1

Reliability of Engineering Systems

1.1. Basic notions and characteristics of reliability

1.1.1. *Basic notions*

The notions described below correspond to the usual terminology used in reliability theory and most of the literature sources on reliability. Reliability theory deals with the following basic notions.

An *object* in reliability theory means a unit (an element or article), an apparatus, an engineering product and any system or its part at all, considering from the point of view of their reliability. Furthermore, the term *unit* is used for simple objects, which is considered a single entity. For complex objects, the term *system* is used and the term *element* means the minimal component of a system.

An *exploitation* of an object (unit or system) means the collection of all its existence phases (creation, transportation, storage, using, maintenance and repair).

Reliability of an object is its complex property, consisting of its possibility to fulfill assign to it functions under given exploitation conditions[1].

1 Under the *quality* of an object it is understood the set of properties, which determine the degree of its possibility to be used for designation. Therefore, reliability is one of the components of the quality of an object.

According to the definition of Gnedenko [GNE 65], *reliability theory* is a scientific discipline about the requirements that should be used for projecting, producing, testing and exploitation of an object in order to get the maximal effect from its use. Reliability theory deals with such notions as: reliability, failure (breakdown), longevity, repair, repair-ability, etc.

Reliability means the possibility of an object to maintain its workability during a given time period under a given exploitation condition.

A *failure* is a partial or full loss of the object's workability. Therefore, we should distinguish *full* and *partial* failures.

In addition, failures are divided into *sudden*, for which the object suddenly (unexpected) loses its workability, *gradual*, for which the workability of an object is lost gradually (usually as a result of some physical parameters of the object going out of the admissible level) and *halting* (temporary loss of the workability).

Longevity is the ability of an object to be used for a long time under needed technical service.

Repair is the procedure that renews objects' reliability.

Repair-ability is the property of an object to predict, detect and remove its failures.

Safety is the property of an object (system, unit) not to allow situations that could be dangerous for people and the environment.

Further notions and definitions are introduced in the chapter if necessary.

Given the complex property of an object, the reliability is described by many different characteristics and indexes. Furthermore, the term *characteristic* is used for complex (functional) reliability characteristics, and the term *index* is usually used for numerical (simple) characteristics.

Among the different reliability characteristics, we first consider those that are used for units and systems which work up to the first failure.

1.1.2. *Reliability of non-renewable units*

In this section, the reliability of an object is studied independently of the reliability of its components as a single entity, and therefore instead of the term "object", here, the term "unit" is used. Suppose that the unit can be in only two states from the point of view of its reliability: "workable" (up) and not workable or "failure" (down). Denote by T the *lifetime* of the unit. It is a random variable (r.v.) and its basic characteristic is its *cumulative distribution function* (c.d.f.) that is the probability that this time is not greater than the fixed time t,

$$F(t) = \mathbf{P}\{T \le t\}. \tag{1.1}$$

Here and later, the symbol $\mathbf{P}\{\cdot\}$ is used for the probability of the event in brackets. In the case of continuous observations for the unit state, this function is a continuous one, but in the case of observations for the unit state in discrete points of time, it is a stepwise one. The function

$$R(t) = 1 - F(t) = \mathbf{P}\{T > t\}, \tag{1.2}$$

in reliability theory is known as *reliability function*[2]. For continuous distribution, the graphs of these functions are shown in Figure 1.1.

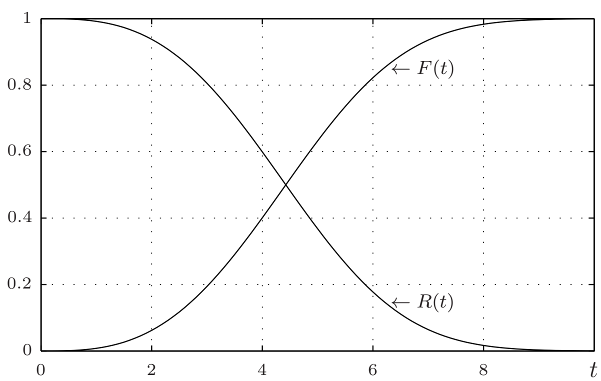


Figure 1.1. *C.d.f. of lifetime and reliability function of some unit*

2 In the biological, medical and actuarial disciplines the term *survival function* is used for this function.

In the case of continuous observation, the r.v. T can also be characterized by its *probability density function* (p.d.f.) $f(t) = F'(t)$. At that lifetime c.d.f. connected with p.d.f. by the equality,

$$F(t) = \int_0^t f(u)\, du. \qquad [1.3]$$

For small values Δt, the quantity $f(t)\Delta t$ is the probability of a unit's failure in time interval $(t, t+\Delta t)$. Because in practice, the probability is measured with frequency, this value is also called *frequency of failures*.

In reliability practice the time is usually measured in discrete units. Therefore, the discrete distributions are more appropriate models for the lifetime's description. However, for theoretical study, the continuous distributions are more convenient. Therefore, according to these reasons, mostly continuous distributions will be used for the units' lifetime distribution description. By the way, when the time is measured in discrete units, the *discrete distribution* can be obtained from the continuous one by discretization of time,

$$f_k = \mathbf{P}\{T = k\Delta\} = \int_{(k-1)\Delta}^{k\Delta} f(u)du \quad k = 1, 2, \ldots, \qquad [1.4]$$

where Δ means the unit of time (in minutes, hours, months or years).

Besides lifetime distribution of a new unit, an important reliability of its characteristic is its *residual lifetime*. Conditional distribution, after its reliable working time t, represents conditional failure probability in time interval $(t, t+x]$ given up to time t a failure does not occur,

$$F(x;t) = \mathbf{P}\{T \le t + x | T > t\} = \frac{\mathbf{P}\{t < T < t + x\}}{\mathbf{P}\{T > t\}} =$$
$$= \frac{F(t+x) - F(t)}{1 - F(t)} = \frac{R(t) - R(t+x)}{R(t)}. \qquad [1.5]$$

For small values of x, we have:

$$F(x;t) = \frac{f(t)}{1 - F(t)} x + o(x) = \lambda(t)x + o(x),$$

where the function $\lambda(t)$ represents a conditional probability density of residual lifetime of a unit under condition that is used without failure during time t. More precisely, this function is determined by the equality,

$$\lambda(t) = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \frac{F(t+\Delta t) - F(t)}{1 - F(t)} = \frac{F'(t)}{1 - F(t)} = \frac{f(t)}{1 - F(t)}, \qquad [1.6]$$

and in reliability literature, it is also known as *hazard rate function* (h.r.f.). This function allows us to evaluate the failure probability of a unit during a small time interval Δt after time t as follows:

$$\mathbf{P}\{t < T \leq t + \Delta t \,|T > t\} = \lambda(t)\Delta t + o(\Delta t)$$

as an area under the curve, as is shown in Figure 1.2.

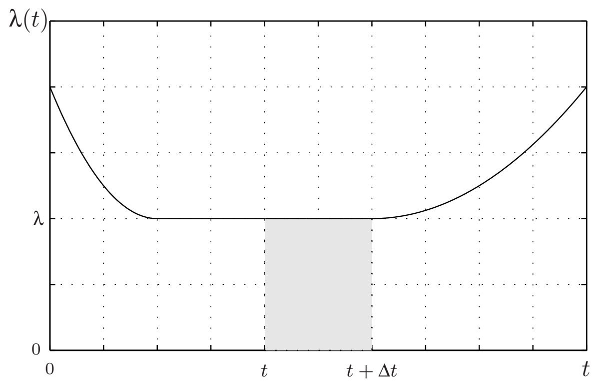


Figure 1.2. *Typical hazard rate function*

Equality [1.6] allows us to represent the c.p.f. of a unit lifetime and its reliability function in terms of its h.r.f. In fact, it can be represented as

$$d \ln(1 - F(t)) = -\lambda(t)\, dt$$

which after integration gives

$$\int_0^t d \ln(1 - F(u))\, du = -\int_0^t \lambda(u)\, du.$$

Supposing that there are no instant failures, which means that $F(+0) = 0$, it gives

$$\ln(1 - F(t)) = -\int_0^t \lambda(u)\,du,$$

or

$$1 - F(t) = R(t) = \exp\left\{-\int_0^t \lambda(u)\,du\right\}. \quad [1.7]$$

Analogously, for conditional lifetime probability in interval $(t, t + x]$, we can find

$$F(x;t) = \mathbf{P}\{T \le t + x | T > t\} = \exp\left\{-\int_t^{t+x} \lambda(u)\,du\right\}. \quad [1.8]$$

Besides functional characteristics in practice, the lifetime of units is also measured with some numerical indexes such as:

– *mean lifetime*, i.e. expectation of lifetime,

$$\mu_T = \mathbf{E}[T] = \int_0^\infty t\,f(t)\,dt = \int_0^\infty (1 - F(t))\;dt = \int_0^\infty R(t)\,dt \quad [1.9]$$

– *variance of lifetime*, which shows the variation of the lifetime around its mean value,

$$\sigma_T^2 = \mathsf{Var}[T] = \mathbf{E}\,[T - \mu_T]^2 = \int_0^\infty (t - \mu_T)^2\; f(t)\,dt. \quad [1.10]$$

Here and later, symbols $\mathbf{E}[\cdot]$ and $\mathsf{Var}[\cdot]$ indicate expectation and variance, respectively.

One of the main problems of reliability theory is elements and unit lifetime distribution modeling. Some parametric families of continuous distributions of non-negative random variables that are usually used for the unit lifetime modeling are presented in the next section. Some of these distributions will also be used later in section 2.3 for modeling of the damage value distributions.

1.1.3. *Some parametric families of continuous distributions of non-negative random variables*

Consider some parametric families of continuous distributions of non-negative random variables along with their indexes.

1.1.3.1. *Exponential distribution*

Exponential lifetime is used for modeling the reliability of units, subject to instantaneous (sudden, unexpected) failures. Its p.d.f. and c.d.f. are

$$f(t) = \lambda e^{-\lambda t}, \quad F(t) = 1 - e^{-\lambda t} \quad \text{for} \quad t \geq 0, \tag{1.11}$$

where $\lambda > 0$ is its parameter. The reliability function of these units is

$$R(t) = e^{-\lambda t}, \tag{1.12}$$

and their h.r.f. is constant and coincides with the distribution parameter λ,

$$\lambda(t) = \frac{f(t)}{R(t)} = \lambda. \tag{1.13}$$

The graphs of these functions are represented in Figure 1.3.

Moreover, the property of h.r.f. to be constant is a *characteristic property* of the exponential reliability law. From relation [1.7] we have

$$R(t) = \exp\left\{-\int_0^t \lambda \, du\right\} = e^{-\lambda t}. \tag{1.14}$$

Another characteristic property of an exponential distribution is its "memoryless" property, which is presented in the following theorem.

THEOREM 1.1.– A unit has an exponential reliability law iff the distribution of its residual lifetime does not depend on the elapsed working time (its age),

$$\mathbf{P}\{T > t + x | T > t\} = \mathbf{P}\{T > x\}. \tag{1.15}$$

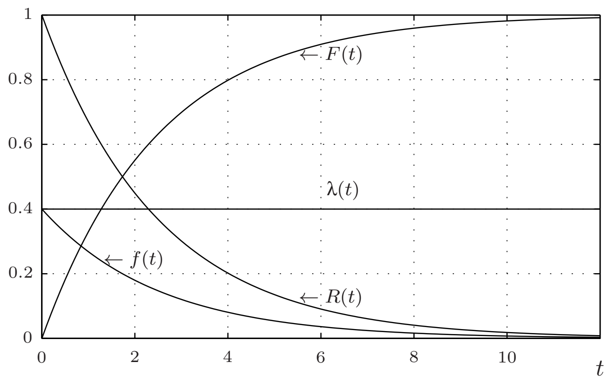


Figure 1.3. *The p.d.f. $f(t)$, the c.d.f. $F(t)$ and the reliability function $R(t)$ of an exponential distribution, $\lambda = 0.4$*

PROOF 1.1.– *Necessity.* Using the conditional probability formula for the exponential reliability law, we have

$$\mathbf{P}\{T > t + x | T > x\} = \frac{\mathbf{P}\{T > t + x,\ T > t\}}{\mathbf{P}\{T > t\}} = \frac{\mathbf{P}\{t > t + x\}}{\mathbf{P}\{T > t\}} =$$

$$= \frac{e^{-\lambda(t+x)}}{e^{-\lambda t}} = e^{-\lambda x} = \mathbf{P}\{T > x\}.$$

Sufficiency. For $R(t) = \mathbf{P}\{T > t\}$ from relation [1.15], we obtain the following equation:

$$R(t + x) = R(t)R(x).$$

For continuous functions under condition $R(0) = 1$, this equation has a unique solution $R(t) = e^{-\lambda t}$ with a positive parameter $\lambda > 0$. ■

NOTE 1.1.– For discrete time distributions (when observations are fixed with discrete intervals) the analogous property characterizes the geometric distribution.

Namely this property and the constancy of the h.r.f allows us to consider this distribution as a distribution of sudden (unexpected) failures because it means that the residual unit's lifetime does not depend on its elapsed time.

Mean and variance of a unit lifetime for this distribution are:

$$\mu_T = \mathbf{E}[T] = \lambda^{-1}, \quad \sigma_T^2 = \mathsf{Var}[T] = \lambda^{-2}. \tag{1.16}$$

Exponential distribution is closely connected with the Poisson process of failures for reparable systems (see, for example, 1.4 in the section 1.2.5).

1.1.3.2. *Shifted exponential distribution*

The p.d.f. of this distribution is

$$f(t) = \lambda\, e^{-\lambda(t-b)} 1_{\{t \geq b\}}, \quad b \geq 0$$

where λ and b are its form and shift parameters. This is represented in Figure 1.4.

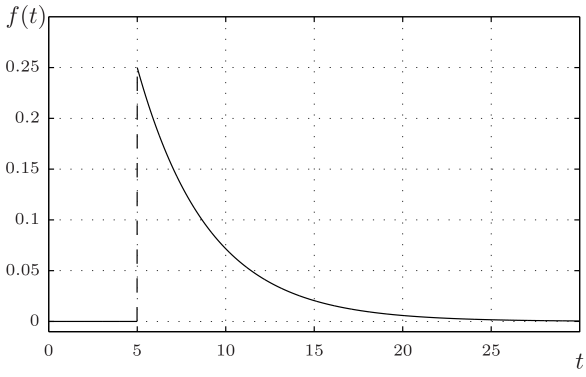


Figure 1.4. *The p.d.f. of a shifted exponential distribution, $\lambda = 0.25$, $b = 5$*

The expectation and the variance of r.v. with this distribution are

$$\mathbf{E}[T] = b + \frac{1}{\lambda}; \quad \mathsf{Var}[T] = \frac{1}{\lambda^2}.$$

This distribution could be used for the sudden failure description in the case when the unit work beginning with some additional time for “warming up” is needed.

1.1.3.3. *Truncated normal distribution*

Truncated normal distribution, contrarily to the exponential one, is used for the description of unit lifetime, subject to gradual failures. The following theorem can explain this assertion.

THEOREM 1.2.– If a failure arises as a result of some physical parameter a of a unit going out admissible limits, and this parameter is changing in time according to some deterministic law $a = f(t, a_0)$, and its initial value a_0 is a r.v., distributed according to the normal law, then the failure time, which is the parameter a destination time to the critical value a_1, has also a normal distribution.

PROOF 1.2.– In fact, under these assumptions, the unit failure time T is a solution of the equation

$$f(T, a_0) = a_1.$$

Denoted by $t = \varphi(a_1, a_0)$ inverse to the $f(t, a_0)$ function. Then, from this equation, we can obtain

$$T = \varphi(a_1, a_0).$$

Expansion of the function $\varphi(a_1, a_0)$ into Taylor series with respect to variable $a = \mathbf{E}[a_0]$ in the neighborhood of point a_1 up to second order members gives:

$$T \approx \varphi(a_1, a) + \varphi'_a(a_1, a)(a - a_0).$$

From here it follows that if the parameter a_0 has a normal distribution, then the time to failure T also has to be normally distributed. Because the lifetime cannot be negative for its description, we can use the truncated normal distribution. For this distribution, the reliability function is

$$R(t) = \mathbf{P}\{T > t | T > 0\} = \frac{1 - \Phi\left(\frac{t-\mu}{\sigma}\right)}{1 - \Phi\left(-\frac{\mu}{\sigma}\right)} = \frac{\Phi\left(\frac{\mu-t}{\sigma}\right)}{\Phi\left(\frac{\mu}{\sigma}\right)}, \qquad t \geq 0, \qquad [1.17]$$

where here and later the notation

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{u^2}{2}} \, du \qquad [1.18]$$

is permanently used for standard normal distribution and its parameters μ and σ are positive. ■

Because, in practice, case $\mu >> \sigma$, the relation $\Phi\left(\frac{\mu}{\sigma}\right) \approx 1$ holds, and we could use an approximate[3]

$$R(t) \approx \Phi\left(\frac{\mu - t}{\sigma}\right).$$

Hazard rate function for this distribution equals

$$\lambda(t) = \frac{1}{\sigma\sqrt{2\pi}} \times \frac{e^{-\frac{(t-\mu)^2}{2\sigma^2}}}{\Phi\left(\frac{\mu-t}{\sigma}\right)}.$$

Using for the function $\lambda(t)$ Taylor's Formula when $t \to \infty$, we get,

$$\lambda(t) \approx \frac{t-\mu}{\sigma^2} + O\left(\frac{1}{t}\right),$$

where symbol $O(\cdot)$ denotes the decreasing rate of the appropriate value. This equality shows that for $t \to \infty$ h.r.f., the truncated normal distribution has a slope asymptote $y = \frac{t-\mu}{\sigma^2}$ (see Figure 1.5).

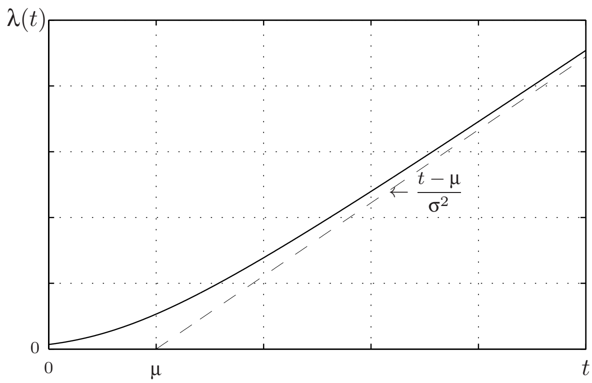


Figure 1.5. *Hazard rate function for truncated normal reliability law*

3 For example, for $\mu \geq 3\sigma$, the approximation $\Phi\left(\frac{\mu}{\sigma}\right) \approx 0.9987$ holds.

Mean and variance of a unit lifetime for this reliability law under the condition $\mu >> \sigma$ are

$$\mu_T = \mathbf{E}[T] = \mu, \quad \sigma_T^2 = \mathsf{Var}[T] = \sigma^2. \qquad [1.19]$$

1.1.3.4. *Gnedenko–Weibull distribution*

The p.d.f. and the reliability function for this reliability law equals

$$f(t) = \lambda\alpha\, t^{\alpha-1} e^{-\lambda\, t^{\alpha}}, \quad R(t) = e^{-\lambda\, t^{\alpha}} \quad \text{for} \;\; t \geq 0 \qquad [1.20]$$

with parameters $\lambda > 0$ and $\alpha > 0$. Its h.r.f. is

$$\lambda(t) = \alpha\, \lambda\, t^{\alpha-1} \quad \text{for} \quad t \geq 0, \qquad [1.21]$$

whose graphs are represented in Figure 1.6.

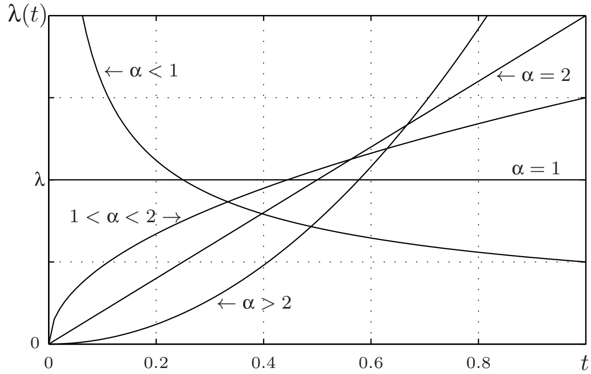


Figure 1.6. *Hazard rate functions for Gnedenko–Weibull reliability law*

Mean lifetime and variance for this law equals

$$\mu_T = \mathbf{E}[T] = \frac{\Gamma\left(1 + \frac{1}{\alpha}\right)}{\lambda^{\frac{1}{\alpha}}},$$

$$\sigma_T^2 = \mathsf{Var}[T] = \frac{\Gamma\left(1 + \frac{2}{\alpha}\right) - \Gamma^2\left(1 + \frac{1}{\alpha}\right)}{\lambda^{\frac{2}{\alpha}}}. \qquad [1.22]$$

Exponential distribution is a special case of the Gnedenko–Weibull distribution when $\alpha = 1$.

The popularity of this distribution in the reliability theory is explained by its property to be the limiting distribution for the maximum and minimum of a series of i.i.d. r.v. Therefore, this distribution arises in the calculation of the reliability characteristics of some complex systems consisting of many elements in the case when the system failure arises when the first of its elements fail or in the case when the system failure arises when many of its elements fail. The details of these situations will be studied in sections 1.4.4 and 1.6.2. Therefore, the following theorem, first obtained by Gnedenko, explains the wide popularity of this distribution in reliability theory.

Let T_i $(i = 1, 2, \dots)$ denotes the sequence of some i.i.d. r.v. (for example lifetimes of elements of some system). Denoted by $T_{(1)} = \min\{T_i : i = \overline{1,n}\}$ minimum and by $T_{(n)} = \max\{T_i : i = \overline{1,n}\}$ maximum of n of these variables. It is supposed that r.v.'s . T_i take any non-negative values (the c.d.f. has a non-bounded domain R_+). In this case, r.v.$T_{(n)}$ unlimitedly increases when n grows. Therefore, in order to provide the existence of some non-degenerate distributions, we need to find some sequence of numbers a_n, b_n such that a limiting distribution of r.v.

$$W_n = a_n T_{(n)} + b_n \qquad \text{for} \quad n \to \infty$$

will be non-degenerate (proper) .

There are two types of limiting distributions for r.v. W_n, which depend on the behavior of "tails" of r.v. T_i distribution,

$$1 - F(t) = \mathbf{P}\{T_i > t\}.$$

We denote

$$1 - G(t) = \lim_{n\to\infty} \mathbf{P}\{W_n > t\}.$$

The following theorem holds:

THEOREM 1.3 (Gnedenko).– It is true

– if $1 - F(t) \approx e^{-t}$ for $t \to \infty$, then $1 - G(t) = e^{-e^t}$;

– if $1 - F(t) \approx t^{-\alpha}$ for $t \to \infty$, then $1 - G(t) = e^{-t^\alpha}$.

PROOF 1.3.– This is out of the scope of our course framework. The details can be found in the special literature (see, for example, [GNE 49, SMI 49]). ■

The second part of this theorem leads to the Gnedenko–Waibull distribution, while the first one leads to the Gomperz distribution, which is often used for modeling human lifetime distributions in actuarial theory.

Another very helpful property of the Gnedenko–Weibull distribution is contained in the following theorem.

THEOREM 1.4.– Let the i.i.d. r.v.'s T_i, $(i = \overline{1,n})$ have Gnedenko–Waibull distribution with the parameters (λ_i, α). Then the r.v.

$$W = \min_{1 \le i \le n} T_i$$

also has Gnedenko–Weibull distribution with parameters (λ, α), where $\lambda = \sum_{i=1}^{n} \lambda_i$. ■

PROOF 1.4.– This can be done with the help of direct calculations and is proposed as exercise 1.1.

EXAMPLE 1.1.– Consider the system consisting of n identical independently failed elements whose lifetimes have the Gnedenko–Weibull distribution with the same parameters λ, α. If the system fails after the failure of at least one element, then the system lifetime also has the Gnedenko–Weibull distribution with the parameters $n\lambda$, α,

$$R_{\text{sys}}(t) = R_1(t) \cdots R_n(t) = \exp\{-(\lambda t^{\alpha} + \cdots + \lambda t^{\alpha})\} = e^{-n\lambda t^{\alpha}}.$$

1.1.3.5. *Gamma-distribution*

The Gamma-distribution is determined by its p.d.f. with parameters $\lambda > 0$ and $\alpha > 0$:

$$f(t) = \frac{\lambda^{\alpha} t^{\alpha-1}}{\Gamma(\alpha)} e^{-\lambda t}, \quad \text{for} \quad t \ge 0, \qquad [1.23]$$

where $\Gamma(\alpha) = \int_0^{\infty} x^{\alpha-1} e^{-x}\, dx$ is a Γ–function. The p.d.f. of Gamma-distribution is represented in Figure 1.7.

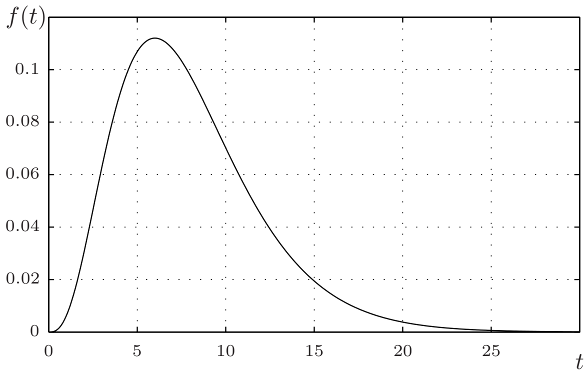


Figure 1.7. *P.d.f. of Gamma-distribution, $\lambda = 0.25$, $\alpha = 2$*

The reliability function for the lifetime Γ-distribution equals

$$R(t) = \int_{\lambda t}^{\infty} \frac{x^{\alpha-1}}{\Gamma(\alpha)} e^{-x}\, dx \quad \text{for} \quad t \geq 0, \tag{1.24}$$

There is no analytical expression for the h.r.f of this distribution, but its mean and variance are

$$\mu_T = \mathbf{E}[T] = \frac{\alpha}{\lambda}, \quad \sigma_T^2 = \mathsf{Var}[T] = \frac{\alpha}{\lambda^2}. \tag{1.25}$$

For $\alpha = 1$, this distribution coincides with the exponential one, and its special case for the integer $\alpha = k$ is known as the Erlang distribution of the order k with parameter λ. Erlang distribution is often used for modeling failures, arising as a result of stress accumulation. Imagine that the unit is subjected to some mechanical shocks that arise after random time intervals having an exponential distribution with parameter λ, and the unit could bear only $k - 1$ such shocks, and after the k-th one, the failure arises. In this case, the lifetime of this unit has an Erlang distribution of the order k with parameter λ.

1.1.3.6. *Log-normal distribution*

This distribution is determined by its p.d.f.

$$f(t) = \frac{1}{t\sigma\sqrt{2\pi}} e^{-\dfrac{(\ln(t-\mu)^2}{2\sigma^2}} \quad \text{for} \quad t \geq 0,$$

with positive parameters shift $\mu > 0$ and form $\sigma > 0$. It is represented in Figure 1.8. The reliability function for the appropriate reliability law is

$$R(t) = \frac{1}{\sqrt{2\pi}} \int_{\frac{1}{\sigma}\ln\frac{t}{\mu}}^{\infty} e^{-\frac{x^2}{2}}\, dx = 1 - \Phi\left(\frac{1}{\sigma}\ln\frac{t}{\mu}\right) = \Phi\left(\frac{\ln\mu - \ln t}{\sigma}\right), \qquad [1.26]$$

and the mean and the variance of this distribution are:

$$\mu_T = \mathbf{E}[T] = \exp\left\{\mu + \frac{\sigma^2}{2}\right\}; \quad \sigma_T^2 = \mathsf{Var}[T] = (e^{\sigma^2} - 1)e^{2\mu+\sigma^2}. \qquad [1.27]$$

Log-normal distribution represents the distribution of r.v. that is an exponent of normally distributed r.v., i.e. if the r.v. X has a normal distribution, then the r.v. $T = e^X$ is log-normally distributed r.v. In other words, the logarithm of log-normally distributed r.v. has a normal distribution. From the definition of this distribution and the Central Limit Theorem of probability theory, it follows that the log-normal distribution arises as a distribution of product generally distributed independent r.v.'s. Since, according to the Central Limit Theorem, the distribution of a sum $X = \sum_{i=1}^n X_i$ of independent uniformly small r.v.'s X_i for a large number n tends to the normal distribution than the r.v.

$$T = e^X = \exp\left\{\sum_{i=1}^n X_i\right\} = \prod_{i=1}^n e^{X_i} = \prod_{i=1}^n T_i$$

should have a log-normal distribution.

1.1.3.7. *Power and Pareto distributions*

The power distribution is determined by its reliability function

$$R(t) = \left(\frac{\mu}{\mu + t}\right)^\alpha \quad \text{for} \quad t > 0 \qquad [1.28]$$

with the positive parameters $\mu > 0$ and $\alpha > 0$. Its p.d.f. and h.r.f. are:

$$f(t) = \frac{\alpha}{\mu}\left(\frac{\mu}{\mu + t}\right)^{\alpha+1}, \quad \lambda(t) = \frac{\alpha}{\mu + t}.$$

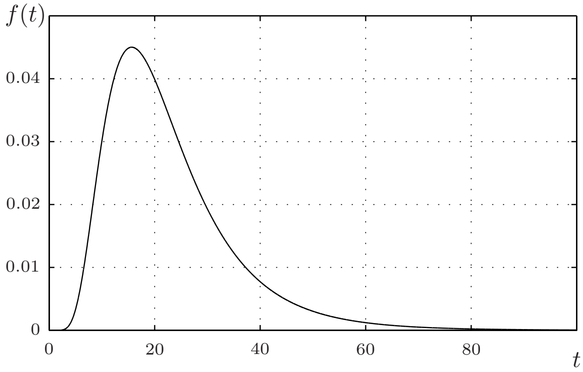


Figure 1.8. *P.d.f. of log-normal distribution, $\mu = 3$, $\sigma = 0.5$*

With the variables changing $\mu = c,\ \mu + t = x$ with $c > 0$, this distribution transforms to the Pareto–distribution with p.d.f.

$$f(x) = \frac{\alpha}{c}\left(\frac{c}{t}\right)^{\alpha+1} \quad \text{for} \quad t \geq c,$$

and parameters $\alpha > 0$, $c > 0$. This is represented in Figure 1.9.

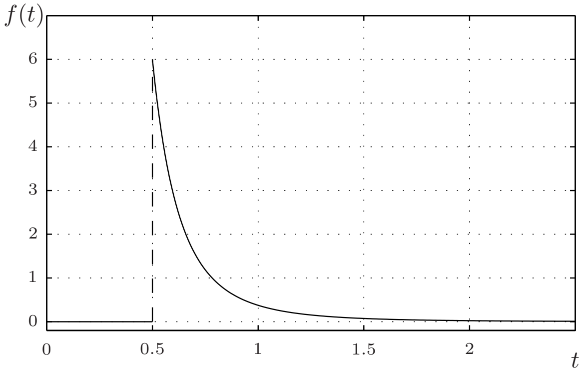


Figure 1.9. *P.d.f. of Pareto distribution, $\alpha = 3$, $c = 0.5$*

The expectation and the variance of a r.v. with this distribution are:

$$\mu_T = \mathbf{E}[T] = \frac{\alpha c}{\alpha - 1} \quad \text{for} \;\; \alpha > 1;$$

$$\sigma_T^2 = \mathsf{Var}[T] = \frac{\alpha c^2}{(\alpha-1)(\alpha-2)} \quad \text{for} \;\; \alpha > 2. \qquad [1.29]$$

We will turn to this distribution in section 2.2.2.6 in connection with the study of damage value distributions.

1.1.3.8. *Relay distribution*

The Relay distribution is determined with its p.d.f.

$$f(t) = \frac{t}{\sigma^2} e^{-\frac{t^2}{2\sigma^2}} \quad \text{for} \quad t \geq 0$$

with the positive parameter $\sigma > 0$. The reliability function for this distribution is:

$$R(t) = e^{-\frac{t^2}{2\sigma^2}} \qquad [1.30]$$

and mean, variance and h.r.f. for this reliability law equal

$$\mu_T = \mathbf{E}T = \sqrt{\frac{\pi}{2}}\sigma, \quad \sigma_T = \frac{4-\pi}{2}\sigma^2, \quad \lambda(t) = \frac{t}{\sigma^2}. \qquad [1.31]$$

NOTE 1.2.– Relay distribution is a special case of the Gnedenko–Weibull distribution if we put $\alpha = 2$ and $\lambda = \frac{1}{2\sigma^2}$.

1.1.3.9. *Uniform distribution*

The uniform distribution is determined with its p.d.f.

$$f(t) = \frac{1}{b-a} 1_{\{a \leq t \leq b\}}, \quad 0 < a < b,$$

with parameters a and b such that $a < b$. This function has a rectangular form (see Figure 1.10), due to which it is often called a *rectangular* distribution.

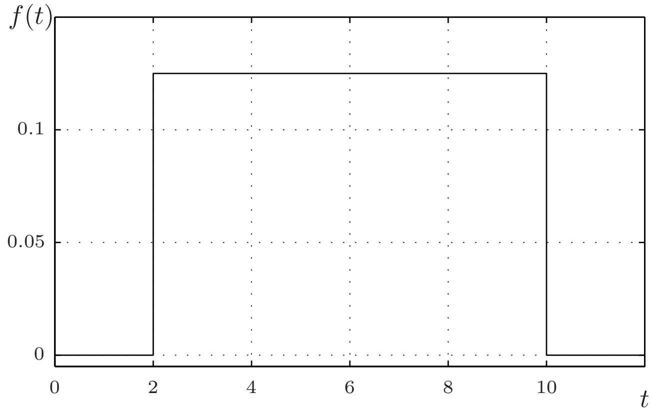


Figure 1.10. *p.d.f. of uniform distribution,* $a = 2, b = 10$

The c.d.f. of this distribution is given by the formula

$$F(t) = \begin{cases} 0, & \text{for } \ t < a; \\ \dfrac{t-a}{b-a}, & \text{for } \ a \leq t \leq b; \\ 1, & \text{for } t > b. \end{cases} \qquad [1.32]$$

The h.r.f. for this reliability law is described by the formula,

$$\lambda(t) = \begin{cases} 0, & t < a; \\ \dfrac{1}{b-t}, & a \leq t \leq b; \\ \text{does not defined}, & t > b, \end{cases} \qquad [1.33]$$

and its mean and the variance equals

$$\mu_T = \mathbf{E}[T] = \frac{a+b}{2}; \quad \sigma_T^2 = \mathsf{Var}[T] = \frac{(b-a)^2}{12}. \qquad [1.34]$$

This distribution is often used for modeling lifetime of units when there is not enough information about it. For example, in the case when only the boundary (minimal and maximal) values of lifetime are known and between these times, failures can arise with equal probability.

1.1.3.10. *Degenerate distribution*

Degenerate is called a distribution of a r.v. that takes only one value, say $b \geq 0$, with probability 1. Its c.d.f. is

$$F(t) = 1_{\{t \geq b\}} = \begin{cases} 0, & \text{for } t < b; \\ 1, & \text{otherwise,} \end{cases}$$

where the function

$$1_{\{t \in A\}} = \begin{cases} 1, & t \in A; \\ 0, & \text{otherwise} \end{cases} \qquad [1.35]$$

is the *indicator function* of the set A. It has a stepwise form with a jump of the value 1 in point b (see Figure 1.11). The expectation and the variance of this r.v. are

$$\mathbf{E}[T] = b; \quad \text{Var}[T] = 0.$$

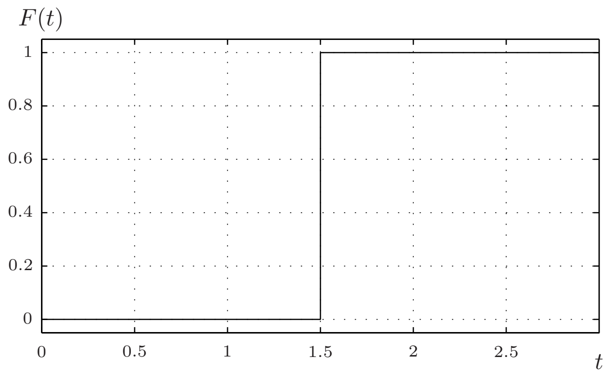


Figure 1.11. *c.d.f. of degenerate distribution, $b = 1.5$*

A mixture of these distributions allows us to construct two-points and any other discrete distributions.

1.1.3.11. *Aging units*

The units with increasing h.r.f. are usually called *aging* units. They are characterized by gradual failures. Using the property of the h.r.f. to increase, we can get useful estimations for the reliability of aging units.

In Tables A1.1 and A1.2 of Appendix 1, the models of the commonly used reliability laws and their appropriate characteristics are presented.

1.1.4. *Examples*

EXAMPLE 1.2.– The lifetime of a gyroscope has the Gnedenko–Weibull distribution [1.20] with parameters $\alpha = 1.5,\ \ \lambda = 10^{-4}(\text{ hours}^{-1})$.

Here we calculate the numerical characteristics of this device up to the time $t = 100$ hours of its operation.

1) To find the probability of the reliable working time using the formula [1.20]

$$R(t) = e^{-\lambda t^{\alpha}}.$$

Substituting the values $\lambda,\ t$ and α from the problem set gives

$$R(100) = \exp\{-10^{-4} \cdot\ 100^{1.5}\} \approx 0.905.$$

2) Frequency and h.r.f. due to formulas [1.3], [1.6] and [1.21] for the Gnedenko–Weibull distribution have the form

$$f(t) = \alpha\,\lambda\,t^{\alpha-1}e^{-\lambda t^{\alpha}}; \quad \lambda(t) = \frac{f(t)}{R(t)} = \alpha\,\lambda\,t^{\alpha-1}.$$

Therefore,

$$f(100) \approx 10^{-4} \cdot\ 1.5 \cdot\ 100^{0.5} \cdot\ 0.905 \approx 0.00136\,\text{hour}^{-1}$$

and

$$\lambda(100) = \frac{f(100)}{R(100)} \approx \frac{0.00136}{0.905} \approx 0.0015\,\text{hour}^{-1}.$$

3) Mean time to the first failure according to formula [1.21] is

$$\mu_T = \lambda^{-1/\alpha} \cdot \Gamma\left(\frac{1}{\alpha} + 1\right) \approx \left(10^{-4}\right)^{-1/1.5} \cdot 0.9027 \approx 419\,\text{hour}.$$

EXAMPLE 1.3.– Suppose that the data about failures of some unit give the following result for the failure frequency:

$$f(t) = c_1\lambda_1 e^{-\lambda_1 t} + c_2\lambda_2 e^{-\lambda_2 t}$$

with some constants $c_1,\ c_2$. Let us calculate all reliability characteristics.

1) Find the reliability function. Based on formulas [1.2], [1.3], we can find

$$\begin{aligned} R(t) &= 1 - \left[\int_0^t c_1\lambda_1 e^{-\lambda_1 u}\,du + \int_0^t c_2\lambda_2 e^{-\lambda_2 u}\,du\right] = \\ &= 1 - \left[-c_1 e^{-\lambda_1 t} + c_1 - c_2 e^{-\lambda_2 t} + c_2\right] = \\ &= 1 - (c_1 + c_2) + c_1 e^{-\lambda_1 t} + c_2 e^{-\lambda_2 t}. \end{aligned}$$

To calculate sum $c_1 + c_2$, we can use the relation $\int_0^\infty f(t)\,dt = 1$. Thus,

$$\int_0^\infty c_1\lambda_1 e^{-\lambda_1 t}\,dt + \int_0^\infty c_2\lambda_2 e^{-\lambda_2 t}\,dt = c_1 + c_2 = 1.$$

and, therefore,

$$R(t) = c_1 e^{-\lambda_1 t} + c_2 e^{-\lambda_2 t}.$$

2) Calculating the h.r.f. according to [1.6] gives

$$\lambda(t) = \frac{f(t)}{R(t)} = \frac{c_1\lambda_1 e^{-\lambda_1 t} + c_2\lambda_2 e^{-\lambda_2 t}}{c_1 e^{-\lambda_1 t} + c_2 e^{-\lambda_2 t}}.$$

3) Using formula [1.9], we can find the mean unit lifetime as

$$\mu_T = \int_0^\infty R(t)\,dt = c_1\int_0^\infty e^{-\lambda_1 t}\,dt + c_2\int_0^\infty e^{-\lambda_2 t}\,dt = \frac{c_1}{\lambda_1} + \frac{c_2}{\lambda_2}.$$

1.1.5. *Exercises*

EXERCISE 1.1.– Let the r.v.'s T_i ($i = \overline{1, n}$) be independent and have the Gnedenko–Weibull distribution with parameters (λ_i, α). Prove theorem 1.4 that the r.v.

$$W = \min_{1 \le i \le n} T_i$$

also has the Gnedenko–Weibull distribution with parameters (λ, α), where $\lambda = \sum_{i=1}^{n} \lambda_i$.

EXERCISE 1.2.– A unit is constant and equals $0.82 \cdot 10^{-3}$ hour^{-1}. Find the reliability of the unit during 6 hours, $R(6)$, frequency of failures in time 100 hours, $f(100)$, and mean unit lifetime μ_T.

Answer: $R(6) = 0.995$, $f(100) = 0.75 \cdot 10^{-3}$ hours^{-1}, $\mu_T = 1220$ hours.

EXERCISE 1.3.– The reliability of an automatic line for the production of cylinders for an automobile engine during 120 hours is 0.9. It is assumed that the exponential reliability law holds. Calculate the h.r.f. and frequency of failures for time $t = 120$ hours and mean lifetime.

Answer: $\lambda = 0.83 \cdot 10^{-3}$ hours^{-1}, $f(120) = 0.747 \cdot 10^{-3}$ hours^{-1}, $\mu_T = 1200$ hours.

EXERCISE 1.4.– Mean lifetime of an automatic control system is 640 hours. It is assumed that the exponential reliability law holds. Find its reliability during $t = 120$ hours, frequency of failures and h.r.f. for this time.

Answer: $R(120) = 0.83$, $f(120) = 1.3 \cdot 10^{-3}$ hours^{-1}, $\lambda(120) = 1.56 \cdot 10^{-3}$ hours^{-1}.

EXERCISE 1.5.– The lifetime of a unit has the truncated normal distribution with parameters $\mu_T = 8000$ hours, $\sigma_T = 1000$ hours. Find its reliability during 8000 hours.

Answer: $R(8000) = 0.5$.

EXERCISE 1.6.– Using the data from exercise 1.5, calculate the frequency of failures for time $t = 6000$ hours.

Answer: $f(6000) = 5.4 \cdot 10^{-5}$ hours^{-1}.

EXERCISE 1.7.– Using the data from exercise 1.5, calculate the h.r.f for time $t = 10000$ hours.

Answer: $\lambda(10000) = 2.35 \cdot 10^{-3}$ hours^{-1}.

EXERCISE 1.8.– In failure data analysis, it has been found that its frequency of failures has a form

$$f(t) = 2\,\lambda\, e^{-\lambda t}(1 - e^{-\lambda t}).$$

Find the reliability characteristics $R(t)$, $\lambda(t)$ and μ_T.

Answer: $R(t) = 2\,e^{-\lambda t} - e^{-2\,\lambda t}$; $\quad \lambda(t) = \dfrac{\left(1 - e^{-\lambda t}\right)\lambda}{1 - \frac{1}{2}e^{-\lambda t}}$; $\quad \mu_T = \dfrac{3}{2\,\lambda}$.

1.2. Reliability of renewable systems

In the previous section reliability characteristics of units (articles) up to the first failure were considered. In this chapter we attempt to study more complex objects that are considered as a single entity, which can be repaired or replaced after failure and, therefore, the term “article” will be used here instead of unit. We start from the case of instantaneous replacement. Taking into account that the replacement time is usually much lower than the article lifetime, we can understand that this model is adequate enough to the real situations.

1.2.1. *Reliability of instantaneously renewable articles*

1.2.1.1. *Renewal process: definition*

Consider a system, consisting of an article, which operates continuously in time and suppose that the failed article is instantaneously replaced by the new identical one. We denote the sequence of the article lifetimes by $\{T_n,\ n = 1, 2, \dots\}$. They are assumed to be independent identically distributed random variables (i.i.d. r.v.’s) and their common cumulative distribution function (c.d.f.) is given by

$$F(t) = \mathbf{P}\{T_i \le t\}$$

with expectation $\mathbf{E}[T_n] = \mu$ and variance $\mathsf{Var}[T_n] = \sigma^2$. Then the values

$$S_1 = T_1, \; S_2 = T_1 + T_2, \cdots, \; S_n = T_1 + T_2 + \cdots + T_n \tag{1.36}$$

form the sequence of the article failure times, and the process

$$N(t) = \max\{n : S_n \leq t\} \tag{1.37}$$

represents the number of replacements up to time t.

DEFINITION 1.1.– The sequence $\{S_n, \; n = 1, 2, \ldots\}$ is called the *failure flow*[4] and the process $\{N(t), \; t \geq 0\}$ is called the *renewal process*.

Sometimes it is necessary to consider the failure flows $\{S_n, \; n = 1, 2, \ldots\}$, in which the first failure time T_1 distribution differs from others (for example, in the case when the observation for the replacement process begins not from the new article). Then, generalized failure flow and the renewal process is introduced.

DEFINITION 1.2.– The sequence $\{S_n, \; n = 1, 2, \ldots\}$ and the process $\{N(t), \; t \geq 0\}$ are called the *general failure flow* appropriate *general renewal process* (or *delayed failure flow* appropriate *delayed renewal process*) if all r.v.'s generating them $\{T_n, \; n = 1, 2, \ldots\}$ are independent and identically distributed except for the first one that has a different c.d.f. $F_1(t)$. The failure flow $\{S_n, \; n = 1, 2, \ldots\}$ and the appropriate process $\{N(t), \; t \geq 0\}$ are called *stationary* (the reason for this name will be discussed later) if

$$F_1(t) = \frac{1}{\mathbf{E}[T_2]} \int_0^t (1 - F(u))du. \tag{1.38}$$

In the case of the article state observation in discrete time, the failure flow and the renewal process transform into discrete time processes and are called *discrete failure flow* and *discrete renewal process* respectively. A special case of discrete renewal processes – arithmetic ones are usually used in practice.

4 In general theory of stochastic processes, this type of process is known as a *point process*.

DEFINITION 1.3.– The renewal process is called arithmetic if r.v's T_n that determine it have an arithmetic distribution.

The distribution $F(t)$ is called *discrete* if it is concentrated at the discrete set of points $\{x_k,\ k = 0, 1, 2, \dots\}$; it is called *lattice* if the points x_k of the set have the form $x_k = a + k\Delta$, and if $a = 0$, then the appropriate distribution is called *arithmetic* and the maximal possible value of Δ is called the *step* of the distribution.

The most interesting characteristics of the failure flow and the renewal process are the distribution of some, say n-th, time to failure, number of failures (and replacements respectively) up to a time t and so on. Consider these characteristics.

1.2.1.2. *Distribution of failure time*

Denote by $F(t)$ the c.d.f of the r.v's T_n, and by μ and σ^2 their mean value and variance and suppose that they satisfy the following assumption.

ASSUMPTION 1.1.– There are no immediate failures. In mathematical terms this means that $F(+0) = 0$.

Because the failure times S_n are the sum of i.i.d. r.v., their distributions are calculated based on the convolution formula:

$$
\begin{aligned}
&F_1(t) \equiv \mathbf{P}\{S_1 \le t\} = \mathbf{P}\{T_1 \le t\} = F(t),\\
&F_n(t) \equiv \mathbf{P}\{S_n \le t\} = \int_0^t F_{n-1}(t-u)\, dF(u) \equiv\\
&\qquad \equiv F^{(*n)}(t), \quad n > 1.
\end{aligned}
\tag{1.39}
$$

The expectation and the variance of these r.v.'s are

$$
\mathbf{E}[S_n] = n\,\mathbf{E}[T_1] = n\,\mu, \quad \mathsf{Var}[S_n] = n\,\mathsf{Var}[T_1] = n\,\sigma^2. \tag{1.40}
$$

1.2.1.3. *Distribution of the failure number*

It is not difficult to find the failure number distribution. Because the events

$\{S_n \le t\} = \{$"time of n-th failure is not greater than t"$\}$

and

$$\{N(t) \geq n\} = \{\text{“it occurs not less than } n \text{ failures up to time } t\text{”}\}$$

are equivalent,

$$\{S_n \leq t\} = \{N(t) \geq n\}, \qquad [1.41]$$

their probabilities coincide, $\mathbf{P}\{S_n \leq t\} = \mathbf{P}\{N(t) \geq n\}$. Therefore,

$$\begin{aligned} p_n(t) \equiv \mathbf{P}\{N(t) = n\} &= \mathbf{P}\{n \leq N(t) < n+1\} = \\ &= \mathbf{P}\{N(t) \geq n\} - \mathbf{P}\{N(t) \geq n+1\} = \\ &= \mathbf{P}\{S_n \leq t\} - \mathbf{P}\{S_{n+1} \leq t\} = \\ &= F^{(*n)}(t) - F^{(*(n+1))}(t), \end{aligned} \qquad [1.42]$$

where the function $F^{(*n)}(t)$ is determined by the equality [1.39].

1.2.1.4. *Moment generating function of the renewal process*

The previous expressions are not convenient enough for calculation. Therefore, the renewal process distribution can be represented in terms of its m.g.f.

$$p(z, t) = \mathbf{E}\left[z^{N(t)}\right] = \sum_{0 \leq n \leq \infty} z^n p_n(t),$$

and its Laplace transform (LT)

$$\tilde{p}(z, s) = \int_0^\infty e^{-st}\, p(z, t)\, dt.$$

Substituting m.g.f. and then LT in formula [1.42], we can obtain

$$\begin{aligned} \tilde{p}(z, s) &= \int_0^\infty e^{-st}\, p(z, t)\, dt = \int_0^\infty e^{-st} \sum_{0 \leq n < \infty} z^n p_n(t) = \\ &= \sum_{0 \leq n < \infty} z^n \left(F^{(*n)}(t) - F^{(*(n+1))}(t), \right). \end{aligned}$$

Taking into account that the LT of the convolution equals the product of LT and its components, and that the LT of a c.d.f. connected with LT of its p.d.f. (Laplace–Stieltjes transform of c.d.f.) by the relation

$$\tilde{F}(s) = \int_0^\infty e^{-st} F(t)\, dt = -\frac{1}{s} e^{-st} F(t) \Big|_0^\infty + \frac{1}{s} \int_0^\infty e^{-st}\, dF(t) =$$

$$= \frac{1}{s} \left[\int_0^\infty e^{-st}\, dF(t) + F(+0) \right] = \frac{1}{s} [\tilde{f}(s) + F(0)],$$

and paying attention to assumption 1.1 about $F(+0) = 0$, we can obtain

$$\tilde{p}(z, s) = \int_0^\infty e^{-st} p(z, t)\, dt = \frac{1}{s} \sum_{0 \le n < \infty} z^n (1 - \tilde{f}(s)) \tilde{f}^n(s) =$$

$$= \frac{1 - \tilde{f}(s)}{s(1 - z\tilde{f}(s))}. \qquad [1.43]$$

We state this result as a theorem.

THEOREM 1.5.– LT of the renewal process m.g.f. is given by formula [1.43]. ■

In general, this expression can be used for the moments of the process $N(t)$ calculation or for its asymptotic analysis. However, sometimes (although not enough often) with its inversion and expansion in Taylor series, we can find explicit expressions for the number of failure distribution. This will be done with the help of an example about the Poisson process in section 1.2.3.

1.2.1.5. *Asymptotic properties of the renewal process*

The Large Number Law (LNL) and the Central Limit Theorem (CLT) for sums of i.i.d. r.v. are known from Probability Theory. Because the intervals between failures are i.i.d. r.v. and the failure flow is their sum, the LNL and CLT are true.

THEOREM 1.6 (LNL for S_n).– It holds

$$\lim_{n\to\infty} \frac{S_n}{n} = \mu,$$

if $\mu < \infty$, then convergence in probability holds, and if $\sigma^2 < \infty$, then the convergence with probability 1 takes place.

THEOREM 1.7 (CLT for S_n).– If $\mu < \infty$ and $\sigma^2 < \infty$, then for $t \to \infty$ time to the n-th failure S_n after the usual normalization tends (in distribution) to normal r.v.,

$$\lim_{n\to\infty} \mathbf{P}\left\{\frac{S_n - n\mu}{\sigma\sqrt{n}} \le x\right\} = \Phi(x),$$

where $\Phi(x)$ is the c.p.f. of standard normal distribution [1.18].

Concerning the asymptotic behavior of the renewal process, we can understand that relation [1.41], connecting the renewal process $N(t)$ with the failure flow S_n, also allows the transformation of these results (LNL and CLT) to the renewal process.

THEOREM 1.8 (LNL for $N(t)$).– The following limiting relation holds

$$\lim_{n\to\infty} \frac{N(t)}{t} = \frac{1}{\mu},$$

if $\mu < \infty$, then convergence in probability holds, and if $\sigma^2 < \infty$, then convergence with probability 1 takes place.

THEOREM 1.9 (CLT for $N(t)$).– If $\mu < \infty$ and $\sigma^2 < \infty$, then for $t \to \infty$, the number of failures $N(t)$ after appropriate normalization tends (in distribution) to normal r.v.,

$$\lim_{n\to\infty} \mathbf{P}\left\{\frac{N(t) - \mu^{-1}t}{\sigma\sqrt{t\mu^{-3}}} \le x\right\} = \Phi(x).$$

As it was mentioned above, the **proofs** of theorems 1.6 and 1.7 are known from Probability Theory; and the proofs of theorems 1.8 and 1.9 can be obtained by using the events equivalence, see formula [1.41].

The last relation allows us to propose enough precise evaluation of the 99% confidence interval for number of failures after enough long time t with the help of "3 σ–rule", namely:

$$\mathbf{P}\{\mu^{-1}t - 3\sigma^2\sqrt{t\mu^{-3}} \le N(t) \le \mu^{-1}t + 3\sigma^2\sqrt{t\mu^{-3}}\} = 0.99$$

1.2.2. *Renewal function*

1.2.2.1. *Definition*

DEFINITION 1.4.– Expectation of the renewal process (mean renewal numbers during time t) is called *renewal function*,

$$H(t) = \mathbf{E}[N(t)]. \quad [1.44]$$

This is one of the most important characteristics of the renewal process. For its calculation, it is convenient to use equality [1.41] and the following relation

$$H(t) = \sum_{n=1}^{\infty} n p_n(t) = \sum_{n=1}^{\infty}\sum_{k\ge n} p_k(t) =$$

$$= \sum_{n=1}^{\infty} \mathbf{P}\{N(t) \ge n\} = \sum_{n=1}^{\infty} F^{*n}(t). \quad [1.45]$$

From this relation, it follows that the renewal function must satisfy the integral equations that are known as *forward* and *backward renewal equations*:

$$H(t) = F(t) + \int_0^t F(t-u)\, dH(u) = \quad [1.46]$$

$$= F(t) + \int_0^t H(t-u)\, dF(u). \quad [1.47]$$

If lifetime of an article has a p.d.f. $f(t) = F'(t)$, then the renewal function $H(t)$ can be differentiable and its derivative

$$h(t) = H'(t) \qquad [1.48]$$

is called *renewal density*. In applications, this function represents (instantaneous) *failure flow intensity*, which is one of the most important reliability characteristics of the renewable articles. Practically, this means the *mean number of failures per unit of time*, or more accurately it can be determined as

$$h(t) = \lim_{\Delta t \to 0} \frac{H(t + \Delta t) - H(t)}{\Delta t}$$

that is the definition of the renewal density.

By differentiation of relations [1.46], [1.47], we can find that the renewal density satisfies the integral equations

$$h(t) = f(t) + \int_0^t f(t-u)h(u)\,du = \qquad [1.49]$$

$$= f(t) + \int_0^t h(t-u)f(u)\,du, \qquad [1.50]$$

which are, respectively, known as *forward* and *backward renewal equations for renewal density*.

For the solution of these equations, we can use the operational method (or the Laplace transform method). In other words, by applying to these equations Laplace transform

$$\tilde{h}(s) = \int_0^\infty e^{-st}h(t)\,dt, \quad \tilde{f}(s) = \int_0^\infty e^{-st}f(t)\,dt,$$

we can obtain the equation[5]

$$\tilde{h}(s) = \tilde{f}(s) + \tilde{f}(s)\tilde{h}(s)$$

and its solution can be obtained in the form

$$\tilde{h}(s) = \frac{\tilde{f}(s)}{1 - \tilde{f}(s)}. \qquad [1.51]$$

This relation can be used in theoretical study of the reliability function as well as in its concrete calculation.

NOTE 1.3.– If the time is measured with discrete units, and the r.v.'s T_n and S_n have an arithmetic distributions, then the reliability function and density are transformed into appropriate sequences H_n and h_n. Analogously to the previous case as an exercise 1.9, it can be proposed to rewrite appropriate reliability equations and their solutions for the arithmetic renewal process.

1.2.2.2. *Renewal theorems*

Study of the renewal process asymptotic behavior for a large time t is one of the most important practical problems. In section 1.2.1.5 some theorems about renewal process $N(t)$ asymptotic behavior have been considered. In this section some results about asymptotic behavior of renewal function will be presented without proof. Proofs of these theorems can be found in the literature (see the bibliography).

We start with the asymptotic behavior of the mean value of the renewal process. The expected number of failures during long time t is a fraction of the time to mean inter-failures time μ,

$$H(t) \approx \frac{t}{\mu} \quad \text{for } t \to \infty.$$

Formally, this property consists of the assertion of the so-called *elementary renewal theorem*.

5 The same result and the following solution can be obtained by applying the Laplace–Stieltjes transform to equations [1.46], [1.47]

THEOREM 1.10 (Elementary renewal theorem).– If $\mathbf{E}[T_n] = \mu < \infty$, then

$$\lim_{t\to\infty} \frac{H(t)}{t} = \frac{1}{\mu}. \qquad [1.52]$$

In the case when lifetime distribution has a density, the theorem assertion can be reinforced.

CONSEQUENCE 1.1.– If c.d.f. $F(t)$ is differentiable (i.e. there exists its p.d.f.), then the renewal function $H(t)$ is also differentiable, and for its density $h(t) = H'(t)$ under theorem 1.10 conditions, the following assertion holds

$$\lim_{t\to\infty} h(t) = \frac{1}{\mu}. \qquad [1.53]$$

The elementary renewal theorem allows significant generalizations for mostly applicable cases of the lifetime absolutely continuous and discrete distributions.

THEOREM 1.11 (Key renewal theorem or Smith's theorem).– If c.d.f. $F(t)$ is differentiable, $\mu < \infty$ and $g(t)$ is some integrable function, i.e. $\int_0^\infty g(u)\,du < \infty$, then the following relation holds

$$\lim_{t\to\infty} \int_0^t g(t-u)\,h(u)\,du = \frac{1}{\mu}\int_0^\infty g(u)\,du. \qquad [1.54]$$

Analogous assertion takes place for arithmetic distributions that is used in the case, when the time is measured in discrete units.

CONSEQUENCE 1.2.– If lifetime has an arithmetic distribution and $g(n)$ is a summing function, i.e. $\sum_{n\geq 0} g(n) < \infty$, then analogous to equality [1.54] holds

$$\lim_{n\to\infty} \sum_{k=0}^{n} g(n-k)\,h_k = \frac{1}{\mu}\sum_{k=0}^{\infty} g(k). \qquad [1.55]$$

Many characteristics of the renewal process can be represented in terms of renewal function. Particularly it can be used for the study of two very important processes, connected with renewable articles.

1.2.3. *Age and residual lifetime of an article*

Considering the renewal process as an article replacement process, we can see that the age and residual lifetime of an article are continuously changing. Thus, its reliability characteristics are also changing. Therefore together with the failure flow $\{S_n,\ n = 1, 2, \dots\}$ and the renewal process $\{N(t),\ t \geq 0\}$, it is also necessary to consider two more processes

$$Z^-(t) = t - S_{N(t)}, \quad Z^+(t) = S_{N(t)+1} - t, \qquad [1.56]$$

the first process is called an *age* and the second process is called a *residual lifetime* of an article operating in time t.

For calculation of the processes $Z^-(t)$ and $Z^+(t)$, one-dimensional distributions are denoted by $G^\pm(t, x)$ c.d.f. of appropriate distributions at the separate random interval T_n,

$$G^\pm(t, x) = \mathbf{P}\{Z^\pm(S_{n-1} + t) \leq x,\ t < T_n\}. \qquad [1.57]$$

The following theorem represents one-dimensional distributions of the processes $Z^\pm(t)$ in terms of its appropriate distributions $G^\pm(t, x)$ at the separate intervals T_n and renewal function $H(t)$.

THEOREM 1.12.– The age and the residual lifetime of an article's one-dimensional distribution have a form

$$\mathbf{P}\{Z^\pm(t) \leq x\} = G^\pm(t, x) + \int_0^t G^\pm(t - u, x)\, dH(u), \qquad [1.58]$$

where $H(u)$ is the renewal function, generated by the failure flow $\{S_n,\ n = 1, 2, \dots\}$.

PROOF 1.5.– The complete probability formula with respect to a system of events $\{S_n \leq t < S_{n+1}\}$ is given by

$$\mathbf{P}\{Z^\pm(t) \leq x\} = G^\pm(t, x) + \sum_{n=1}^{\infty} \mathbf{P}\{Z^\pm(t) \leq x,\ S_n \leq t < S_{n+1}\}.$$

Then using the complete probability formula with respect to r.v.'s S_n in the second summand, changing the order of summing and integration and taking into account that $H(t) = \sum_{n=1}^{\infty} \mathbf{P}\{S_n \le t\}$, we can obtain

$$\mathbf{P}\{Z^{\pm}(t) \le x\} = G^{\pm}(t,x) +$$

$$+ \sum_{n=1}^{\infty} \int_0^t \mathbf{P}\{Z^{\pm}(t) \le x, \ S_n \le t < S_{n+1} \mid S_{=}u\} \, d\mathbf{P}\{S_k \le u\} =$$

$$= G^{\pm}(t,x) +$$

$$+ \int_0^t \sum_{n=1}^{\infty} \mathbf{P}\{Z^{\pm}(t-u) \le x, \ t-u < T_{n+1}\} \, d\mathbf{P}\{S_n \le u\} =$$

$$= G^{\pm}(t,x) + \int_0^t G^{\pm}(t-u,x) \, dH(u)$$

which proves the theorem. ■

We now consider the distributions of the processes at the separate inter-failure times T_n.

LEMMA 1.1.– For the processes $Z^{\pm}(t)$ at separate failure time T_n, the following representations hold

$$G^{-}(t,x) = \mathbf{P}\{Z^{-}(S_{n-1}+t) \le x, \ t < T_n\} =$$

$$= 1_{\{t \le x\}}(1 - F(t)); \qquad [1.59]$$

$$G^{+}(t,x) = \mathbf{P}\{Z^{+}(S_{n-1}+t) \le x, \ t < T_n\} =$$

$$= F(t+x) - F(t), \qquad [1.60]$$

where $1_{\{t \le x\}}$ denotes the indicator function of the set $\{t \le x\}$ defined by formula [1.35] in section 1.1.3.10,

$$1_{\{t \le x\}} = \begin{cases} 1, & t \le x; \\ 0, & \text{otherwise.} \end{cases}$$

PROOF 1.6.– Indeed, for the processes $Z^-(t)$, $Z^+(t)$ at the separate interval T_n, i.e. jointly with the event $\{t < T_n\}$, the following representations hold

$$\{Z^-(S_{n-1}+t),\ t < T_n\} = \{t,\ t < T_n\},$$
$$\{Z^+(S_{n-1}+t),\ t < T_n\} = \{T_n - t,\ t < T_n\}.$$

Thus, for $G^{\pm}(t,x)$, it is true:

$$\begin{aligned} G^-(t,x) &= \mathbf{P}\{Z^-(S_{n-1}+t) \le x,\ t < T_n\} = \\ &= \mathbf{P}\{t \le x,\ t < T_n\} = 1_{\{t \le x\}}(1 - F(t)); \\ G^+(t,x) &= \mathbf{P}\{Z^+(S_{n-1}+t) \le x,\ t < T_n\} = \\ &= \mathbf{P}\{T_n - t \le x,\ t < T_n\} = F(t+x) - F(t), \end{aligned}$$

which proves the lemma. ■

For $t \to \infty$, the age and the residual lifetime processes tend to a stationary regime, or in other words, there exist the limits

$$\lim_{t\to\infty} \mathbf{P}\{Z^{\pm}(t) \le x\}.$$

The limiting distributions for both processes coincide and have the form of the initial time distribution [1.38] for a stationary renewal process.

THEOREM 1.13.– When $t \to \infty$, the distributions of the age and the residual lifetime processes converge to the stationary one, whose distribution is given by

$$\lim_{t\to\infty} \mathbf{P}\{Z^{\pm}(t) \le x\} = \frac{1}{\mu}\int_0^x (1 - F(u))\,du. \qquad [1.61]$$

PROOF 1.7.– Because $F(t) \to 1$ for $t \to \infty$, then it is evident that there exists the limit

$$\lim_{t\to\infty} G^{\pm}(t,x) = 0.$$

Moreover, the functions $G^{\pm}(t,x)$ satisfy Smith's key renewal theorem 1.11. Thus, there exist the limits:

$$\lim_{t\to\infty} \mathbf{P}\{Z^-(t) \le x\} = \frac{1}{\mu}\int_0^\infty 1_{\{t\le x\}}\,(1 - F(t))\;dt = \frac{1}{\mu}\int_0^x (1 - F(t))\;dt;$$

$$\lim_{t\to\infty} \mathbf{P}\{Z^+(t) \le x\} = \frac{1}{\mu}\int_0^\infty (F(t+x) - F(t))\;dt =$$

$$= \frac{1}{\mu}\int_0^\infty \Big[(1 - F(t)) - (1 - F(t+x))\Big]\,dt = \frac{1}{\mu}\int_0^x (1 - F(u))\;du.$$

The theorem is thus proved. ■

1.2.4. *Reliability characteristics with regard to replacement time*

In reality, for fault detecting, localization and real repair or replacement of some article, additional time is needed. Suppose that besides the article lifetime $\{T_n,\ n = 1,2,\ldots\}$, there is also some sequence of renovation (repair, replacement) times $\{T'_n,\ n = 1,2,\ldots\}$, which are i.i.d. r.v.. Their common c.d.f. is denoted by

$$G(t) = \mathbf{P}\{T'_n \le t\}$$

From the point of view of workability of a renewable article, the main characteristic of its reliability is the availability coefficient, $K_{\mathrm{av}}(t)$.

DEFINITION 1.5.– An *availability coefficient* of some article is part of the time for which the article is in up state during the whole active time t,

$$K_{\mathrm{av}}(t) = \frac{\text{Up time of an article time during active time } t}{t}.$$

An additional value

$$K_{\mathrm{fail}}(t) = \frac{\text{Down time of an article time during active time } t}{t} =$$
$$= 1 - K_{\mathrm{av}}(t)$$

is known as *failure coefficient* of an article.

With increasing time t to infinity according to the ergodic theorems of the probability theory, these values tend to some non-random stationary values

$$\lim_{t\to\infty} K_{\text{av}}(t) = K_{\text{av}} = \frac{\mathbf{E}[T_1]}{\mathbf{E}[T_1] + \mathbf{E}[T_1']}, \qquad [1.62]$$

$$\lim_{t\to\infty} K_{\text{fail}}(t) = K_{\text{fail}} = \frac{\mathbf{E}[T_1']}{\mathbf{E}[T_1] + \mathbf{E}[T_1']}. \qquad [1.63]$$

Another reliability characteristics of renewable articles or systems and the methods of their investigation will be considered in section 1.6, where reliability renewable redundant systems are considered.

1.2.5. *Examples*

EXAMPLE 1.4 (Poisson process).– We consider the failure flow (and appropriate renewal process) whose lifetimes have the exponential distribution with p.d.f. $f(t) = \lambda e^{-\lambda t}$. Failure times S_n in this case are the sums of independent identically exponentially distributed r.v.'s and according to [1.39] they have Erlang distributions,

$$\mathbf{P}\{S_n \le t\} \equiv F^{(*n)}(t) = 1 - e^{-\lambda t} \sum_{k=0}^{n-1} \frac{(\lambda t)^k}{k!}. \qquad [1.64]$$

As in exercise 1.12, it is proposed to check this formula.

Then the distribution of the renewal (replacements number) process $N(t)$, calculated by formula [2.7], is given by

$$\mathbf{P}\{N(t) = n\} \equiv p_n(t) = \frac{(\lambda t)^n}{n!} e^{-\lambda t}. \qquad [1.65]$$

This is a Poisson distribution and therefore the appropriate process is called the *Poisson* process.

According to [1.51], the Laplace transform of the renewal density of this process is given by

$$\tilde{h}(s) = \frac{\tilde{f}(s)}{1 - \tilde{f}(s)} = \frac{\lambda/(\lambda + s)}{1 - \lambda/(\lambda + s)} = \frac{\lambda}{s}. \qquad [1.66]$$

Using an inverse Laplace transform from this relation, we can obtain the expression for renewal density and by integration, we can also obtain the expression for renewal function

$$h(t) = \lambda, \quad H(t) = \lambda t. \qquad [1.67]$$

Thus, the intensity of failures for the Poisson process coincides with the h.r.f. of the separate unit and the parameter of its exponential lifetime distribution.

Finally, calculations with formula [1.58], according to formulas [1.59] and [1.60], show that the age and residual lifetime distributions of an article with this lifetime distribution do not depend on the observation time t and coincide with the initial distribution,

$$\mathbf{P}\{Z^{\pm}(t) \le x\} = 1 - e^{-\lambda x}. \qquad [1.68]$$

At this point, it is reasonable to focus on some paradox, connected with the processes $Z^{\pm}(t)$. In fact, because the values $Z^{\pm}(t)$ have the same distributions that coincide with the failure lifetime distribution, according to the very evident relation

$$Z^{-}(t) + Z^{+}(t) = T_{N(t)+1}, \qquad [1.69]$$

it follows, for example,

$$\mathbf{E}[Z^{-}(t)] + \mathbf{E}[Z^{+}(t)] = \mathbf{E}[T_{N(t)+1}]$$

and thus it does not follow that $\mathbf{E}Z^{-}(t) \le \mu$, $\mathbf{E}Z^{+}(t) \le \mu$ or $\mathbf{E}\left(Z^{-}(t) + Z^{+}(t)\right) = \mu$. It follows that in the right side of relation [1.69], the random interval has a random index, and the expectation of the inter-failure interval, covering the fixed point t, is twice the usual failure interval. Thus, $\mathbf{E}[T_{N(t)+1}] \neq \mathbf{E}[T_n] = \lambda^{-1}$, because the intervals with the random index distributed different from intervals with the fixed index.

EXAMPLE 1.5.– We now calculate the failure flow intensity for a renewable article with failure frequency from example 1.3 of section 1.1.4.

In order to use formula [1.51], we first find the LT of the failure frequency $f(t)$:

$$\tilde{f}(s) = \int_0^\infty f(t)e^{-st}\,dt = \int_0^\infty c_1\lambda_1 e^{-\lambda_1 t}e^{-st}dt + \int_0^\infty c_2\lambda_2 e^{-\lambda_2 t}e^{-st}\,dt =$$
$$= c_1\lambda_1 \int_0^\infty e^{-t(\lambda_1+s)}\,dt + c_2\lambda_2 \int_0^\infty e^{-t(\lambda_2+s)}\,dt = \frac{c_1\lambda_1}{\lambda_1 + s} + \frac{c_2\lambda_2}{\lambda_2 + s}.$$

Substitution of this expression into formula [1.51] gives

$$\tilde{h}(s) = \frac{\tilde{f}(s)}{1-\tilde{f}(s)} = \frac{s(c_1\lambda_1 + c_2\lambda_2) + \lambda_1\lambda_2}{s[s+\lambda_1(1-c_1)+\lambda_2(1-c_2)]} =$$
$$= \frac{c_1\lambda_1 + c_2\lambda_2}{s+\lambda_1(1-c_1)+\lambda_2(1-c_2)} + \frac{\lambda_1\lambda_2}{s\,[s+\lambda_1(1-c_1)+\lambda_2(1-c_2)]}.$$

Using inverse Laplace transform tables, for example, we can find

$$h(t) = (c_1\lambda_1 + c_2\lambda_2)e^{-[\lambda_1(1-c_1)+\lambda_2(1-c_2)]t} +$$
$$+ \lambda_1\lambda_2\left[\frac{1}{\lambda_1(1-c_1)+\lambda_2(1-c_2)} - \frac{e^{-[\lambda_1(1-c_1)+\lambda_2(1-c_2)]t}}{\lambda_1(1-c_1)+\lambda_2(1-c_2)}\right]$$

After some simple algebra, it is given by

$$h(t) = \frac{1}{\lambda_1 c_2 + \lambda_2 c_1}\left[\lambda_1\lambda_2 + c_1c_2(\lambda_1-\lambda_2)^2 e^{-(\lambda_1c_2+\lambda_2c_1)t}\right].$$

EXAMPLE 1.6 (Alternating renewal process).– We consider a renewable system with regard to the replacement times (see section 1.2.3). We denote by $F(t)$ and $G(t)$ the c.d.f.'s of the unit lifetimes T_n and replacement times T_n', respectively,

$$F(t) = \mathbf{P}\{T_n \le t\}, \quad G(t) = \mathbf{P}\{T_n' \le t\}.$$

The system behavior from its reliability point of view can be described by the process $\{X(t),\ t \geq 0\}$, with two states:

$$X(t) = \begin{cases} 1, & \text{if the system is up;} \\ 0, & \text{otherwise.} \end{cases}$$

This relation stochastic process with two states is called *the alternating renewal process*.

Supposing that in the initial time, the system is in the "up" state, its failures occur in random times

$$S_1 = T_1, S_2 = (T_1 + T_1') + T_2, \ldots, S_{n+1} = \sum_{i=1}^{n}(T_i + T_i') + T_{n+1},$$

and its renovations, respectively, occur in time epochs

$$S_1' = T_1 + T_1', S_2' = (T_1 + T_1') + (T_2 + T_2'), \ldots, S_{n+1}' = \sum_{i=1}^{n+1}(T_i + T_i').$$

In general, the times between failures and renovations are i.i.d. r.v.'s

$$T_n'' = T_n + T_n',$$

while the sequence $\{S_n,\ n = 1, 2, \ldots\}$ is a delayed and the sequence $\{S_n',\ n = 1, 2, \ldots\}$ is a simple failure flow.

We denote by $\tilde{f}(s)$ and $\tilde{g}(s)$ the m.g.f.'s of life and replacement times (LT of their p.d.f.'s), respectively,

$$\tilde{f}(s) = \mathbf{E}\, e^{-sT} = \int_0^\infty e^{-st} f(t)\, dt,$$

$$\tilde{g}(s) = \mathbf{E}\, e^{-sT'} = \int_0^\infty e^{-st} g(t)\, dt.$$

Then according to [1.51] and taking into account that the m.g.f. of sum of the independent r.v.'s is equal to the product of the m.g.f. of its summands, the LT of the renewal density of this process is given by

$$\tilde{h}(s) = \frac{\tilde{f}(s)\tilde{g}(s)}{1 - \tilde{f}(s)\tilde{g}(s)}.$$

With the help of this expression, some other characteristics of this process can be calculated.

To calculate time-dependent state probabilities of the considered process, we denote by $\pi_0(t)$ and $\pi_1(t)$ the probability of its states,

$$\pi_i(t) = \mathbf{P}\{X(t) = i\}, \quad (i \in \{0, 1\}).$$

Suppose that in the initial time, the system is in the up state. Then in some time epoch t, it can be in the up state in the following ways:

i) up to time t there is no failures;

ii) the last failure before time t occurs in time $u \le t$, and after that there are no failures.

Then, with the help of the same argumentation that has been used in section 1.2.2, for the probability $\pi_1(t)$, we can obtain the following expression

$$\pi_1(t) = \mathbf{P}\{0 \le t \le T_1\} + \int_0^t \mathbf{P}\{t - u \le T_1\} h(u)\, du. \qquad [1.70]$$

The expression for $\pi_0(t)$ can be obtained by analogous reasons or by using an evident relation

$$\pi_0(t) + \pi_1(t) = 1.$$

Passing to LT in relation [1.70] and taking into account that the LT of the reliability function

$$\mathbf{P}\{0 \le t \le T_1\} = 1 - F(t) = R(t)$$

is

$$\tilde{R}(s) = \frac{1 - \tilde{f}(s)}{s},$$

we can find

$$\tilde{\pi}_1(s) = \frac{1 - \tilde{f}(s)}{s(1 - \tilde{f}(s)\tilde{g}(s))}. \qquad [1.71]$$

Finally, using the connection between asymptotic behavior, a function at infinity and its Laplace transform in zero, we can find, for example, by using L'Hospital's rule

$$\lim_{t\to\infty} \pi_1(t) = \lim_{s\to 0} s\,\tilde{\pi}_1(s) = \frac{\mu_F}{\mu_F + \mu_G}. \qquad [1.72]$$

REMARK 1.1.– The same result can be obtained using the Smith theorem 1.11 by passing to limit when $t \to \infty$ in formula [1.70].

1.2.6. *Exercises*

EXERCISE 1.9.– Find the mean number of renovations of an article during time t and per unit of time if its life and repair times have exponential distributions with parameters λ and μ, respectively. Perform a calculation for the case, when $t = 10\,000$ h, $\lambda^{-1} = 1000$ h, $\mu^{-1} = 10$ h.

EXERCISE 1.10.– Solve the previous exercise in the case if the article lifetime has Erlang distribution with p.d.f.

$$f(t) = \lambda^2 t\, e^{-\lambda t}.$$

EXERCISE 1.11.– Find a stationary availability coefficient K_{av} of some computer that consists of n blocks. Life and repair times of the i-th block have the exponential distributions with parameters λ_i and μ_i ($i = \overline{1, n}$), respectively. All blocks have to work simultaneously and have enough repair facilities.

Answer:

$$K_{\text{av}} = Av = \prod_{i=1}^{n} \frac{\mu_i}{\lambda_i + \mu_i}.$$

EXERCISE 1.12.– Check formula [1.64].

1.3. Statistical analysis of reliability characteristics

1.3.1. *Introductory notes*

The main problem of reliability analysis consists of obtaining the initial information about it. Because the reliability of complex systems depends on the reliability of their components, thus to calculate the reliability characteristics of complex system, some initial information about the reliability of its components (elements, units) is usually needed. Usually the information about minimal components is preferable; however, sometimes the knowledge about reliability characteristics of some more complicated components is also sufficient. Therefore, the term "unit" for the reliability object will be used in this section.

The initial information is very rarely known in practice. Some theoretical reasons (such as constancy of h.r.f. or theorem 1.1 for exponential distribution, theorem about truncated normal distributions 1.2, or Gnedenko theorem 1.3) can only give some assumptions about the class of lifetime distribution. The element producers can usually give only a very approximate information about mean lifetimes of their production and they are also needed in special methods for obtaining this information. Mathematical statistics is the instrument for measuring probabilistic characteristics. It deals with elaborations of the methods for the estimation of unknown distributions and their parameters, and different hypothesis testing based on the observation. The observations about units' reliability data represent the material for their statistical analysis. Therefore, the initial information about system reliability can be obtained from collection and statistical analysis of the observations for lifetimes of units and systems. It is too expensive and not effective enough to subject complex systems to statistical analysis. Thus, the main direction of the complex systems reliability evaluation is its calculation based on the reliability of their components. Therefore, the collection and analysis of statistical data about lifetime of units (articles and elements) is needed for obtaining the initial reliability information.

There are many specific problems in the statistical analysis of reliability data, and the trials for reliability data organization. Thus, some statistical procedures for the estimation of elements reliability indexes and characteristics should be included in the textbook on reliability.

It is necessary to note that the lifetime of elements for highly reliable systems is usually commensurable with or greater than their obsolescence time that leads to impossibility of long time experiments for the necessary statistical material collection. It stimulates to develop some special models for accelerated testing of the units and special methods of the appropriate data elaboration (for details and further references, see [BAG 02]). This problem represents a separate direction of the reliability theory, so it might be a scope of special issue that should be included in this series.

In any case, the methods of statistical data collection and retrieving from it the maximal information about units' reliability characteristics depend on the information available and the methods of the unit's reliability testing. There are many different procedures of the element's reliability indexes and characteristic observations and methods of their elaboration. In this section only some basic notions and traditional methods of elements and articles for statistical analysis of reliability characteristics and indexes will be considered. We first consider some plans of the trials for obtaining reliability data.

1.3.2. *Observations and the plans of reliability trials*

The statistical material about element lifetimes can be obtained in the following different ways: by simple uncontrollable *passive* observations or by specially organized *observation plan.* In spite of the seemingly evident advantages of collection and elaboration of data about failures of operating equipment, the analysis of these data has significant disadvantages, of which the main advantage is heterogeneity of appropriate data that demands some special additional methods for the data separation. Thus, to obtain the trustworthy information about unit lifetime distributions, we should use some specially planned and organized trials with elements. Therefore, different plans for the experiments are possible, whose results also demand different elaboration methods. Briefly, we consider some basic plans of the experiment realization and their results processing methods. There are different ways of carrying out reliability trials: dealing with non-renewable or renewable units. In both cases, a fixed number, say n units, is subject to test, and in the first plan, the units are not replaced after the failure of each of them, while in the second plan, each failed unit is replaced by a new one with the same properties; moreover, trials with only one unit $n = 1$ is also possible. In both plans, the different possibilities of observations can be used:

– Up to all unit failure (only for the non-renewable plan), this plan will be called a *basic plan* (BP); according to this plan, the fixed number n of unit lifetimes t_i $(i = \overline{1, n})$ is observed.

– During a fixed time, say t_0, this plan will be called as a *fixed time plan* (FTP); according to this plan the random number $N(t_0)$ of units lifetimes t_i $(i = \overline{1, N(t_0)})$ are observed.

– Up to the time of some fixed, say m-th failure, this plan will be called as a *fixed number plan* (FNP); according to this plan, the fixed number m of unit lifetimes t_i $(i = \overline{1, m})$ is observed during the random time $T_{(m)}$ to m-th failure.

As a result of observations, different data may be fixed:

– the *lifetimes* t_i,

– *number of failures* $n_i = n(\Delta t_i)$ during some time intervals Δt_i.

NOTE 1.5.– The basic plan can also be used in the case of passive observations under operating equipment with renewable units. However, in this case, the observable information represents enough complex functionals from the initial lifetime distributions and it is not so easy to extract useful information from these observations.

All these differences in observation of reliability data demand different types for their elaboration. Here only a short review of the reliability trial plans and of the elaboration of appropriate statistical data will be presented. More detailed and in-depth information on this topic can be found in the literature, especially in monographs [BAR 75, HER 00, GBS 65, BAG 02] and so on. We start with statistical analysis of non-renewable units.

1.3.3. *Statistical analysis of reliability characteristics for trials under the basic plan*

1.3.3.1. *Lifetime observations*

Let us consider the basic plan of trials when n units are tested up to failure of each of them. Let there be independent measurement $t_1, t_2, \ldots, t_n$ of n identical unit lifetimes. For statistical analysis of these data, it is very convenient to first represent these data in their increment order

$$t_{(1)} \leq t_{(2)} \leq \cdots \leq t_{(n)}.$$

The set of data arranged in order of increasing is called the *variation series* and its elements are called the *order statistics* or *variants*.

Then the statistical estimation of an unit lifetime c.d.f. is its *empirical* or *sample distribution function* (s.d.f.). Let us denote by $N(t)$ the number of units that failed at the time t. Then the s.d.f. is determined by the relation (note that all statistical estimations of some function or parameter are usually indicated by cap.)

$$\hat{F}_n(t) = \frac{N(t)}{n}. \qquad [1.73]$$

Accordingly, *empirical (sample) reliability function* is determined by the number of workable articles at time t (see Figure 1.12),

$$\hat{R}_n(t) = \frac{n - N(t)}{n} = 1 - \hat{F}_n(t). \qquad [1.74]$$

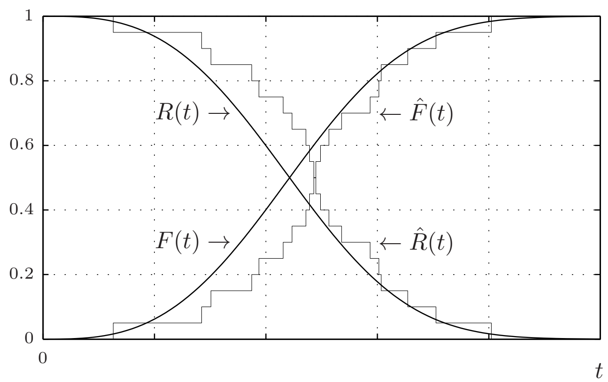


Figure 1.12. *Sample distribution function of an unit lifetime and its sample reliability function*

The statistical estimation of the p.d.f. $f(t)$ is the *histogram* $\hat{f}(t)$, which in reliability statistics is called the *failure frequency*. For its construction, the interval between minimal and maximal observations (its length $R = t_{(n)} - t_{(1)}$ is known as a *sample span*) is divided for some number $k < n$ (usually of equal length) segments Δ_i, $(i = \overline{1, k})$, and a stepwise function $\hat{f}(t)$ as shown in Figure 1.13, is constructed,

$$\hat{f}(t) = \frac{n_i}{n\,|\Delta_i|} \quad \text{for } t \in \Delta_i, \qquad [1.75]$$

where n_i is the number of observations that lies in the interval Δ_i, and $|\Delta_i|$ is its length.

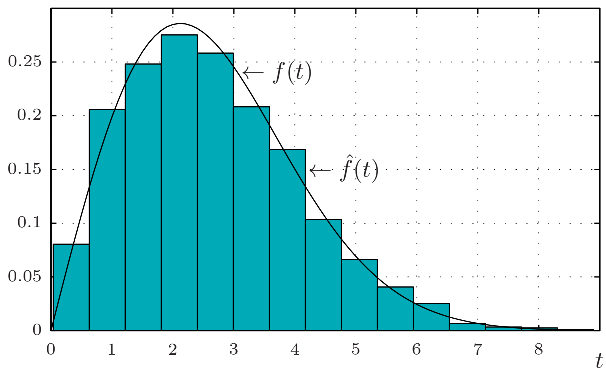


Figure 1.13. *Failure frequency $\hat{f}(t)$ and p.d.f. of some article lifetime $f(t)$*

The statistical estimation of a h.r.f. can be calculated according to the formula

$$\hat{\lambda}(t) = \frac{\hat{f}(t)}{\hat{R}(t)} = \frac{n_i}{n(t)|\Delta_i|} \quad \text{for } t \in \Delta_i, \qquad [1.76]$$

where $n(t)$ is the number of articles that do not fail at time t (beginning of the interval Δ_i) and by $|\Delta_i|$, as before, the length of the time interval Δ_i is denoted.

The examples of the h.r.f. $\hat{\lambda}(t)$ and failure frequency $\hat{f}(t)$, estimation for data from example 1.7 from section 1.3.5, is shown in Figure 1.15

(see section 1.3.5). Finally, the estimations of the expectation and the variance are given by the formulas

$$\hat{\mu} = m = \bar{t} = \frac{1}{n}\sum_{i=1}^{n} t_i; \quad \hat{\sigma}^2 = S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(t_i - \bar{t})^2. \qquad [1.77]$$

1.3.3.2. *Number of failure observations*

However, in real practice, and in the case of bench testing on reliability, the registration of exact times of failures is often quite impossible. Instead, only the numbers n_i of failed units during some time intervals $\Delta_i = [t_{i-1}, t_i)$, are observed. This means that the statistical material for elaboration is, in fact, represented in groups for the histogram construction. Therefore, for the calculation of failure frequency and h.r.f. estimation, the same formulas [1.75] and [1.76] are used, while for the calculation of the sample lifetime c.d.f. and sample reliability function as well as the estimation of lifetime mean and variance, some corrections are needed. These corrections are reduced as follows. It is advisable to smooth (linearly interpolate) the s.d.f. between points t_{i-1} and t_i, and for the calculation of sample mean and variance, all observations should be put in the center of the appropriate interval $\hat{t}_i = \frac{1}{2}(t_{i-1} + t_i)$; as a result, it leads to the formulas

$$\hat{\mu} = m = \bar{t} = \frac{1}{n}\sum_{i=1}^{n} n_i\hat{t}_i; \quad \hat{\sigma}^2 = S^2 = \frac{1}{n-1}\sum_{i=1}^{n} n_i(\hat{t}_i - \bar{t})^2, \qquad [1.78]$$

where n_i is the number of observations in the interval $(t_{i-1},\ t_i]$.

According to one of the basic probability theory laws – Large Number Law – these sample characteristics converge to the appropriate theoretical values when the sample size n tends to infinity that allows us to measure reliability characteristics and indexes with statistical methods.

Another plans when the observations are going on not for the last failure, but during fixed time (FTP), or up to fixed failure number (FNP) leads to biased in side of decreasing estimations because in the estimation participate only failed units and therefore the reliable units are mostly excluded from the estimation process.

The method of reliability characteristics and indexes of non-renewable article calculation based on the statistical data is demonstrated in example 1.7

from section 1.3.5. In the same section some additional statistical data for the estimation of non-renewable articles' reliability characteristics are proposed.

1.3.4. *Statistical estimation of the reliability characteristics and indexes for trials with renewable units*

Analogously to the trials with non-renewable units, the basic reliability characteristics can also be estimated with passive observations using the results of failure registration of operating units or with active specially organized bench trials.

Note that the basic difference between trials with renewable and non-renewable units involves the fact that in the first case, a fixed number $n = n_0$ of patterns (specimens) participate in the trials, while in the second case, the number $n(t)$ of patterns, participating in the experiment, varies because during trials some of the fixed $n(0) = n_0$ in the beginning of the experiment patterns fail and leave the experiment.

For independent observations of some patterns lifetimes $t_1, t_2, \ldots t_n$, the estimations of main reliability characteristics such as c.d.f. of lifetime, reliability function, frequency of failure, etc. for trials with renewal and non-renewal patterns of unit coincide. Turns to the differences. The basic additional characteristic of the renewable system is its failure flow intensity $h(t)$. Its estimation is calculated by the formula

$$\hat{h}(t) = \frac{n_i}{n\,|\Delta_i|} \quad \text{for} \quad t \in \Delta_i, \qquad [1.79]$$

where n_i is the number of failed patterns in the time interval Δ_i, $|\Delta_i|$ is its length, and n is the constant number of patterns, participating in the experiment. Note that in spite of this formula formally coinciding with the analogous formula [1.75] for the frequency of failure estimation of non-renewable units, substantially these formulas lead to different results. As was pointed out above, for the frequency of failure estimation, the fixed number n_0 of patterns decreases with time in the beginning of the experiment, because the number n_i failed in the interval Δ_i patterns leave the experiment, while for the intensity of failure flow, the number of patterns during the experiment remains constant due to the replacement of the failed specimens with the new one.

For the estimation of the stationary intensity of failure flow, we can use asymptotic properties of the renewal density function (see section 1.2.1.5), according to which

$$\lim_{t\to\infty} h(t) = \lim_{t\to\infty} \frac{n(t)}{t} = h = \frac{1}{\mu}.$$

Thus, if there are information about number $n(t)$ of failures during the "long" time interval t, then

$$\hat{h} \approx \frac{n(t)}{t}, \quad \text{from which } \hat{\mu} = \frac{1}{\hat{h}} \approx \frac{t}{n(t)}. \qquad [1.80]$$

Another important characteristic of failure flow, observed under the renewable units plan, is the estimation of mean time between failures (MTBF) and appropriate estimation of variance of the time between failures (VTBF). This index can also be calculated by formula [1.77]; however, it coincides with the estimation of mean to the first failure in essence only for "simple" units, for which the failures are homogeneous, because for "complex" units (articles), the failures could be the results of different reasons (with failures of its different components) that leads to heterogeneity of statistical data.

In the case of MTBF and VTBF estimation by trials with several specimens, formulas [1.77] should be replaced by another:

$$\hat{\mu} = m = \bar{t} = \frac{\sum_{j=1}^{k}\sum_{i=1}^{n_j} t_{ij}}{\sum_{j=1}^{k} n_j}; \quad \hat{\sigma^2} = S^2 = \frac{\sum_{j=1}^{k}\sum_{i=1}^{n_j} (t_{ij} - \bar{t})^2}{\sum_{j=1}^{k} n_j}, \qquad [1.81]$$

where k is the number of testing specimens, t_{ij} is the time between $(i-1)$-th and i-th failures of the j-th pattern and n_j is the number of the j-th pattern failures during the testing time t.

Finally, for the estimation of availability and failure coefficients, additional observations are needed for separate specimen replacement times $t'_1, t'_2, \ldots, t'_n$;

therefore, the appropriate estimations can be calculated by the following formulas:

$$\hat{K}_{\text{av}} = \frac{\sum_{i=1}^{n} t_i}{\sum_{i=1}^{n} (t_i + t_i')}; \quad \hat{K}_{\text{fail}} = \frac{\sum_{i=1}^{n} t_i'}{\sum_{i=1}^{n} (t_i + t_i')}. \qquad [1.82]$$

1.3.5. *Examples*

EXAMPLE 1.7.– Suppose that $n_0 = 1000$ specimens of some unit are tested under non-renewal up to fixed time plan. The failures are fixed over each $\Delta t = 100$ hours. The data about number n_i of failures at the i-th time interval are given in Table A2.1 in Appendix 2. As the reliability characteristics of the unit, the estimations of reliability function $R(t)$, failure frequency $f(t)$, hazard rate function $\lambda(t)$, mean μ and variance σ^2 of lifetime are calculated.

The estimations $\bar{t}$ and S^2 of mean μ and variance σ^2 of lifetime, calculated by formulas [1.78], taking into account only the failed patterns, which is $N_{\text{fail}} = 575$, give the following results:

$$\bar{t} = \frac{1}{N_{\text{fail}}} \sum_{i=1}^{n} n_i \hat{t}_i = \frac{50 \cdot 50 + 40 \cdot 150 + \cdots + 40 \cdot 2950}{575} = 1400 \text{ hours}.$$

$$\hat{\sigma^2} = S^2 = \frac{1}{N_{\text{fail}}} \sum_{i=1}^{n} n_i (\hat{t}_i - \bar{t})^2 \approx 40860 \text{ hours}^2.$$

Thus, for both indexes, a low estimate will be obtained, because in the calculation, the most reliable patterns that operated more than 3000 hours were not taken into account. There are different possible ways to improve these estimations; however, these possibilities are not discussed here, and the reader is referred to the special literature (see [GNE 65]). The graphs of functional characteristics for statistical estimations, calculated according to formulas [1.73]–[1.76], are shown in Figures 1.14 and 1.15.

EXAMPLE 1.8.– During a time interval $[t_1, t_2]$, the large number $n >> 1$ of failures of a renewable article has been fixed. Estimate the mean time between failures (MTBF) $\bar{t}$.

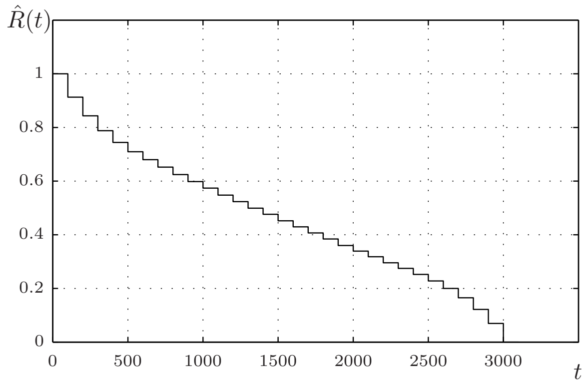


Figure 1.14. *Sample reliability function (example 1.7)*

Considering the number of failures, it is possible to believe that the articles are observed for a long time. Therefore, using relation [1.80], we can obtain

$$\hat{h} = \frac{n(t_2) - n(t_1)}{t_2 - t_1}, \quad \text{or} \quad \bar{t} = \frac{t_2 - t_1}{n(t_2) - n(t_1)}.$$

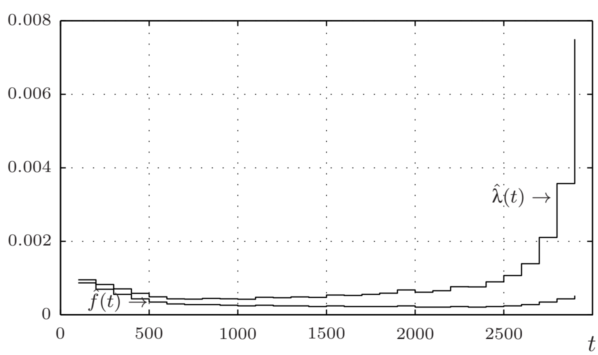


Figure 1.15. *Frequency $\hat{f}(t)$ and h.r.f. estimation $\hat{\lambda}(t)$ (example 1.7)*

EXAMPLE 1.9.– Suppose that k renewable patterns of an article are tested over a period of time. Each example is observed for t_i hours and failed n_i times. Find the mean lifetime of the unit using observations for all pattern operations.

Because several patterns of the same type of article are tested, for the MTBF estimation, we should use relation [1.81]. Because the summary operating time of the i-th pattern equals to t_i, the last formula gives

$$\hat{\mu}_T = \bar{t} = \frac{\sum_{i=1}^{k} t_i}{\sum_{i=1}^{k} n_i}.$$

1.3.6. *Exercises*

The data for calculation are given in Table A2.2 in Appendix 2.

EXERCISE 1.13.– A renewable unit has been observed during the time Δt, and it has been fixed $n(\Delta t)$ failures. Before the start of observations, the unit operates during t_1 hours, and the full operating time to the end of observations was t_2 hours.

Find the mean lifetime (MTBF). The data for the exercise solution are given in Table A2.4.

EXERCISE 1.14.– Find the estimation of the variance using the data from Table A2.4.

EXERCISE 1.15.– During a period of time, some number of patterns of some renewable article are tested. The i-th pattern is observed over t_i hours and failed (and changed) n_i times. The data for the exercise solution are given in Table A2.5.

Find the MTBF using the observation data for the operation of all patterns.

1.4. Structural reliability

In this and the following two sections of this chapter, the system reliability depending on the reliability of their parts is studied. Therefore, instead of the term “unit” (or “article”), the term “system” is used here for complex object, the terms “component” or “subsystem” is used for their parts and the term “elements” is used for their minimal non-divisible parts.

1.4.1. *System structure function*

Let us consider a system consisting of n components (or elements as a special case), each of which can be in one of two states in the sense of their reliability: workable (up) and non-workable, or failed (down). We denote by x_i $(i = \overline{1,n})$ the indicator of the i-th component state,

$$x_i = \begin{cases} 0, & \text{if } i\text{-th component down;} \\ 1, & \text{otherwise.} \end{cases}$$

DEFINITION 1.6.– *The structure function* of a system (in the sense of its reliability) is called the function, which shows whether the system is workable or not depending on the states of its components,

$$\varphi(x_1, \ldots, x_n) = \begin{cases} 0, & \text{if the system is down;} \\ 1, & \text{otherwise.} \end{cases}$$

Let $\mathbf{x} = (x_1, \ldots, x_n)$ denote the vector of the system element's state. The relation $\mathbf{x} < \mathbf{y}$ means that $x_i \leq y_i$ for all i, and for at least one of its components, say j-th, the inequality is strong, $x_j < y_j$.

EXAMPLE 1.10 (Series connection).– The scheme of a system from n components in series (in the sense of its reliability) is shown in Figure 1.16.

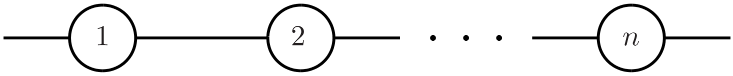


Figure 1.16. *System in series*

This system is workable if all its elements are in up states. Therefore, the system structure function is

$$\varphi(x_1, \ldots, x_n) = \prod_{i=1}^{n} x_i. \qquad [1.83]$$

EXAMPLE 1.11 (Parallel connection).– The scheme of a system from n components connected in parallel (in sense of reliability) in Figure 1.17 is shown.

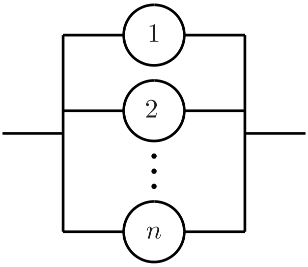


Figure 1.17. *Parallel system*

This system is workable if at least one of its components is workable. Therefore, its structure function is

$$\varphi(x_1, \ldots, x_n) = 1 - \prod_{i=1}^{n}(1 - x_i). \qquad [1.84]$$

EXAMPLE 1.12 ("k from n"—system).– Consider the system that is workable if at least any k of its n components are workable. Its structure function is

$$\varphi(x_1, \ldots, x_n) = \begin{cases} 1, & \text{if } k \text{ of its } n \text{ components are workable,} \\ 0, & \text{otherwise.} \end{cases}$$

The structure scheme of the special case, when $n = 3,\ k = 2$, is shown in Figure 1.18; the structure function in this case can be represented analytically as

$$\varphi(x_1, x_2, x_3) = x_1x_2x_3 + x_1x_2(1 - x_3) + x_1(1 - x_2)x_3 + (1 - x_1)x_2x_3. \qquad [1.85]$$

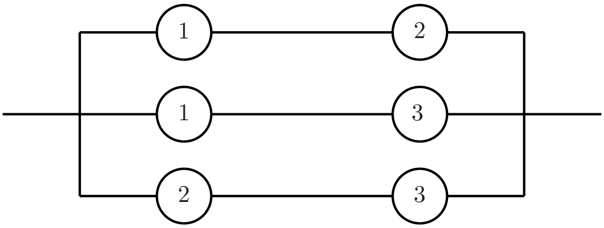


Figure 1.18. *Structure scheme of the "2 from 3"–system*

EXAMPLE 1.13 (Cable TV transmitter).– Consider a system, whose structure scheme is shown in Figure 1.19. It can be treated as a simplified model of a cable TV transmission. Here S is a central station and S_1, S_2, S_3 are three local stations. All stations are connected by cables numbered by integers from 1 to 5. The system is workable if all stations are connected directly or through another station to the central one.

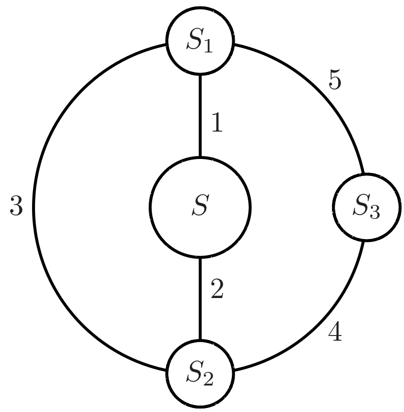


Figure 1.19. *Cable TV system*

It is possible to show (the methods for that will be presented later) that the system structure function equals

$$\varphi(\mathbf{x}) = 1 - (1 - x_2x_3x_5)(1 - x_2x_4x_5)(1 - x_2x_3x_4) \times$$
$$\times(1 - x_1x_3x_4)(1 - x_1x_3x_5)(1 - x_1x_2x_5)(1 - x_1x_2x_4).$$

1.4.2. *Monotone structures*

Say that the system has a *monotone structure*, if

– it is not operable, when all its elements fail;

– it is operable if all its elements are operable;

– the system state cannot have value if any of its elements change its state from “down” to “up”;

– all system elements are essential.

DEFINITION 1.7.– An element is *non-essential* for the system if its structure function does not depend on its state. Otherwise, the element is known as *essential* for the system.

The notion of the monotone structure shows that all real systems are monotone. Formally, the monotone structure can be defined in the following way:

DEFINITION 1.8.– The system is *monotone* if its structure function satisfies the following conditions:

i) $\varphi(0,\ldots,0) = 0; \quad \varphi(1,\ldots,1) = 1,$

ii) $\mathbf{x} < \mathbf{y} \Rightarrow \varphi(\mathbf{x}) \leq \varphi(\mathbf{y})$.

For the investigation and construction of a system structure function, some additional notions will be needed.

DEFINITION 1.9.– A state vector $\mathbf{x}$ is called a *cut vector* if $\varphi(\mathbf{x}) = 0$. Then, the set $C(\mathbf{x}) = \{i : x_i = 0\}$ is called a *cut set*. If additionally for any $\mathbf{y}$, such that $\mathbf{y} > \mathbf{x}$, the relation $\varphi(\mathbf{y}) = 1$ holds, then the cut set is called a *minimal cut set*.

DEFINITION 1.10.– A state vector $\mathbf{x}$ is called a *path vector* if $\varphi(\mathbf{x}) = 1$. Then the set $A(\mathbf{x}) = \{i : x_i = 1\}$ is called a *path set*. If additionally for any $\mathbf{y}$, such that $\mathbf{y} < \mathbf{x}$, the relation $\varphi(\mathbf{y}) = 0$ holds, then the corresponding path set is called a *minimal path set*.

A minimal cut set is a minimal set of components whose failure causes the failure of the whole system. On the other hand, a minimal path set is a minimal set of elements whose workability provides the workability of the whole system. If all elements of a path set are up then the system is up. The minimal path set cannot be reduced, as it has no redundant elements.

An important property of structure function is given below.

THEOREM 1.14 (Structure function representation).– Let $A_1,\ldots,A_s$ be the minimal path sets of some system. Then

$$\varphi(\mathbf{x}) = 1 - \prod_{j=1}^{s}\left(1 - \prod_{i\in A_j} x_i\right). \qquad [1.86]$$

Let $C_1, \dots, C_k$ be the minimal cut sets of a system. Then

$$\varphi(\mathbf{x}) = \prod_{j=1}^{k} \left(1 - \prod_{i \in C_j} (1 - x_i) \right). \qquad [1.87]$$

PROOF 1.8.– If there exists at least one minimal path set, say A_1, with all workable elements, then $\prod_{i \in A_1} x_i = 1$. Therefore, $\varphi(\mathbf{x}) = 1$. Otherwise, if the system is workable, then there exists at least one minimal path set, whose elements are workable. Thus, the right hand side of [1.86] equals 1. Therefore, $\varphi(\mathbf{x}) = 1$ if there exists at least one minimal path set, whose elements are workable. This proves [1.86]. The proofs of formula [1.87] are analogous. ■

This theorem shows that any monotone system can be represented in two equivalent ways:

– as a series connection of parallel subsystems with each being a minimal cut set, or

– as a parallel connection of series subsystems with each being a minimal path set.

After some simplifications, both representations become identical. It is necessary to note here that after structure function transformations, it can contain the powers of some structure variables such as $x_i^{k_i}$, which should be changed to x_i, because for binary variables, the following property holds $x_i^2 = x_i$. Thus, the final form of the structure function should not contain powers of variables.

In order to exclude the structure variable powers, we can use the following *decomposition rule*. Let us denote

$$(1_i, \mathbf{x}) = (x_1, \dots, x_{i-1}, 1, x_{i+1}, \dots, x_n);$$

$$(0_i, \mathbf{x}) = (x_1, \dots, x_{i-1}, 0, x_{i+1}, \dots, x_n).$$

THEOREM 1.15.– The following decomposition rule holds:

$$\varphi(\mathbf{x}) = x_i \varphi(1_i, \mathbf{x}) + (1 - x_i) \varphi(0_i, \mathbf{x}). \qquad [1.88]$$

PROOF 1.9.– This can simply can be checked with the substitution appropriate values $x_i = 0$ or $x_i = 1$. ■

By repeating the decomposition procedure, any structure function $\varphi(\mathbf{x})$ can be represented in the form

$$\varphi_D(\mathbf{x}) = \sum_{\mathbf{a}} \varphi(\mathbf{a}) \prod_{i=1}^{n} x_i^{a_i}(1 - x_i)^{(1-a_i)}, \qquad [1.89]$$

where the vector **a** takes all the possible values of the vector **x**.

In Boolean algebra, this representation is known as *normal disjunctive form* (NDF) for a Boolean function.

EXAMPLE 1.14.– For the system "bridge", shown in Figure 1.20, the decomposition rule with respect to the third element gives

$$\varphi(\mathbf{x}) = x_3[1 - (1 - x_1)(1 - x_2)][1 - (1 - x_4)(1 - x_5)] +$$
$$+ (1 - x_3)[1 - (1 - x_1 x_4)(1 - x_2 x_5)].$$

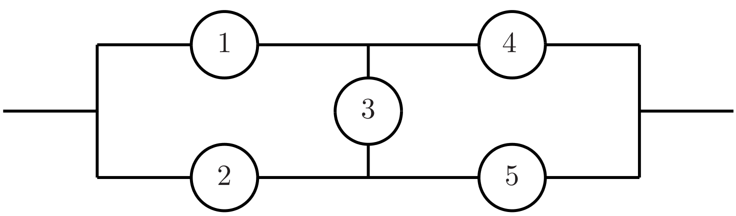


Figure 1.20. *Scheme "bridge"*

For checking this representation, we should understand that if the third element is up, then the system looks like a series of parallel systems (1, 2) and (4, 5), but if it fails, then the system becomes the parallel connections of subsystems (1, 4) and (2, 5).

With exercises 1.16, 1.17 given at the end of this section, the structure function of this system can be represented in the normal disjunctive form.

1.4.3. *Reliability of monotone systems from independent elements*

In this section it is supposed that the system operates in a stationary regime and that the structure variables are binary r.v. X_i, taking the values 1 and 0 with probabilities p_i and $q_i = 1 - p_i$, respectively,

$$p_i = \mathbf{P}\{X_i = 1\}; \quad q_i = \mathbf{P}\{X_i = 0\} = 1 - p_i.$$

This means that the i-th component is up with probability p_i and down with probability q_i. Since the whole system is up, when $\varphi(\mathbf{x}) = 1$, then the probability of its workability is

$$p_{\text{sys}} = \mathbf{P}\{\varphi(X_1, \dots, X_n) = 1\} = \mathbf{E}\varphi(X_1, \dots, X_n). \tag{1.90}$$

Due to the properties of expectation, the last formula is very useful for the calculation of system reliability. However, we should take into account that for the real systems, the structure function construction and thus the use of this formula is not a simple problem, for which special methods and computer tools are used for solving. Let us consider some simple examples of system reliability calculation.

Example 1.15 (Reliability of a system in series).– For the system in series, shown in Figure 1.16, the structure function is [1.83]

$$\varphi(x_1, \dots, x_n) = \prod_{i=1}^{n} x_i,$$

reliability of the system according to [1.90] is

$$p_{\text{sys}} = \mathbf{E}\prod_{i=1}^{n} X_i = \prod_{i=1}^{n} \mathbf{P}\{X_i = 1\} = \prod_{i=1}^{n} p_i \tag{1.91}$$

and for the case of equally reliable components, $p_1 = p_2 = \cdots = p_n = p$ takes the form $p_{\text{sys}}(p) = p^n$.

Example 1.16 (Reliability of a system in parallel).– For the system in parallel, shown in Figure 1.17, the structure function is [1.84],

$$\varphi(x_1, \dots, x_n) = 1 - \prod_{i=1}^{n}(1 - x_i).$$

Thus, according to [1.90], the reliability of such a system is

$$p_{sys} = \mathbf{E}[1 - \prod_{i=1}^{n}(1 - X_i)] = 1 - \prod_{i=1}^{n}\mathbf{E}(1 - X_i) =$$
$$= 1 - \prod_{i=1}^{n}\mathbf{P}\{X_i = 0\} = 1 - \prod_{i=1}^{n}(1 - p_i) \qquad [1.92]$$

and for the case of equally reliable components, $p_1 = p_2 = \cdots = p_n = p$ takes the form $p_{\text{sys}}(p) = 1 - (1 - p)^n$.

Let us denote by $\mathbf{p} = (p_1, \ldots, p_n)'$ the probability vector of the system components up states. Then, using the system structure decomposition formula [1.88], we can obtain the following result:

THEOREM 1.16.–
$$p_{\text{sys}} = p_i\varphi(1_i, \mathbf{p}) + (1 - p_i)\varphi(0_i, \mathbf{p}). \qquad [1.93]$$

PROOF 1.10.– Using the independence of the components to be in their states, we can obtain

$$p_{\text{sys}} = \mathbf{E}\varphi(\mathbf{X}) = \mathbf{E}[X_i\varphi(1_i, \mathbf{X})] + \mathbf{E}[(1 - X_i)\varphi(0_i, \mathbf{X})] =$$
$$= p_i\varphi(1_i, \mathbf{p}) + (1 - p_i)\varphi(0_i, \mathbf{p}),$$

which proves the theorem. ■

In general, for the reliability of the monotone system with independent (in the sense of their reliability) elements, the following theorem holds.

THEOREM 1.17.– The reliability of the monotone system with independent (in the sense of its reliability) components equals the value of its structure function in NDF, in which instead of structure variables, the probabilities of their up states are substituted,

$$p_{\text{sys}} = \varphi_D(\mathbf{p}).$$

PROOF 1.11.– Using NDF of the system structure function due to independent system elements (and structure variables), we can obtain

$$p_{sys} = \mathbf{E}\varphi_D(\mathbf{X}) = \sum_{\mathbf{a}} \varphi(\mathbf{a})\mathbf{E}\left[\prod_{i=1}^{n} X_i^{a_i}(1 - X_i)^{(1-a_i)}\right] =$$

$$= \sum_{\mathbf{a}} \varphi(\mathbf{a}) \prod_{i=1}^{n} \mathbf{E}\left[X_i^{a_i}(1 - X_i)^{(1-a_i)}\right].$$

Since a_i equals 0 or 1, the right hand side of this formula takes the form

$$\mathbf{E}\left[X_i^{a_i}(1 - X_i)^{(1-a_i)}\right] = p_i^{a_i} q_i^{(1-a_i)},$$

and, therefore,

$$p_{sys} = \sum_{\mathbf{a}} \varphi(\mathbf{a}) \prod_{i=1}^{n} p_i^{a_i} q_i^{(1-a_i)} = \varphi_D(\mathbf{p}) \qquad [1.94]$$

which proves the theorem. ■

1.4.4. *Reliability function for monotone structures*

The result considered above for stationary system reliability can also be used for time-dependent reliability. For this case, it is sufficient for any time t to fix each component probability up state to this time $p_i = R_i(t)$ and to use them as stationary component probabilities to substitute into corresponding formulas of their reliability functions (element reliability for a fixed time t).

Especially for the reliability function $R_{\text{sys}}(t)$ of a system from n components in series with reliability functions $R_i(t)$, $(i = \overline{1, n})$, according to [1.84], we can obtain

$$R_{\text{sys}}(t) = \prod_{i=1}^{n} R_i(t).$$

Representing components reliability functions $R_i(t)$ in terms of their h.r.f. $\lambda_i(t)$ from this relation, we have

$$R_{\text{sys}}(t) = \exp\left\{-\int_0^t \lambda_{\text{sys}}(x)\,dx\right\} = \exp\left\{-\int_0^t \sum_{i=1}^n \lambda_i(x)\,dx\right\}.$$

This representation gives a very simple and useful rule for the h.r.f. of the reliability system in series (which is commonly known as a *main connection*),

$$\lambda_{\text{sys}}(t) = \sum_{i=1}^n \lambda_i(t).$$

In other words, this rule can be formulated as follows: *under the main connections of the system h.r.f. equals the sum of its components' h.r.f.'s.*

This formula is more simplified for the immediate failures (when h.r.f. is constant), $\lambda_i(t) = \lambda_i = \text{const}$,

$$\lambda_{\text{sys}} = \sum_{i=1}^n \lambda_i.$$

EXAMPLE 1.17.– Let us calculate the reliability of the system "2 from 3-õ", considered as an example 1.12 and shown in Figure 1.18. The structure function of this system is

$$\varphi(x_1, x_2, x_3) = x_1x_2x_3 + x_1x_2(1 - x_3) + x_1(1 - x_2)x_3 + (1 - x_1)x_2x_3.$$

Therefore, the reliability of this system under assumption about equal reliability of its elements is

$$p_{\text{sys}}(p) = p^3 + 3p^2(1 - p) = p^2(3 - 2p).$$

1.4.5. *Exercises*

EXERCISE 1.16.– Opening the parenthesis, find NDF of this system "bridge" structure function.

EXERCISE 1.17.– Find the same result with the help of minimal path and cut sets.

EXERCISE 1.18.– Scheme for some system reliability calculation is shown in Figure 1.21. Write the system structure function and calculate the system reliability p_{sys} if reliability of its elements equal

$$p_1 = 0.9,\ p_2 = 0.8,\ p_3 = 0.85,\ p_4 = 0.94.$$

Answer: $p_{\text{sys}} = 1 - (1 - p_1 p_2)(1 - p_3 p_4) \approx 0.944.$

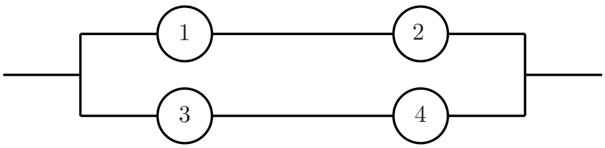


Figure 1.21. *Scheme for reliability calculation for exercises 1.18 and 1.19*

EXERCISE 1.19.– Scheme for some system reliability calculation is shown in Figure 1.21. Calculate the system mean time to the first failure, its reliability function and hazard rate at the time $t = 100$ hours if its elements' hazard rates are constant and equal to

$$\lambda_1 = \lambda_3 = 0.3 \cdot 10^{-3}\,\text{hours}^{-1}; \quad \lambda_2 = \lambda_4 = 0.7 \cdot 10^{-3}\,\text{hours}^{-1}.$$

Answer:

$$R_{\text{sys}}(t) = 1 - \left[1 - e^{-(\lambda_1+\lambda_2)t}\right]^2; \quad R_{\text{sys}}(100) \approx 0.99;$$

$$\mu_{\text{sys}} = \frac{3}{2(\lambda_1 + \lambda_2)} = 1500\,\text{hours};$$

$$f_{sys}(t) = 2(\lambda_1 + \lambda_2)e^{-(\lambda_1+\lambda_2)}\left[1 - e^{-(\lambda_1+\lambda_2)t}\right];$$

$$f_{\text{sys}}(100) \approx 1.8 \cdot 10^{-4}\,\text{hours}^{-1};$$

$$\lambda_{\text{sys}}(100) = \frac{f_{sys}(100)}{R_{sys}(100)} \approx 1.8 \cdot 10^{-4}\,\text{hours}^{-1}.$$

EXERCISE 1.20.– Scheme for some system reliability calculation is shown in Figure 1.22. Write the system structure function and calculate the system reliability p_{sys} if failure probabilities of its elements equal

$$q_1 = q_3 = 0.1; \quad q_2 = q_4 = 0.2.$$

Answer: $p_{\text{sys}} = (1 - q_1^2)(1 - q_2^2) = 0.950.$

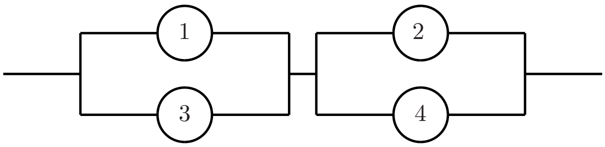


Figure 1.22. *Scheme for reliability calculation for exercises 1.20 and 1.21*

EXERCISE 1.21.– Scheme for some system reliability calculation is shown in Figure 1.22. Calculate the system mean time to the first failure, its reliability function and hazard rate at the time $t = 100$ hours if its elements' hazard rates are constant and equal to

$$\lambda_1 = \lambda_3 = 0.3 \cdot 10^{-3}\,\text{hours}^{-1}; \quad \lambda_2 = \lambda_4 = 0.7 \cdot 10^{-3}\,\text{hours}^{-1}.$$

Answer:

$$R_{sys}(t) = \left[1 - \left(1 - e^{-\lambda_1 t}\right)^2\right]\left[1 - \left(1 - e^{-\lambda_2 t}\right)^2\right];$$

$$R_{\text{sys}}(100) \approx 0.995;$$

$$f_{\text{sys}}(t) = 2\,e^{-(\lambda_1+\lambda_2)t}\Big[(\lambda_1 + \lambda_2)(2 + e^{-(\lambda_1+\lambda_2)t})$$

$$- (2\,\lambda_1 + \lambda_2)e^{-\lambda_1 t} - (\lambda_1 + 2\,\lambda_2)e^{-\lambda_2 t}\Big];$$

$$f_{\text{sys}}(100) \approx 1.05 \cdot 10^{-4}\ \text{hours}^{-1};$$

$$\lambda_{\text{sys}}(100) = \frac{f_{sys}(100)}{R_{sys}(100)} \approx 1.05 \cdot 10^{-4}\,\text{hours}^{-1};$$

$$\mu_{\text{sys}} = \frac{4.5}{(\lambda_1 + \lambda_2)} - 2\left(\frac{1}{2\,\lambda_1 + \lambda_2} + \frac{1}{\lambda_1 + 2\,\lambda_2}\right) \approx 1785\,\text{hours}.$$

EXERCISE 1.22.– A scheme for a system reliability calculation is shown in Figure 1.23. Write the system structure function and calculate the system reliability p_{sys} if failure probabilities of its elements equal

$$q_1 = q_3 = q_5 = 0.05; \quad q_2 = q_4 = q_6 = 0.1.$$

Answer: $p_{\text{sys}} = 1 - [1 - (1 - q_1)(1 - q_2)]^3 = 0.997$.

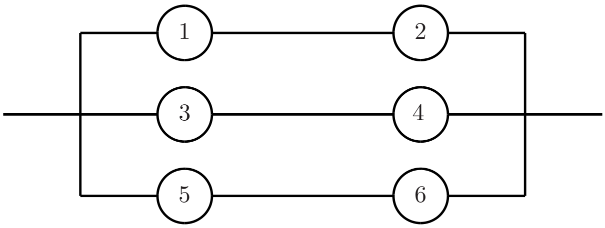


Figure 1.23. *Scheme for the system reliability calculation for exercise 1.22*

EXERCISE 1.23.– A scheme for a system reliability calculation is shown in Figure 1.24. Write the system structure function and calculate the system reliability p_{sys} and failure probability q_{sys} if reliability of its elements is equal to

$$p_1 = p_2 = p_3 = 0.9; \quad p_4 = p_5 = p_6 = 0.8.$$

Answer: $p_{\text{sys}} = [1 - (1 - p_1)^3][1 - (1 - p_4)^3] \approx 0.991$;
$q_{\text{sys}} = 1 - p_{sys} = 0.009$.

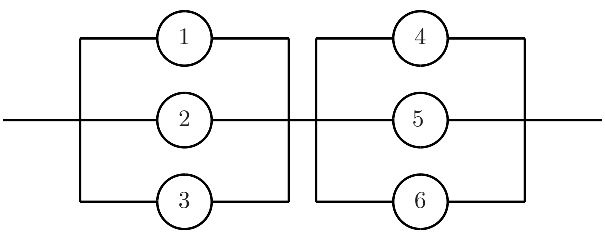


Figure 1.24. *Scheme for the system reliability calculation for exercise 1.23*

EXERCISE 1.24.– An automobile engine has four candles, one for each cylinder. The h.r.f. of each candle is constant and equals $\lambda = 10^{-3}$hours^{-1}. It is supposed that if one of the candles fails, the automobile can move.

1) Calculate the reliability function of the engine and show its grahical representation.

2) Find the probability that the traveling during $t = 20$ hours will be successful (without candles changing).

1.5. System life tree and its structure function

In this section one of methods for system structure function construction is considered. However, we start with more general notion of the *event tree* that has more wide applications not only in reliability theory.

1.5.1. *Event tree*

In many applications, connected with investigations of complex phenomena function, such as a complex object in technique, biology, medicine, business, etc., it is very convenient to use the event tree notion.

DEFINITION 1.11.– An *event tree* is a graph of the turned over tree type, the root of which is a resulting event. Its branches represent generated events and leaves represent minimal initial events.

For the system reliability analysis of an event tree, the *fault tree*, which fixes non-workable system states, is usually used. Thus, the failures of the system, its subsystems, components and elements are considered as events. However, in this section, which is devoted to system reliability, we will consider workability of elements, components, subsystems and the whole system as events, and therefore the *life tree* will be used as an appropriate tree. The elements, subsystem and the whole system operable states are considered as the events in the life tree. The indicators of appropriate events are structure variables for the elements, subsystem and the whole system. This tree is a dual to the fault tree (which is often also considered in reliability theory) and it allows a direct calculation of the structure function and different reliability characteristics and indexes of a system. Later in the section devoted to the technological risk analysis, the analogous approach will be used for risk tree construction.

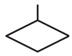
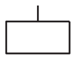
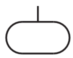
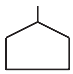
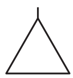
#	Symbol	Name	Description
1		Circle	Basic event with sufficient data
2		Diamond	Undeveloped event
3		Rectangle	Event represented by a gate
4		Oval	Conditional event used with inhibit gate
5		House	House event. Either occurring or not occurring
6		Triangle (input from above)	Transfer symbol
7		Triangle (input from the left)	Transfer symbol

Table 1.1. *The event symbols*

For any event tree, especially for the life tree, construction of the special notations of different types of events and connections has been proposed, as shown in Tables 1.1 and 1.2, following the book of Henley and Kumamoto [HEN 91].

In the next section the problems of the system life tree construction, system structure function and its basic reliability characteristics and index calculation are considered.

1.5.2. *An object structure scheme*

For the life tree of some object construction, it is very convenient to begin with its structure scheme consideration and the following analysis of its elements, components and subsystems workability. Most of the up-to-date complex technical systems have an hierarchical structure, which should be taken into account for the system reliability investigation. The life tree, in fact, duplicates the system structure, and the structure function can be simply calculated using the life tree.

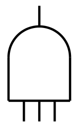
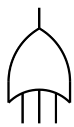
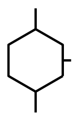
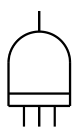
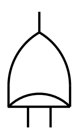
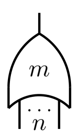
#	Gate symbol	Gate name: structure function $\phi(\mathbf{x})$	Causal relation
1		AND gate; $\prod_{i=1}^{n} x_i$	Output events occurs if all input events occur simultaneously.
2		OR gate; $1 - \prod_{i=1}^{n}(1 - x_i)$	Output events occurs if any one of input events occurs.
3		Inhibit gate; $x \cdot u$	Input x produced output when conditional event u occurs.
4		Priority AND gate; $\sum_{k=1}^{n} x_k \prod_{i=1}^{k-1} x_i$	Output events occurs if all input events occur in the order from left to right.
5		Exclusive OR gate; $x_1(1 - x_2) + (1 - x_1)x_2$	Output event occurs if one, but not both, of the input events occur.
6		"m out of n" gates (voting or sample gate); $1_{\left\{\sum_{i=1}^{n} x_i \geq m\right\}}$	Output event occurs if m out of n input events occur.

Table 1.2. *The gate symbols*

A hierarchical structure of the system can be visualized as a turned over tree-type graph, whose root is the system, vertex is its subsystems of different levels, and arcs show the connections between the system and different level subsystems in the sense of their workability. For the complex systems, this procedure can be divided into the life trees of different subsystem constructions.

For representation of a complex hierarchical system, the vector notation will be used. Let us denote the system elements by vectors $\mathbf{i} = (i_1, i_2, \ldots, i_r)$, where i_1 is the number of the first subsystem system considered, i_2 is the number of its subsystem (i.e. the subsystem of the second level) to which the element belongs; i_r is the number of the element of the $(r-1)$-st level subsystem $\mathbf{i}_{r-1} = (i_1, i_2, \ldots, i_{r-1})$, and r is the hierarchical level of considering element (its rank), in which the ranks of different elements can be different. The k-th level subsystems will be denoted by truncated vectors $\mathbf{i}_k = (i_1, i_2, \ldots, i_k)$, and the j-th component of subsystem $\mathbf{i}_k$ will be denoted by $j(\mathbf{i}_k)$. Thus, the system elements are identified with the vectors $\mathbf{i} = (i_1, i_2, \ldots, i_r)$ and appropriate subsystems with truncated vectors $\mathbf{i}_k = (i_1, i_2, \ldots, i_k)$.

1.5.3. *An example: the auto engine structure scheme*

Consider a simplified model of an auto engine that contains only two subsystems: electric and fuel supply subsystems. In Figure 1.25, a simplified scheme of an auto engine is shown with appropriate notations. For example, the following notations are used:

(0) – the system: auto engine;

(1) – the electric equipment subsystem:

- $(1, 1)$ – accumulator battery component;
- $(1, 2)$ – starter component:
  - $(1, 2, 1)$ – relay element,
  - $(1, 2, 2)$ – motor element,
  - $(1, 2, 3)$ – bendix element;
- $(1, 3)$ – ignition component:

(1, 3, 1) – ignition coil element,

(1, 3, 2) – bearing element;

(1, 4) – four candles (four homogeneous elements);

(2) – fuel supply subsystem:

(2, 1) – fuel pump element,

(2, 2) – filter element,

(2, 3) – carburetor element,

(2, 4) – gasoline tank.

In the next section this example will be used to demonstrate the system structure function construction.

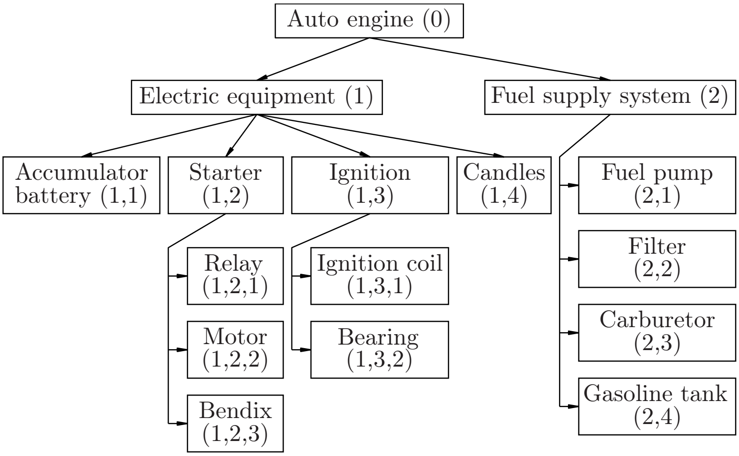


Figure 1.25. *The structure scheme of an auto engine*

1.5.4. *Life tree and the system structure function*

It is very convenient to begin a system reliability study with its structure function construction, which involves the analysis of the system and its subsystems of different levels up to elementary one workability conditions. These conditions should be represented in the form of appropriate connections (gates) and these connections in the algebraic form are described in Table 1.2. If workability of some units, components or subsystems depends on several reasons (failures of different types are possible), then for construction of the life tree based on the system structure scheme, we should add appropriate events that provide workability of corresponding chains.

In Figure 1.26, the life tree of an auto engine is shown which structure scheme in Figure 1.25 has been shown. Based on the analysis of separate subsystem workability, “simple connections” are changed to corresponding gates. Because the workability of components such as the accumulator battery, the ignition subsystem and the candles depends on several reasons (these components allow several types of failures), the life tree is implemented with additional conditions for the workability of these elements, namely the events $A_{(1,1,1)}$ $A_{(1,1,2)}$ $A_{(1,2,3)}$ component $(1,1)$ (accumulator battery), the events $A_{(1,3,3)}$ and $A_{(1,3,4)}$ for ignition subsystem $(1,3)$ and events $A_{(1,4,k,i)}$ $(k = \overline{1,4})$ for each candle $i = \overline{1,4}$ (see below).

For different events (up states) of elements, components, subsystem and the whole system, the notations $A_{(\mathbf{i}_k)}$ are used, with an index corresponding to the appropriate element, components, subsystems and the whole system[6]. Therefore, for the auto engine life tree, the notations shown below should be used; however, for the sake of simplicity and space limitation, the notations for events without brackets and commas are used in the figure.

$A_{(0)}$ – auto engine is up;

$A_{(1)}$ – electric equipment is up:

$A_{(1,1)}$ – accumulator battery is up:

$A_{(1,1,1)}$ – battery is charged;

$A_{(1,1,2)}$ – no circuiting between plates;

$A_{(1,1,3)}$ – there is contact in clam terminals;

$A_{(1,2)}$ – starter is up:

6 Usually for complex systems the number of some level subsystems can be more than 10; therefore, the only integers may not be enough for subsystem notations, and thus the commas are needed in elements and subsystem notations. Nevertheless, for the sake of simplicity and space limitation, the notations without brackets and commas are used in figures considered in the example below.

$A_{(1,2,1)}$ – relay is up;

$A_{(1,2,2)}$ – motor is up;

$A_{(1,2,3)}$ – bendix is up;

$A_{(1,3)}$ – ignition is up:

$A_{(1,3,1)}$ – ignition coil is up;

$A_{(1,3,2)}$ – bearing is up;

$A_{(1,3,3)}$ – the gap is not broken;

$A_{(1,3,4)}$ – the ignition is not broken;

$A_{(1,4)}$ – at least three of four candles are up, i.e. for each of them:

$A_{(1,4,1,i)}$ – no closure, $i = \overline{1,4}$;

$A_{(1,4,2,i)}$ – gap is not brought down;

$A_{(1,4,3,i)}$ – no dirt;

$A_{(1,4,4,i)}$ – contacts are up;

$A_{(2)}$ – fuel supply system is workable:

$A_{(2,1)}$ – fuel pump is up;

$A_{(2,2)}$ – the filter is not dirty;

$A_{(2,3)}$ – carburetor is up;

$A_{(2,4)}$ – there is gasoline.

The next step consists of the system structure function calculation. It should be noted that the element and subsystem structure variables $x_{\mathbf{i}_k}$ are indicator functions appropriate events $x_{\mathbf{i}_k} = 1_{A_{\mathbf{i}_k}}$ and are equal to

$$x_{\mathbf{i}_k} = \begin{cases} 1, & \text{if subsystem } \mathbf{i}_k \text{ is up;} \\ 0, & \text{otherwise.} \end{cases}$$

Calculation of the structure function for subsystems and the whole system follows the reliability theory rules, beginning from the top level. For the structure function calculation, we can use Table 1.2, where appropriate symbols and corresponding structure functions in terms of their structure variables are represented. Calculation of a subsystem $\mathbf{i}_{k-1}$ structure function is performed by changing the gate to an appropriate algebraic operation,

$$x_{\mathbf{i}_{k-1}} = \varphi_{\mathbf{i}_k}(x_{(\mathbf{i}_k,1)}, \ldots, x_{(\mathbf{i}_k, n(\mathbf{i}_k))}).$$

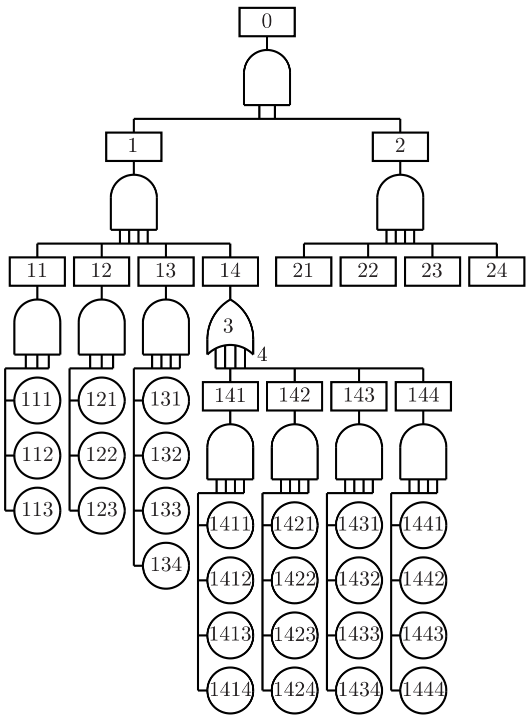


Figure 1.26. *Auto engine life tree*

We demonstrate the system structure function calculation with the help of the example auto engine, whose life tree is shown in Figure 1.26. Beginning from the top level for the structure function of the system, we have

$$\varphi = \varphi_{(1)}\varphi_{(2)};$$

$$\varphi_{(1)} = \varphi_{(1,1)}\varphi_{(1,2)}\varphi_{(1,3)}\varphi_{(1,4)}; \quad \varphi_{(2)} = x_{(2,1)}x_{(2,2)}x_{(2,3)}x_{(2,4)};$$

$$\varphi_{(1,1)} = x_{(1,1,1)} x_{(1,1,2)} x_{(1,1,3)}; \qquad \varphi_{(1,2)} = x_{(1,2,1)} x_{(1,2,2)} x_{(1,2,3)};$$

$$\varphi_{(1,3)} = x_{(1,3,1)} x_{(1,3,2)} x_{(1,3,3)} x_{(1,3,4)}.$$

However, because the reliability of subsystem $(1,4)$ depends on the state each of four elements, its structure function is

$$\varphi_{(1,4)} = \prod_{i=1}^{4} \hat{\varphi}_{(1,4,i)} + \sum_{j=1}^{4} (1 - \hat{\varphi}_{(1,4,j)}) \prod_{i=1, i \neq j}^{4} \hat{\varphi}_{(1,4,i)},$$

where

$$\hat{\varphi}_{(1,4,i)} = \prod_{j=1}^{4} x_{(1,4,i,j)}$$

which is the structure function of one of the candles due to homogeneity of subsystem (1,4) elements, and $x_{(1,4,i,j)}$ are its structure variables.

1.5.5. *Calculation of the system reliability*

In order to calculate the system reliability characteristics, the life tree should be provided with the necessary information. Depending on the investigation goals and admissible data, it may be:

– elements' reliability during fixed time interval;

– elements' reliability functions.

The system reliability characteristics and indexes are calculated according to the rules described in the previous section by changing structure variables with appropriate reliability indexes. In particular, we demonstrate the calculation of the auto engine reliability with the help of the above example, using initial information given at the elementary level. In this case it is supposed that the probabilities of workable (up) states are given, $p_{\mathbf{i}} = \mathbf{P}\{A_{\mathbf{i}}\}$:

(1,1) accumulator battery reliability:

– $p_{(1,1,1)}$ – the probability of the battery is charged;

– $p_{(1,1,2)}$ – the probability of plates are up;

– $p_{(1,1,3)}$ – the probability of contacts present in terminals;

(1,2) starter reliability:

- $p_{(1,2,1)}$ – relay up probability;
- $p_{(1,2,2)}$ – starter motor up probability;
- $p_{(1,2,3)}$ – probability of bendix serviceability;

(1,3) reliability of the ignition system components:

- $p_{(1,3,1)}$ – the probability of ignition coil up;
- $p_{(1,3,2)}$ – the probability of bearing up;
- $p_{(1,3,3)}$ – the probability of gap;
- $p_{(1,3,4)}$ – the probability of ignition serviceability;

(1,4) candles reliability:

- $p_{(1,4,i,1)}$ – probability of circuit absence $(i = \overline{1,4})$;
- $p_{(1,4,i,2)}$ – probability of gap up $(i = \overline{1,4})$;;
- $p_{(1,4,i,3)}$ – probability of dirt absence $(i = \overline{1,4})$;;
- $p_{(1,4,i,4)}$ – probability of contacts are up $(i = \overline{1,4})$;;

(2) reliability of the fuel system elements:

- $p_{(2,1)}$ – probability of gasoline pump up;
- $p_{(2,2)}$ – probability that the filter is not dirty;
- $p_{(2,3)}$ – probability of carburetor up;
- $p_{(2,4)}$ – probability that the tank contains enough gasoline.

Thus, according to theorem 1.17 of the previous section and due to monotonicity of the system under consideration, the appropriate reliability system characteristics are calculated by substitution of components' reliability indexes instead of respective structure variables, namely beginning from the lower level:

(1,1) accumulator battery reliability (probability up state) is

$$p_{(1,1)} = \varphi_{(1,1)}(p_{(1,1,1)}, p_{(1,1,2)}, p_{(1,1,3)}) = p_{(1,1,1)}p_{(1,1,2)}p_{(1,1,3)};$$

(1,2) reliability (probability up state) of the starter is

$$p_{(1,2)} = \varphi_{(1,2)}(p_{(1,2,1)},\ p_{(1,2,2)},\ p_{(1,2,3)}) = p_{(1,2,1)}p_{(1,2,2)}p_{(1,2,3)};$$

(1,3) reliability (probability up state) of ignition is

$$p_{(1,3)} = \varphi_{(1,3)}(p_{(1,3,1)},\ p_{(1,3,2)},\ p_{(1,3,3)},\ p_{(1,3,4)}) =$$
$$= p_{(1,3,1)}p_{(1,3,2)}p_{(1,3,3)}p_{(1,3,4)};$$

(1,4) probability of at least three of four candles are operable is

$$p_{(1,4)} = \hat{p}^4_{(1,4,j)} + 4(1 - \hat{p}_{(1,4,j)})\hat{p}^3_{(1,4,j)},$$

where $\hat{p}_{(1,4,j)} = p_{(1,4,j,1)}p_{(1,4,j,2)}p_{(1,4,j,3)}p_{(1,4,j,4)}$ is the reliability one of the candles;

(2) fuel supply system reliability is

$$p_{(2)} = \varphi_{(2)}(p_{(2,1)},\ p_{(2,2)},\ p_{(2,3)},\ p_{(2,4)}) = p_{(2,1)}p_{(2,2)}p_{(2,3)}p_{(2,4)}.$$

Denoting by $\mathbf{p}_1 = (p_{(1,1)}, p_{(1,2)}, p_{(1,3)}, p_{(1,4)})$ and $\mathbf{p}_2 = (p_{(2,1)}, p_{(2,2)}, p_{(2,3)}, p_{(2,4)})$ vectors of the first and the second subsystems' reliability, we can find that:

(1) the reliability of electric equipment is

$$p_1 = \varphi_1(\mathbf{p}_1) = \varphi_1(p_{(1,1)}, p_{(1,2)}, p_{(1,3)}, p_{(1,4)}) = \prod_{i=1}^{4} \varphi_{1,i} = \prod_{i=1}^{4} p_{1,i};$$

(2) the reliability of the fuel supply subsystem is

$$p_2 = \varphi_2(\mathbf{p}_2) = \varphi_2(p_{(2,1)}, p_{(2,2)}, p_{(2,3)}, p_{(2,4)}) = \prod_{i=1}^{4} \varphi_{2,i} = \prod_{i=1}^{4} p_{2,i};$$

(0) and thus the reliability of the auto engine is

$$p_{sys} = p_1 p_2 = \varphi_1(\mathbf{p}_1)\varphi_{(2)}(\mathbf{p}_2).$$

In the next section the case when the initial information about the system is given in the form of their elements' reliability functions will be considered.

1.5.6. *System reliability function calculation*

As has been shown in section 1.4.4, the system reliability function is also simply calculated with the help of structure function of a system by substitution of appropriate elements and components reliability functions instead of their structure variables. We then illustrate the procedure of a system reliability function $R_{\text{sys}}(t)$ calculation with the help of the example about an auto engine using the elements reliability functions $R_{\mathbf{i}}(t)$ as an initial information.

For numerical calculation, special numerical values of these indexes are determined. Let us denote by $\mu_{\mathbf{i}}$ the mean **i**-th element lifetime. The numerical data do not represent any real situation, but are used only to illustrate the numerical calculations.

(1,1) mean lifetime of an accumulator battery:

a) $\mu_{(1,1,1)} = 95$ hours is the mean intercharging battery time;

b) $\mu_{(1,1,2)} = 187$ hours is the mean plates lifetime;

c) $\mu_{(1,1,3)} = 215$ hours is the mean miscommunication terminal time;

(1,2) mean lifetime of starter elements:

a) $\mu_{(1,2,1)} = 225$ hours is the mean relay lifetime;

b) $\mu_{(1,2,2)} = 178$ hours is the mean of the starter's motor lifetime;

c) $\mu_{(1,2,3)} = 315$ hours is the mean bendix lifetime;

(1,3) mean lifetime of ignition system elements;

a) $\mu_{(1,3,1)} = 295$ hours is the mean of an ignition coil lifetime;

b) $\mu_{(1,3,2)} = 415$ hours is the bearing lifetime;

c) $\mu_{(1,3,3)} = 170$ hours is the mean gap inter-correction time;

d) $\mu_{(1,3,4)} = 280$ hours is the mean ignition inter-correction time;

(1,4) candles mean lifetime:

a) $\mu_{(1,4,i,1)} = 193$ hours is the mean time to circuit, $i = \overline{1,4}$;

b) $\mu_{(1,4,i,2)} = 427$ hours is the mean time gap violation;

c) $\mu_{(1,4,i,3)} = 115$ hours is the mean time to dirty of candles;

d) $\mu_{(1,4,i,4)} = 217$ hours is the mean time to contact violation;

(2) mean lifetime of the fuel subsystem elements:

a) $\mu_{(2,1)} = 193$ hours is the mean gasoline pump lifetime;

b) $\mu_{(2,2)} = 157$ hours is the mean time to filter dirty;

c) $\mu_{(2,3)} = 281$ hours is the mean carburetor lifetime;

d) $\mu_{(2,4)} = 301$ hours is the mean time to absence of gasoline.

Under the assumption of elements' exponential reliability law, their parameters (hazard rates) are equal to $\lambda_{\mathbf{i}} = 1/\mu_{\mathbf{i}}$:

$$\lambda_{(1,1)} = \lambda_{(1,1,1)} + \lambda_{(1,1,2)} + \lambda_{(1,1,3)} \approx 0.021;$$

$$\lambda_{(1,2)} = \lambda_{(1,2,1)} + \lambda_{(1,2,2)} + \lambda_{(1,2,3)} \approx 0.013;$$

$$\lambda_{(1,3)} = \lambda_{(1,3,1)} + \lambda_{(1,3,2)} + \lambda_{(1,3,3)} + \lambda_{(1,3,4)} \approx 0.015;$$

$$\lambda_{(2)} = \lambda_{(2,1)} + \lambda_{(2,2)} + \lambda_{(2,3)} + \lambda_{(2,4)} \approx 0.018.$$

Concerning subsystem $(1,4)$, it is necessary to note that the failure of each candle occurs due to one of the four reasons, and under the assumption that their exponential distribution time to one candle failure also has an exponential distribution with parameter

$$\lambda_{(1,4,i,\cdot)} = \sum_{1\le j\le 4} \lambda_{(1,4,i,j)} \approx 0.021.$$

The reliability functions of all subsystems, except for subsystem $(1,4)$, are exponential with the above given parameters:

$$R_{(1,1)}(t) = \prod_{i=1}^{3} R_{(1,1,i)}(t) = \prod_{i=1}^{3} e^{-\lambda_{(1,1,i)}t} = e^{-\lambda_{(1,1)}t};$$

$$R_{(1,2)}(t) = \prod_{i=1}^{3} R_{(1,2,i)}(t) = \prod_{i=1}^{3} e^{-\lambda_{(1,2,i)}t} = e^{-\lambda_{(1,2)}t};$$

$$R_{(1,3)}(t) = \prod_{i=1}^{4} R_{(1,3,i)}(t) = \prod_{i=1}^{4} e^{-\lambda_{(1,3,i)}}t = e^{-\lambda_{(1,3)}t};$$

$$R_{(2)}(t) = \prod_{i=1}^{3} R_{(2,i)}(t) = \prod_{i=1}^{3} e^{-\lambda_{(2,i)}t} = e^{-\lambda_{(2)}t}.$$

However, subsystem $(1,4)$ reliability function due to its structure is equal to

$$\begin{aligned}R_{(1,4)}(t) &= R^4_{(1,4,i)}(t) + 4(1 - R_{(1,4,i)}(t))R^3_{(1,4,i)}(t) = \\ &= e^{-4\,\lambda_{(1,4,i)}t} + (1 - e^{-\lambda_{(1,4,i)}t})e^{-3\,\lambda_{(1,4,i)}t} = \\ &= 4\,e^{-3\,\lambda_{(1,4,i)}t} - 3\,e^{-4\,\lambda_{(1,4,i)}t},\end{aligned}$$

where $R_{(1,4,i)}(t)$ is the reliability function of the i-th candle, which is also exponential

$$R_{(1,4,i)}(t) = \prod_{i=1}^{4} R_{(1,4,i,j)}(t) = \prod_{i=1}^{4} e^{-\lambda_{(1,4,i,j)}t} = e^{-\lambda_{(1,4,i)}t},$$

with parameter

$$\lambda_{(1,4,i)} = \lambda_{(1,4,i,1)} + \lambda_{(1,4,i,2)} + \lambda_{(1,4,i,3)} + \lambda_{(1,4,i,4)} \approx 0.021.$$

Thus, the reliability function of the electric equipment subsystem is

$$\begin{aligned}R_{(1)}(t) &= R_{(1,1)}(t)R_{(1,2)}(t)R_{(1,3)}(t)R_{(1,4)}(t) = \\ &= e^{-\lambda_{(1)}t}\left(4\,e^{-3\,\lambda_{(1,4,i)}t} - 3\,e^{-4\,\lambda_{(1,4,i)}t}\right),\end{aligned}$$

where $\lambda_{(1)} = \lambda_{(1,1)} + \lambda_{(1,2)} + \lambda_{(1,3)} \approx 0.049.$

Therefore the reliability function of the whole system is

$$
\begin{aligned}
R_{sys}(t) &= R_{(1)}(t)R_{(2)}(t) = \\
&= R_{(1,1)}(t)R_{(1,2)}(t)R_{(1,3)}(t)R_{(1,4)}(t)R_{(2)}(t) \approx \\
&\approx 4\,e^{-0.130\,t} - 3\,e^{-0.151\,t}.
\end{aligned}
$$

In particular, the system reliability over 10 hours is $R_{sys}(10) \approx 0.43$.

In section 2.3, which is devoted to risk analysis, a more detailed analysis of reliability indexes will be proposed, and in section 2.3.5.11, this example investigation will be continued.

1.6. Non-renewable redundant systems

One of the basic possibilities to increase the system reliability involves the creation of redundancy (reserve). Nature provides us with numerous examples of redundancy: we have two eyes, two ears, two legs and arms, most of animals have four legs, etc. These redundancies provide the reliability and safety of organisms.

By constructing new articles and systems, we follow nature and provide them with some redundancy. In this and the next section we consider different redundancy models and methods for reliability calculation of redundant systems. In this section the methods of structure reliability calculation are used for the calculation of non-renewable redundant systems. The next section deals with the reliability analysis of renewable redundant systems. However, we start with the classification of redundancy means.

1.6.1. *Basic redundancy means – terms*

DEFINITION 1.12.– The *redundancy* is a means to increase the reliability of units (elements, articles and systems) by using a reserve. under the term *redundancy* it is understood some additional facilities and possibilities over the minimal needed for fulfillment by the unit of its functions.

With respect to facilities, we should distinguish structural, times, functional and other types of redundancy.

DEFINITION 1.13.– *Structural* redundancy is a redundancy for which some additional (reserve) units, components or subsystems are used.

Functional redundancy uses the system or its components (units) capability to fulfill some additional functions except the basic ones.

Time redundancy uses an object free time for fulfillment of some additional functions.

With respect to level, we should distinguish common and separate redundancy (reservation).

DEFINITION 1.14.– *Common* redundancy reserves the whole system (or article) while *separate* redundancy reserves some components of the system (its subsystems or elements).

With respect to switching means we should distinguish hot, cold and warm redundancy.

DEFINITION 1.15.– *Hot* redundancy uses the reserve (standby) unit in the same regime as the basic one. In this case, each unit has the same failure rate regardless of whether it is in standby or in operation.

Cold redundancy switches reserve (standby) units only after the failure of the basic unit. In this case, components in standby do not fail.

Lastly, *warm* redundancy reserves the components that are not used jointly with the basic one, but partially spend their resources in standby. This option is used when switching on the standby components that demand some additional time, and warm redundancy tends to decrease the switching time for reserve elements. The standby components can fail, but their failure rates are smaller than those of the basic component.

Another type of redundancy is proposed in the next definition.

DEFINITION 1.16.– *Group-wise redundancy* is those for which the function of some group of basic units can be fulfilled by one or several standby components, each of which can change any of the failed basic components of the group.

With respect to further using the failed units, we consider non-renewable and renewable redundancies.

DEFINITION 1.17.– In the case of *non-renewable* redundancy, the failed unit is lost and the system operates only for a short period of time, and thus all reserves will be exhausted. In the case of *renewable* redundancy, the failed unit is restored or replaced with the new one that has the same characteristics and the system continues to work.

Different possibilities of restoration should also be taken into account: only one, some part or all failed elements could be restored simultaneously depending on the number of restoration facilities.

We cannot consider all possible redundancy schemes, and thus focus only on the main ones beginning from the non-renewable redundancy scheme.

1.6.2. *Hot redundancy*

The structure scheme of the hot redundancy system, shown in Figure 1.27, coincides with the parallel connection components of the system.

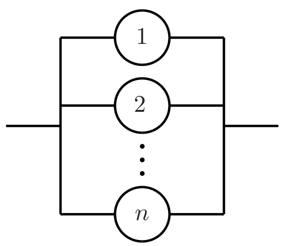


Figure 1.27. *The structure scheme of hot redundancy*

The structure function of such a system (see example 1.11) is

$$\varphi(x_1, x_2, \ldots, x_n) = 1 - \prod_{i=1}^{n}(1 - x_i).$$

Thus (see section 1.4), the reliability function $R_{sys}(t)$ of the hot redundant system from n components with the reliability functions $R_i(t)$ is

$$R_{sys}(t) = 1 - \prod_{i=1}^{n}(1 - R_i(t)). \qquad [1.95]$$

On the other hand, it is clear that the lifetime T_{sys} of such a system is equal to the maximal lifetimes T_i of its components,

$$T_{sys} = \max\{T_1, T_2, \ldots, T_n\}.$$

Thus, the system lifetime distribution is

$$F_{sys}(t) = \mathbf{P}\{T_{sys} \le t\} = \mathbf{P}\{\max_{1\le i\le n} T_i \le t\} =$$

$$= \mathbf{P}\{T_1 \le t, \ldots, T_n \le t\} = \prod_{i=1}^{n} F_i(t). \qquad [1.96]$$

Of course, both approaches give the same result. All the other system reliability characteristics can be obtained from this result.

1.6.3. *Cold redundancy*

The structure scheme of the non-renewable cold redundant system is shown in Figure 1.28.

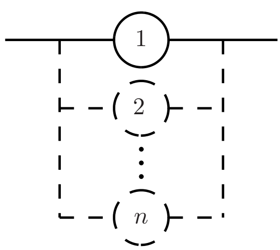


Figure 1.28. *The structure scheme of cold redundancy*

For this redundancy scheme, there exists only one basic operating component and $n - 1$ redundant components in standby that are switching on and passing in the operating state one by one only when the basic (operating) component fails. The system operating time T_{sys} for the cold redundant system is equal to the sum of operating times of its components T_i,

$$T_{\text{sys}} = \sum_{i=1}^{n} T_i.$$

Therefore, the system lifetime c.d.f. is calculated by using the convolution formula

$$F_{\text{sys}}(t) = \mathbf{P}\{T_{\text{sys}} \le t\} = \mathbf{P}\{T_1 + \cdots + T_n \le t\} = F^{(*n)}(t), \qquad [1.97]$$

where the function $F^{(*n)}(t)$ is given by the equality

$$F^{(*1)}(t) = \mathbf{P}\{T_1 \le t\} = F(t),$$

$$F^{(*n)}(t) = \int_0^t F^{(*(n-1))}(t-u)\, dF(u), \quad n > 1. \qquad [1.98]$$

Appropriate mean and variance of the system lifetime are

$$\mathbf{E}[T_{\text{sys}}] = \sum_{i=1}^{n} \mathbf{E}[T_i] = n\,\mu; \quad \mathsf{Var}[T_{\text{sys}}] = \sum_{i=1}^{n} \mathsf{Var}[T_i] = n\,\sigma^2, \qquad [1.99]$$

where μ and σ^2 are expectation and variance of the components' lifetime, respectively.

1.6.4. *Markov process for system reliability investigations*

In this section, one more approach based on the theory of Markov random processes for the system reliability investigation will be considered, and here it will be applied for the non-renewable warm redundant model investigations, while in the next section the renewable system will be studied based on this approach.

Consider a warm redundant system, containing one basic and $n-1$ standby units, and suppose that the lifetimes of basic T and standby $T^{(\mathrm{st})}$ units have exponential distributions with parameters λ and ν, respectively,

$$F(t) = \mathbf{P}\{T_n \le t\} = 1 - e^{-\lambda t},$$
$$F^{(\mathrm{st})}(t) = \mathbf{P}\{T_n^{(\mathrm{st})} \le t\} = 1 - e^{-\nu t}. \qquad [1.100]$$

It is necessary to note that the models for hot and cold redundant systems follow from this model if we put $\nu = 0$ or $\nu = \lambda$, respectively. Let us denote by $X(t)$ the number of non-workable (failed) units of the system in time t[7]. It means that $X = \{X(t),\ t \ge 0\}$ is a random process with a finite set of states $E = \{0, 1, \ldots, n\}$. Under the above assumption that the element lifetimes have an exponential distribution, due to its memoryless property (see theorem 1.1) in the section 1.1.3.1, the process X is a homogeneous Markov one [GNE 65, CHU 60, ROS 96]. These processes are characterized by the property that their future behavior does not depend on the past given presence state and its *transition probabilities*

$$p_{jk}(s,t) = \mathbf{P}\{X(s+t) = k \,\big|\, X(s) = j\}$$

and also does not depend on time s, but only on the interval t and the presence and future states, namely:

$$p_{jk}(s,t) = p_{jk}(t).$$

The matrix $P(t) = \left[p_{ij}(t)\right]_{i,j\in E}$ is known as the *transition matrix* of the process X. Any Markov process is fully determined by its transition matrix and the initial state distribution. Moreover, due to the Markov property, the transition matrix satisfies the semi-group property, $P(s+t) = P(s)P(t)$, and as a result of this property, it also satisfies the equality

$$P(t) = P^n\left(\frac{t}{n}\right), \quad \text{for any} \quad n = 1, 2, \ldots.$$

7 It is also possible to use the dual process "number of workable at time t units".

The last equality means that the transition matrix can be determined by its value in infinity small neighboring of its value in zero, namely by the right hand side derivatives (if they exists) of transition probabilities,

$$\lambda_{ij} = \lim_{h \to +0} \frac{\delta_{ij} - p_{ij}(h)}{-h} = \frac{d}{dt} p_{ii}(t)\Big|_{t=+0}. \qquad [1.101]$$

DEFINITION 1.18.– If the right hand side derivatives λ_{ij} exist, they are called *transition intensities*, which are derived from the matrix $\Lambda = [\lambda_{ij}]_{i,j \in E}$, which is known as the *transition intensity matrix* or *infinitesimal matrix* of the process. The process in this case is known as the *standard* Markov process.

REMARK 1.2.– It is possible to show (see [CHU 60]) that the assumption about differentiability of transition probabilities is not required; for its differentiability, it is sufficient to take (not show) the natural assumption about the continuity of the transition probabilities at zero.

$$\lim_{t \to +0} P(t) = P(0) = I.$$

Transition probabilities of the standard Markov processes satisfy the Kolmogorov differential equations

$$\frac{d}{dt} P(t) = \Lambda P(t) = P(t)\Lambda$$

where the initial condition $P(0) = I$ indicates that this process is determined by its transition intensity matrix.

Together with the transition intensity matrix, the Markov process is also convenient to determine with the help of the so-called *marked transition graph*. It is the oriented graph, whose vertices are the process states and the edges show the possible direct transitions while the marks of edges indicate the appropriate transition intensities.

In our case of a non-renewable warm redundant system, there is only one possibility of reaching directly to the state $k + 1$, namely from the state k, because in the case of some element failure, the process X is increased by 1. Thus, only $\lambda_{k,k+1} \neq 0$, i.e. for the transition intensities, only one index is needed, $\lambda_k = \lambda_{k,k+1}$. This kind of Markov process is called the *birth process*, because the process was first used for animal population investigation.

It is possible to show that from warm redundancy appropriate transition intensities equal $\lambda_k = \lambda + (n - k - 1)\,\nu$, the cold and hot redundant systems can be obtained if we put $\nu = 0$ or $\nu = \lambda$, respectively. The marked transition graph of the process X is shown in Figure 1.29.

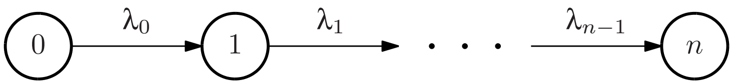


Figure 1.29. *Marked transition graph for non-renewal system from n elements*

Denote by $p_k(t)$ the probability of the k-th state of the process at time t,

$$p_k(t) = \mathbf{P}\{X(t) = k\}.$$

Using the complete probability formula and Markov property of the process X for its state probabilities, we can get the following difference equations:

$$p_k(t + \Delta t) = \lambda_{k-1}\Delta t\, p_{k-1}(t) + (1 - \lambda_k \Delta t)\, p_k(t) + o(\Delta t). \qquad [1.102]$$

To explain these equations, we should take into account that in order to process X that occurs in time $t + \Delta t$ in the state k, it should be in time t in the state k (the probability of this event is $p_k(t)$) and should not leave this state during the time interval Δt (with probability $1 - \lambda_k \Delta t$), or should be in time t in the state $k - 1$ (whose probability is $p_{k-1}(t)$) and pass to the state k (with probability $\lambda_{k-1}\Delta t$).

After the simple algebra, equation [1.102] gives

$$\frac{p_k(t + \Delta t) - p_k(t)}{\Delta t} = \lambda_{k-1} p_{k-1}(t) - \lambda_k p_k(t) + o(1),$$

and passing to the limit when $\Delta t \to 0$, we get

$$\frac{dp_k(t)}{dt} = \lambda_{k-1} p_{k-1}(t) - \lambda_k p_k(t) \quad (k \in E). \qquad [1.103]$$

The following rule allows us to write the differential equations for the probability states by directly using the marked transition graph for the

Markov process. *The derivative of a state probability equals the algebraic sum of product state probabilities by transition intensities with sign "+" for input arrows and with sign "−" for output from this state arrows.*

Under the assumption that in the initial time all components are in the up state, the solution of system [1.103] with the initial conditions

$$p_0(0) = 1, \quad p_k(0) = 0 \quad \text{for } k \in E \qquad [1.104]$$

gives the possibility of finding the system reliability function. Really, the probability $p_n(t)$ represents the c.d.f. of the system lifetime that coincides with the distribution of the sum of n independent exponentially distributed with the parameters $\lambda_i, \ (i = \overline{1,n})$ r.v.'s. The formula for this distribution in the general case is cumbersome; however, its m.g.f. $\phi(s)$ is the product of summand m.g.f.'s $\phi_i(s)$ and has a simple form

$$\phi(s) = \prod_{1 \le i \le n} \phi_i(s) = \prod_{1 \le i \le n} \frac{\lambda_i}{s + \lambda_i}.$$

The lifetime c.d.f. for the cold redundant non-renewable system coincides with the Erlang distribution (see section 1.1.3.1). The reliability function in this case has the form

$$R(t) = e^{-\lambda t} \sum_{i=0}^{n} \frac{(\lambda t)^i}{i!}. \qquad [1.105]$$

For the hot redundant system and the more general cases of redundancy, when there exist several basic and several standby units, the formulas are cumbersome. We can find them in some reference books, for example in [KOZ 75].

1.6.5. *Reliability properties of redundant systems*

Consider some properties of the redundant systems.

1.6.5.1. *Dependence of the system reliability on the redundancy level*

Consider the problem of the redundancy level influence on the system reliability. In Figure 1.30, two schemes of redundancy are presented: more high and more low redundancy levels.

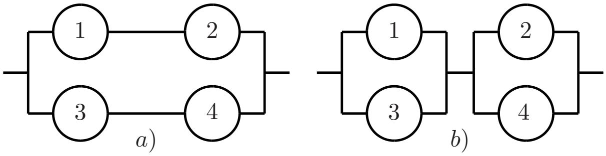


Figure 1.30. *More high a) and more low b) redundancy levels*

For the parallel-series redundancy (Figure 1.30(a)), the structure functions of components in series are

$$\varphi_1(x_1, x_2) = x_1 x_2; \quad \varphi_2(x_3, x_4) = x_3 x_4.$$

Therefore, the structure function $\varphi_a(\mathbf{x})$ of this system is

$$\varphi_a(\mathbf{x}) = 1 - (1 - \varphi_1(x_1, x_2))(1 - \varphi_2(x_3, x_4)) = 1 - (1 - x_1 x_2)(1 - x_3 x_4).$$

In the case when $p_i = p$ for the reliability of the a-system $p^a_{\text{sys}}(p)$, we can get the expression

$$p^a_{\text{sys}}(p) = 1 - (1 - p^2)^2 = p^2(2 - p^2). \qquad [1.106]$$

Analogous reasons for the second system (Figure 1.30(b)) give

$$\varphi_b(\mathbf{x}) = \varphi_1(x_1, x_2)\varphi_2(x_3, x_4) = (1 - (1 - x_1)(1 - x_3))(1 - (1 - x_2)(1 - x_4)),$$

and thus,

$$p^b_{sys}(p) = (1 - (1 - p)^2)^2 = p^2(2 - p)^2. \qquad [1.107]$$

The graphs of functions $p^a_{\text{sys}}(p)$ and $p^b_{\text{sys}}(p)$, shown in Figure 1.31, indicate that increasing the redundancy level decreases the system reliability.

Consider one more example of systems with different levels of redundancy.

EXAMPLE 1.18.– Consider the system from three equal-reliable units, as shown in Figure 1.32.

The reliability $p_{\text{sys}}(p)$ of the system is

$$p_{\text{sys}}(p) = p\left(1 - (1 - p)^2\right) = p^2(2 - p), \qquad [1.108]$$

where p is the reliability of a separate unit. Consider two variants of the system redundancy.

a) Reservation of the whole system. In this case, the reliability of the system is

$$p_{\text{sys}}^{(a)}(p) = 1 - \left(1 - (p^2(2-p))\right)^2 = p^2(2-p)\left(2 - p^2(2-p)\right). \qquad [1.109]$$

b) Reservation of each unit. In this case, the system reliability is

$$p_{\text{sys}}^{(b)}(p) = \left(2p - p^2\right)\left(1 - (1-p)^4\right) = p^2(2-p)^2\,(2 - p(2-p)). \qquad [1.110]$$

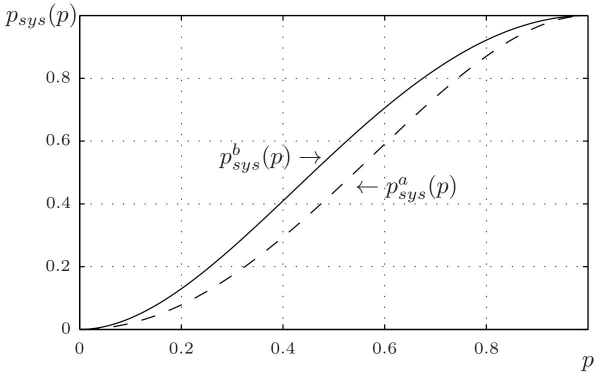


Figure 1.31. *The graphs of system reliability versus its unit reliability for different redundancy levels*

The graphs of the functions $p_{\text{sys}}^{(a)}(p)$ and $p_{\text{sys}}^{(b)}(p)$ have an S-type form and are analogous to the graphs shown in Figure 1.31. As in exercise 1.25, it is proposed to the reader to check formulas [1.109, 1.110] and draw the graphs of these functions.

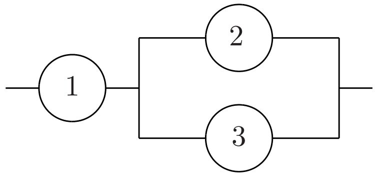


Figure 1.32. *Three-element system*

1.6.5.2. *Dependence of the system reliability on the number of components*

Analogous situation takes place with increasing number of the system units. Consider the system consisting of k units in series and denote its reliability by p. If all units have the same reliability, then the reliability of each unit has to be equal to $\sqrt[k]{p}$. The reliability of the analogous system with such double redundant units is

$$p_{sys}(p) = \left[1 - (1 - \sqrt[k]{p})^2\right]^k = \left[1 - \left(1 - \sqrt[k]{1 - (1 - p)}\right)^2\right]^k.$$

From here it follows that in the case of enough reliable units, $p = 1 - \varepsilon$ for sufficiently small ε, the Taylor expansion of the expression $\sqrt[k]{1 - (1 - p)}$ gives

$$p_{sys}(p) \approx \left[1 - \left(\frac{\varepsilon}{k}\right)^2\right]^k \approx e^{-\frac{\varepsilon^2}{k}} \to 1 \quad \text{for} \quad k \to \infty. \qquad [1.111]$$

The last expression indicates that the reliability of complex systems, consisting of a large enough number of reliable units, could be made as high (close to one) as needed. It can also be represented as the assertion.

THEOREM 1.18.– Redundancy increases the reliability of systems, consisting of enough reliable units.

Note that the above arguments can also be used for the evaluation of the system's time-dependent reliability.

1.6.5.3. *Separate and common reservation*

Consider two variants of a redundant system, consisting of k units: separate (individual) and common (joint) redundancy. These two types of reservations are shown in Figures 1.33 and 1.34.

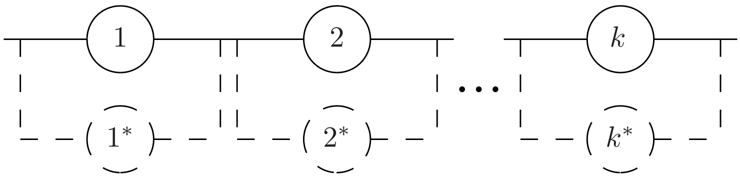


Figure 1.33. *Separate redundancy*

Find the mean system lifetime for these methods of redundancy.

1) For the system with separate redundancy units, having an exponential lifetime distribution with parameter λ for each unit, the mean system lifetime is

$$\mu_{\text{sep}} = \frac{1}{\lambda}\frac{(k+1)!}{k^k}\sum_{i=0}^{k}\frac{k^i}{i!}. \qquad [1.112]$$

2) In the case of common redundancy, units in standby begin to work in the case of some basic unit failure, and the system fails in time when all k units in standby fail. Under the assumption about exponential unit lifetime distribution, the mean system lifetime is

$$\mu_{\text{com}} = \frac{k+1}{\lambda k}, \qquad [1.113]$$

where λk is the summary element failure rate.

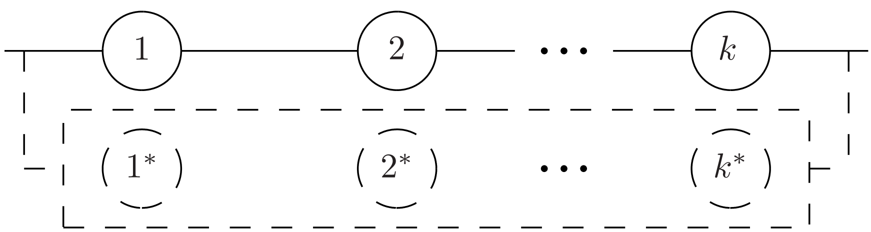


Figure 1.34. *Common redundancy*

To compare these two methods of redundancy, we denote by W_k the efficiency coefficient

$$W_k = \frac{\mu_{\text{com}}}{\mu_{\text{sep}}} = \frac{(k+1)k^k}{k!\sum_{i=0}^{k}\frac{k^i}{i!}}. \qquad [1.114]$$

The values of the efficiency coefficient for $k = 1, 2, 3$ are

$$W_1 = 1; \quad W_2 = 1.2; \quad W_3 = 1.38.$$

It is possible to show that the efficiency coefficient W_k increases when the number of elements k increases. Moreover, the following asymptotic formula can be used when $k \to \infty$:

$$W_k \approx \sqrt{\frac{k}{2\pi}}. \tag{1.115}$$

The results of this section show that the system reliability increases with joining of the units in standby.

1.6.5.4. *Rate of reliability increasing under reservation*

Consider a redundant system, consisting of k identical units. Thus, in the best case of the cold redundancy, the system lifetime T_{sys} will be equal to the sum of all standby unit lifetimes,

$$T_{\text{sys}} = \sum_{i=1}^{n} T_i, \tag{1.116}$$

The mean system lifetime is

$$\mu_{\text{sys}} = n\mu, \tag{1.117}$$

where μ is the mean lifetime of each unit. This equality shows the linear growth of the mean lifetime of the system under the cold redundancy. This means that the non-renewal redundant systems can increase their reliability not more than linearly to the number of standby units.

1.6.6. *A unit warranty operating time calculation*

During the projection of industrial objects, the constructor should predict with given probability its warranty operating time. We define the $(1-\alpha)$-*warranty operating time* t_{war} of an object as the time that it can operate without failure with probability $1-\alpha$. It is expected that after this time, the object is degraded and must be cardinally upgraded or replaced. To find this time, it is necessary to determine the limiting failure probability (the greatest allowed failure probability) α or the smallest allowed reliability $1-\alpha$, and solve one of the two equations

$$F(t) = \alpha \quad \text{or} \quad R(t) = 1-\alpha. \tag{1.118}$$

The values of α depend on the destination of the object and can be varied between $10^{-6} \leq \alpha \leq 10^{-2}$

Solutions c_α of both equations are the same value that is known as *α-quantile*, or – *100*$(1 - \alpha)$*-percent point*.

$$t_{\text{war}} = c_\alpha = F^{-1}(\alpha) = R^{-1}(1 - \alpha)$$

For continuous distributions, the solutions of these equations exist and are unique. For stepwise or discrete distributions as α-quantile, the following value is usually considered:

$$c_\alpha = \inf\{t : F(t) = \alpha\}. \qquad [1.119]$$

However, the quantile is very rarely represented in a closed form. In this case, the numerical solution can be obtained, for example, with the help of some special computer tools, for example MS Excel, Statistica, MatLab, MatCad and so on.

The tables of $100\alpha\%$ percentiles for commonly used distributions and some values of α can be found in most books on probability theory, statistics and reliability theory.

EXAMPLE 1.19.– Find the $(1-\alpha)$-warranty operating time for $\alpha = 0.01$ of a unit, which has a constant h.r.f. and mean lifetime equal to 200 years. Constant h.r.f. means that the unit lifetime distribution is exponential. For $\mu = 200$ years, we find $\lambda = \frac{1}{\mu} = 0.005\,\text{year}^{-1}$. Equation [1.119]

$$R(t) = e^{-\lambda t} = 1 - \alpha,$$

indicates that the 99%-warranty operating time t_{war} value is equal to

$$t_{\text{war}} = -\frac{1}{\lambda}\ln(1 - \alpha) = -200\ln(0.99) \approx 2\,\text{year}.$$

1.6.7. *Exercises*

EXERCISE 1.25.– Please check formulas [1.109, 1.110] and draw the graphs of the functions represented there.

EXERCISE 1.26.– The cold redundant system consists of two units: basic and standby. When switching on the standby unit, the unit can fail with probability $p = 0.025$. Calculate the system reliability function, if the unit's h.r.f. is constant and equal to 0.003^{-1}.

EXERCISE 1.27.– The system consists of three units: one of the redundant units is in hot standby and the other is in standby. Draw the graphs of the system reliability function and h.r.f., if the unit's h.r.f. is constant and equal to 0.002^{-1}.

EXERCISE 1.28.– Cold redundant system contains of eight identical units. Lifetime of each unit has the exponential distribution with parameter $\lambda = 0.001\, hour^{-1}$. Draw the graphs of the system reliability function.

EXERCISE 1.29.– Consider two unit hot redundant systems. Find the mean system lifetime (in years), if the lifetime of each unit has the exponential distribution with parameter $\lambda = 0.13.46 \cdot 10^{-7}\, hours^{-1}$.

EXERCISE 1.30.– Find 99% warranty operating time of an object that has normal reliability law with parameters $\mu = 300$ years and $\sigma = 50$ years.

1.7. Renewable redundant systems

1.7.1. *The model*

Here we consider the redundant system under the assumption that the failed units can be renewed (repaired or replaced). Concerning the redundancy method, the previous assumptions are preserved, i.e. the redundancy can be hot, cold or warm. Concerning the number of simultaneously repaired units, different assumptions can be used: only one, several (limited number) or all failed units can be repaired simultaneously.

Here, we demonstrate the methods of renewable redundant systems reliability investigation for the model of warm double redundant system with only one repair facility (when only one of failed units can be repaired simultaneously). The repaired unit is returned back to the system as a standby unit.

For renewable redundant system, the lifetimes of basic, standby units as well as their renewal times form sequences of r.v.'s. Denote these r.v.'s by T_n,

$T_n^{(st)}$ and $T_n^{(ren)}$, respectively, and suppose that they all are independent and identical for each sequence distributed r.v.'s, which have exponential distributions with parameters λ, ν and μ respectively,

$$F(t) = \mathbf{P}\{T_n < t\} = 1 - e^{-\lambda t};$$
$$F^{(st)}(t) = \mathbf{P}\{T_n^{(st)} < t\} = 1 - e^{-\nu t};$$
$$G(t) = \mathbf{P}\{T_n^{(ren)} < t\} = 1 - e^{-\mu t}.$$

To construct the mathematical model of the considered phenomenon, denote as before by $X(t)$ the number of failed (down) elements in time t[8]. Under the given assumptions, the random process $X = \{X(t),\ t \geq 0\}$ is a Markov one, i.e. the process, whose future behavior does not depend on its past behavior given its present state. This property follows from theorem 1.1 (see section 1.1.3.1) about the memoryless property of the exponential distribution.

As was mentioned in section 1.6, it is convenient to determine the Markov processes with the help of its marked transition graph. For the considered model of the warm double redundant system with one renewal facility, the marked transition graph is shown in Figure 1.35.

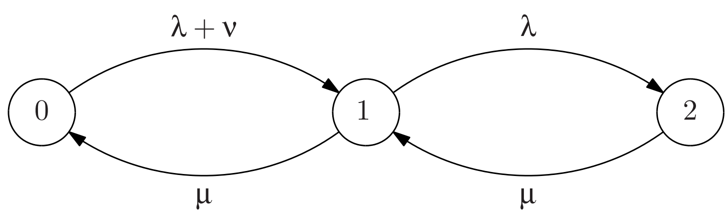


Figure 1.35. *Marked transition graph for the warm redundancy system with one renewal facility*

8 As in previous section it is also possible to use the dual process "number of workable at time t units".

1.7.2. *Equations for probabilities of the system states*

For the double redundant system ($n = 2$), we denote by

$$p_k(t) = \mathbf{P}\{X(t) = k\} \quad (k \in \{0, 1, 2\})$$

the state probabilities of the system. The transitions of the process from any state are possible only into neighboring states. Therefore, the difference equations for the considered warm redundancy system has a form

$$\begin{aligned}
p_0(t + \Delta t) &= p_0(t)\,(1 - (\lambda + \nu)\Delta t) + p_1(t)\,\mu\,\Delta t + o(\Delta t),\\
p_1(t + \Delta t) &= p_0(t)\,(\lambda + \nu)\Delta t + p_1(t)\,(1 - (\lambda + \mu)\Delta t) + p_2(t)\,\mu\,\Delta t + o(\Delta t),\\
p_2(t + \Delta t) &= p_1(t)\,\lambda\,\Delta t + p_2(t)\,(1 - \mu\,\Delta t) + (\Delta t).
\end{aligned}$$

The equations are explained as follows.

First equation. In order for the process to occur in state "0" in time $t + \Delta t$, it must be in this state in time t with probability $p_0(t)$ and should not leave it with the probability of $1 - (\lambda + \nu)\Delta t$ or it has to be in time t in the state "1" with probability $p_1(t)$ and move to the state "0" during the small time interval Δt with probability $\mu\Delta t$.

Second equation. In order for the process to occur in state "1" in time $t + \Delta t$ it must be in this state in time t with probability $p_1(t)$ and should not leave it with probability $1 - (\lambda + \mu)\Delta t$ or it has to be in time t in the state "0" with probability $p_0(t)$ and move to the state "1" during the small time interval Δt with probability $(\lambda+\nu)\Delta t$ or it has to be in time t in the state "2" with probability $p_2(t)$ and move to the state "1" during small time interval Δt with probability $\mu\Delta t$.

Third equation. In order for the process to occur in state "2" in time $t + \Delta t$ it must be in this state in time t with probability $p_2(t)$ and should not leave it with probability $1 - \mu\Delta t$ or it has to be in time t in the state "1" with probability $p_1(t)$ and move to the state "2" during the small time interval Δt with probability $\lambda\Delta t$.

From here, after some simple algebra and passing to the limit when $\Delta t \to 0$, we can get

$$\left.\begin{aligned}
p_0'(t) &= -(\lambda + \nu)\,p_0(t) + \mu\,p_1(t),\\
p_1'(t) &= (\lambda + \nu)\,p_0(t) - (\lambda + \mu)p_1(t) + \mu\,p_2(t),\\
p_2'(t) &= \lambda\,p_1(t) - \mu\,p_2(t).
\end{aligned}\right\} \qquad [1.120]$$

The last system should be solved under some initial conditions that for the fully workable system in the initial time $t = 0$ are:

$$p_0(0) = 1, \quad p_1(0) = p_2(0) = 0. \qquad [1.121]$$

This system can also be obtained directly from the marked transition graph (Figure 1.35) with the help of the rule for the differential equation composition, given in section 1.6.4.

Analogous equations for systems with hot and cold reserves can be obtained from the above equations if we put $\nu = \lambda$ and $\nu = 0$, respectively.

1.7.3. *Steady state probabilities: system failure probability*

Denote by

$$\pi_k = \lim_{t\to\infty} p_k(t), \quad (k \in \{0, 1, 2\}),$$

the limiting (stationary) state probabilities. The system of equations for the steady-state probabilities π_k satisfy to the algebraic system of equations [1.122].

$$\left.\begin{aligned} -(\lambda+\nu)\,\pi_0 + \mu\,\pi_1 &= 0,\\ (\lambda+\nu)\,\pi_0 - (\lambda+\mu)\,\pi_1 + \mu\,\pi_2 &= 0,\\ \lambda\pi_1 - \mu\,\pi_2 &= 0. \end{aligned}\right\} \qquad [1.122]$$

This system is a homogeneous system with degenerated matrix. In order to get a unique solution, it is necessary to add some additional conditions that do not change under passing to the limit; an example is the following *normalizing condition*:

$$\pi_0 + \pi_1 + \pi_2 = 1. \qquad [1.123]$$

In the system from fourth equations [1.122] and [1.123] one from three first is extra. Representing the first and third part of equations [1.122], the values π_0 and π_1 in terms of π_2 give

$$\pi_1 = \frac{\mu}{\lambda}\,\pi_2; \quad \pi_0 = \frac{\mu}{\lambda+\nu}\,\pi_1 = \frac{\mu^2}{(\lambda+\nu)\,\lambda}\,\pi_2.$$

Denote by

$$\gamma = \frac{\mu}{\lambda} \qquad [1.124]$$

the *unit renewal coefficient*, and by

$$\kappa = \frac{\nu}{\lambda} \qquad [1.125]$$

the *reliability reserve coefficient*. In these notations, system [1.122] with normalizing condition [1.123] solution is

$$\begin{aligned}
\pi_0 &= \frac{\gamma^2}{(1+\gamma)(1+\kappa)+\gamma^2},\\
\pi_1 &= \frac{\gamma(1+\kappa)}{(1+\gamma)(1+\kappa)+\gamma^2},\\
\pi_2 &= \frac{1+\kappa}{(1+\gamma)(1+\kappa)+\gamma^2}.
\end{aligned} \qquad [1.126]$$

Finally, because the state 2 coincides with the system failure state, the system failure probability p_{fail} is

$$p_{fail} = \pi_2 = 1 - K_{av} = \frac{1+\kappa}{(1+\gamma)(1+\kappa)+\gamma^2}, \qquad [1.127]$$

where K_{av} is the so-called availability coefficient that was introduced before (see definition 1.5 in section 1.2.4). Appropriate steady state characteristics for the hot and cold double redundant systems can be obtained from the above equation if we put $\kappa = 1$ and $\kappa = 0$, respectively.

1.7.4. *Reliability function for renewable systems*

Note that the system lifetime T is defined as the time from the absolutely intact (new) system exploitation beginning up to its first failure. Denote also by $\hat{T}$ the time between two successive failures. The distributions of these r.v.'s differ by the initial state of the system. For the calculation of these distributions, we modify the process X by the transformation of the state "2" into an absorbing state, which means that the output from this state is impossible, and denote the new process by $\hat{X} = \{\hat{X}(t),\ t \geq 0\}$. Thus, the distribution of the time to the first system failure coincides with the probability of the modified process to be in the state “2”,

$$F(t) = \mathbf{P}\{T \le t\} = \mathbf{P}\{\hat{X}(t) = 2\}. \qquad [1.128]$$

The marked transition graph for the warm double redundant system with one repair facility and absorption in the state “2” is shown in Figure 1.36.

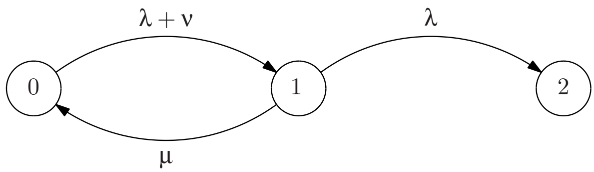


Figure 1.36. *Marked transition graph for the modified process* $\hat{X}(t)$

Using the rule for differential equation construction (see section 1.6.4), we can get the following system of differential equations for the process $\hat{X}$ probability states:

$$\left.\begin{aligned} p_0'(t) &= -(\lambda+\nu)\,p_0(t) + \mu\,p_1(t),\\ p_1'(t) &= (\lambda+\nu)\,p_0(t) - (\lambda+\mu)\,p_1(t),\\ p_2'(t) &= \lambda\,p_1(t), \end{aligned}\right\} \qquad [1.129]$$

which should be solved with the above initial conditions

$$p_0(0) = 1; \quad p_1(0) = p_2(0) = 0. \qquad [1.130]$$

For the system solution, the operational method, which is based on the system Laplace transforms, will be used. As the last equation of system [1.129] is solved by integration, we apply Laplace transforms to the first two equations of [1.129]. Denoting:

$$\tilde{p}_k(s) = \int_0^\infty e^{-st}\,p_k(t)\,dt,$$

taking into account that

$$\int_0^\infty e^{-st}\, p_k'(t)\, dt = e^{-st}\, p_k(t)\Big|_0^\infty + \int_0^\infty e^{-st}\, p_k(t)\, dt = s\tilde{p}_k(s) - p_k(0),$$

and using initial conditions [1.130] for the functions $\tilde{p}_k(s)$, we can obtain the following system of algebraic equations:

$$\left.\begin{aligned} s\tilde{p}_0(s) - 1 &= -(\lambda+\nu)\,\tilde{p}_0(s) + \mu\,\tilde{p}_1(s),\\ s\tilde{p}_1(s) &= (\lambda+\nu)\,\tilde{p}_0(s) - (\lambda+\mu)\,\tilde{p}_1(s). \end{aligned}\right\}$$

Rewriting the system in the form

$$\left.\begin{aligned} (\lambda+\nu+s)\,\tilde{p}_0(s) - \mu\,\tilde{p}_1(s) &= 1;\\ -(\lambda+\nu)\,\tilde{p}_0(s) + (\lambda+\mu+s)\,\tilde{p}_1(s) &= 0, \end{aligned}\right\} \qquad [1.131]$$

we can find its solution, for example, with the help of the Kramer rule,

$$\begin{aligned} \tilde{p}_0(s) &= \frac{\lambda+\mu+s}{(\lambda+\nu+s)(\lambda+\mu+s) - (\lambda+\nu)\,\mu},\\ \tilde{p}_1(s) &= \frac{\lambda+\nu}{(\lambda+\nu+s)(\lambda+\mu+s) - (\lambda+\nu)\,\mu}. \end{aligned} \qquad [1.132]$$

Because for finding the problem solution for the c.p.f. to the first failure we only need the function $p_1(t)$, for the inverse transform of this function calculation, we expand the function $\tilde{p}_1(s)$ into the simple fractions with unknown coefficients $A_1,\ A_2$,

$$\tilde{p}_1(s) = \frac{A_1}{s+s_1} + \frac{A_2}{s+s_2}, \qquad [1.133]$$

where the values $s_1,\ s_2$ are the absolute values of the roots of the characteristic equation of system [1.131] (the denominator of fraction for $\tilde{p}_1(s)$),

$$\begin{aligned} &(\lambda+\nu+s)\,(\lambda+\mu+s) - (\lambda+\nu)\,\mu =\\ &= s^2 + (2\lambda+\nu+\mu)\,s + \lambda\,(\lambda+\nu) \;=\; 0. \end{aligned}$$

In terms of dimensionless coefficients γ and κ, the values are given by

$$s_{1,2} = \frac{\lambda}{2}\left(2 + \gamma + \kappa \mp \sqrt{(\gamma + \kappa)^2 + 4\gamma}\right). \qquad [1.134]$$

Calculate the coefficients of expansion [1.133] from the system of equations

$$\begin{aligned} A_1 + A_2 &= 0, \\ A_1 s_2 + A_2 s_1 &= \lambda + \nu, \end{aligned} \qquad [1.135]$$

which can be obtained by comparison of the numerators of equation [1.133]. Its solution yields the values

$$A_1 = -A_2 = \frac{\lambda + \nu}{s_2 - s_1}.$$

Now the inverse transform of [1.133] gives

$$p_1(t) = A_1 e^{-s_1 t} + A_2 e^{-s_2 t}.$$

From this expression, by integrating, we can find the lifetime c.d.f., which after substitution of the coefficients A_1 and A_2 takes the form

$$\begin{aligned} F(t) = p_2(t) &= \lambda \int_0^t p_1(u)\,du = \\ &= 1 - \frac{\lambda^2(1+\kappa)}{s_1 r} e^{-s_1 t}\left(1 - \frac{s_1}{s_2} e^{-rt}\right), \end{aligned} \qquad [1.136]$$

where $r = s_2 - s_1 = \lambda\sqrt{(\gamma + \kappa)^2 + 4\gamma}$. Finally, for the system reliability function, we can get the following expression:

$$R(t) = 1 - p_2(t) = \frac{\lambda^2(1+\kappa)}{s_1 r} e^{-s_1 t}\left(1 - \frac{s_1}{s_2} e^{-rt}\right). \qquad [1.137]$$

REMARK 1.3.– The c.d.f. of the time $\hat{T}$ between failures can be obtained by using the same approach, but the solution of the system of differential equations [1.129] should be solved under the initial conditions

$$p_1(0) = 1; \quad p_0(0) = p_2(0) = 0.$$

As in exercise 1.31, it is proposed to find the c.d.f. of the time $\hat{T}$ between failures for the warm double redundant system.

By calculation, we can find that the mean values of the warm redundant systems with and without renewal are

$$\mathbf{E}[T] = \frac{1}{\lambda + \nu} + \frac{1}{\lambda} = \frac{1}{\lambda}\left(\frac{2+\kappa}{1+\kappa}\right) \quad \text{– without renewal;}$$

$$\mathbf{E}[\hat{T}] = \frac{2+\gamma+\kappa}{\lambda(1+\kappa)} = \frac{1}{\lambda}\left(\frac{2+\gamma+\kappa}{1+\kappa}\right) \text{– with renewal.}$$

Thus, the renewal coefficient γ shows the efficiency of the renewable system with respect to those of the non-renewable system.

Note that expressions [1.136] and [1.137] for the c.d.f. of the time up to the first system failure and for reliability function as well as the mean values of the warm redundant systems with and without renovation are also applicable for the hot and cold redundant systems. Only the characteristic equation [1.135] and appropriate values [1.134] of its roots are changed.

1.7.5. *Exercises*

EXERCISE 1.31.– Find the c.d.f. of the time $\hat{T}$ between failures for the warm double redundant system.

EXERCISE 1.32.– Consider the cold double redundant system. Each component of the system consists of two units in series. The units have constant h.r.f. $\lambda_1 = 1 \cdot 10^{-2}\, hours^{-1}$ and $\lambda_2 = 3 \cdot 10^{-2}\, hours^{-1}$. After the failure, the system is renewed and the renewal time can be neglected.

a) Find the mean number of failures during $t = 100$ hours,

b) Draw the graphs of failure intensity (mean number of failure per unit of time) $h(t)$ versus working time t

Answer:

$$H(100) = \frac{\lambda t}{2} - \frac{1}{4} + \frac{1}{4}e^{-2\lambda t} \approx 1.75\, failure;$$

$$h(t) = \frac{\lambda}{2}(1 - e^{-2\lambda t}),$$

where $\lambda = \lambda_1 + \lambda_2$.

The graph of the failure intensity $h(t)$ is shown in Figure 1.37.

EXERCISE 1.33.– Using the conditions of exercise 1.32:

a) find the mean number of cycles "failure + renovation" of the system during time t if the summary renovation time of both components has an exponential c.d.f. with parameter $\mu = 0.3\, hours^{-1}$;

b) draw the graphs of the cycles' mean number per unit of time versus working time t.

Answer:

$$H(t) = \frac{\lambda\,\mu\,t}{2\,\mu + \lambda} - \frac{\mu\,(\mu + 2\,\lambda)}{(2\,\mu + \lambda)^2} - \frac{\lambda^2\,\mu}{(s_2 - s_1)}\left(\frac{e^{-s_2\,t}}{s_2^2} - \frac{e^{-s_1\,t}}{s_1^2}\right);$$

$$h(t) = \frac{\lambda\,\mu}{2\,\mu + \lambda} + \frac{\lambda^2\,\mu}{s_2 - s_1}\left(\frac{e^{-s_2\,t}}{s_2} - \frac{e^{-s_1\,t}}{s_1}\right);$$

$H(100) \approx 1.60\, cycles$,

$h(100) \approx 0.0187\, cycles/hour$,

where $s_{1,2} = \frac{1}{2}\left(2\,\lambda + \mu \pm \sqrt{\mu^2 - 4\,\lambda\,\mu}\right)$ and $\lambda = \lambda_1 + \lambda_2 = 0.04\, hours^{-1}$.

The graph of the function $h(t)$ is shown in Figure 1.37.

EXERCISE 1.34.– Solve the previous exercise if components are renovated alternatively (in two stage).

Answer:

$$H(t) = \frac{\lambda\,\mu\,t}{2\,(\lambda + \mu)} - \frac{\lambda\,\mu}{2\,(s_2 - s_1)}\left(\frac{e^{-s_2\,t}}{s_2} - \frac{e^{-s_1\,t}}{s_1}\right) - \lambda\,\mu\,\frac{1 - e^{-(\lambda+\mu)t}}{2\,(\lambda + \mu)^2} - \frac{1}{4};$$

$$h(t) = \frac{\lambda\,\mu}{2}\left(\frac{e^{-s_2\,t} - e^{-s_1\,t}}{s_2 - s_1} + \frac{1 - e^{-(\lambda+\mu)t}}{\lambda + \mu}\right);$$

$H(100) \approx 1.46\, cycles$;

$h(100) \approx 0.0176\, cycles/hour$,

where $s_{1,2} = \frac{1}{2}\left(\lambda + \mu \pm \sqrt{\lambda^2 - 6\lambda\mu + \mu^2}\right)$ and $\lambda = \lambda_1 + \lambda_2$.

EXERCISE 1.35.– Under the conditions of exercise 1.32, consider the case of two redundant components. Draw the graph of mean number of failures per unit of time versus system working time t.

Answer:

$$H(t) = \frac{1}{3}\left(\lambda t - 1 + e^{-\frac{3}{2}\lambda t}\left(\cos\alpha + \frac{1}{\sqrt{3}}\sin\alpha\right)\right);$$

$$h(t) = \frac{\lambda}{3}\left(1 - e^{-\frac{3}{2}\lambda t}\left(\cos\alpha + \sqrt{3}\sin\alpha\right)\right);$$

$$H(100) \approx 1\, failure;$$

$$h(100) \approx 0.0134\, failures/hour,$$

where $\lambda = \lambda_1 + \lambda_2$, $\alpha = \dfrac{\lambda t\sqrt{3}}{2}$.

Graph of the function $h(t)$ versus the system working time is shown in Figure 1.37.

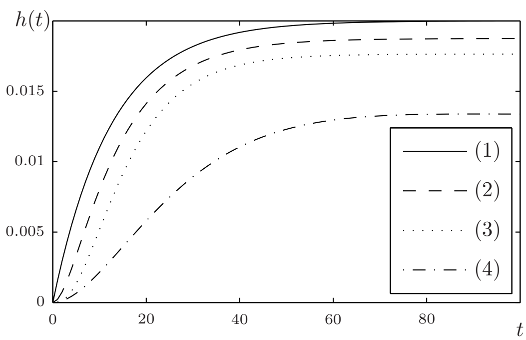


Figure 1.37. *The graphs of functions $h(t)$ for exercises 1–4: (1) – instant renovation (1 redundant); (2) – one-stage renovation (1 redundant); (3) – two-stage renovation (1 redundant); (4) – instant renovation (2 redundant)*

EXERCISE 1.36.– Calculate the availability coefficient K_{av} and the mean number to the first failure for hot double redundant system if:

a) there exists only one repair facility,

b) there exist two repair facilities.

1.8. Bibliographical comments

1.8.1. *Section 1.1*

The basic notions considered in this book correspond to the usual reliability theory terminology and most of the literature sources on reliability. Some definitions of this section follow the book [GNE 65]. The families of parametric c.d.f. of non-negative r.v.'s in most of the books on reliability theory are considered (see [BAR 75, GER 00]). The Gnedenko–Weibull distribution was first introduced by Frechet in 1927, and used for description of particle sizes in 1933. It was considered in detail by Weibull in 1951. However, in 1949, Gnedenko obtained this distribution as a limiting for maximum series independent r.v.'s. [GNE 49].

Some of the exercises for this and other sections of this chapter are based on the problems from the book [POL 72].

1.8.2. *Section 1.2*

The material about renewal processes follows the book of Cox [COX 61] and the classical paper of Smith [SMI 58]. Theorems about Large Number Law and Central Limit Theorems for the sums of i.i.d. r.v.'s can be found in any course of the Probability Theory. Appropriate theorems for the renewal processes as well as renewal theorems and theorems about processes of age and residual time are also discussed in detail in [COX 61, SMI 58] and other books.

1.8.3. *Section 1.3*

In this short review, we mostly follow the book [GNE 65]. However, the problem of statistical investigations for the system reliability needs a special (and more wide) consideration. The new direction of the investigation connected with the accelerated trials and appropriate data elaboration is a very interesting and useful topic in the reliability statistic. This approach is studied in the books and papers of Bagdonavicius and Nikulin (see [BAG 02]).

1.8.4. *Section 1.4*

Structural reliability is one of the most popular topics of reliability theory, which is represented in almost all the books on reliability. The material of this section partially follows I. Gertsbakh [GER 00]. The monotone structures and their properties were first summarized in [BAR 75]. Some of the exercises are taken from [POL 72].

1.8.5. *Section 1.5*

First, the notion of the event tree was introduced in the early 1960s by H.A. Watson from Bell Laboratory for the system reliability analysis.The failures of the elements, subsystems and the whole system are considered as events. Due to this, the notion *failure tree* appears. Later, analogous methods were also used for risk analysis in technique, medicine, finance, insurance and others processes.

For any event tree, especially for the life tree, constructing the special notations of different types of events and connections was proposed, and here they arepresented in tables (Tables 1.1 and 1.2) following the book of Henley and Kumamoto [HEN 91].

1.8.6. *Sections 1.6 and 1.7*

The material of these sections is also the traditional topic for reliability theory, which can be found in most books on reliability theory (see [GNE 65, GER 00] and other books). The Markov process is part of the stochastic process theory and all its necessary information can be found in any books on the Stochastic Processes (see [CHU 60, ROS 96, SER 09]. For the application of the Markov process to the redundant system, see also [GNE 65].