# 1

# Location of Solid Bodies

Mechanics is the science of relations between:

– the motion of an arbitrary physical system $(\mathscr{D})$ in the system of reference $\langle \lambda \rangle$ from which it is observed;

– and the forces that act upon it.

But the motion of a physical system in space is a very relative notion, which depends in an essential manner on the observer. Before any development concerning the physical system, it is, therefore, fundamental to locate it relative to its observer. This is the focus of the present chapter, in which two considerations will be examined, namely how to represent it in order to locate it, and then how to determine its location.

For all that follows, the physical system considered is the solid body, which is a continuous and rigid (this latter notion will be subsequently specified) material set.

## 1.1. The notion of system of reference

The system of reference is used in mechanics to set a reference, based on which it is possible to locate a mechanical element (solid body or part of an elastic medium), then to observe the evolution of its location and study its motion.

At the terrestrial scale, the favored domain of mechanics is the physical space seen from a geometrical perspective, as an affine space of a three-dimensional vector space, represented by a *basis* consisting of three independent vectors; this representation is important because any vector in the geometrical space of terrestrial mechanics can be expressed as a linear combination of these three vectors of the basis.

The principle of the reference system of an element located in this space thus relies on two simultaneous measurements:

– that of the distance between this element and a point in space considered as fixed reference for the motion examined;

– that of the orientation of the segment "origin-element" relative to the three independent directions that form the reference trihedron.

The set comprising the point of reference and the system of three independent axes forms a system of reference.

## 1.2. Frame of reference

### 1.2.1. *Setting up a frame of reference*

When a system of reference is used for the study of a motion, it is also called a frame of reference. This notion is of particular importance in the study of motion, since its choice depends on:

– the observer, meaning the context in which the motion evolves;

– the forces that act on the mechanical system under study, as the set of these forces should be inside the environment of reference, in order to be able to apply to this motion the principles that govern it.

#### 1.2.1.1. *Choosing the elements of a frame of reference*

In principle, the frame of reference used to locate a solid body and follow its evolution should consist of elements that are independant of the body and of the motion to be described. When the latter takes place in the terrestrial environment, the ideal frame of reference meets the following conditions:

– the point of reference chosen as origin of the system of reference is the center of inertia of the solar system.

– its independent directions are symbolized by three axes that are defined by three stars considered to be fixed relative to the motion of the Earth.

It is the solar frame of reference, noted $\langle g \rangle$, which ultimately illustrates the fundamental notion of the Galilean system of reference, which will be specified later on in this chapter.

This frame of reference is, however, poorly suited for the vast majority of motions taking place in the terrestrial environment, if only because of the measurement of the distance "origin-element", which fixes a scale that is completely disproportionate relative to the dimensions of the body whose motion is studied or to its field of evolution.

Moreover, since the Earth is moving relative to this system of reference, its axes, which would provide three independent directions, cannot be considered fixed for the motion studied. If we resorted to such a frame of reference, we would constantly be in *relative motion*, which would further complicate the formulation of the problem even though, for most of the motions studied in the terrestrial space, the *drive* terms are negligible compared with the *relative* ones; but this is not the case for slow motions such as those of icebergs or the continental drift.

Other systems of reference should, therefore, be considered. An example could be a system of reference linked to the Earth, having the origin in its center and the axes alongside three fixed directions of the terrestrial globe. But, this would pose the same scale difficulty mentioned above. Consequently, the study of the motion of a body requires a frame of reference adapted to its field of evolution, meaning a point in this field whose fixed position does not depend on the motion, so that it can serve as the origin, and three axes that are in their turn considered independent.

We shall subsequently see that these conditions for locating a moving body are not sufficient for the application without precautions of what constitutes the basis of the mechanical expression of solids, the *fundamental law of dynamics*, which is essentially expressed in the Galilean frame of reference.

#### 1.2.1.2. *Units of measurement*

The study of the motion of a body begins by locating it in the frame of reference chosen as best adapted to its evolution and, among other things, by expressing the "origin-element" distance according to the three directions of reference. It is, therefore, necessary to define the origin of the system of reference and a unit of measurement in each direction.

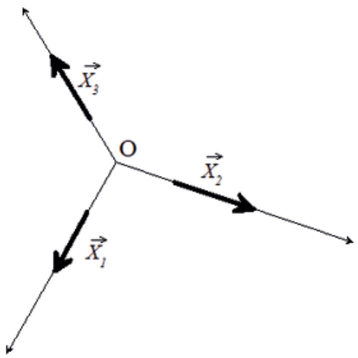



**Figure 1.1.** *Reference trihedron*

The three directions, defined in general by three independent vectors $\left(\overrightarrow{X_1}, \overrightarrow{X_2}, \overrightarrow{X_3}\right)$ – which are neither all three of them coplanar, nor at least two of them collinear – are then represented as oriented axes with a common origin, thus forming a trihedron.

#### 1.2.1.3. *Normed system of reference*

A vector is associated with each of these axes, and its norm can serve as a unit for measuring the projection of the "origin-element" vector on the axis. It is desirable, and even advisable, that measurements performed along each axis should be comparable, in the sense that the same unit should be taken on each axis unless otherwise required; in this case, the axes are characterized by the following unit vectors, respectively:

$$\vec{x}_1 = \frac{\overrightarrow{X_1}}{\left\|\overrightarrow{X_1}\right\|}; \quad \vec{x}_2 = \frac{\overrightarrow{X_2}}{\left\|\overrightarrow{X_2}\right\|}; \quad \vec{x}_3 = \frac{\overrightarrow{X_3}}{\left\|\overrightarrow{X_3}\right\|},$$

where $\vec{x}_1^{\,2} = \vec{x}_1 \cdot \vec{x}_1 = 1; \quad \vec{x}_2^{\,2} = \vec{x}_2 \cdot \vec{x}_2 = 1; \quad \vec{x}_3^{\,2} = \vec{x}_3 \cdot \vec{x}_3 = 1$.

In this case, the system of reference or the frame of reference, denoted as $\left\langle \mathrm{O} \mid \vec{x}_1\ \vec{x}_2\ \vec{x}_3 \right\rangle$, is said to be *normed*.

### 1.2.1.4. *Orthogonal system of reference*

The measure of the "origin-element" segment is expressed in units defined below by projecting this segment on the three axes of the system of reference; the direction along which the projection is made should be chosen for each axis. This choice should preferably lead to setting simple and direct relations between these projections and the projected quantity.

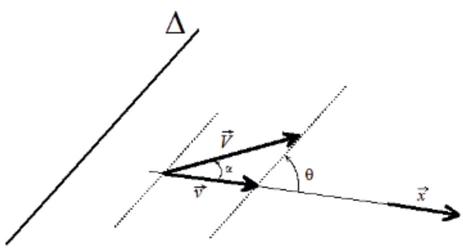



**Figure 1.2.** *Projection parallel to an axis*

Let us consider the plane defined by a vector $\vec{V}$ and an axis of unit vector $\vec{x}$, with an angle $\alpha$ between these two vectors; let us further consider a direction $\Delta$ at an angle $\theta$ with the axis. According to the above diagram, the projection of vector $\vec{V}$ on this axis, parallel to $\Delta$, leads to a quadratic relation between the modules of vector $\vec{V}$ and its projection $\vec{v}$. Based on relations in general and right triangles, the following formula can be established

$$\left\|\vec{v}\right\|^2 - 2\left\|\vec{V}\right\| \times \left\|\vec{v}\right\| \cos\alpha + \left\|\vec{V}\right\|^2 \left(1 - \frac{\sin^2\alpha}{\sin^2\theta}\right) = 0 \text{, where}$$

$$\left\|\vec{v}\right\| = \left\|\vec{V}\right\| \cos\alpha ,$$

which is in line with the result provided by trigonometry for orthogonal projection $(\theta = \pi / 2)$, though not quite convenient to use.

Since, the conditions in which vectors are projected on the three axes of a system of reference should preferably be simplified and standardized, the choice will go as a rule (except for very specific cases that are not the object of this work) for a frame of reference with three mutually orthogonal directions that form a trirectangular trihedron; the orthogonal projections of the "origin-element" vector are easily expressed using the basic trigonometric functions of a right triangle.

Such a system of reference is called *orthogonal*; it is characterized by three vectors $\overrightarrow{x_1}$, $\overrightarrow{x_2}$, $\overrightarrow{x_3}$ that are connected by the relations

$$\overrightarrow{x_1} \cdot \overrightarrow{x_2} = 0, \quad \overrightarrow{x_2} \cdot \overrightarrow{x_3} = 0, \quad \overrightarrow{x_3} \cdot \overrightarrow{x_1} = 0$$

### 1.2.1.5. *Direct orthonormal system of reference – properties*

In conclusion to the above analyses, a system of reference that is adequate for studying the motion of bodies is at the same time normed and orthogonal, i.e. *orthonormal.*

But, in order to accurately account for the motion of bodies, their location should be perfected by taking into consideration various facets of their motion. As a rule, we decompose it into two essential elementary phases whose combination gives the motion of bodies: translation and rotation. While translation can be readily represented by a vector that defines its direction and orientation, the axis about which rotation takes place and its sense should be specified.

It is, therefore, important to specify how we define orientation in the previously introduced system of three axes, and it does not take an oenologist to memorize, and all the more so to use, Maxwell's corkscrew rule, which is presented below.

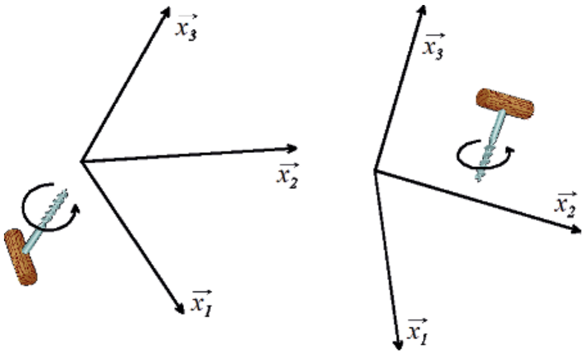



**Figure 1.3.** *Illustration of how Maxwell's corkscrew rule is applied*

Being placed perpendicularly to the plane $\Pi\left(\overrightarrow{x_1},\overrightarrow{x_2}\right)$, the corkscrew is rotated in the sense defined by the oriented right angle $\widehat{\overrightarrow{x_1},\overrightarrow{x_2}}$; the orientation of its advance then defines the orientation of the third axis $\overrightarrow{x_3}$ of the system of reference.

Such a system of reference is called *direct*. Under these conditions, the normed and mutually orthogonal basis vectors satisfy the three vector relations

$$\overrightarrow{x_1} \wedge \overrightarrow{x_2} = \overrightarrow{x_3}\ , \quad \overrightarrow{x_2} \wedge \overrightarrow{x_3} = \overrightarrow{x_1}\ , \quad \overrightarrow{x_3} \wedge \overrightarrow{x_1} = \overrightarrow{x_2}$$

The direct orthonormal system of reference is commonly used in solid mechanics, with the exception of some particular cases that would require the use of other types of systems of reference. In particular, this choice makes it easier to use vector calculus because of the standardization of the interpretation of its results. It is also advisable to have a clear summary of the properties that characterize the direct orthonormal system of reference, namely the three relations previously introduced

$$\begin{cases} \overrightarrow{x_1}^2 = 1, & \overrightarrow{x_2}^2 = 1, & \overrightarrow{x_3}^2 = 1 \\ \overrightarrow{x_1} \cdot \overrightarrow{x_2} = 0, & \overrightarrow{x_2} \cdot \overrightarrow{x_3} = 0, & \overrightarrow{x_3} \cdot \overrightarrow{x_1} = 0 \\ \overrightarrow{x_1} \wedge \overrightarrow{x_2} = \overrightarrow{x_3}, & \overrightarrow{x_2} \wedge \overrightarrow{x_3} = \overrightarrow{x_1}, & \overrightarrow{x_3} \wedge \overrightarrow{x_1} = \overrightarrow{x_2} \end{cases}$$

### 1.2.1.6. *Symbolic formulation*

The first two sets of the above relations can be symbolically written as a single expression

$$\overrightarrow{x_i} \cdot \overrightarrow{x_j} = \delta_{ij},$$

and the last one as

$$\overrightarrow{x_i} \wedge \overrightarrow{x_j} = \varepsilon_{ijk} \overrightarrow{x_k},$$

where the indices $i, j, k$ take the three values $1, 2 \text{ or } 3$ and the terms $\delta_{ij}$ and $\varepsilon_{ijk}$ are defined as follows:

– the Kronecker symbol $\delta_{ij}$ defined by

$$\begin{matrix} \delta_{ij} = 1 & \text{if } i = j \\ \delta_{ij} = 0 & \text{if } i \neq j \end{matrix};$$

– $\varepsilon_{ijk}$, the three-index permutation symbol defined by:

- $\varepsilon_{ijk} = 1$ if the values of three indices $i, j, k$ are not a circular permutation of the ordered values of the triplet 1,2,3,

- $\varepsilon_{ijk} = -1$ if the values of three indices $i, j, k$ are a circular permutation of the ordered values of the triplet 1,3,2,

- $\varepsilon_{ijk} = 0$ if the values of at least two of the indices $i, j, k$ are equal.

### 1.2.2. *Various types of frames of reference*

The expression of the equations that reflect the fundamental law of dynamics depends on the frame of reference relative to which the motion is considered; corrective terms can be introduced to account for the real environment in which the observer is located and for the environment in which the observed motion unfolds, and for the influence that these environments can have on the observed motion.

Up until the 18th Century, mechanics was the domain of astronomers and mathematicians who were essentially interested in the study of celestial bodies. Newton's second law or the *fundamental law of dynamics* aimed at expressing the motion of planets in the solar system under the action of forces of attraction exerted by the other celestial bodies. Akin to celestial mechanics, which relied heavily on time measurement, the field of watchmaking was also rapidly expanding, owing to the study of pendular motions.

The industrial development during the 19th Century and the invention of the machine as the main production tool led the mathematicians and physicists into studying the mechanics of systems. The fundamental law had already been formulated, but it had to be applied to a different environment, as the top priority was the local observation of motion, and this went beyond the field of watchmaking, which had nevertheless opened the path.

The study of motion had to take into consideration other external forces in addition to those exerted by the gravitational field; they were either acting at a distance, such as the electromagnetic forces, or by contact, when the physical system in motion was subject to stresses, or still when there were physical links between the system and its external environment or among the components of the system.

Foucault's works, most notably those related to the gyroscope and pendulum motion, have revealed the important role played by the choice of the system of reference. In particular, they have shown that when a system was isolated from its close environment and, more specifically, freed from the actions that the environment could exert on it, the observed motion of the gyroscope or pendulum could not be explained in a system of reference linked to the place of observation.

In the case of the pendulum, the locally observed rotation of its oscillation plane could only be explained when the description of its motion took into account the motion of the Earth.

The pendulum devised by Foucault, and used in the demonstration he conducted in the Pantheon in 1857, consisted of a heavy sphere hanging from a high ceiling on a 67 m long silver wire in such a way that the disruption of the pendulum oscillation because of the stress exerted by its anchor would be minimal. In addition to the fact that the suspension wire was chosen to minimize the deformation stresses, the weight of the sphere was such that it served the same purpose. The pendulum was, therefore, isolated from where its motion apparently took place.

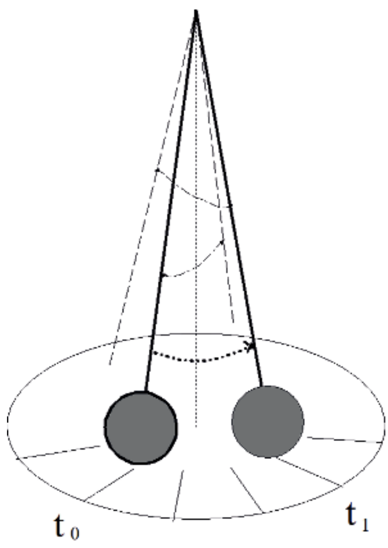



**Figure 1.4.** *The principle of Foucault's pendulum*

The pendulum was put into oscillation in a vertical plane located at the moment $t_0$ and during its pendular motion the behavior of its oscillation plane was observed, which indicated that it rotated about the vertical axis.

This observation can only be explained if the motion of the pendulum takes place in a system of reference that differs from the one of the observer linked to a local system of reference, and relative to which it rotates. But, the only rotation undergone by the local system of reference is the rotation of the Earth, to which it is connected; the observation indicates that the motion of the pendulum takes place in a system of reference with axes oriented independently of the local rotation, being, for example, collinear with the fixed axes of the solar frame of reference. This explanation could be corroborated by an adequate interpretation of the measurement of the angle of rotation of the pendulum oscillation plane during the time interval $[t_0, t_1]$.

In the mechanics it is, therefore, important to have a relevant classification of the systems of reference in order to conveniently express the consequences of the application of the fundamental law.

#### 1.2.2.1. *Galilean frame of reference*

The fundamental law of dynamics states that there is at least one geometric system of reference (called *Galilean*) and a system of time reference (*privileged time scale*) for which:

– *a material point* (that is a body whose dimensions are negligible at the scale of observations and whose motion can at each moment be assimilated to a translation) *which is not subjected to any force is moving with acceleration zero, therefore has a rectilinear and uniform motion;*

– a material point subjected to one or several forces has, at each instant, an acceleration that is proportional to the resultant of these forces;

– the coefficients of proportionality between acceleration and the resultant of forces are constants characteristic to various materials that can be considered. By definition, these constants are the masses.

This notion of the Galilean system of reference is of paramount importance in the formulation of the mechanics, so its meaning must be clearly determined, as it is intimately connected to the nature of the motion under consideration.

The choice of the Galilean system of reference, indispensable for the application of the fundamental laws of dynamics, depends on the motion to be observed. All the actuators susceptible to influence the motion of the mechanical system $(\mathscr{D})$ to be described and studied should be included in the system of reference, in the sense that it should be possible to define their location and describe their motion as well; and in the absence of these actions, the mechanical system considered should have a rectilinear and uniform motion.

The system is then said to be mechanically isolated for the motions described.

For the study of motions in the terrestrial environment, the ideal Galilean system of reference has its origin in the center $G$ of the solar system and the directions of its axes given by the three stars most distant from this system and whose angular position relative to an observer of the solar system would be considered invariable. It is worth noting here that the closest star to the Sun is already distant enough to accurately define an axis direction.

#### 1.2.2.2. *Pseudo-Galilean frame of reference*

The approach of most problems in mechanics requires that they should be expressed in such systems of reference that the orders of magnitude of the variables involved can be compared with the other orders of magnitude of the problem; furthermore, all the elements involved in the study of a motion should be part of it, and clearly identified and located. This point is particularly important, since it heavily influences the accuracy of results and their interpretation, in the case of very high or very low values, which are not related to the physical environment of the study, and in disproportion with the others.

The need for a frame of reference that is better adapted to solving a problem of mechanics has led to the introduction of a concept of the pseudo-Galilean system of reference, in order to apply the fundamental law of dynamics under more convenient conditions; this should, therefore, have properties comparable to those of the Galilean system of reference, and more specifically, a material point that is not subjected to any force in this system should also have a rectilinear uniform motion.

Ideally, this system of reference should, therefore, have fixed orientations of axes and undergo uniform translation relative to the Galilean system of reference; in reality, few practical systems of reference meet such criteria. This means that:

– to the extent that forces are inevitably acting, to account for the relative motion of the pseudo-Galilean system of reference relative to the Galilean system of reference, which is not strictly translation nor a rectilinear one,

– and because rotations inevitably generate accelerations, corrective terms are added to the expressions of the fundamental law.

But, if the orders of magnitude of these corrective terms generated by the motion of the working system of reference are negligible relative to those of the problem, and do not significantly influence the study of a motion, then the system of reference can be considered pseudo-Galilean and can serve as basic frame of reference for the application of the fundamental law.

#### 1.2.2.3. *Other local systems of reference*

The study of the motion of a body starts necessarily by determining its location in a properly chosen system of reference that allows for a correct description, and which in practice is almost never an ideal Galilean system of reference, and does not even meet the criteria of the pseudo-Galilean system of reference.

Such a system of reference is nevertheless critically important, as it allows for the practical location of the solid body whose motion is to be studied. But, it is obviously not adapted to the direct application of the fundamental law; it is, therefore, important to introduce in the formulation of this law the corrective terms that are essential for describing the motion proper to this system of reference relative to the frame of reference chosen for studying the motion.

Chapter 2 of this volume provides the information required for expressing these complementary terms.

## 1.3. Location of a solid body

In order to illustrate this statement, we shall consider the relative location of two rectangular parallelepipeds considered solid, and therefore impenetrable, $(S_1)$ and $(S_2)$.

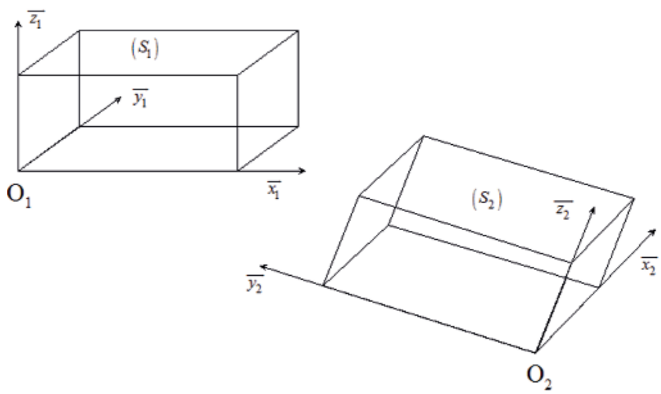



**Figure 1.5.** *The principle of locating a solid body*

To the vertex $O_1$ of $(S_1)$, we connect a direct orthonormal system of axes with the basis $\left(\overrightarrow{x_1}\ \overrightarrow{y_1}\ \overrightarrow{z_1}\right)$; to the vertex $O_2$ of $(S_2)$, we similarly connect a direct orthonormal system of axes $\left(\overrightarrow{x_2}, \overrightarrow{y_2}, \overrightarrow{z_2}\right)$ and its associated basis $\left(\overrightarrow{x_2}\ \overrightarrow{y_2}\ \overrightarrow{z_2}\right)$.

In order to locate $(S_2)$ relative to $(S_1)$, it is sufficient:

– to locate the position of $O_2$ relative to $O_1$ by the *position vector* $\overrightarrow{O_1O_2}$;

– to express the orientation of the $\left(\overrightarrow{x_2}\,\overrightarrow{y_2}\,\overrightarrow{z_2}\right)$ basis relative to $\left(\overrightarrow{x_1}\,\overrightarrow{y_1}\,\overrightarrow{z_1}\right)$.

NOTE 1.1.– While representing series of vectors inside brackets, in order to differentiate the notation of a sequence of vectors, for example the system $\left(\overrightarrow{x_1},\overrightarrow{y_1},\overrightarrow{z_1}\right)$ composed of these three vectors separated by comma, from the reference entity that forms a basis, in the rest of this work we shall use for the latter the notation $\left(\overrightarrow{x_1}\,\overrightarrow{y_1}\,\overrightarrow{z_1}\right)$ without separators between the vectors to express the basis associated with the solid $(S_1)$, for example, which will be written $(S_1) \equiv \left(\overrightarrow{x_1}\,\overrightarrow{y_1}\,\overrightarrow{z_1}\right)$.

Similarly, the system of reference associated with the solid $(S_1)$, consisting of the origin $\mathrm{O}_1$ and the basis $(S_1)$, is denoted $\langle S_1 \rangle \equiv \left\langle \mathrm{O}_1 \,|\, \overrightarrow{x_1}\,\overrightarrow{y_1}\,\overrightarrow{z_1} \right\rangle$.

### 1.3.1. *The principle of locating a solid*

In order to locate a solid $(S)$ in a system of reference $\langle \lambda \rangle$ – an essential condition for studying the motion in this system of reference – the solid should, therefore, be connected to a system of reference $\langle S \rangle$ that characterizes it. This implies choosing a fixed origin $\mathrm{O}_S$ in the solid and associating with it a direct orthonormal system of axes with the function of basis $(S) \equiv \left(\overrightarrow{x_S}\,\overrightarrow{y_S}\,\overrightarrow{z_S}\right)$.

### 1.3.2. *Location parameters of a solid*

The location parameters of a solid result from the above considerations; they are of two types, linear or angular when the point taken as origin of the system of reference $\langle S \rangle$ is located, which means that the components of its position vector are determined, angular when the system of reference needs to be oriented relative to

$\langle \lambda \rangle$, which serves as frame of reference for determining and expressing the location of the solid.

### 1.3.3. *Coordinates of the position vector*

Let us, therefore, consider a system of reference $\langle \lambda \rangle \equiv \left\langle \mathrm{O}_\lambda \,|\, \overrightarrow{x_\lambda}\ \overrightarrow{y_\lambda}\ \overrightarrow{z_\lambda} \right\rangle$ chosen for the study of motion and another system $\langle S \rangle \equiv \left\langle \mathrm{O}_S \,|\, \overrightarrow{x_S}\ \overrightarrow{y_S}\ \overrightarrow{z_S} \right\rangle$ connected to the solid. The components of the position vector $\overrightarrow{\mathrm{O}_\lambda \mathrm{O}_S}$ depend on the chosen representation.

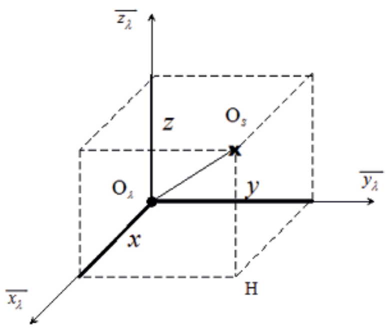



**Figure 1.6.** *The Cartesian coordinates*

#### 1.3.3.1. *Cartesian coordinates x,y,z*

$$\overrightarrow{\mathrm{O}_\lambda \mathrm{O}_S} = x\,\overrightarrow{x_\lambda} + y\,\overrightarrow{y_\lambda} + z\,\overrightarrow{z_\lambda}$$

#### 1.3.3.2. *Cylindrical or cylindrical-polar coordinates r,$\alpha$,z*

$$\overrightarrow{\mathrm{O}_\lambda \mathrm{O}_S} = r\,\overrightarrow{u}(\alpha) + z\,\overrightarrow{z_\lambda}$$

**H** being the orthogonal projection of **$\mathbf{O_S}$** on the plane $\Pi\left(\mathrm{O}_\lambda \,|\, \overrightarrow{x_\lambda}, \overrightarrow{y_\lambda}\right)$, that is

$\overrightarrow{\mathrm{O}_\lambda \mathrm{O}_S} = \overrightarrow{\mathrm{O}_\lambda \mathrm{H}} + \overrightarrow{\mathrm{H}\mathrm{O}_S}$ with $\overrightarrow{\mathrm{O}_\lambda \mathrm{H}} = x\,\overrightarrow{x_\lambda} + y\,\overrightarrow{y_\lambda}$ and $\overrightarrow{\mathrm{H}\mathrm{O}_S} = z\,\overrightarrow{z_\lambda}$, and $\vec{u}(\alpha)$ is the unit vector collinear with $\overrightarrow{\mathrm{O}_\lambda\mathrm{H}}$, oriented from $\mathrm{O}_\lambda$ to $\mathrm{H}$, or

$$\vec{u}(\alpha)=\frac{\overrightarrow{\mathrm{O}_\lambda\mathrm{H}}}{\left\|\overrightarrow{\mathrm{O}_\lambda\mathrm{H}}\right\|} \quad \text{with} \quad \left\|\overrightarrow{\mathrm{O}_\lambda\mathrm{H}}\right\|=r=\sqrt{x^2+y^2}>0,$$

then,

$$\vec{u}(\alpha)=\frac{x\overrightarrow{x_\lambda}+y\overrightarrow{y_\lambda}}{\sqrt{x^2+y^2}}=\cos\alpha\,\overrightarrow{x_\lambda}+\sin\alpha\,\overrightarrow{y_\lambda}.$$

The angle $\alpha$ is modulo $2\pi$ defined by the following two relations

$$\cos\alpha=\frac{x}{\sqrt{x^2+y^2}} \quad , \quad \sin\alpha=\frac{y}{\sqrt{x^2+y^2}}.$$

NOTE 1.2.– The notation $\vec{u}(\alpha)$ is conventional; it expresses the vector in the plane $\Pi\left(\overrightarrow{x_\lambda},\overrightarrow{y_\lambda}\right)$ which makes an oriented angle $\alpha$ with $\overrightarrow{x_\lambda}$, namely $\alpha\equiv\widehat{\vec{x}_\lambda,\vec{u}(\alpha)}$.

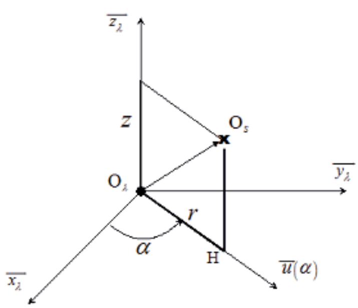



**Figure 1.7.** *Cylindrical-polar coordinates*

Likewise, we define the following vectors

$$\vec{u}\left(\alpha+\pi/2\right) \ , \ \vec{u}(\alpha+\pi)=-\vec{u}(\alpha) \ , \ \vec{u}\left(\alpha+3\pi/2\right)=-\vec{u}\left(\alpha+\pi/2\right),$$

according to Figure 1.8.

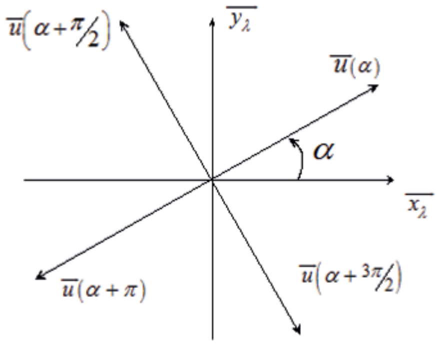



**Figure 1.8.** *Principle of the notation* $\vec{u}(\alpha)$

### 1.3.3.3. *Spherical coordinates R,α,β*

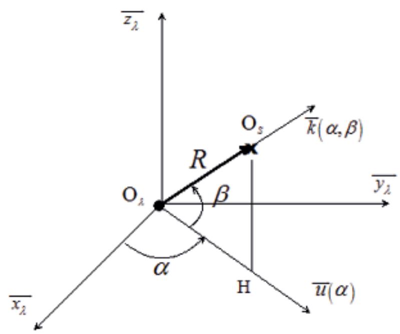



**Figure 1.9.** *Spherical coordinates*

$$\overrightarrow{O_\lambda O_S} = R\,\vec{k}(\alpha,\beta)$$

where, $\vec{k}(\alpha,\beta)$ is the unit vector collinear with $\overrightarrow{O_\lambda O_S}$ oriented from $O_\lambda$ to $O_S$, which is

$$\vec{k}(\alpha,\beta) = \frac{\overrightarrow{O_\lambda O_S}}{\left\|\overrightarrow{O_\lambda O_S}\right\|} \quad \text{with} \quad \left\|\overrightarrow{O_\lambda O_S}\right\| = R = \sqrt{x^2+y^2+z^2} > 0\,,$$

$$\Rightarrow \quad \vec{k}(\alpha,\beta) = \frac{x\,\overrightarrow{x_\lambda} + y\,\overrightarrow{y_\lambda} + z\,\overrightarrow{z_\lambda}}{\sqrt{x^2+y^2+z^2}} = \frac{x\,\overrightarrow{x_\lambda} + y\,\overrightarrow{y_\lambda}}{\sqrt{x^2+y^2+z^2}} + \frac{z\,\overrightarrow{z_\lambda}}{\sqrt{x^2+y^2+z^2}}.$$

The angle $\beta$ $\left(-\frac{\pi}{2} < \beta < \frac{\pi}{2}\right)$ is chosen as follows

$$\sin\beta = \frac{z}{\sqrt{x^2+y^2+z^2}} \quad \text{and} \quad \cos\beta = \sqrt{1-\sin^2\beta} = \frac{\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+z^2}} > 0.$$

Hence, we deduce

$$\frac{1}{\sqrt{x^2+y^2+z^2}} = \frac{\cos\beta}{\sqrt{x^2+y^2}},$$

and the expression of vector $\vec{k}(\alpha,\beta)$

$$\vec{k}(\alpha,\beta) = \cos\beta\,\frac{x\,\overrightarrow{x_\lambda} + y\,\overrightarrow{y_\lambda}}{\sqrt{x^2+y^2}} + \sin\beta\,\overrightarrow{z_\lambda}.$$

Using the relations established for $\alpha$ in cylindrical-polar coordinates, we obtain

$$\vec{k}(\alpha,\beta) = \cos\beta\left(\cos\alpha\,\overrightarrow{x_\lambda} + \sin\alpha\,\overrightarrow{y_\lambda}\right) + \sin\beta\,\overrightarrow{z_\lambda} = \cos\beta\,\vec{u}(\alpha) + \sin\beta\,\overrightarrow{z_\lambda}.$$

The relations that express the Cartesian coordinates $(x,y,z)$ as a function of spherical coordinates $(R,\alpha,\beta)$ are:

$$\begin{cases} x = R\cos\beta\cos\alpha \\ y = R\cos\beta\sin\alpha \\ z = R\sin\beta \end{cases}.$$

NOTE 1.3.– It is considered by convention that $\vec{K}(\alpha,\beta) = \vec{k}(\alpha,\beta+\pi/2)$.

### 1.3.3.4. *General coordinates*

It is also possible to locate the position in the system of reference $\langle \lambda \rangle$ of the point $O_S$ relative to $O_\lambda$ with an arbitrary system of three parameters $Q_1, Q_2, Q_3$, which can be expressed as

$$\overrightarrow{O_\lambda O_S} = \vec{F}(Q_1, Q_2, Q_3).$$

The use of this system of coordinates is of particular interest in software applications, when the formalism developed in this manner is designed to apply to a high number of problems and to offer the possibility of choosing the system of coordinates to be used.

An illustration of this case is provided by the example below, in which $O_\lambda$ is located in the middle of the segment PN. Two groups of three parameters are thus obtained, $(\alpha, r_1, r_2)$ and $(\alpha, \alpha_1, \alpha_2)$, and they both allow for locating the point $O_S$.

This type of coordinate, called bifocal, is used in particular in the space domain. The angle $\alpha_1$ represents the 1st azimuth of the $O_S$ to be located.

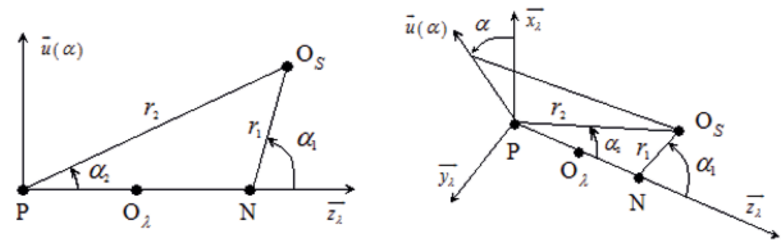



**Figure 1.10.** *Example of general coordinates*

## 1.3.4. *Exercises*

### 1.3.4.1. *Exercise 1 – Systems of coordinates*

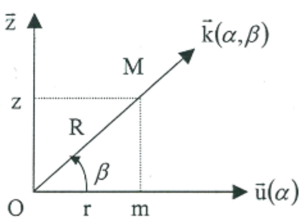
In the system of reference $\langle 0 \rangle \equiv \langle O | \vec{x}\, \vec{y}\, \vec{z} \rangle$, let M be a point identified by its spherical coordinates $(R, \alpha, \beta)$ which take the values

$$R = 10m,\ \alpha = \beta = \frac{\pi}{4} rad$$

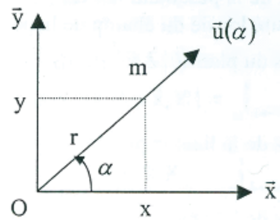
*Question*: Determine the cylindrical-polar coordinates $(R, \alpha, z)$ and the Cartesian coordinates $(x, y, z)$ of point M .

$$\vec{k}(\alpha, \beta) = \cos\beta\, \vec{u}(\alpha) + \sin\beta\, \vec{z} \text{ with } \vec{u}(\alpha) = \cos\alpha\, \vec{x} + \sin\alpha\, \vec{y},$$

$$\overrightarrow{OM} = R\,\vec{k}(\alpha, \beta) = r\,\vec{u}(\alpha) + z\,\vec{z} = x\,\vec{x} + y\,\vec{y} + z\,\vec{z}\,.$$



Meridian Plane



Equatorial plane

Hence, we deduce that

$$r = R\cos\beta = 10m \times \frac{\sqrt{2}}{2} = 5\sqrt{2}\,m$$

$$z = R\sin\beta = 5\sqrt{2}\,m$$

$$x = r\cos\alpha = 5\sqrt{2}\,m \times \frac{\sqrt{2}}{2} = 5m$$

$$y = r\sin\alpha = 5m$$

### 1.3.4.2. *Exercise 2 – System of coordinates*

In the system of reference $\langle 0 \rangle \equiv \left\langle O \middle| \vec{x}\ \vec{y}\ \vec{z} \right\rangle$, let M be a point identified by its Cartesian coordinates $(x, y, z)$ that take the values

$$x = 3m,\ \ y = -4m,\ \ \ z = 5m\,.$$

*Question*: Determine its cylindrical-polar coordinates $(R, \alpha, z)$ and then its spherical coordinates $(R, \alpha, \beta)$.

$$\overrightarrow{\text{OM}} = x\vec{x} + y\vec{y} + z\vec{z} = r\cos\alpha\,\vec{x} + r\sin\alpha\,\vec{y} + z\vec{z}$$

$$\Rightarrow \quad (r\cos\alpha)^2 + (r\sin\alpha)^2 = r^2 = x^2 + y^2; \quad \tan\alpha = \frac{y}{x}.$$

According to the above figure, we also have

$$R = \sqrt{r^2 + z^2}$$

$$\cos\beta = \frac{r}{R}, \quad \sin\beta = \frac{z}{R}.$$

Therefore, we obtain

$$r = \sqrt{9+16} = 5m, \tan\alpha = -\frac{4}{3} \quad \Rightarrow \quad \alpha = -53{,}13° = -0{,}92\,rad$$

$$R = \sqrt{25+25} = 5\sqrt{2}, \sin\beta = \frac{5}{5\sqrt{2}} = \frac{\sqrt{2}}{2} = \cos\beta \quad \Rightarrow \quad \beta = \frac{\pi}{4}.$$

## 1.4. Positioning of a system of reference connected to a solid

### 1.4.1. *Several examples of location systems of reference*

Before we approach this aspect of the question of locating a solid before any study of its motion, it may help to take a look at the systems of reference that could be used in the locating stage. In the section above we have sketched the configuration of three of them, directly connected to the common systems of coordinates described above.

In certain configurations of motion on a given curve or surface, one may tend to favor the description starting from local data that characterize it. This leads to using the attached local trihedrons. This is the case of the Frenet trihedron, which is locally defined in a point of a curve, or that of Darboux–Ribaucour trihedron, in the case of a surface.

### 1.4.1.1. *The Cartesian system of reference*

This is the system of reference $\left\langle O_\lambda | \overrightarrow{x_\lambda}\, \overrightarrow{y_\lambda}\, \overrightarrow{z_\lambda} \right\rangle$.

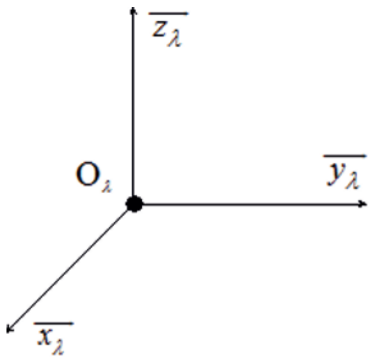



**Figure 1.11.** *The Cartesian system of reference*

### 1.4.1.2. *Cylindrical-polar system of reference*

This is the system of reference $\left\langle O_\lambda | \vec{u}(\alpha)\ \vec{u}\left(\alpha + \pi/_2\right)\ \overrightarrow{z_\lambda} \right\rangle$.

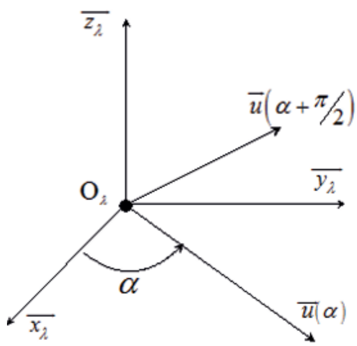



**Figure 1.12.** *Cylindrical-polar system of reference*

### 1.4.1.3. *Spherical system of reference*

This is the system of reference $\left\langle O_\lambda | \vec{k}(\alpha,\beta)\ \vec{u}\left(\alpha + \pi/_2\right)\ \vec{K}(\alpha,\beta) \right\rangle$.

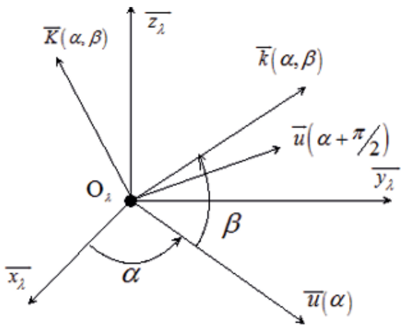



**Figure 1.13.** *Spherical system of reference*

### 1.4.1.4. *Frenet trihedron*

Defined in a point M of a curve $\Gamma$, the Frenet trihedron consists of three unit vectors forming a basis, with which two scalar quantities are associated, which define the curve in that given point. These elements are defined as follows: their presentation will be completed in the chapter dedicated to left curved lines in Volume 2 [BOR 16a].

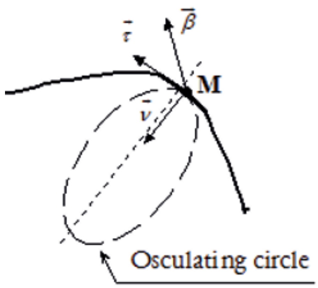



**Figure 1.14.** *Diagram of the Frenet trihedron*

– $\vec{\tau}$ unit vector tangent in M to the curve $\Gamma$;

– in a first approximation, if in the vicinity of the point M the curve is equivalent to an arc of circle of radius $R$ (called radius of curvature), when passing to the limit, the length of the arc tending to 0, the limit plane that contains this arc is the osculating plane;

– $\vec{\nu}$ unit vector orthogonal to $\vec{\tau}$ in M, contained in the osculating plane and oriented from point M to the curvature center of the arc;

– $\vec{\beta}$ unit vector orthogonal in M to the osculating plane $\Pi\left(\mathrm{M}|\vec{\tau},\vec{\nu}\right)$, such that $\vec{\beta}=\vec{\tau}\wedge\vec{\nu}$;

– $\left\langle \mathrm{M}|\vec{\tau}\,\vec{\nu}\,\vec{\beta}\right\rangle$ is the local Frenet system of reference.

### 1.4.1.5. *Darboux–Ribaucour trihedron*

A surface is represented by a vector function of two variables $\vec{f}(q_1,q_2)$. If one set of values is fixed for the parameter $q_1$, then to each one corresponds a curve $\Gamma_{q_2}$ along which evolves the parameter $q_2$; similarly, if we fix a set of values for $q_2$, then to each one corresponds a curve $C_{q_1}$.

To each point M on the surface $S$ studied, there corresponds a pair of values $(q_1,q_2)$, and consequently a curve $C_{q_1}$ and a curve $\Gamma_{q_2}$, a vector $\vec{\tau_1}$ tangent in M to the former and a vector $\vec{\tau_2}$ tangent in M to the latter. We then consider the vector $\vec{n}$, perpendicular to the surface $S$, such that $\vec{n}=\vec{\tau_1}\wedge\vec{\tau_2}$.

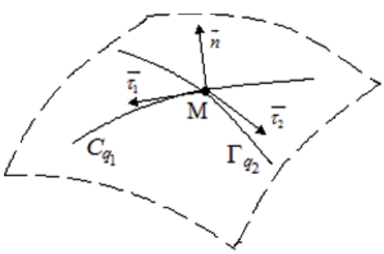



**Figure 1.15.** *Diagram of the Darboux–Ribaucour trihedron*

$\left\langle \mathrm{M}|\vec{\tau_1}\,\vec{\tau_2}\,\vec{n}\right\rangle$ is the local Darboux–Ribaucour system of reference.

The establishment of Frenet and Darboux–Ribaucour trihedrons will be further elaborated in the respective chapters of Volume 2 [BOR 16a] in relation to vector functions of 1 or 2 variables.

### 1.4.2. *General location parameters*

In order to define a principle for establishing a set of parameters representative for their relative positioning, let us consider two direct orthonormal bases $(x) \equiv \left(\overrightarrow{x_1}\,\overrightarrow{x_2}\,\overrightarrow{x_3}\right)$ and $(X) \equiv \left(\overrightarrow{X_1}\,\overrightarrow{X_2}\,\overrightarrow{X_3}\right)$. By assigning them to the same origin $\mathrm{O}$, we shall attempt to locate one relative to the other, by means of three angular parameters $Q_4, Q_5, Q_6$, the two systems of reference $\left\langle \mathrm{O} \middle| \overrightarrow{x_1}\,\overrightarrow{x_2}\,\overrightarrow{x_3} \right\rangle$ and $\left\langle \mathrm{O} \middle| \overrightarrow{X_1}\,\overrightarrow{X_2}\,\overrightarrow{X_3} \right\rangle$ in the general case when the two systems of axes are completely distinct $\left(\overrightarrow{X_i} \neq \overrightarrow{x_j}, \forall i, j = 1,2 \text{ or } 3\right)$.

Let $\Pi\left(\overrightarrow{X_i}, \overrightarrow{x_j}\right)$ be a plane defined by the two vectors $\overrightarrow{X_i}$ and $\overrightarrow{x_j}$, and the line $\Delta_{ij}$ orthogonal to this plane, of unit vector $\overrightarrow{n_{ij}}$. The angle $Q_5$ formed by the two vectors $\overrightarrow{X_i}$ and $\overrightarrow{x_j}$ is defined by the relation

$$\overrightarrow{X_i} \wedge \overrightarrow{x_j} = \overrightarrow{n_{ij}} \sin Q_5 .$$

The following diagram indicates the choice of angles.

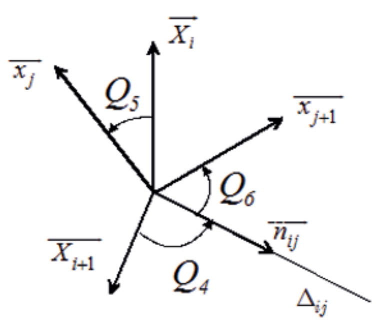



**Figure 1.16.** *Orientation of one basis relative to the other*

NOTE 1.4.– The indices i and j of various unit vectors of the two considered bases vary strictly according to a cyclic permutation of the numbers 1, 2 and 3 (for example, if j = 3, j + 1=1).

The transformation that leads from the basis $(X)$ to the basis $(x)$ results from three successive rotations:

– rotation by angle $Q_4 = \widehat{\overrightarrow{X_{i+1}}, \overrightarrow{n_{ij}}}$ about the axis $\overrightarrow{X_i}$. It is said to be of axis $\overrightarrow{X_i}$;

– rotation by angle $Q_5 = \widehat{\overrightarrow{X_i}, \overrightarrow{x_j}}$ of axis $\overrightarrow{n_{ij}}$;

– rotation by angle $Q_6 = \widehat{\overrightarrow{n_{ij}}, \overrightarrow{x_{j+1}}}$ of axis $\overrightarrow{x_j}$.

It can be noted that the transformation introduces two intermediary bases $(1)$ and $(2)$ to get from $(X)$ to $(x)$. By setting $\overrightarrow{v_{ij}} = \overrightarrow{X_i} \wedge \overrightarrow{n_{ij}}$ and $\overrightarrow{\mu_{ij}} = \overrightarrow{x_j} \wedge \overrightarrow{n_{ij}}$, the two intermediary bases are

$$\begin{cases} (1) \equiv \left( \overrightarrow{n_{ij}}\ \overrightarrow{v_{ij}}\ \overrightarrow{X_i} \right) \\ (2) \equiv \left( \overrightarrow{n_{ij}}\ \overrightarrow{\mu_{ij}}\ \overrightarrow{x_j} \right) \end{cases},$$

according to the three successive passage schemes given in Figure 1.17.

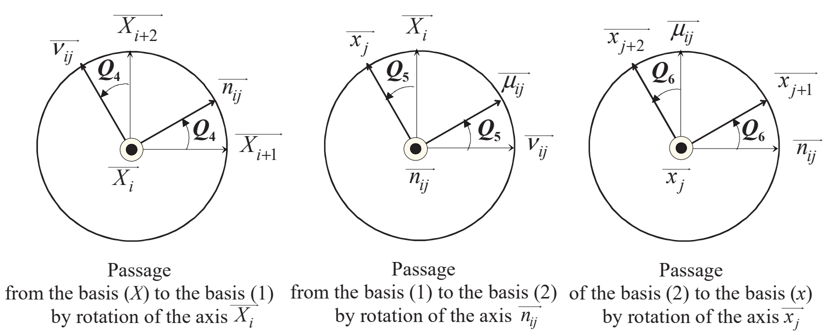



**Figure 1.17.** *General diagram of the passage from one basis to another*

By successive projections in the basis $(1)$, then $(2)$, then $(x)$, the formulation of the projection of this latter basis on the basis $(X)$ is obtained.

$$\begin{cases} \overrightarrow{x_j} = \sin Q_4 \sin Q_5 \overrightarrow{X_{i+1}} - \cos Q_4 \sin Q_5 \overrightarrow{X_{i+2}} + \cos Q_5 \overrightarrow{X_i} \\ \overrightarrow{x_{j+1}} = (\cos Q_4 \cos Q_6 - \sin Q_4 \cos Q_5 \sin Q_6) \overrightarrow{X_{i+1}} \ldots \\ \qquad \ldots + (\sin Q_4 \cos Q_6 + \cos Q_4 \cos Q_5 \sin Q_6) \overrightarrow{X_{i+2}} + \sin Q_5 \sin Q_6 \overrightarrow{X_i}. \\ \overrightarrow{x_{j+2}} = (-\cos Q_4 \sin Q_6 - \sin Q_4 \cos Q_5 \cos Q_6) \overrightarrow{X_{i+1}} \ldots \\ \qquad \ldots + (-\sin Q_4 \sin Q_6 + \cos Q_4 \cos Q_5 \cos Q_6) \overrightarrow{X_{i+2}} + \sin Q_5 \cos Q_6 \overrightarrow{X_i} \end{cases}$$

### 1.4.3. *Euler angles*

The question is now to apply the above described principle to define the location of a solid $(S)$ to which the basis $\left(\overrightarrow{x_S}\ \overrightarrow{y_S}\ \overrightarrow{z_S}\right)$ is associated, which has to be located relative to a reference basis $\left(\overrightarrow{x_\lambda}\ \overrightarrow{y_\lambda}\ \overrightarrow{z_\lambda}\right)$.

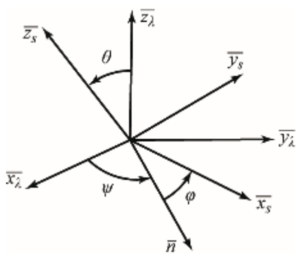



**Figure 1.18.** *Euler angles diagram*

Commonly used Euler angles have specific names; they are defined as follows:

$$\overrightarrow{X_i} \equiv \overrightarrow{z_\lambda} \quad \Rightarrow \quad \overrightarrow{X_{i+1}} \equiv \overrightarrow{x_\lambda} \quad , \quad \overrightarrow{X_{i+2}} \equiv \overrightarrow{y_\lambda} \ ,$$
$$\overrightarrow{x_j} \equiv \overrightarrow{z_S} \quad \Rightarrow \quad \overrightarrow{x_{j+1}} \equiv \overrightarrow{x_S} \quad , \quad \overrightarrow{x_{j+2}} \equiv \overrightarrow{y_S} \ .$$

This leads to identifying the angles as

$Q_4 \equiv \psi$ *precession angle*

$Q_5 \equiv \theta$ *nutation angle,*

$Q_6 \equiv \varphi$ *spin angle,*

according to Figure 1.18, and defining them by:

$$\begin{cases} \overrightarrow{z_\lambda} \wedge \overrightarrow{z_S} = \vec{n}\sin\theta \\ \overrightarrow{z_\lambda} \cdot \overrightarrow{z_S} = \cos\theta \end{cases} \qquad \begin{cases} \overrightarrow{x_\lambda} \cdot \vec{n} = \cos\psi \\ \overrightarrow{y_\lambda} \cdot \vec{n} = \sin\psi \end{cases} \qquad \begin{cases} \vec{n} \cdot \overrightarrow{x_S} = \cos\varphi \\ \vec{n} \cdot \overrightarrow{y_S} = -\sin\varphi \end{cases},$$

where $\vec{n}$ is the *nodal vector*.

NOTE 1.5.– The previous angular representation has a limitation in the processes used in numerical calculation. Indeed, when the angle $\theta$ (or $Q_5$) is very small, which happens when the planes $\Pi\left(\overrightarrow{x_\lambda}, \overrightarrow{y_\lambda}\right)$ and $\Pi\left(\overrightarrow{x_S}, \overrightarrow{y_S}\right)$ merge or become parallel, it becomes impossible to distinguish precession $\psi$ from spin $\varphi$. This is particularly the case of numerical calculation where, should such a situation occur, it may lead to distorted results. It may be required to choose another angular representation, provided that the three chosen angles are independent.

### 1.4.4. *Changes of basis in the Euler representation*

The requirement to locate a solid before expressing its motion, and then to study it, leads to choosing systems of reference that are best suited for conducting each of these stages. It is, therefore, important to be able to make the changes of basis that help getting to the essential stage, which is the application of the fundamental law of dynamics in the adequate Galilean or pseudo-Galilean system of reference.

To make ideas clear, let us consider two bases

$$(\lambda) \equiv \left(\overrightarrow{x_\lambda}\ \overrightarrow{y_\lambda}\ \overrightarrow{z_\lambda}\right) \quad \text{and} \quad (S) \equiv \left(\overrightarrow{x_S}\ \overrightarrow{y_S}\ \overrightarrow{z_S}\right),$$

with a common origin $\mathrm{O}_\lambda$ and a point $\mathrm{M}$ belonging to the physical system being considered $(\mathscr{D})$.

The position vector $\overrightarrow{O_\lambda M}$ can be projected in the two bases

$$\overrightarrow{O_\lambda M} = x_\lambda \overrightarrow{x_\lambda} + y_\lambda \overrightarrow{y_\lambda} + z_\lambda \overrightarrow{z_\lambda} = x_S \overrightarrow{x_S} + y_S \overrightarrow{y_S} + z_S \overrightarrow{z_S} .$$

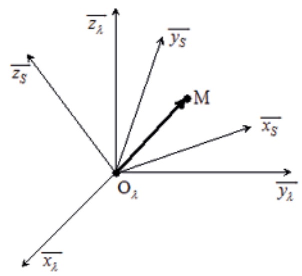



**Figure 1.19.** *Euler angles, location of the systems of reference*

The task is then to express the relations that connect the components of the position vector in the two bases as a function of the three angles that define the orientation of the basis $(S)$ relative to the basis $(\lambda)$.

The change of basis $(\lambda) \to (S)$ results from three successive rotations, by the corresponding angles $\psi$, $\theta$ and $\varphi$.

### 1.4.4.1. *The first rotation by angle ψ, or precession*

This rotation about the axis $\overrightarrow{z_\lambda}$, called precession, makes the passage from the basis $(\lambda)$ to the basis $(1) \equiv \left(\vec{n}\ \overrightarrow{n_1}\ \overrightarrow{z_\lambda}\right)$ according to Figure 1.20.

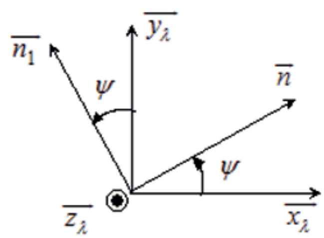



**Figure 1.20.** *Precession*

The projection of basis $(1)$ on the basis $(\lambda)$ yields:

$$\begin{cases} \vec{n} = \cos\psi\,\overrightarrow{x_\lambda} + \sin\psi\,\overrightarrow{y_\lambda} \\ \overrightarrow{n_1} = -\sin\psi\,\overrightarrow{x_\lambda} + \cos\psi\,\overrightarrow{y_\lambda}\,, \\ \qquad\qquad \overrightarrow{z_\lambda} \end{cases}$$

a result that can be represented by the following table of the change of basis:

| $(\lambda)\,\|\,(1)$ | $\vec{n}$ | $\overrightarrow{n_1}$ | $\overrightarrow{z_\lambda}$ |
|---|---|---|---|
| $\overrightarrow{x_\lambda}$ | $\cos\psi$ | $-\sin\psi$ | $0$ |
| $\overrightarrow{y_\lambda}$ | $\sin\psi$ | $\cos\psi$ | $0$ |
| $\overrightarrow{z_\lambda}$ | $0$ | $0$ | $1$ |

,

and by the associated matrix

$$M(\lambda,1) = \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The relation between the two bases is then written in matrix form:

$$\begin{bmatrix} \overrightarrow{x_\lambda} \\ \overrightarrow{y_\lambda} \\ \overrightarrow{z_\lambda} \end{bmatrix} = M(\lambda,1) \begin{bmatrix} \vec{n} \\ \overrightarrow{n_1} \\ \overrightarrow{z_\lambda} \end{bmatrix}.$$

If the vector $\overrightarrow{\mathrm{O}_\lambda\mathrm{M}}$ is expressed in both bases

$$\overrightarrow{\mathrm{O}_\lambda\mathrm{M}} = x_1\vec{n} + y_1\overrightarrow{n_1} + z_1\overrightarrow{z_\lambda} = x_\lambda\overrightarrow{x_\lambda} + y_\lambda\overrightarrow{y_\lambda} + z_\lambda\overrightarrow{z_\lambda}\,,$$

$$\overrightarrow{\mathrm{O}_\lambda\mathrm{M}} = x_1\left(\cos\psi\overrightarrow{x_\lambda} + \sin\psi\overrightarrow{y_\lambda}\right) + y_1\left(-\sin\psi\overrightarrow{x_\lambda} + \cos\psi\overrightarrow{y_\lambda}\right) + z_1\overrightarrow{z_\lambda}\,,$$

$$\overrightarrow{O_\lambda M} = (x_1 \cos\psi - y_1 \sin\psi)\overrightarrow{x_\lambda} + (x_1 \sin\psi + y_1 \cos\psi)\overrightarrow{y_\lambda} + z_1 \overrightarrow{z_\lambda}.$$

$\overrightarrow{O_\lambda M}$ can be written in matrix form

$$\begin{bmatrix} x_\lambda \\ y_\lambda \\ z_\lambda \end{bmatrix} = \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = M(\lambda,1) \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}.$$

It can be noted that since it is possible to write

$$\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} x_\lambda \\ y_\lambda \\ z_\lambda \end{bmatrix} = M^{-1}(\lambda,1) \begin{bmatrix} x_\lambda \\ y_\lambda \\ z_\lambda \end{bmatrix},$$

this yields

$$M^{-1}(\lambda,1) = M(1,\lambda) = \overline{M(\lambda,1)},$$

where $\overline{M(\lambda,1)}$ is the transpose matrix of $M(\lambda,1)$.

### 1.4.4.2. *The second rotation by angle $\theta$, or nutation*

This rotation about the axis $\vec{n}$, called nutation, makes the passage from the basis $(1)$ to the basis $(2) \equiv \left(\vec{n}\, \vec{w}\, \vec{z_S}\right)$ according to Figure 1.21.

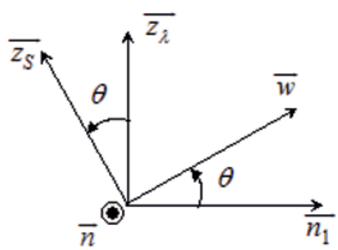



**Figure 1.21.** *Nutation*

The table of the change of basis and the associated matrix are

| $\overline{(1)}\,\|\,\underline{(2)}$ | $\vec{n}$ | $\vec{w}$ | $\vec{z_S}$ |
|---|---|---|---|
| $\vec{n}$ | 1 | 0 | 0 |
| $\vec{n_1}$ | 0 | $\cos\theta$ | $-\sin\theta$ |
| $\vec{z_\lambda}$ | 0 | $\sin\theta$ | $\cos\theta$ |

,

$$M(1,2)=\begin{bmatrix}1 & 0 & 0\\ 0 & \cos\theta & -\sin\theta\\ 0 & \sin\theta & \cos\theta\end{bmatrix},$$

hence the relation between the two bases is

$$\begin{bmatrix}\vec{n}\\ \vec{n_1}\\ \vec{z_\lambda}\end{bmatrix}=M(1,2)\begin{bmatrix}\vec{n}\\ \vec{w}\\ \vec{z_S}\end{bmatrix}.$$

If the vector $\overrightarrow{O_\lambda M}$ is expressed in the basis $(2)$ by $\overrightarrow{O_\lambda M}=x_2\vec{n}+y_2\vec{w}+z_2\vec{z_S}$, then the matrix form is obtained:

$$\begin{bmatrix}x_1\\ y_1\\ z_1\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ 0 & \cos\theta & -\sin\theta\\ 0 & \sin\theta & \cos\theta\end{bmatrix}\times\begin{bmatrix}x_2\\ y_2\\ z_2\end{bmatrix}=M(1,2)\begin{bmatrix}x_2\\ y_2\\ z_2\end{bmatrix},$$

and moreover

$$M^{-1}(1,2)=M(2,1)=\overline{M(1,2)}.$$

### 1.4.4.3. *The third rotation by angle $\varphi$, or spin*

This rotation about the axis $\vec{z_S}$, called spin, makes the passage from the basis (2) to the basis (S) according to Figure 1.22.

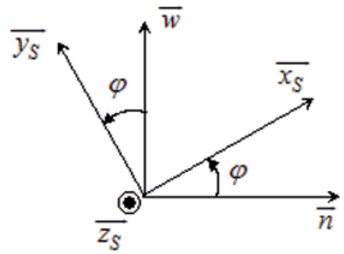



**Figure 1.22.** *Spin*

hence the table of the change of basis and the associated matrix are

| $(2)\|(S)$ | $\overrightarrow{x_S}$ | $\overrightarrow{y_S}$ | $\overrightarrow{z_S}$ |
|---|---|---|---|
| $\overrightarrow{n}$ | $\cos\varphi$ | $-\sin\varphi$ | 0 |
| $\overrightarrow{w}$ | $\sin\varphi$ | $\cos\varphi$ | 0 |
| $\overrightarrow{z_S}$ | 0 | 0 | 1 |

,

$$M(2,S)=\begin{bmatrix}\cos\varphi & -\sin\varphi & 0\\ \sin\varphi & \cos\varphi & 0\\ 0 & 0 & 1\end{bmatrix},$$

and the two matrix relations are

$$\begin{bmatrix}\overrightarrow{n}\\ \overrightarrow{w}\\ \overrightarrow{z_S}\end{bmatrix}=M(2,S)\begin{bmatrix}\overrightarrow{x_S}\\ \overrightarrow{y_S}\\ \overrightarrow{z_S}\end{bmatrix}$$

$$\begin{bmatrix}x_2\\ y_2\\ z_2\end{bmatrix}=\begin{bmatrix}\cos\varphi & -\sin\varphi & 0\\ \sin\varphi & \cos\varphi & 0\\ 0 & 0 & 1\end{bmatrix}\times\begin{bmatrix}x_S\\ y_S\\ z_S\end{bmatrix}=M(2,S)\begin{bmatrix}x_S\\ y_S\\ z_S\end{bmatrix},$$

and moreover

$$M^{-1}(2,S) = M(S,2) = \overline{M(2,S)}.$$

#### 1.4.4.4. *Relation between (λ) and (S) bases*

Based on these results, the following relations between the vectors of the two bases are obtained:

$$\begin{bmatrix} \overrightarrow{x_\lambda} \\ \overrightarrow{y_\lambda} \\ \overrightarrow{z_\lambda} \end{bmatrix} = M(\lambda,1)\times M(1,2)\times M(2,S)\begin{bmatrix} \overrightarrow{x_S} \\ \overrightarrow{y_S} \\ z_S \end{bmatrix} = M(\lambda,S)\begin{bmatrix} \overrightarrow{x_S} \\ \overrightarrow{y_S} \\ z_S \end{bmatrix}$$

or

$$\begin{bmatrix} \overrightarrow{x_\lambda} \\ \overrightarrow{y_\lambda} \\ \overrightarrow{z_\lambda} \end{bmatrix} = \begin{bmatrix} \cos\psi\cos\varphi - \sin\psi\cos\theta\sin\varphi & -\cos\psi\sin\varphi - \sin\psi\cos\theta\cos\varphi & \sin\psi\sin\theta \\ \sin\psi\cos\varphi + \cos\psi\cos\theta\sin\varphi & -\sin\psi\sin\varphi + \cos\psi\cos\theta\cos\varphi & -\cos\psi\sin\theta \\ \sin\theta\sin\varphi & \sin\theta\cos\varphi & \cos\theta \end{bmatrix}\begin{bmatrix} \overrightarrow{x_S} \\ \overrightarrow{y_S} \\ z_S \end{bmatrix},$$

Therefore, the relation between the components of vector $\overrightarrow{\mathrm{O}_\lambda\mathrm{M}}$ in these two bases is:

$$\begin{bmatrix} x_\lambda \\ y_\lambda \\ z_\lambda \end{bmatrix} = M(\lambda,1)\times M(1,2)\times M(2,S)\begin{bmatrix} x_S \\ y_S \\ z_S \end{bmatrix},$$

which leads to the relation between the passage matrices

$$M(\lambda,S) = M(\lambda,1)\times M(1,2)\times M(2,S).$$

NOTE 1.6.– To ensure coherence of the terms of a matrix of change of basis, and considering that this is a relation between unit vectors, the sum of squared terms of the same row or the same column of this matrix should always be equal to 1. For example, it can be verified that $\overrightarrow{x_\lambda}^2 = 1$ or

$$(\cos\psi\cos\varphi-\sin\psi\cos\theta\sin\varphi)^2+(-\cos\psi\sin\varphi-\sin\psi\cos\theta\cos\varphi)^2\ldots$$
$$\ldots+\sin^2\psi\sin^2\theta=1$$

and the sum of the member-by-member products of the terms of two rows or of two columns should be zero in order to express that the basis is orthogonal, for example $\overrightarrow{y_\lambda}\cdot\overrightarrow{z_\lambda}=0$ or

$$(\sin\psi\cos\varphi+\cos\psi\cos\theta\sin\varphi)\sin\theta\sin\varphi\ldots$$
$$\ldots+(-\sin\psi\sin\varphi+\cos\psi\cos\theta\cos\varphi)\sin\theta\cos\varphi-\cos\psi\sin\theta\cos\theta=0\,.$$

## 1.4.5. *Exercises*

### 1.4.5.1. *Exercise 3 – Euler angles*

Let $(S)$ be a solid and two vectors $\overrightarrow{x_S}$ and $\overrightarrow{y_S}$ associated with it in the direct orthonormal basis $(\lambda)\equiv\left(\overrightarrow{x_\lambda}\,\overrightarrow{y_\lambda}\,\overrightarrow{z_\lambda}\right)$.

*Question 1(a)*: Given the vectors

$$\overrightarrow{x_S}=\frac{\sqrt{6}}{4}\overrightarrow{x_\lambda}+\frac{\sqrt{2}}{4}\overrightarrow{y_\lambda}+\frac{\sqrt{2}}{2}\overrightarrow{z_\lambda};\quad \overrightarrow{y_S}=-\frac{\sqrt{6}}{4}\overrightarrow{x_\lambda}-\frac{\sqrt{2}}{4}\overrightarrow{y_\lambda}+\frac{\sqrt{2}}{2}\overrightarrow{z_\lambda}$$

verify that they are unit and orthogonal vectors. Calculate $\overrightarrow{z_S}=\overrightarrow{x_S}\wedge\overrightarrow{y_S}$.

$\overrightarrow{x_S}$ and $\overrightarrow{y_S}$ are unit vectors $\Rightarrow$ $\overrightarrow{x_S}^2=1$ , $\overrightarrow{y_S}^2=1$,

$\overrightarrow{x_S}$ and $\overrightarrow{y_S}$ are orthogonal $\overrightarrow{x_S}\cdot\overrightarrow{y_S}=0$

$$\frac{6}{16}+\frac{2}{16}+\frac{2}{4}=\frac{16}{16}=1\quad;\quad -\frac{\sqrt{6}}{4}\times\frac{\sqrt{6}}{4}-\frac{\sqrt{2}}{4}\times\frac{\sqrt{2}}{4}+\frac{\sqrt{2}}{2}\times\frac{\sqrt{2}}{2}=0$$

Therefore, the two vectors are unit and orthogonal vectors

$$\vec{z_S} = \vec{x_S} \wedge \vec{y_S} = \left( \frac{\sqrt{6}}{4}\vec{x_\lambda} + \frac{\sqrt{2}}{4}\vec{y_\lambda} + \frac{\sqrt{2}}{2}\vec{z_\lambda} \right) \wedge \left( -\frac{\sqrt{6}}{4}\vec{x_\lambda} - \frac{\sqrt{2}}{4}\vec{y_\lambda} + \frac{\sqrt{2}}{2}\vec{z_\lambda} \right).$$

$$\text{" "} \qquad = \frac{1}{2}\vec{x_\lambda} - \frac{\sqrt{3}}{2}\vec{y_\lambda}$$

It can be verified that this vector is also a unit vector orthogonal to $\vec{x_S}$ and $\vec{y_S}$.

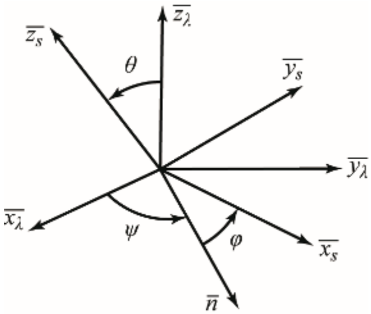
*Question 1(b)*: Write the matrix $[\,p(\lambda,S)\,]$ of passage from $(\lambda)$ to the basis $(S) \equiv \left(\vec{x_S}\ \vec{y_S}\ \vec{z_S}\right)$ connected to the solid $(S)$.

The table of passage from the basis (λ) to the basis (S) can be written:

$$p(\lambda,S) \quad \Leftrightarrow \quad \begin{array}{c|ccc} & \vec{x_S} & \vec{y_S} & \vec{z_S} \\ \hline \vec{x_\lambda} & \frac{\sqrt{6}}{4} & -\frac{\sqrt{6}}{4} & \frac{1}{2} \\ \vec{y_\lambda} & \frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} & -\frac{\sqrt{3}}{2} \\ \vec{z_\lambda} & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \end{array},$$

Therefore, the matrix of passage is

$$[\,p(\lambda,S)\,] = \begin{bmatrix} \frac{\sqrt{6}}{4} & -\frac{\sqrt{6}}{4} & \frac{1}{2} \\ \frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \end{bmatrix}.$$




*Question 1(c)*: Define the orientation of the basis $(S)$ relative to (λ) by means of the Euler angles. Calculate the value of these angles, and specify the nodal vector $\vec{n}$ :

– $\overrightarrow{z_\lambda}\cdot\overrightarrow{z_S} = \cos\theta = 0 \quad \Rightarrow \quad \theta = \dfrac{\pi}{2} + k\pi \quad$ with $\quad \theta \in [-\pi, \pi]$,

$$\Rightarrow \begin{cases} \theta = \dfrac{\pi}{2} & \Rightarrow \quad \sin\theta = 1 \\ \theta = -\dfrac{\pi}{2} & \Rightarrow \quad \sin\theta = -1 \end{cases},$$

– $\overrightarrow{z_\lambda} \wedge \overrightarrow{z_S} = \vec{n}\sin\theta$

$$\overrightarrow{z_\lambda} \wedge \left(\frac{1}{2}\overrightarrow{x_\lambda} - \frac{\sqrt{3}}{2}\overrightarrow{y_\lambda}\right) = \frac{\sqrt{3}}{2}\overrightarrow{x_\lambda} + \frac{1}{2}\overrightarrow{y_\lambda} = \cos\frac{\pi}{6}\overrightarrow{x_\lambda} + \sin\frac{\pi}{6}\overrightarrow{y_\lambda} = \vec{u}\left(\frac{\pi}{6}\right) = \vec{n}\sin\theta$$

$$\Rightarrow \begin{cases} \sin\theta = 1 & \Rightarrow \quad \vec{n} = \cos\dfrac{\pi}{6}\overrightarrow{x_\lambda} + \sin\dfrac{\pi}{6}\overrightarrow{y_\lambda} = \vec{u}\left(\dfrac{\pi}{6}\right) \\ \sin\theta = -1 & \Rightarrow \quad \vec{n} = \cos\left(\pi + \dfrac{\pi}{6}\right)\overrightarrow{x_\lambda} + \sin\left(\pi + \dfrac{\pi}{6}\right)\overrightarrow{y_\lambda} = \vec{u}\left(\pi + \dfrac{\pi}{6}\right) \end{cases},$$

or

$$\frac{\sqrt{2}}{2}\left(\overrightarrow{x_S} + \overrightarrow{y_S}\right) \wedge \overrightarrow{z_S} = \frac{\sqrt{2}}{2}\overrightarrow{x_S} - \frac{\sqrt{2}}{2}\overrightarrow{y_S} = \cos\frac{\pi}{4}\overrightarrow{x_S} - \sin\frac{\pi}{4}\overrightarrow{y_S} = \vec{n}\sin\theta$$

$$\Rightarrow \begin{cases} \sin\theta = 1 & \Rightarrow \quad \vec{n} = \cos\dfrac{\pi}{4}\overrightarrow{x_S} - \sin\dfrac{\pi}{4}\overrightarrow{y_S} \\ \sin\theta = -1 & \Rightarrow \quad \vec{n} = \cos\left(\pi + \dfrac{\pi}{4}\right)\overrightarrow{x_S} - \sin\left(\pi + \dfrac{\pi}{4}\right)\overrightarrow{y_S} \end{cases},$$

$$-\begin{cases} \vec{n}\cdot\overrightarrow{x_\lambda} = \cos\psi = \begin{cases} \cos\dfrac{\pi}{6} \\ \cos\left(\pi+\dfrac{\pi}{6}\right) \end{cases} \\ \vec{n}\cdot\overrightarrow{y_\lambda} = \sin\psi = \begin{cases} \sin\dfrac{\pi}{6} \\ \sin\left(\pi+\dfrac{\pi}{6}\right) \end{cases} \end{cases} \Rightarrow \begin{cases} \sin\theta = 1 \quad , \quad \psi = \dfrac{\pi}{6}+2k\pi \\ \sin\theta = -1 \quad , \quad \psi = \pi+\dfrac{\pi}{6}+2k\pi \end{cases},$$

$$-\begin{cases} \vec{n}\cdot\overrightarrow{x_S} = \cos\varphi = \begin{cases} \cos\dfrac{\pi}{4} \\ \cos\left(\pi+\dfrac{\pi}{4}\right) \end{cases} \\ \vec{n}\cdot\overrightarrow{y_S} = \sin\varphi = \begin{cases} \sin\dfrac{\pi}{4} \\ \sin\left(\pi+\dfrac{\pi}{4}\right) \end{cases} \end{cases} \Rightarrow \begin{cases} \sin\theta = 1 \quad , \quad \varphi = \dfrac{\pi}{4}+2k\pi \\ \sin\theta = -1 \quad , \quad \phi = \pi+\dfrac{\pi}{4}+2k\pi \end{cases}.$$

*Question 2*: Answer the same questions for the vectors

$$\overrightarrow{x_S} = \frac{1}{\sqrt{3}}\overrightarrow{x_\lambda} - \frac{1}{\sqrt{3}}\overrightarrow{y_\lambda} - \frac{1}{\sqrt{3}}\overrightarrow{z_\lambda}; \quad \overrightarrow{y_S} = \frac{1}{\sqrt{2}}\overrightarrow{x_\lambda} + \frac{1}{\sqrt{2}}\overrightarrow{z_\lambda}.$$

Using the same approach employed above, it can be verified that

$$\begin{aligned} &\overrightarrow{x_S}^2 = \frac{3}{3} = 1 \quad , \quad \overrightarrow{y_S}^2 = \frac{2}{2} = 1 \quad , \quad \overrightarrow{x_S}\cdot\overrightarrow{y_S} = \frac{1}{\sqrt{6}} - \frac{1}{\sqrt{6}} = 0 \\ &\overrightarrow{z_S} = \overrightarrow{x_S}\wedge\overrightarrow{y_S} = -\frac{1}{\sqrt{6}}\overrightarrow{x_\lambda} - \frac{2}{\sqrt{6}}\overrightarrow{y_\lambda} + \frac{1}{\sqrt{6}}\overrightarrow{z_\lambda} \end{aligned},$$

$$[p(\lambda,S)] = \begin{bmatrix} \dfrac{1}{\sqrt{3}} & \dfrac{1}{\sqrt{2}} & -\dfrac{1}{\sqrt{6}} \\ -\dfrac{1}{\sqrt{3}} & 0 & -\dfrac{2}{\sqrt{6}} \\ -\dfrac{1}{\sqrt{3}} & \dfrac{1}{\sqrt{2}} & \dfrac{1}{\sqrt{6}} \end{bmatrix},$$

$$\overrightarrow{z_\lambda}\cdot\overrightarrow{z_S} = \cos\theta = \frac{1}{\sqrt{6}} = 0{,}408 \quad , \quad \sin\theta = \pm\sqrt{1-\cos^2\theta} = \pm\sqrt{\frac{5}{6}} = \pm 0{,}913 \quad ,$$

$$\overrightarrow{z_\lambda}\wedge\overrightarrow{z_S} = \vec{n}\sin\theta = \frac{2}{\sqrt{6}}\overrightarrow{x_\lambda} - \frac{1}{\sqrt{6}}\overrightarrow{y_\lambda} = \frac{1}{\sqrt{2}}\overrightarrow{x_S} + \frac{1}{\sqrt{3}}\overrightarrow{y_s}$$

$$\Rightarrow \quad \theta = 1{,}15\, rd\,.$$

$$\begin{cases} \vec{n}\cdot\overrightarrow{x_\lambda} = \cos\psi = \dfrac{2}{\sqrt{6}\sin\theta} = 0.894 \\ \vec{n}\cdot\overrightarrow{y_\lambda} = \sin\psi = -\dfrac{1}{\sqrt{6}\sin\theta} = -0{,}447 \end{cases} \qquad \begin{cases} \vec{n}\cdot\overrightarrow{x_S} = \cos\varphi = \dfrac{1}{\sqrt{2}\sin\theta} = 0{,}775 \\ \vec{n}\cdot\overrightarrow{y_S} = -\sin\varphi = \dfrac{1}{\sqrt{3}\sin\theta} = -0{,}632 \end{cases},$$

$$\Rightarrow \quad \psi = 0{,}465\, rd, \qquad \varphi = -0{,}685\, rd$$

### 1.4.5.2. *Exercise 4 – Euler angles*

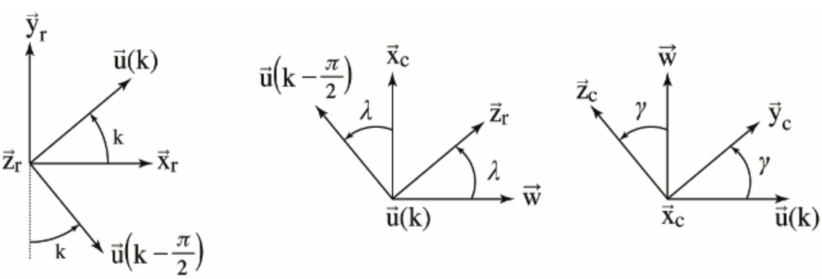
The basis $(e)$ connected to the cutting face of a tool is located relative to the basis $(r)$ connected to the body of the tool, by means of the angles:

– $k$, cutting edge angle measured over $\vec{z}$;

– $\lambda$, cutting edge inclination angle measured over $\vec{u}(k)$;

– $\gamma$, *cutting angle* measured over $\overrightarrow{x_c}$,

as indicated by the diagrams below.



The following passage tables can be associated with these three changes of basis:

| | $\overrightarrow{x_r}$ | $\overrightarrow{y_r}$ | $\overrightarrow{z_r}$ |
|---|---|---|---|
| $\vec{u}\left(k-\frac{\pi}{2}\right)$ | $\sin k$ | $-\cos k$ | $0$ |
| $\vec{u}(k)$ | $\cos k$ | $\sin k$ | $0$ |
| $\overrightarrow{z_r}$ | $0$ | $0$ | $1$ |

| | $\vec{u}(k)$ | $\overrightarrow{z_r}$ | $\vec{u}\left(k-\frac{\pi}{2}\right)$ |
|---|---|---|---|
| $\vec{u}(k)$ | $1$ | $0$ | $0$ |
| $\vec{w}$ | $0$ | $\cos\lambda$ | $-\sin\lambda$ |
| $\overrightarrow{x_c}$ | $0$ | $\sin\lambda$ | $\cos\lambda$ |

,

| | $\overrightarrow{x_c}$ | $\vec{u}(k)$ | $\vec{w}$ |
|---|---|---|---|
| $\overrightarrow{x_c}$ | $1$ | $0$ | $0$ |
| $\overrightarrow{y_c}$ | $0$ | $\cos\gamma$ | $\sin\gamma$ |
| $\overrightarrow{z_c}$ | $0$ | $-\sin\gamma$ | $\cos\gamma$ |

.

*Question 1*: Determine the vectors $\overrightarrow{x_r}, \overrightarrow{y_r}, \overrightarrow{z_r}$ by their components in the basis $(c) \equiv \left(\overrightarrow{x_c}\ \overrightarrow{y_c}\ \overrightarrow{z_c}\right)$ using the angle parameters $k, \lambda, \gamma$.

According to the previous tables of change of basis

$$\overrightarrow{x_r} = \cos k\,\vec{u}(k) + \sin k\,\vec{u}\left(k-\frac{\pi}{2}\right)$$
$$\overrightarrow{y_r} = \sin k\,\vec{u}(k) - \cos k\,\vec{u}\left(k-\frac{\pi}{2}\right),$$

$$\overrightarrow{z_r} = \cos\lambda\,\vec{w} + \sin\lambda\,\overrightarrow{x_c}$$
$$\vec{u}\left(k-\frac{\pi}{2}\right) = -\sin\lambda\,\vec{w} + \cos\lambda\,\overrightarrow{x_c},$$

$$\begin{aligned}\vec{u}(k) &= \cos\gamma\,\overrightarrow{y_c} - \sin\gamma\,\overrightarrow{z_c}\\ \vec{w} &= \sin\gamma\,\overrightarrow{y_c} + \cos\gamma\,\overrightarrow{z_c}\end{aligned},$$

$$\Rightarrow \begin{cases}\overrightarrow{x_r} = \sin k\cos\lambda\,\overrightarrow{x_c} + (\cos k\cos\gamma - \sin k\sin\lambda\sin\gamma)\,\overrightarrow{y_c}\ \cdots\\ \qquad\cdots - (\cos k\sin\gamma + \sin k\sin\lambda\cos\gamma)\,\overrightarrow{z_c}\\ \overrightarrow{y_r} = -\cos k\cos\lambda\,\overrightarrow{x_c} + (\sin k\cos\gamma + \cos k\sin\lambda\sin\gamma)\,\overrightarrow{y_c}\ \cdots\\ \qquad\cdots - (\sin k\sin\gamma - \cos k\sin\lambda\cos\gamma)\,\overrightarrow{z_c}\\ \overrightarrow{z_r} = \sin\lambda\,\overrightarrow{x_c} + \cos\lambda\sin\gamma\,\overrightarrow{y_c} + \cos\lambda\cos\gamma\,\overrightarrow{z_c}\end{cases}.$$

*Question 2(a)*: Next, let us consider $k = \dfrac{\pi}{4},\ \lambda = 0,\ \gamma = \dfrac{\pi}{6}$. Establish the passage table of change of basis from $(r)$ to $(c)$.

The change of basis from $(r)$ to $(c)$ is expressed by the table

| | $\overrightarrow{x_c}$ | $\overrightarrow{y_c}$ | $\overrightarrow{z_c}$ |
|---|---|---|---|
| $\overrightarrow{x_r}$ | $\sin k\cos\lambda$ | $\cos k\cos\gamma - \sin k\sin\lambda\sin\gamma$ | $-\cos k\sin\gamma - \sin k\sin\lambda\cos\gamma$ |
| $\overrightarrow{y_r}$ | $-\cos k\cos\lambda$ | $\sin k\cos\gamma + \cos k\sin\lambda\sin\gamma$ | $-\sin k\sin\gamma + \cos k\sin\lambda\cos\gamma$ |
| $\overrightarrow{z_r}$ | $\sin\lambda$ | $\cos\lambda\sin\gamma$ | $\cos\lambda\cos\gamma$ |

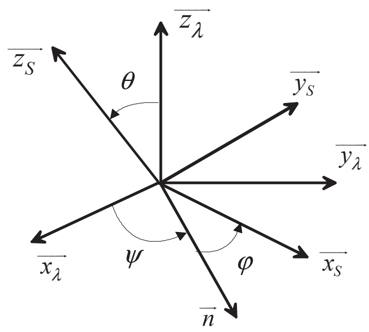
,

which for the given particular values is written,

| | $\overrightarrow{x_c}$ | $\overrightarrow{y_c}$ | $\overrightarrow{z_c}$ |
|---|---|---|---|
| $\overrightarrow{x_r}$ | $\dfrac{\sqrt{2}}{2}$ | $\dfrac{\sqrt{6}}{4}$ | $-\dfrac{\sqrt{2}}{4}$ |
| $\overrightarrow{y_r}$ | $-\dfrac{\sqrt{2}}{2}$ | $\dfrac{\sqrt{6}}{4}$ | $-\dfrac{\sqrt{2}}{4}$ |
| $\overrightarrow{z_r}$ | $0$ | $\dfrac{1}{2}$ | $\dfrac{\sqrt{3}}{2}$ |

.

*Question 2(b)*: Deduce from the previous table the value of the angles of precession $\psi$, nutation $\theta$ and spin $\varphi$ of the Eulerian representation of the orientation of basis $(c)$ relative to $(r)$. Determine the nodal vector $\vec{n}$.




By analogy with this diagram of the principle of definition of the Euler angles, we have

$$\begin{cases} \overrightarrow{z_c} \wedge \overrightarrow{z_r} = \vec{n}\sin\theta \\ \overrightarrow{z_c} \cdot \overrightarrow{z_r} = \cos\theta \end{cases}, \quad \begin{cases} \overrightarrow{x_c} \cdot \vec{n} = \cos\psi \\ \overrightarrow{y_c} \cdot \vec{n} = \sin\psi \end{cases}, \quad \begin{cases} \vec{n} \cdot \overrightarrow{x_r} = \cos\varphi \\ \vec{n} \cdot \overrightarrow{y_r} = -\sin\varphi \end{cases}.$$

According to the previous table

$$\begin{cases} \overrightarrow{z_c} \wedge \overrightarrow{z_r} = \dfrac{\sqrt{2}}{4}\overrightarrow{y_r} - \dfrac{\sqrt{2}}{4}\overrightarrow{x_r} = -\dfrac{1}{2}\overrightarrow{x_c} = \vec{n}\sin\theta \;\Rightarrow\; \vec{n} = \pm\overrightarrow{x_c} \quad , \quad \sin\theta = \mp\dfrac{1}{2} \\ \overrightarrow{z_c} \cdot \overrightarrow{z_r} = \dfrac{\sqrt{3}}{2} = \cos\theta \end{cases},$$

$$\begin{cases} \overrightarrow{x_c} \cdot \vec{n} = \pm 1 = \cos\psi \\ \overrightarrow{y_c} \cdot \vec{n} = 0 = \sin\psi \end{cases},$$

$$\begin{cases} \vec{n} \cdot \overrightarrow{x_r} = \pm\dfrac{\sqrt{2}}{2} = \cos\varphi \\ \vec{n} \cdot \overrightarrow{y_r} = \mp\dfrac{\sqrt{2}}{2} = -\sin\varphi \end{cases}.$$

These results lead to two sets of values

$$- \vec{n} = \overrightarrow{x_c}, \quad \theta = -\frac{\pi}{6}, \quad \psi = 0, \quad \varphi = \frac{\pi}{4},$$

$$- \vec{n} = -\overrightarrow{x_c}, \quad \theta = \frac{\pi}{6}, \quad \psi = \pi, \quad \varphi = -\frac{3\pi}{4}.$$

## 1.5. Vector rotation $\mathscr{R}_{\overline{u},\alpha}$

Let $\mathscr{R}_{\vec{u},\alpha}$ be the rotation by angle $\alpha$ of a vector $\vec{a}$ about the axis $\Delta$ with the direction vector $\vec{u}$. Let $\vec{b}$ be the vector resulting from the rotation of $\vec{a}$.

Let $\Pi$ be the plane perpendicular to vector $\vec{u}$ and passing through the point $\mathrm{A}$, the origin of vector $\vec{a}$. The projection on the axis $\vec{u}$ and on plane $\Pi$ can be written

$$\begin{aligned} \vec{a} &= \left(\vec{u} \cdot \vec{a}\right)\vec{u} + \vec{u} \wedge \left(\vec{a} \wedge \vec{u}\right) \\ \vec{b} &= \left(\vec{u} \cdot \vec{b}\right)\vec{u} + \vec{u} \wedge \left(\vec{b} \wedge \vec{u}\right) \end{aligned}.$$

Since $\vec{b}$ results from $\vec{a}$ by rotation about the axis $\left(\mathrm{A}|\vec{u}\right)$, the two vectors have the same orthogonal projection on this axis, which gives

$$\left(\vec{u} \cdot \vec{a}\right)\vec{u} = \left(\vec{u} \cdot \vec{b}\right)\vec{u}.$$

The projections of the two vectors on the plane $\Pi$, orthogonal to $\vec{u}$, have the same module.

$$\left\|\vec{u} \wedge \left(\vec{a} \wedge \vec{u}\right)\right\| = \left\|\vec{u} \wedge \left(\vec{b} \wedge \vec{u}\right)\right\|.$$

The problem posed is to determine the vector $\vec{b}$ as a function of the known elements, namely $\vec{a}$, $\vec{u}$ and $\alpha$. Taking into account the above, this comes down to determining the expression $\vec{u} \wedge \left(\vec{b} \wedge \vec{u}\right)$.

Let us then consider the local system of reference connected to the two vectors $\vec{u}$ and $\vec{a}$, whose unit vectors are

$$\vec{u}, \quad \vec{k} = \frac{\vec{u} \wedge \left(\vec{a} \wedge \vec{u}\right)}{\left\|\vec{u} \wedge \left(\vec{a} \wedge \vec{u}\right)\right\|}, \quad \vec{l} = \frac{\vec{u} \wedge \vec{a}}{\left\|\vec{u} \wedge \vec{a}\right\|},$$

The three vectors $\vec{u}$, $\vec{a}$ and $\vec{k}$ are indeed coplanar. Therefore, $\vec{u} \wedge \vec{a}$ defines a direction orthogonal to this plane, which leads to the vector $\vec{l}$.

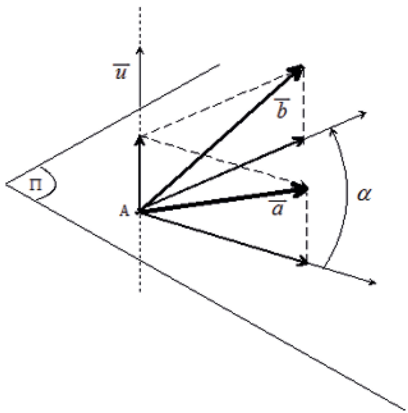



**Figure 1.23.** *Vector rotation* $\mathscr{R}_{\overline{u},\alpha}$

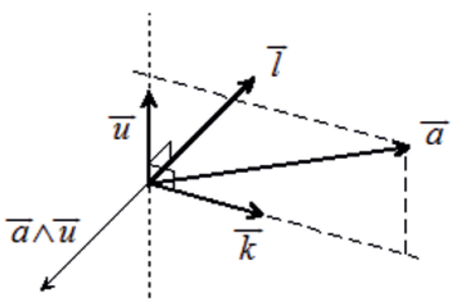



**Figure 1.24.** *System of reference associated with the vector rotation* $\mathscr{R}_{\overline{u},\alpha}$

It can be noted that

$$\left\|\vec{u}\wedge\left(\vec{a}\wedge\vec{u}\right)\right\|=\left\|\vec{u}\right\|\times\left\|\vec{a}\wedge\vec{u}\right\|\times\left|\sin\left(\vec{u},\vec{a}\wedge\vec{u}\right)\right|,$$

with $\vec{u}\perp\left(\vec{a}\wedge\vec{u}\right)$ and $\left\|\vec{u}\right\|=1$,

hence, $\left\|\vec{u}\wedge\left(\vec{a}\wedge\vec{u}\right)\right\|=\left\|\vec{a}\wedge\vec{u}\right\|=\left\|\vec{u}\wedge\vec{a}\right\|$.

Since, the vector to be determined $\vec{u}\wedge\left(\vec{b}\wedge\vec{u}\right)$ is orthogonal to $\vec{u}$, it belongs to the plane $\Pi\left(\vec{k},\vec{l}\right)$; it can, therefore, be projected orthogonally on these two vectors, which leads to

$$\vec{u}\wedge\left(\vec{b}\wedge\vec{u}\right)=\left\{\left[\vec{u}\wedge\left(\vec{b}\wedge\vec{u}\right)\right]\cdot\vec{k}\right\}\vec{k}+\left\{\left[\vec{u}\wedge\left(\vec{b}\wedge\vec{u}\right)\right]\cdot\vec{l}\right\}\vec{l},$$

with $\left\|\vec{u}\wedge\left(\vec{b}\wedge\vec{u}\right)\right\|=\left\|\vec{u}\wedge\left(\vec{a}\wedge\vec{u}\right)\right\|=\left\|\vec{u}\wedge\vec{a}\right\|$ since $\left\|\vec{a}\right\|=\left\|\vec{b}\right\|$,

$$\begin{aligned}\Rightarrow\quad \vec{u}\wedge\left(\vec{b}\wedge\vec{u}\right)&=\cos\alpha\left\|\vec{u}\wedge\left(\vec{a}\wedge\vec{u}\right)\right\|\vec{k}+\sin\alpha\left\|\vec{u}\wedge\vec{a}\right\|\vec{l}\\ \text{'' ''}\qquad&=\cos\alpha\left[\vec{u}\wedge\left(\vec{a}\wedge\vec{u}\right)\right]+\sin\alpha\left[\vec{u}\wedge\vec{a}\right]\end{aligned}.$$

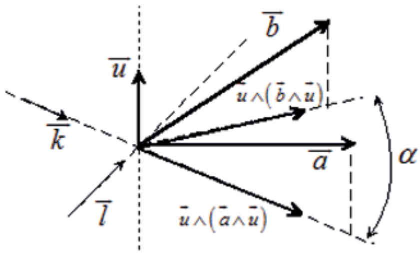



**Figure 1.25.** *Diagram of projection in the vector rotation* $\mathscr{R}_{\overline{u},\alpha}$

This leads us to deduce the general expression of $\vec{b}$

$$\vec{b}=\left(\vec{u}\cdot\vec{a}\right)\vec{u}+\left[\vec{u}\wedge\left(\vec{a}\wedge\vec{u}\right)\right]\cos\alpha+\left(\vec{u}\wedge\vec{a}\right)\sin\alpha.$$

But, since

$$\left(\vec{u}\cdot\vec{a}\right)\vec{u} = \vec{a} - \vec{u} \wedge \left(\vec{a} \wedge \vec{u}\right) = \vec{a} + \vec{u} \wedge \left(\vec{u} \wedge \vec{a}\right),$$

the vector $\vec{b}$, deduced from vector $\vec{a}$ by the rotation by angle $\alpha$ about the axis of the direction vector $\vec{u}$, has the general expression

$$\vec{b} = \mathscr{R}_{\vec{u},\alpha}\left(\vec{a}\right) = \vec{a} + \left[\vec{u} \wedge \left(\vec{u} \wedge \vec{a}\right)\right](1-\cos\alpha) + \left(\vec{u} \wedge \vec{a}\right)\sin\alpha.$$

The matrix representation $[\mathbf{F}]$ of the vector rotation by angle $\alpha$ about the axis of unit vector $\vec{u} = u_i \overrightarrow{x_i}$ can be written in the basis $(e) \equiv \left(\overrightarrow{x_1}\, \overrightarrow{x_2}\, \overrightarrow{x_3}\right)$

$$[\mathbf{F}]\big/_e = \begin{bmatrix} \cos\alpha + (1-\cos\alpha)u_1^2 & (1-\cos\alpha)u_1u_2 - \sin\alpha\, u_3 & (1-\cos\alpha)u_3u_1 + \sin\alpha\, u_2 \\ (1-\cos\alpha)u_1u_2 + \sin\alpha\, u_3 & \cos\alpha + (1-\cos\alpha)u_2^2 & (1-\cos\alpha)u_2u_3 - \sin\alpha\, u_1 \\ (1-\cos\alpha)u_3u_1 - \sin\alpha\, u_2 & (1-\cos\alpha)u_2u_3 + \sin\alpha\, u_1 & \cos\alpha + (1-\cos\alpha)u_3^2 \end{bmatrix},$$

which, by posing $\vec{a} = a_i \overrightarrow{x_i}$ and $\vec{b} = b_i \overrightarrow{x_i}$, leads us to write

$$\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} \cos\alpha + (1-\cos\alpha)u_1^2 & (1-\cos\alpha)u_1u_2 - \sin\alpha\, u_3 & (1-\cos\alpha)u_3u_1 + \sin\alpha\, u_2 \\ (1-\cos\alpha)u_1u_2 + \sin\alpha\, u_3 & \cos\alpha + (1-\cos\alpha)u_2^2 & (1-\cos\alpha)u_2u_3 - \sin\alpha\, u_1 \\ (1-\cos\alpha)u_3u_1 - \sin\alpha\, u_2 & (1-\cos\alpha)u_2u_3 + \sin\alpha\, u_1 & \cos\alpha + (1-\cos\alpha)u_3^2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}.$$

## 1.5.1. *Exercises*

### 1.5.1.1. *Exercise 5 – Vector rotation*

In the direct orthonormal basis $\left(\overrightarrow{X_1}\, \overrightarrow{X_2}\, \overrightarrow{X_3}\right)$, let $\mathscr{R}_{\vec{u},\alpha}$ be the vector rotation represented by the matrix

$$\left[\mathscr{R}_{\vec{u},\alpha}\right] = \frac{1}{3}\begin{bmatrix} -2 & -1 & 2 \\ 2 & -2 & 1 \\ 1 & 2 & 2 \end{bmatrix}.$$

*Question*: Determine the unit vector $\vec{u}$ of the axis of the vector rotation and its angle $\alpha$.

We pose

$$\left[\mathscr{R}_{\vec{u},\alpha}\right]_{/(X)}=\frac{1}{3}\begin{bmatrix}-2 & -1 & 2\\ 2 & -2 & 1\\ 1 & 2 & 2\end{bmatrix}=\begin{bmatrix}p_{11} & p_{12} & p_{13}\\ p_{21} & p_{22} & p_{23}\\ p_{31} & p_{32} & p_{33}\end{bmatrix}.$$

This matrix has the property

$$Trace\left[\mathscr{R}_{\vec{u},\alpha}\right]=p_{ii}=-\frac{2}{3}-\frac{2}{3}+\frac{2}{3}=-\frac{2}{3}=1+2\cos\alpha\,,$$

$$\cos\alpha=-\frac{5}{6}\ \text{with}\ \ \alpha\in[0,\pi]\quad\Rightarrow\quad\alpha=146,44°\,,$$

$$\sin\alpha=\sqrt{1-\cos^2\alpha}=\frac{\sqrt{11}}{6}.$$

The vector $\vec{u}=u_i\overrightarrow{X_i}$ has the following components in the basis ($X$)

$$\begin{cases} u_1=\dfrac{p_{32}-p_{23}}{2\sin\alpha}=\dfrac{1}{3}\times\dfrac{6}{2\sqrt{11}}\times 1=\dfrac{\sqrt{11}}{11}\\ u_2=\dfrac{p_{13}-p_{31}}{2\sin\alpha}=\dfrac{1}{3}\times\dfrac{6}{2\sqrt{11}}\times 1=\dfrac{\sqrt{11}}{11}\\ u_3=\dfrac{p_{21}-p_{12}}{2\sin\alpha}=\dfrac{1}{3}\times\dfrac{6}{2\sqrt{11}}\times 3=\dfrac{3\sqrt{11}}{11}\end{cases}.$$

In the basis $(X)$, the vector rotation $\mathscr{R}_{\vec{u},\alpha}$ is, therefore, defined by

$$\mathscr{R}_{\vec{u},\alpha}\equiv\left[\vec{u}=\frac{\sqrt{11}}{11}\left(\overrightarrow{X_1}+\overrightarrow{X_2}+3\overrightarrow{X_3}\right)\middle|\,\alpha=146,44°\right].$$

### 1.5.1.2. *Exercise 6 – Vector rotation*

In the direct orthonormal basis $(X)\equiv\left(\overrightarrow{X_1}\,\overrightarrow{X_2}\,\overrightarrow{X_3}\right)$, let $\mathscr{R}_{\vec{u}(\alpha),\alpha}$ be a vector rotation with $\vec{u}(\alpha)=\cos\alpha\overrightarrow{X_1}+\sin\alpha\overrightarrow{X_2}$, which makes the passage from the basis $(X)$ to the basis $(x) \equiv \left(\overrightarrow{x_1}\,\overrightarrow{x_2}\,\overrightarrow{x_3}\right)$, which is also a direct orthonormal basis.

*Question 1*: Write as a function of $\alpha$ the vectors $\overrightarrow{x_1}, \overrightarrow{x_2}, \overrightarrow{x_3}$ with their components in the basis $(X)$.

The general expression of the vector rotation that leads us to deduce the vector $\vec{b}$ from $\vec{a}$ by the vector rotation $\mathscr{R}_{\vec{u},\alpha}$ is given by the relation

$$\vec{b} = \mathscr{R}_{\vec{u},\alpha}\left(\vec{a}\right) = \vec{a} + \left[\vec{u} \wedge \left(\vec{u} \wedge \vec{a}\right)\right](1-\cos\alpha) + \left(\vec{u} \wedge \vec{a}\right)\sin\alpha .$$

As a preliminary stage in obtaining the result, we shall consider that the vector $\overrightarrow{x_i}$, $\forall i = 1,2,3$ is the transform by rotation of vector $\overrightarrow{X_i}$; it is possible to readily repeat the calculation with another correspondence hypothesis, and this would lead in the end to a coincidence of trihedrons of the resulting bases, simply by cyclic permutation of the identities of the unit vectors of the new basis.

$$\begin{cases} \overrightarrow{x_1} = \mathscr{R}_{\vec{u},\alpha}\left(\overrightarrow{X_1}\right) = \overrightarrow{X_1} + \left[\vec{u}(\alpha) \wedge \left(\vec{u}(\alpha) \wedge \overrightarrow{X_1}\right)\right](1-\cos\alpha) + \left(\vec{u}(\alpha) \wedge \overrightarrow{X_1}\right)\sin\alpha \\ \overrightarrow{x_2} = \mathscr{R}_{\vec{u},\alpha}\left(\overrightarrow{X_2}\right) = \overrightarrow{X_2} + \left[\vec{u}(\alpha) \wedge \left(\vec{u}(\alpha) \wedge \overrightarrow{X_2}\right)\right](1-\cos\alpha) + \left(\vec{u}(\alpha) \wedge \overrightarrow{X_2}\right)\sin\alpha \\ \overrightarrow{x_3} = \mathscr{R}_{\vec{u},\alpha}\left(\overrightarrow{X_3}\right) = \overrightarrow{X_3} + \left[\vec{u}(\alpha) \wedge \left(\vec{u}(\alpha) \wedge \overrightarrow{X_3}\right)\right](1-\cos\alpha) + \left(\vec{u}(\alpha) \wedge \overrightarrow{X_3}\right)\sin\alpha \end{cases},$$

$$\vec{u}(\alpha) \wedge \overrightarrow{X_i} = \cos\alpha\,\overrightarrow{X_1} \wedge \overrightarrow{X_i} + \sin\alpha\,\overrightarrow{X_2} \wedge \overrightarrow{X_i},$$

$$\begin{aligned} \vec{u}(\alpha) \wedge \left(\vec{u}(\alpha) \wedge \overrightarrow{X_i}\right) = \cos^2\alpha\,\overrightarrow{X_1} \wedge \left(\overrightarrow{X_1} \wedge \overrightarrow{X_i}\right) & \ldots \\ \ldots + \cos\alpha\sin\alpha\,\overrightarrow{X_1} \wedge \left(\overrightarrow{X_2} \wedge \overrightarrow{X_i}\right) & \ldots \\ \ldots + \sin\alpha\cos\alpha\,\overrightarrow{X_2} \wedge \left(\overrightarrow{X_1} \wedge \overrightarrow{X_i}\right) & \ldots \\ \ldots + \sin^2\alpha\,\overrightarrow{X_2} \wedge \left(\overrightarrow{X_2} \wedge \overrightarrow{X_i}\right) & \end{aligned},$$

$$\vec{u}(\alpha)\wedge\left(\vec{u}(\alpha)\wedge\overrightarrow{X_i}\right)=\cos^2\alpha\left[\left(\overrightarrow{X_1}\cdot\overrightarrow{X_i}\right)\overrightarrow{X_1}-\overrightarrow{X_i}\right]$$
$$\ldots+\sin^2\alpha\left[\left(\overrightarrow{X_2}\cdot\overrightarrow{X_i}\right)\overrightarrow{X_2}-\overrightarrow{X_i}\right]\ldots \quad .$$
$$\ldots+\sin\alpha\cos\alpha\left[\left(\overrightarrow{X_2}\cdot\overrightarrow{X_i}\right)\overrightarrow{X_1}+\left(\overrightarrow{X_1}\cdot\overrightarrow{X_i}\right)\overrightarrow{X_2}\right]$$

Therefore, the following relation is obtained

$$\overrightarrow{x_i}=\mathscr{R}_{\vec{u},\alpha}\left(\overrightarrow{X_i}\right)=\overrightarrow{X_i}+\cos^2\alpha(1-\cos\alpha)\left[\left(\overrightarrow{X_1}\cdot\overrightarrow{X_i}\right)\overrightarrow{X_1}-\overrightarrow{X_i}\right]\ldots$$
$$\ldots+\sin^2\alpha(1-\cos\alpha)\left[\left(\overrightarrow{X_2}\cdot\overrightarrow{X_i}\right)\overrightarrow{X_2}-\overrightarrow{X_i}\right]\ldots$$
$$\ldots+\sin\alpha\cos\alpha(1-\cos\alpha)\left[\left(\overrightarrow{X_2}\cdot\overrightarrow{X_i}\right)\overrightarrow{X_1}+\left(\overrightarrow{X_1}\cdot\overrightarrow{X_i}\right)\overrightarrow{X_2}\right]\ldots ,$$
$$\ldots+\left(\cos\alpha\overrightarrow{X_1}\wedge\overrightarrow{X_i}+\sin\alpha\overrightarrow{X_2}\wedge\overrightarrow{X_i}\right)\sin\alpha$$

or

$$\begin{cases}\overrightarrow{x_1}=\mathscr{R}_{\vec{u},\alpha}\left(\overrightarrow{X_1}\right)=\left[1-\sin^2\alpha(1-\cos\alpha)\right]\overrightarrow{X_1}\ldots\\ \qquad\ldots+\sin\alpha\cos\alpha(1-\cos\alpha)\overrightarrow{X_2}-\sin^2\alpha\overrightarrow{X_3}\\ \overrightarrow{x_2}=\mathscr{R}_{\vec{u},\alpha}\left(\overrightarrow{X_2}\right)=\sin\alpha\cos\alpha(1-\cos\alpha)\overrightarrow{X_1}\ldots\\ \qquad\ldots+\left[1-\cos^2\alpha(1-\cos\alpha)\right]\overrightarrow{X_2}+\sin\alpha\cos\alpha\overrightarrow{X_3}\\ \overrightarrow{x_3}=\mathscr{R}_{\vec{u},\alpha}\left(\overrightarrow{X_3}\right)=\sin^2\alpha\overrightarrow{X_1}-\sin\alpha\cos\alpha\overrightarrow{X_2}+\cos\alpha\overrightarrow{X_3}\end{cases}.$$

*Question 2*: Establish the matrix representation $[p(X,x)]$ of this vector rotation.

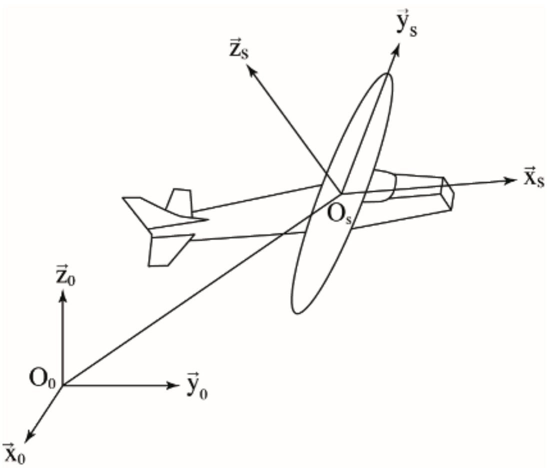
The matrix representation of this vector rotation is the matrix of change of basis from $(X)$ to $(x)$, hence

$$\left[\mathscr{R}_{\vec{u},\alpha}\right]=[p(X,x)]=\begin{bmatrix}1-\sin^2\alpha(1-\cos\alpha) & \sin\alpha\cos\alpha(1-\cos\alpha) & \sin^2\alpha\\ \sin\alpha\cos\alpha(1-\cos\alpha) & 1-\cos^2\alpha(1-\cos\alpha) & -\sin\alpha\cos\alpha\\ -\sin^2\alpha & \sin\alpha\cos\alpha & \cos\alpha\end{bmatrix}.$$

## 1.6. Other exercises

### 1.6.1. *Exercise 7 – Location of an airplane – Euler angles*

The orientation of the airplane $(S)$ should be determined relative to a system of reference $\langle 0 \rangle \equiv \left\langle O_0 \,|\, \overrightarrow{x_0}\; \overrightarrow{y_0}\; \overrightarrow{z_0} \right\rangle$.




The basis connected to the airplane is defined by:

– $\overrightarrow{x_S}$ , unit vector along the fuselage axis and oriented forwards;

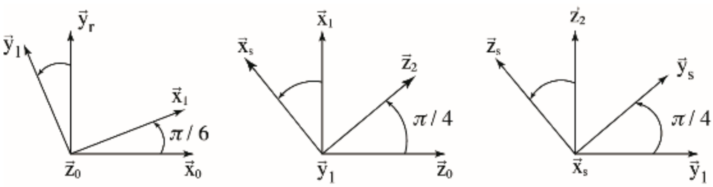
– $\overrightarrow{y_S}$ , unit vector along the longitudinal axis of the wings and oriented towards the left of the pilot so that the third axis of the trihedron directly connected to the airplane should be oriented towards the upper part of it;

– $\overrightarrow{z_S} = \overrightarrow{x_S} \wedge \overrightarrow{y_S}$ , unit vector oriented “upwards” relative to the pilot.

The three angles $\alpha, \beta, \gamma$ that describe the orientation of the airplane are associated with three plane rotations defined below

$$\left(\overrightarrow{x_0}\ \overrightarrow{y_0}\ \overrightarrow{z_0}\right) \xrightarrow{\mathscr{R}_{\overrightarrow{z_0},\alpha=\frac{\pi}{6}}} \left(\overrightarrow{x_1}\ \overrightarrow{y_1}\ \overrightarrow{z_0}\right) \xrightarrow{\mathscr{R}_{\overrightarrow{y_1},\beta=\frac{\pi}{4}}} \left(\overrightarrow{x_S}\ \overrightarrow{y_1}\ \overrightarrow{z_2}\right) \xrightarrow{\mathscr{R}_{\overrightarrow{x_S},\gamma=\frac{\pi}{4}}} \left(\overrightarrow{x_S}\ \overrightarrow{y_S}\ \overrightarrow{z_S}\right)$$

*Question 1*: Draw the three plane diagrams that illustrate these 3 rotations.




*Question 2*: Express $\overrightarrow{x_S}$, $\overrightarrow{y_S}$ and $\overrightarrow{z_S}$ by their components in the basis $(0)$.

$$\overrightarrow{x_S} = \cos\frac{\pi}{4}\overrightarrow{x_1} - \sin\frac{\pi}{4}\overrightarrow{z_0} = \cos\frac{\pi}{4}\left(\cos\frac{\pi}{6}\overrightarrow{x_0} + \sin\frac{\pi}{6}\overrightarrow{y_0}\right) = \frac{\sqrt{6}}{4}\overrightarrow{x_0} + \frac{\sqrt{2}}{4}\overrightarrow{y_0} - \frac{\sqrt{2}}{2}\overrightarrow{z_0},$$

$$\overrightarrow{y_s} = \cos\frac{\pi}{4}\overrightarrow{y_1} + \sin\frac{\pi}{4}\overrightarrow{z_2} = \cos\frac{\pi}{4}\overrightarrow{y_1} + \sin\frac{\pi}{4}\left(\cos\frac{\pi}{4}\overrightarrow{z_0} + \sin\frac{\pi}{4}\overrightarrow{x_1}\right)$$

$$'' = \sin^2\frac{\pi}{4}\left(\cos\frac{\pi}{6}\overrightarrow{x_0} + \sin\frac{\pi}{6}\overrightarrow{y_0}\right) + \cos\frac{\pi}{4}\left[-\sin\frac{\pi}{6}\overrightarrow{x_0} + \cos\frac{\pi}{6}\overrightarrow{y_0}\right] + \sin\frac{\pi}{4}\cos\frac{\pi}{4}\overrightarrow{z_0},$$

$$'' = \left(\sin^2\frac{\pi}{4}\cos\frac{\pi}{6} - \cos\frac{\pi}{4}\sin\frac{\pi}{6}\right)\overrightarrow{x_0} + \left(\sin^2\frac{\pi}{4}\sin\frac{\pi}{6} + \cos\frac{\pi}{4}\cos\frac{\pi}{6}\right)\overrightarrow{y_0} + \sin\frac{\pi}{4}\cos\frac{\pi}{4}\overrightarrow{z_0}$$

$$\Rightarrow\ \overrightarrow{y_S} = \frac{\sqrt{3}-\sqrt{2}}{4}\overrightarrow{x_0} + \frac{1+\sqrt{6}}{4}\overrightarrow{y_0} + \frac{1}{2}\overrightarrow{z_0},$$

$$\vec{z}_S = -\sin\frac{\pi}{4}\vec{y}_1 + \cos\frac{\pi}{4}\vec{z}_2 = -\sin\frac{\pi}{4}\vec{y}_1 + \cos\frac{\pi}{4}\left(\cos\frac{\pi}{4}\vec{z}_0 + \sin\frac{\pi}{4}\vec{x}_1\right)$$

$$'' = \sin\frac{\pi}{4}\cos\frac{\pi}{4}\left(\cos\frac{\pi}{6}\vec{x}_0 + \sin\frac{\pi}{6}\vec{y}_0\right) - \sin\frac{\pi}{4}\left[-\sin\frac{\pi}{6}\vec{x}_0 + \cos\frac{\pi}{6}\vec{y}_0\right] + \cos^2\frac{\pi}{4}\vec{z}_0 ,$$

$$'' = \left(\sin\frac{\pi}{4}\cos\frac{\pi}{4}\cos\frac{\pi}{6} + \sin\frac{\pi}{4}\sin\frac{\pi}{6}\right)\vec{x}_0 + \left(\sin\frac{\pi}{4}\cos\frac{\pi}{4}\sin\frac{\pi}{6} - \sin\frac{\pi}{4}\cos\frac{\pi}{6}\right)\vec{y}_0 + \cos^2\frac{\pi}{4}\vec{z}_0$$

$$\Rightarrow \overrightarrow{z_S} = \frac{\sqrt{3}+\sqrt{2}}{4}\overrightarrow{x_0} + \frac{1-\sqrt{6}}{4}\overrightarrow{y_0} + \frac{1}{2}\overrightarrow{z_0} .$$

*Question 3*: Deduce from it the matrix of passage $[\,p(0,S)\,]$.

The table of passage from the reference basis $(0)$ to the basis $(S)$ connected to the airplane can be written

$$p(0,S) \Leftrightarrow \begin{array}{c|ccc} & \overrightarrow{x_S} & \overrightarrow{y_S} & \overrightarrow{z_S} \\ \hline \overrightarrow{x_0} & \frac{\sqrt{6}}{4} & \frac{\sqrt{3}-\sqrt{2}}{4} & \frac{\sqrt{3}+\sqrt{2}}{4} \\ \overrightarrow{y_0} & \frac{\sqrt{2}}{4} & \frac{1+\sqrt{6}}{4} & \frac{1-\sqrt{6}}{4} \\ \overrightarrow{z_0} & -\frac{\sqrt{2}}{2} & \frac{1}{2} & \frac{1}{2} \end{array},$$

and the matrix of passage is

$$[\,p(0,S)\,] = \begin{bmatrix} \frac{\sqrt{6}}{4} & \frac{\sqrt{3}-\sqrt{2}}{4} & \frac{\sqrt{3}+\sqrt{2}}{4} \\ \frac{\sqrt{2}}{4} & \frac{1+\sqrt{6}}{4} & \frac{1-\sqrt{6}}{4} \\ -\frac{\sqrt{2}}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

*Question 4*: Determine the corresponding values of the Euler angles $\psi,\theta,\varphi$ for the orientation of $(S)$ relative to $(0)$

– $\vec{z_0}\cdot\vec{z_S} = \cos\theta = \frac{1}{2} = \cos\frac{\pi}{3} \quad\Rightarrow\quad \theta = \pm\frac{\pi}{3} + 2k\pi \quad\Rightarrow\quad \sin\theta = \pm\frac{\sqrt{3}}{2}$,

– $\vec{z_0}\wedge\vec{z_S} = \vec{n}\sin\theta$,

$$\begin{cases} \vec{z}_0 \wedge \left(\frac{\sqrt{3}+\sqrt{2}}{4}\vec{x}_0 + \frac{1-\sqrt{6}}{4}\vec{y}_0 + \frac{1}{2}\vec{z}_0\right) = -\frac{1-\sqrt{6}}{4}\vec{x}_0 + \frac{\sqrt{3}+\sqrt{2}}{4}\vec{y}_0 = \vec{n}\sin\theta \\ \left(-\frac{\sqrt{2}}{2}\vec{x}_S + \frac{1}{2}\vec{y}_S + \frac{1}{2}\vec{z}_S\right)\wedge\vec{z}_S = \frac{1}{2}\vec{x}_S + \frac{\sqrt{2}}{2}\vec{y}_S = \vec{n}\sin\theta \end{cases},$$

$$\begin{cases} \text{for } \theta = \frac{\pi}{3}, \ \vec{n} = -\frac{1-\sqrt{6}}{2\sqrt{3}}\vec{x_0} + \frac{\sqrt{3}+\sqrt{2}}{2\sqrt{3}}\vec{y_0} = \frac{\sqrt{3}}{3}\vec{x_S} + \frac{\sqrt{6}}{3}\vec{y_S} \\ \text{for } \theta = -\frac{\pi}{3}, \ \vec{n} = \frac{1-\sqrt{6}}{2\sqrt{3}}\vec{x_0} - \frac{\sqrt{3}+\sqrt{2}}{2\sqrt{3}}\vec{y_0} = -\frac{\sqrt{3}}{3}\vec{x_S} - \frac{\sqrt{6}}{3}\vec{y_S} \end{cases},$$

$$- \begin{cases} \vec{n}\cdot\vec{x_0} = \cos\psi = \mp\frac{1-\sqrt{6}}{2\sqrt{3}} \\ \vec{n}\cdot\vec{y_0} = \sin\psi = \pm\frac{\sqrt{3}+\sqrt{2}}{2\sqrt{3}} \end{cases} \quad \begin{cases} \vec{n}\cdot\vec{x_S} = \cos\varphi = \pm\frac{\sqrt{3}}{3} \\ \vec{n}\cdot\vec{y_S} = -\sin\varphi = \pm\frac{\sqrt{6}}{3} \end{cases}.$$

We shall verify that $\sin^2\psi + \cos^2\psi = 1$ , $\sin^2\varphi + \cos^2\varphi = 1$. In summary

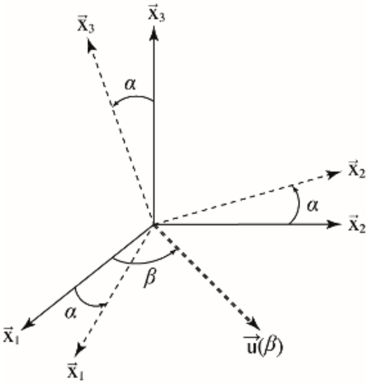
$$- \ \theta = \frac{\pi}{3}, \quad \begin{cases} \cos\psi = -\frac{1-\sqrt{6}}{2\sqrt{3}} = 0{,}418 \ , \ \sin\psi = \frac{\sqrt{3}+\sqrt{2}}{2\sqrt{3}} = 0{,}908 \\ \cos\varphi = \frac{\sqrt{3}}{3} = 0{,}577 \ , \ \sin\varphi = -\frac{\sqrt{6}}{3} = -0{,}817 \end{cases},$$

$\Rightarrow \quad \psi = 1{,}139\, rd, \quad \varphi = -0{,}955\, rd$

$$- \ \theta = -\frac{\pi}{3}, \quad \begin{cases} \cos\psi = \frac{1-\sqrt{6}}{2\sqrt{3}}, \sin\psi = -\frac{\sqrt{3}+\sqrt{2}}{2\sqrt{3}} \\ \cos\varphi = -\frac{\sqrt{3}}{3} \ , \ \sin\varphi = \frac{\sqrt{6}}{3} \end{cases}.$$

### 1.6.2. *Exercise 8 – Vector rotation*

Let $\mathscr{R}_{\vec{u}(\beta),\alpha}$ be a vector rotation with $\vec{u}(\beta) = \cos\beta\,\overrightarrow{X_1} + \sin\beta\,\overrightarrow{X_2}$, which transforms the direct orthonormal basis $(X) \equiv \left(\overrightarrow{X_1}\,\overrightarrow{X_2}\,\overrightarrow{X_3}\right)$ into $(x) \equiv \left(\overrightarrow{x_1}\,\overrightarrow{x_2}\,\overrightarrow{x_3}\right)$, which is also a direct orthonormal basis.




*Question 1*: Write the matrix representation $\left[\mathscr{R}_{\vec{u}(\beta),\alpha}\right]$ of this transformation.

The matrix representation $[\mathbf{F}]$ of the vector rotation by angle $\alpha$ about the axis of unit vector $\vec{u} = u_i\overrightarrow{x_i}$ can be written in the basis $(e) \equiv \left(\overrightarrow{x_1}\,\overrightarrow{x_2}\,\overrightarrow{x_3}\right)$

$$[\mathbf{F}]_{/e} = \begin{bmatrix} \cos\alpha + (1-\cos\alpha)u_1^2 & (1-\cos\alpha)u_1u_2 - \sin\alpha\,u_3 & (1-\cos\alpha)u_3u_1 + \sin\alpha\,u_2 \\ (1-\cos\alpha)u_1u_2 + \sin\alpha\,u_3 & \cos\alpha + (1-\cos\alpha)u_2^2 & (1-\cos\alpha)u_2u_3 - \sin\alpha\,u_1 \\ (1-\cos\alpha)u_3u_1 - \sin\alpha\,u_2 & (1-\cos\alpha)u_2u_3 + \sin\alpha\,u_1 & \cos\alpha + (1-\cos\alpha)u_3^2 \end{bmatrix},$$

and consequently, the vector rotation matrix can be written in the basis $(X)$

$$\left[\mathscr{R}_{\vec{u}(\beta),\alpha}\right]_{/(X)} = \begin{bmatrix} \cos\alpha + (1-\cos\alpha)\cos^2\beta & (1-\cos\alpha)\sin\beta\cos\beta & \sin\alpha\sin\beta \\ (1-\cos\alpha)\sin\beta\cos\beta & \cos\alpha + (1-\cos\alpha)\sin^2\beta & -\sin\alpha\cos\beta \\ -\sin\alpha\sin\beta & \sin\alpha\cos\beta & \cos\alpha \end{bmatrix}.$$

*Question 2*: Deduce from this, as a function of $\alpha$ and $\beta$, the Euler angles and the nodal vector $\vec{n}$ that define the orientation of $(x)$ relative to $(X)$.

The table of passage from the basis $(X)$ to the basis $(x)$ can be expressed as

$$p(X,x) \Leftrightarrow \begin{array}{c|ccc} & \vec{x}_1 & \vec{x}_2 & \vec{x}_3 \\ \hline \vec{X}_1 & \cos\alpha + (1-\cos\alpha)\cos^2\beta & (1-\cos\alpha)\sin\beta\cos\beta & \sin\alpha\sin\beta \\ \vec{X}_2 & (1-\cos\alpha)\sin\beta\cos\beta & \cos\alpha + (1-\cos\alpha)\sin^2\beta & -\sin\alpha\cos\beta \\ \vec{X}_3 & -\sin\alpha\sin\beta & \sin\alpha\cos\beta & \cos\alpha \end{array}.$$

In order to define the Euler representation of the relative location of these two bases, the following elements are available:

–

$$\overrightarrow{X_3}\cdot\overrightarrow{x_3} = \cos\theta = \cos\alpha \quad \Rightarrow \quad \theta = \pm\alpha + 2\mathrm{k}\pi \quad \Rightarrow \quad \sin\theta = \pm\sin\alpha,$$

– $\overrightarrow{X_3}\wedge\overrightarrow{x_3} = \vec{n}\sin\theta$,

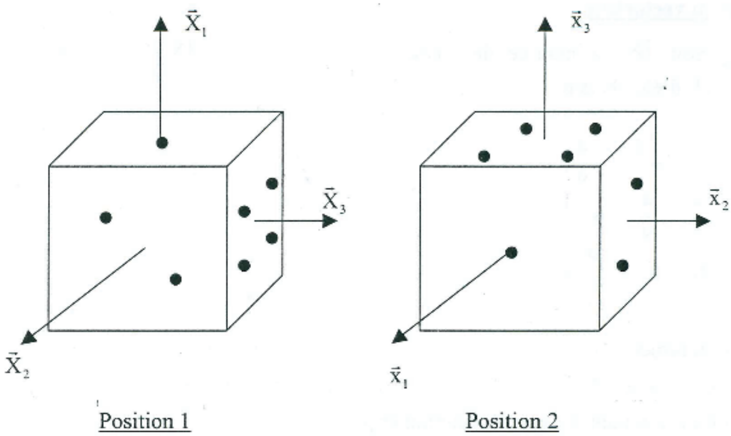
$$\Rightarrow \quad \begin{cases} \vec{X}_3 \wedge \left(\sin\alpha\sin\beta\,\vec{X}_1 - \sin\alpha\cos\beta\,\vec{X}_2 + \cos\alpha\,\vec{X}_3\right) = \sin\alpha\cos\beta\,\vec{X}_1 + \sin\alpha\sin\beta\,\vec{X}_2 \\ \qquad\qquad\qquad '' \quad '' \qquad\qquad\qquad = \pm\vec{n}\sin\alpha \\ \left(-\sin\alpha\sin\beta\,\vec{x}_1 + \sin\alpha\cos\beta\,\vec{x}_2 + \cos\alpha\,\vec{x}_3\right)\wedge\vec{x}_3 = \sin\alpha\cos\beta\,\vec{x}_1 + \sin\alpha\sin\beta\,\vec{x}_2 \\ \qquad\qquad\qquad '' \quad '' \qquad\qquad\qquad = \pm\vec{n}\sin\alpha \end{cases},$$

$$\begin{cases} \text{for } \theta = \alpha, \ \vec{n} = \cos\beta\,\overrightarrow{X_1} + \sin\beta\,\overrightarrow{X_2} = \cos\beta\,\overrightarrow{x_1} + \sin\beta\,\overrightarrow{x_2} \\ \text{for } \theta = -\alpha, \ \vec{n} = -\left(\cos\beta\,\overrightarrow{X_1} + \sin\beta\,\overrightarrow{X_2}\right) = -\left(\cos\beta\,\overrightarrow{x_1} + \sin\beta\,\overrightarrow{x_2}\right) \end{cases},$$

$$- \begin{cases} \vec{n}\cdot\overrightarrow{X_1} = \cos\psi = \pm\cos\beta \\ \vec{n}\cdot\overrightarrow{X_2} = \sin\psi = \pm\sin\beta \end{cases} \quad \Rightarrow \quad \begin{cases} \text{for } \theta = \alpha, \ \psi = \beta + 2k\pi \\ \text{for } \theta = -\alpha, \ \psi = \pi + \beta + 2k\pi \end{cases},$$

$$- \begin{cases} \vec{n}\cdot\overrightarrow{x_1} = \cos\varphi = \pm\cos\beta \\ \vec{n}\cdot\overrightarrow{x_2} = -\sin\varphi = \pm\sin\beta \end{cases} \quad \Rightarrow \quad \begin{cases} \text{for } \theta = \alpha, \ \varphi = -\beta + 2k\pi \\ \text{for } \theta = -\alpha, \ \varphi = \pi - \beta + 2k\pi \end{cases}.$$

### 1.6.3. *Exercise 9 – Vector rotation*




Let us consider the two orientations of the dice $(S)$ defined by the above positions 1 and 2, with which the two direct orthonormal bases $(X) \equiv \left(\overrightarrow{X_1}\ \overrightarrow{X_2}\ \overrightarrow{X_3}\right)$ and $(x) \equiv \left(\overrightarrow{x_1}\ \overrightarrow{x_2}\ \overrightarrow{x_3}\right)$ are associated.

*Question 1*: Determine the matrix of change of basis $[\,p(X,x)\,]$.

In the basis $(X)$, we have

$$\begin{aligned}\overrightarrow{x_3} &= \overrightarrow{X_1}\\ \overrightarrow{x_1} &= \overrightarrow{X_2}\\ \overrightarrow{x_2} &= \overrightarrow{X_3}\end{aligned} \quad \text{hence } [\,p(X,x)\,] = \begin{bmatrix} 0 & 0 & 1\\ 1 & 0 & 0\\ 0 & 1 & 0\end{bmatrix} = \begin{bmatrix} p_{11} & p_{12} & p_{13}\\ p_{21} & p_{22} & p_{23}\\ p_{31} & p_{32} & p_{33}\end{bmatrix}.$$

*Question 2*: Determine the vector rotation $\mathscr{R}_{\vec{u},\alpha}$ for passing from position 1 to position 2

Since the above matrix of change of basis is the vector rotation matrix $\mathscr{R}_{\vec{u},\alpha}$, we obtain

$$Trace\left[\mathscr{R}_{\vec{u},\alpha}\right] = 1 + 2\cos\alpha = Trace[\,p(X,x)\,] = p_{ii} = 0\,,$$

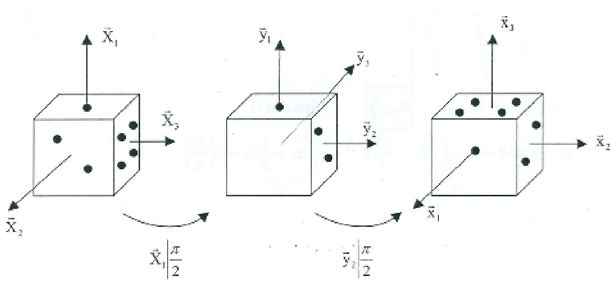
$$\cos\alpha = -\frac{1}{2} \quad \text{with} \quad \alpha \in [0,\pi] \quad \Rightarrow \quad \alpha = \frac{2\pi}{3} \quad \Rightarrow \quad \sin\alpha = \frac{\sqrt{3}}{2}.$$

The vector $\vec{u} = u_i \overrightarrow{X_i}$ is such that $\begin{cases} u_1 = \dfrac{p_{32} - p_{23}}{2\sin\alpha} = \dfrac{\sqrt{3}}{3} \\ u_2 = \dfrac{p_{13} - p_{31}}{2\sin\alpha} = \dfrac{\sqrt{3}}{3}, \\ u_3 = \dfrac{p_{21} - p_{12}}{2\sin\alpha} = \dfrac{\sqrt{3}}{3} \end{cases}$

$$\Rightarrow \quad \mathscr{R}_{\vec{u},\alpha}(X,x) = \left[ \vec{u} = \frac{\sqrt{3}}{3}\left(\overrightarrow{X_1} + \overrightarrow{X_2} + \overrightarrow{X_3}\right) \middle| \alpha = \frac{2\pi}{3} \right].$$

*Question 3*: Show that it is equally possible to obtain the position 2 of the dice by two successive plane rotations starting from position 1. Write the matrices of passage associated to these two rotation planes.

An intermediary position of the dice, to which the basis $(y) \equiv \left(\overrightarrow{y_1}\ \overrightarrow{y_2}\ \overrightarrow{y_3}\right)$ is assigned, should be considered. A first vector rotation is applied, such that $\vec{u} = \overrightarrow{X_1}$ and the angle is $\alpha = \dfrac{\pi}{2}$, then a second one such that $\vec{u} = \overrightarrow{y_2}$ and $\alpha = \dfrac{\pi}{2}$.




Knowing that

$$\vec{b} = \mathscr{R}_{\vec{u},\alpha}\left(\vec{a}\right) = \vec{a} + \left[\vec{u} \wedge \left(\vec{u} \wedge \vec{a}\right)\right](1 - \cos\alpha) + \left(\vec{u} \wedge \vec{a}\right)\sin\alpha,$$

– the first vector rotation $\begin{cases} \overrightarrow{y_1} = \mathscr{R}_{\overrightarrow{X_1},\frac{\pi}{2}}\left(\overrightarrow{X_1}\right) = \overrightarrow{X_1} \\ \overrightarrow{y_2} = \mathscr{R}_{\overrightarrow{X_1},\frac{\pi}{2}}\left(\overrightarrow{X_2}\right) = \overrightarrow{X_2} - \overrightarrow{X_2} + \overrightarrow{X_3} = \overrightarrow{X_3} \\ \overrightarrow{y_3} = \mathscr{R}_{\overrightarrow{X_1},\frac{\pi}{2}}\left(\overrightarrow{X_3}\right) = \overrightarrow{X_3} - \overrightarrow{X_3} - \overrightarrow{X_2} = -\overrightarrow{X_2} \end{cases}$ ,

– the second vector rotation $\begin{cases} \overrightarrow{x_1} = \mathscr{R}_{\overrightarrow{y_2},\frac{\pi}{2}}\left(\overrightarrow{y_1}\right) = \overrightarrow{y_1} - \overrightarrow{y_1} - \overrightarrow{y_3} = -\overrightarrow{y_3} \\ \overrightarrow{x_2} = \mathscr{R}_{\overrightarrow{y_2},\frac{\pi}{2}}\left(\overrightarrow{y_2}\right) = \overrightarrow{y_2} \\ \overrightarrow{x_3} = \mathscr{R}_{\overrightarrow{y_2},\frac{\pi}{2}}\left(\overrightarrow{y_3}\right) = \overrightarrow{y_3} - \overrightarrow{y_3} + \overrightarrow{y_1} = \overrightarrow{y_1} \end{cases}$ .

We then obtain the two matrices of passage

$$[p(X,y)] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad [p(y,x)] = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix},$$

and it can be verified that

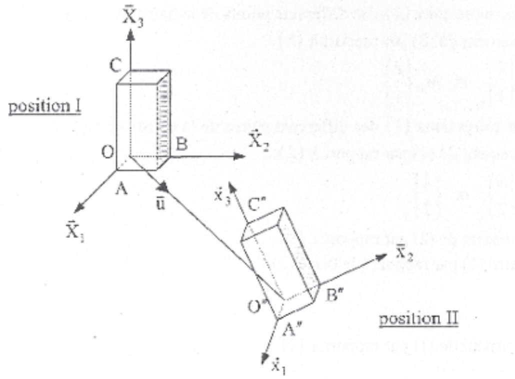
$$[p(X,x)] = [p(X,y)] \times [p(y,x)].$$

### 1.6.4. *Exercise 10 – Vector rotation*

Let us consider two positions of the solid $(S)$ represented by a rectangular parallelepiped with sides $a$, $2a$ and $3a$.

The position I of this solid is identified by the system of reference connected to it $\langle R_1 \rangle \equiv \left\langle \mathrm{O} \middle| \overrightarrow{X_1}\, \overrightarrow{X_2}\, \overrightarrow{X_3} \right\rangle$, which is obtained starting from three of its edges and one of its vertices and defined, according to the diagram, by

$$\overrightarrow{\mathrm{OA}} = a\overrightarrow{X_1}, \quad \overrightarrow{\mathrm{OB}} = 2a\overrightarrow{X_2}, \quad \overrightarrow{\mathrm{OC}} = 3a\overrightarrow{X_3}\ .$$




During time its location changes. Its position II is characterized by the system of reference connected to it $\langle R_2 \rangle \equiv \left\langle O'' | \overrightarrow{x_1}\, \overrightarrow{x_2}\, \overrightarrow{x_3} \right\rangle$ so that

$$\begin{cases} \overrightarrow{x_1} = \dfrac{1}{18}\left[ 10\overrightarrow{X_1} + \left(2 - 6\sqrt{3}\right)\overrightarrow{X_2} - \left(2 + 6\sqrt{3}\right)\overrightarrow{X_3} \right] \\ \overrightarrow{x_2} = \dfrac{1}{18}\left[ \left(2 + 6\sqrt{3}\right)\overrightarrow{X_1} + 13\overrightarrow{X_2} + \left(-4 + 3\sqrt{3}\right)\overrightarrow{X_3} \right] \\ \overrightarrow{x_3} = \dfrac{1}{18}\left[ \left(-2 + 6\sqrt{3}\right)\overrightarrow{X_1} - \left(4 + 3\sqrt{3}\right)\overrightarrow{X_2} + 13\overrightarrow{X_3} \right] \end{cases}.$$

The passage from one position to the other, executed by a robot, is a combination of vector rotation $\mathscr{R}_{\vec{u},\alpha}$ and translation $\overline{OO''} = 6a\vec{u}$ where $\vec{u}$ is a unit vector.

*Question 1*: Write the matrix of passage $[\,p(X,x)\,]$ and deduce from it $\vec{u}$ and $\alpha$.

$$p(X,x) \Leftrightarrow \begin{array}{c|ccc} & \overrightarrow{x_1} & \overrightarrow{x_2} & \overrightarrow{x_3} \\ \hline \overrightarrow{X_1} & \dfrac{5}{9} & \dfrac{1+3\sqrt{3}}{9} & \dfrac{-1+3\sqrt{3}}{9} \\ \overrightarrow{X_2} & \dfrac{1-3\sqrt{3}}{9} & \dfrac{13}{18} & -\dfrac{4+3\sqrt{3}}{18} \\ \overrightarrow{X_3} & -\dfrac{1+3\sqrt{3}}{9} & \dfrac{-4+3\sqrt{3}}{18} & \dfrac{13}{18} \end{array} \cdot$$

$$Trace[\,p(X,x)\,]=\frac{5}{9}+\frac{13}{18}+\frac{13}{18}=2=1+2\cos\alpha \quad\Rightarrow\quad \cos\alpha=\frac{1}{2},$$

For $\alpha\in[0,\pi]$, $\alpha=\frac{\pi}{3}$ and $\sin\alpha=\frac{\sqrt{3}}{2}$, we obtain the vector $\vec{u}=u_i\,\overrightarrow{X_i}$

$$\begin{cases} u_1=\dfrac{p_{32}-p_{23}}{2\sin\alpha}=\dfrac{1}{3}\\ u_2=\dfrac{p_{13}-p_{31}}{2\sin\alpha}=\dfrac{2}{3}\\ u_3=\dfrac{p_{21}-p_{12}}{2\sin\alpha}=-\dfrac{2}{3}\end{cases} \quad\Rightarrow\quad \vec{u}=\frac{1}{3}\left(\overrightarrow{X_1}+2\overrightarrow{X_2}-2\overrightarrow{X_3}\right).$$

*Question 2*: Determine the coordinates in $\langle R_1\rangle$ of the point $\mathrm{A}'$, which is the position of the vertex A after vector rotation.

After the vector rotation $\mathscr{R}_{\vec{u},\alpha}$ the point A is in the position $\mathrm{A}'$ so that

$$\overrightarrow{\mathrm{OA}'}=\mathscr{R}_{\vec{u},\alpha}\left(\overrightarrow{\mathrm{OA}}\right)=\mathscr{R}_{\vec{u},\alpha}\left(a\overrightarrow{X_1}\right)=a\vec{x}_1=a\left(\frac{5}{9}\overrightarrow{X_1}+\frac{1-3\sqrt{3}}{9}\overrightarrow{X_2}-\frac{1+3\sqrt{3}}{9}\overrightarrow{X_3}\right).$$

*Question 3*: Determine the coordinates in $\langle R_1\rangle$ of the point $\mathrm{A}''$, position of the vertex A as a result of the two transformations.

$$\overrightarrow{\mathrm{OA}''}=\overrightarrow{\mathrm{OA}'}+\overrightarrow{\mathrm{A}'\mathrm{A}''}$$

$$\text{'' ''}=a\left(\frac{5}{9}\overrightarrow{X_1}+\frac{1-3\sqrt{3}}{9}\overrightarrow{X_2}-\frac{1+3\sqrt{3}}{9}\overrightarrow{X_3}\right)+2a\left(\overrightarrow{X_1}+2\overrightarrow{X_2}-2\overrightarrow{X_3}\right).$$

$$\text{'' ''}=a\left(\frac{23}{9}\overrightarrow{X_1}+\frac{37-3\sqrt{3}}{9}\overrightarrow{X_2}-\frac{37+3\sqrt{3}}{9}\overrightarrow{X_3}\right)$$

*Question 4*: Determine the displacement vector of A, $\overrightarrow{\mathrm{AA}''}$, by its components in the basis $(R_1)$.

$$\overrightarrow{\mathrm{AA''}} = \overrightarrow{\mathrm{AO}} + \overrightarrow{\mathrm{OA'}} + \overrightarrow{\mathrm{A'A''}}$$

$$\text{" "} = -a\overrightarrow{X_1} + a\left(\frac{5}{9}\overrightarrow{X_1} + \frac{1-3\sqrt{3}}{9}\overrightarrow{X_2} - \frac{1+3\sqrt{3}}{9}\overrightarrow{X_3}\right) + 2a\left(\overrightarrow{X_1} + 2\overrightarrow{X_2} - 2\overrightarrow{X_3}\right)$$

$$\text{" "} = a\left(\frac{14}{9}\overrightarrow{X_1} + \frac{37-3\sqrt{3}}{9}\overrightarrow{X_2} - \frac{37+3\sqrt{3}}{9}\overrightarrow{X_3}\right)$$