(0)\cdot\frac{1}{r^*}\cdot e^{r^*t}\Big]_0^T = \frac{N(0)\cdot e^{r^*t}]_0^T}{r^*} \\ &= \frac{N(T)-N(0)}{r^*}\end{aligned} \tag{1.10}$$

Substituting expression (1.10) for person-years lived between 0 and T into equation (1.9) gives:

$$CGR[0,T] = \frac{N(T)-N(0)}{\left[\dfrac{N(T)-N(0)}{r^*}\right]} = r^*$$

In the case of a constant growth rate, we also have:

$$\bar{r}[0,T] = \frac{1}{T}\int_0^T r^*\,dt = r^*$$

So in the case of constant growth rates – and, except for rare circumstances, only in this case – the crude growth rate will equal $\bar{r}$. Differences between the two will normally be trivial in size unless the growth rate sequence is extremely erratic and the time period (0 to T) very long, say a decade or more.

If one wants to ensure that the crude growth rate calculated by (1.9) is in fact equal to the mean of annualized growth rates, then a simple rule for computing person-years is indicated: compute person-years lived during the period as though the growth rate were constant throughout. Under this circumstance, the denominator for calculating all crude rates would be:

$$\int_0^T N(t)\,dt = \begin{cases} \dfrac{N(T)-N(0)}{\bar{r}[0,T]} = \dfrac{[N(T)-N(0)]\cdot T}{\ln\left(\dfrac{N(T)}{N(0)}\right)}, & \text{if } \bar{r} \neq 0 \\ T\cdot N(0), & \text{if } \bar{r} = 0 \end{cases}$$

Although we defined the "mean annualized growth rate" as the average of period rates, in equation (1.8) it does not have person-years in the denominator, which was said to be a typical feature of a demographic rate. In this format, it shares the characteristic of many rates in common usage, such as a mean rate of speed or mean rate of inflation. But under the simplifying assumption that the "mean annualized growth rate" is constant during the interval of measurement, its value is in fact identical to that of the crude growth rate, which does explicitly contain person-years in the denominator.

1.7 Estimating Period Person-years

The above argument suggests that, if one knew nothing about the path of $N(t)$, or $r(t)$, during a particular year, one should assume constancy of the growth rate during the period and estimate person-years lived during the year as:

$$PY[0,1] = \frac{N(1) - N(0)}{r[0,1]} = \frac{N(1) - N(0)}{\ln\left[\frac{N(1)}{N(0)}\right]}$$

More generally, when the period is not necessarily one year long,

$$PY[0,T] = \frac{[N(T) - N(0)] \cdot T}{\ln\left[\frac{N(T)}{N(0)}\right]} \qquad \textbf{(1.11)}$$

Using equation (1.11) to estimate person-years has the advantage of forcing consistency between the crude growth rate for the period and the mean annualized growth rate for that period, and it would be exactly correct if the growth rate were constant during the period. But it does require observations on population size at the beginning and end of the period. It is often the case (e.g., in the United States) that population size estimates are only available at mid-year. It will usually be perfectly acceptable to use the mid-year population size as an estimate of person-years lived during the year. The mid-year approximation to person-years will be exactly correct if the $N(t)$ sequence is linear between the beginning and end of the year, as demonstrated in figure 1.3. Even if the $N(t)$ sequence is a product of a constant growth rate, the error in using the mid-year approximation will be very small. For example, if $r = 0.03$ (rapid by historical standards), the ratio of true person-years lived in a year to the mid-year population will be 1.00004. The mid-year population will always underestimate the true number of person-years lived if the population is changing at a constant rate, whether positive or negative.

More caution is necessary in using mid-period approximations to estimate person-years when the interval of time for which an estimate is sought extends far beyond a year. For example, if we estimate the person-years lived during a 10-year period in a population growing at 3 percent a year by taking the mid-period population times 10 (i.e., mid-height times width), then the ratio of true person-years lived to our estimated person-years will be 1.0038. This error of about four-tenths of 1 percent is too large to ignore for most purposes. Note that if we had used the arithmetic mean of beginning and end-period populations (times 10) as our estimate of person-years lived in this example, we would have *overestimated* true person-years by the factor 1.0075. So this procedure provides an even poorer estimate of person-years than does the mid-period population in a population having a constant positive growth rate.

If mid-year population estimates are available for each year during a 10-year period, a sensible way to estimate person-years lived during the period would be simply to add up the 10 estimates. If observations are available at the beginning, middle, and end of the period, then it is possible to ascertain whether growth is more nearly linear or exponential and to use the corresponding approximation for each half-period.

Although it is convenient and fairly accurate to estimate person-years lived during a particular year as the population size in the middle of the year, it is important to remember that the resulting demographic rate should not be expressed as a number of occurrences divided by a

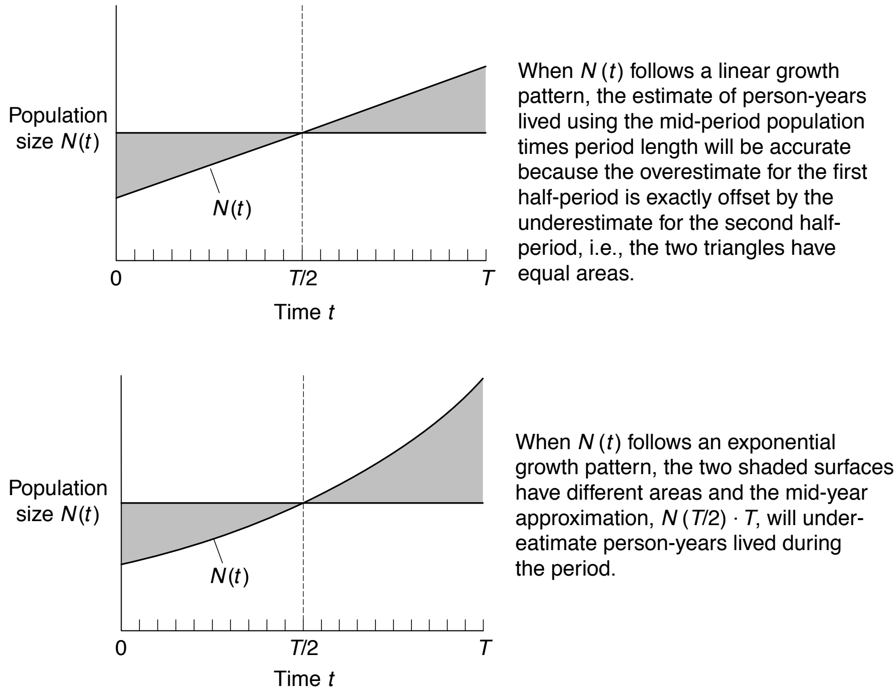


Figure 1.3 Approximation of person-years lived by midperiod population times period length

number of people. The unit in which exposure-time is measured (usually, person-years) must not disappear, or confusion is inevitable. We are using the mid-year population as an *estimate* of person-years lived during the period, and not as a *substitute* for person-years. The risk of confusion is greatest when an annualized rate is being estimated for a period that is not one year in length. Box 1.3 illustrates the computation of growth rates and person-years lived during a 10-year period in a hypothetical population with a constant annualized growth rate of 0.03.

1.8 The Concept of a Cohort

Almost as important to demography as the concept of a population is the concept of a cohort. A cohort is the aggregate of all units that experience a particular demographic event during a specific time interval. As in the case of a population, a cohort always has some specific geographic referent, whether it is explicit or implicit. A cohort usually consists of people, but it may also consist of entities (e.g., marriages) formed by a demographic event. The cohort is usually identified verbally both by the event itself and by the time period in which it is experienced. Some examples of cohorts are:

"US birth cohort of 1942," which refers to all persons born as US citizens in calendar year 1942;

Box 1.3 *Illustration of Calculation of Growth Rates and Person-years*

Suppose that a population had 100,000 persons at time 0 and that it grew at a constant annualized growth rate of 0.03. Then:

$$N(0) = 100,000$$

$$N(5) = 100{,}000 \cdot e^{5\cdot .03} = 116{,}183$$

$$N(10) = 100{,}000 \cdot e^{10\cdot .03} = 134{,}986$$

1. Calculating the mean annualized growth rate between $t = 0$ and $t = 10$:

$$\bar{r}[0, 10] = \frac{\ln\left(\frac{N(10)}{N(0)}\right)}{10} = \frac{\ln\left(\frac{134{,}986}{100{,}000}\right)}{10} = 0.0300$$

2. Estimating person-years lived between $t = 0$ and $t = 10$:

   a) Assuming a constant growth rate:

$$PY[0, T] = \frac{N(T) - N(0)}{\bar{r}[0, T]} = \frac{N(10) - N(0)}{\bar{r}[0, 10]} = \frac{134{,}986 - 100{,}000}{0.03} = 1{,}166{,}200$$

   b) Assuming growth is linear and using the mid-period approximation:

$$PY[0, T] = N(T/2) \cdot T$$

$$PY[0, 10] = N(5) \cdot 10 = 116{,}183 \cdot 10 = 1{,}161{,}830$$

   c) Assuming growth is linear and using the mean of initial and final population sizes:

$$PY[0, T] = \left[\frac{N(0) + N(T)}{2}\right] \cdot T$$

$$PY[0, 10] = \left[\frac{N(0) + N(10)}{2}\right] \cdot 10$$

$$= \left[\frac{100{,}000 + 134{,}986}{2}\right] \cdot 10 = 1{,}174{,}930$$

3. Calculating crude growth rates based upon various estimates of person-years lived:

   a) $CGR[0, T] = \dfrac{34{,}986}{1{,}166{,}200} = 0.0300$

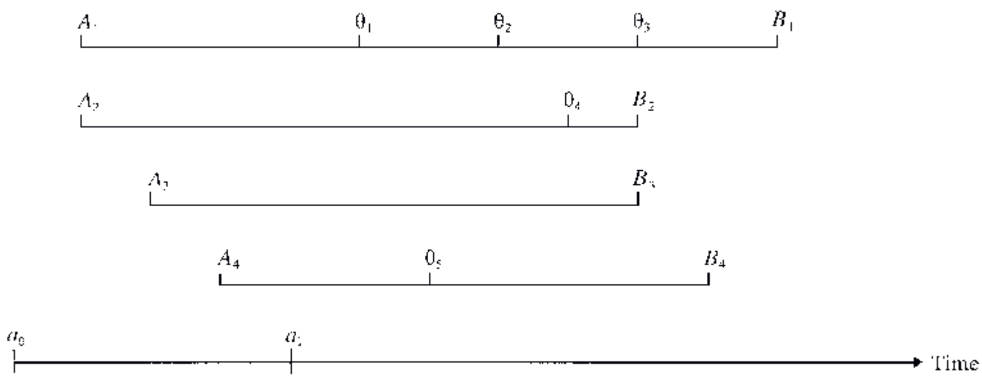
   b) $CGR[0, T] = \dfrac{34{,}986}{1{,}161{,}830} = 0.0301$

   c) $CGR[0, T] = \dfrac{34{,}986}{1{,}174{,}930} = 0.0298$

"French marriage cohort of 1990," which refers to all marriages contracted in France during the calendar year 1990;
"French female marriage cohort of 1990," which refers to all women who married in France in 1990;
"Austrian immigrant cohort of 1995," which refers to all immigrants into Austria in 1995.

The most frequently encountered type of cohort is a birth cohort. Persons who are born during the same period are destined to pass through life together, in the sense that they will reach their xth birthday during a period exactly x years beyond that which defined their cohort membership. For the US birth cohort of 1942, all would reach their 10th birthday (assuming that they survived) in 1952, their 15th birthday in 1957, and so on. The time period that circumscribes the cohort need not be one year in length; it is common to deal with such entities as the US birth cohort of 1918–22, for example.

To calculate a rate for a cohort, we simply restrict the counting of occurrences and person-years of exposure to people who were born during the period that defines membership in the cohort. The lines below show the counting schema for a birth cohort defined by birth in the period a_0 to a_1:



Although those life-lines refer to a birth cohort, the concept can clearly be extended to other types of cohorts.

1.9 Probabilities of Occurrence of Events

We can define an additional concept for a cohort that is impossible for a population: the concept of a probability. The term is used in demography in a manner similar to its usage in statistics. It refers to the *chance* that some event will occur, rather than to the rate at which it occurs. Thus, for example, we may compute the probability that a marriage would end in a divorce for a given birth cohort by counting, over all members of the cohort, the number of marriages and the number of divorces over the cohort's lifetime:

$$q^D = \frac{Number\ of\ Divorces}{Number\ of\ Marriages}$$

In doing so, we have used a "relative frequency" approach to estimating the probability of divorce. We have said, in effect, that our best guess about the true underlying probability of divorce in the cohort is the observed frequency of divorce. The situation is analogous to drawing balls out of a very large urn. If we draw a sample of 10 balls and 2 of them are red, then the relative frequency of red balls in that drawing is 0.2. This relative frequency is also the maximum likelihood estimator of the true proportion of balls in the entire urn that are red, assuming that the outcome of drawings is independent. That is, a proportion in the entire urn of 0.2 is more likely than any other proportion to have given rise to the observed sample of 10 balls of which 2 are red. Many introductory statistics texts contain a clear discussion of maximum likelihood estimation.

The structure of a probability in demography is thus quite different from the structure of a rate:

$$Rate = \frac{Number\ of\ Occurrences}{Number\ of\ Person\text{-}years\ Lived}$$

$$Probability = \frac{Number\ of\ Occurrences}{Number\ of\ Preceding\ Events\ or\ Trials}$$

The denominator of the probability indicates that it is not possible to define a probability unless there is some *event* or *trial* (equivalent to the act of drawing balls out of an urn). Since each occurrence in the numerator (e.g., divorce) must be preceded by an event in the denominator (marriage), the number of occurrences cannot exceed the number of preceding events. Thus the probability cannot exceed one and, since we are only dealing with positive quantities, probabilities cannot be negative.

Populations do not have probabilities except insofar as they pertain to cohorts that are included in the population. Although we could count the number of marriages in a population during some calendar year and the number of divorces during that year, the two numbers combined do not give a sensible estimate of the probability of divorce because they don't apply to the same cohort. We are, in effect, counting events (or trials) in one urn and occurrences in another. If we happened to choose a year in a small population where no one married but there was a divorce, our population's probability of divorce q^D would be $1/0 = \infty$, an obviously absurd outcome. Only when we count the events pertaining to the cohort at risk of the event can we properly define a probability.

The concepts of cohorts and of probabilities that certain events will occur to cohorts can be applied to a vast number of situations extending well beyond demography's customary range. They are central to all analysis of longitudinal data in the social and health sciences. Perhaps their major utility derives from the fact that they translate aggregate-level measures into implications for individuals. They help "locate" the individual in an otherwise amorphous and undifferentiated population.

Despite its conceptual simplicity, analysis of data on actual cohorts suffers from several major practical limitations. First, computing cohort rates and probabilities requires complete information on each individual until he or she has died (or at least has ceased to be "at risk" of the event of interest). We may lose track of some individuals, for instance, when they move out of the area of the study. Out-migration is part of a more general problem called "loss to follow-up." We deal with one way of coping with this problem in chapter 4. A more serious practical problem is that, by the time the cohort's experience is completely observed, much of the experience may be ancient. In order to provide more timely information, demographers rely primarily on data for recent periods. The measures that are constructed from period rates include life expectancy, expected years to be lived in the single state, total fertility rate, net reproduction rate and gross reproduction rate. They also include probabilities of dying, giving birth, migrating, and so on. In constructing these and other measures, demographers rely on the concept of a cohort, but adapt that concept to deal with data pertaining to a period. The principal adaptation is the introduction of "hypothetical cohorts," a concept that will be encountered frequently in the remainder of this volume.