1

Vector Calculus

The vector is the basic tool in the formalism of mechanics because it brings together in one concept two fundamental ideas, that is the size of the used parameter or the studied phenomenon and the direction in which it must be considered or in which it applies. The calculus rules that describe it are continuously exploited in the mathematical expression of the motion of bodies. This chapter lists them and develops them for the ease of use.

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1.1. Vector space

1.1.1. *Definition*

The vector space E is a set with two operating laws: an internal law, from $E \to E$, which confers an Abelian group structure (commutative), and an external law, the multiplication by a scalar. The elements of a vector space are called *vectors* and, in the formalism of mechanics, are generally represented by an alphabetical symbol topped with an arrow: $\vec{u}$.

1.1.1.1. *Properties of the internal composition law*

The internal composition law in the formalism of mechanics is the vector addition, denoted as +, and that has the following properties:

– if $\vec{u}, \vec{v} \in E$, so, $\vec{u} + \vec{v} \in E$;

– it is commutative: $\forall \vec{u},\vec{v} \in E, \vec{u}+\vec{v}=\vec{v}+\vec{u}$;

– it is associative: $\forall \vec{u},\vec{v},\vec{w} \in E, \vec{u}+\left(\vec{v}+\vec{w}\right)=\left(\vec{u}+\vec{v}\right)+\vec{w}$;

– it has a neutral element denoted as $\vec{0}$ so that: $\forall \vec{u} \in E, \vec{u}+\vec{0}=\vec{0}+\vec{u}=\vec{u}$;

– any element has an inverse (or opposite), that is to say: $\forall \vec{u} \in E,\ \exists -\vec{u} \in E,$ such as: $\vec{u}+\left(-\vec{u}\right)=\vec{u}-\vec{u}=\vec{0}$.

1.1.1.2. *Properties of the external composition law*

The external composition law is identified in the mechanical formalism as the multiplication by a scalar $\lambda \in \mathbb{R}$ of a vector $\vec{u} \in E$, such that:

$$\lambda \times \vec{u} = \lambda\vec{u} \in E.$$

This law has the following properties:

– there is a neutral element, the scalar 1 as: $1 \times \vec{u} = \vec{u}$;

– the law is distributive with respect to addition and in relation to the addition and multiplication in $\mathbb{R}$, as:

$$\begin{cases} \forall \lambda \in \mathbb{R}, \forall \vec{u},\vec{v} \in E : \lambda \times \left(\vec{u}+\vec{v}\right) = \lambda\vec{u}+\lambda\vec{v} \\ \forall \lambda,\mu \in \mathbb{R}, \forall \vec{u} \in E : (\lambda+\mu)\times\vec{u} = \lambda\vec{u}+\mu\vec{u} \\ \forall \lambda,\mu \in \mathbb{R}, \forall \vec{u} \in E : \lambda \times \left(\mu \times \vec{u}\right) = (\lambda\mu)\vec{u} \end{cases}$$

1.1.2. *Vector space – dimension – basis*

We say that n vectors $\overrightarrow{u_1},\overrightarrow{u_2},\ldots,\overrightarrow{u_n} \in E$ are linearly independent if the relation:

$$\lambda_1\overrightarrow{u_1}+\lambda_2\overrightarrow{u_2}+\ldots+\lambda_n\overrightarrow{u_n} = \lambda_\alpha\overrightarrow{u_\alpha} = \vec{0}$$

has only the following solution:

$$\lambda_\alpha = 0, \quad \forall \alpha = 1, \ldots, n .$$

We say that a vector space E is of *dimension* n when it holds at most n linearly independent vectors; any other vector element of this space is then expressed as a linear combination with coefficients $\lambda_\alpha \in \mathbb{R}$ of these n vectors. Any other vector of the vector space E of dimension n can be expressed from only these n linearly independent vectors, which constitute the *basis* of what we call *space*.

Thus, if this basis consists of n linearly independent vectors $\overrightarrow{u_1}, \overrightarrow{u_2}, \ldots, \overrightarrow{u_n} \in E$, any vector $\vec{V}$ of E can be written as:

$$\vec{V} = \mu_1 \overrightarrow{u_1} + \mu_2 \overrightarrow{u_2} + \ldots + \mu_n \overrightarrow{u_n} ,$$

where the coefficients $\mu_1, \mu_2, \ldots, \mu_n \in \mathbb{R}$ are the *components* of $\vec{V}$ in the considered basis.

1.1.3. *Affine space*

The first area to be considered as the frame of the movement of physical bodies is terrestrial space. This can be represented by geometrical parameters and its properties fit well with the concept of vector space, a mathematical entity to which it is advised to give a reality; hence the notion of affine space of vector space.

This notion follows a precise mathematical definition that is not necessary to repeat here; it merely gives the physical space the mathematical specificity, which is necessary to make it the mechanical workplace, and thus the characteristics of the vector space rules applied to it.

1.2. Affine space of dimension 3 – free vector

The geometric space of the mechanics of rigid solids is a conventional three-dimensional space in which the entire contents of this book through its various volumes will unfold. This is the affine space of a vector space of dimension 3 with groups of no more than three linearly independent vectors, which form the bases.

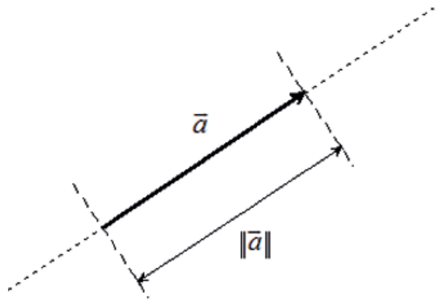

Figure 1.1. *Representation of the concept of vector in the affine space*

In geometric space, considered as affine space of a vector space of dimension 3, the vector is represented by a line segment oriented as an arrow. This representation contains three characteristics:

– an intensity that is illustrated by the segment length: intensity of a force and magnitude of a speed. The length of a vector $\vec{a}$ of the geometric space is called *norm* and recorded $\left\|\vec{a}\right\|$;

– a direction, indicated by the supporting line of the arrow, along which the physical phenomenon that represents the vector (strength, speed) is exerted;

– the orientation of application of this phenomenon is given by the orientation of this arrow.

These three characteristics are essential when it comes to studying phenomena that are exerted in the space of classical mechanics.

The vectors of the geometric affine space that have the same characteristics of norm, direction and orientation, regardless of the point that indicates their origin in this space (their point of application, it will be said), are equivalent and form a family or equivalence class called a *free vector*. Any vector of this family is likely to replace another of the same class in operations that do not require to specify the point of application; this precision is essential, however, for the calculation of moments at one point.

We will subsequently represent, in an illustration of an operation such as a sum, a scalar product or a vector product, two vectors from the same origin, the choice of which does not affect the result.

1.3. Scalar product $\vec{a} \cdot \vec{b}$

The scalar product of two vectors $\vec{a}$ and $\vec{b}$ is a scalar, represented by the operator symbol "$\cdot$"; it is defined by the relation:

$$\vec{a} \cdot \vec{b} = \left\|\vec{a}\right\| \times \left\|\vec{b}\right\| \times \cos\left(\widehat{\vec{a},\vec{b}}\right) \quad \text{where} \quad -\pi \leq \widehat{\vec{a},\vec{b}} \leq \pi\,.$$

Considering the property of the cosine function: $\cos(-\alpha) = \cos\alpha$, it is not necessary that the considered angle $\widehat{\vec{a},\vec{b}}$ is oriented; in these conditions, more effectively we can write the expression of the scalar product in the form:

$$\vec{a} \cdot \vec{b} = \left\|\vec{a}\right\| \times \left\|\vec{b}\right\| \times \cos\theta, \quad -\pi \leq \theta \leq \pi\,.$$

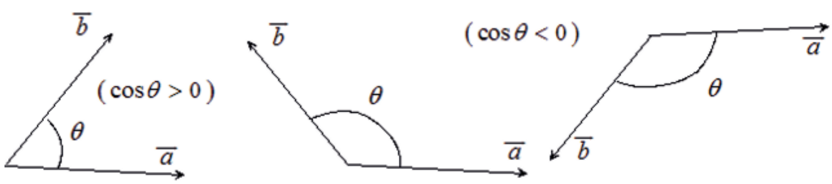

Figure 1.2. *Angle of vectors in the scalar product*

The sign of the scalar product that is an algebraic quantity depends on whether θ is acute $(\cos\theta > 0)$ or obtuse $(\cos\theta < 0)$.

1.3.1. *Properties of the scalar product*

Vis-à-vis the vector operations, specific to vector spaces, scalar product has the following properties:

– it is commutative:

$$\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}\,,$$

– it is distributive right and left with respect to the vector addition:

$$\vec{a}\cdot\left(\vec{b_1}+\vec{b_2}\right) = \vec{a}\cdot\vec{b_1}+\vec{a}\cdot\vec{b_2} \text{ and } \left(\vec{a_1}+\vec{a_2}\right)\cdot\vec{b} = \vec{a_1}\cdot\vec{b}+\vec{a_2}\cdot\vec{b}\,,$$

– its multiplication by a scalar gives:

$$\lambda\left(\vec{a}\cdot\vec{b}\right) = \left(\lambda\vec{a}\right)\cdot\vec{b} = \vec{a}\cdot\left(\lambda\vec{b}\right),$$

– the scalar product of two linear combinations of the vectors unfolds as follows:

$$\left(\sum_{i=1}^{m}\lambda_i\vec{a_i}\right)\cdot\left(\sum_{j=1}^{n}\mu_j\vec{b_j}\right) = \sum_{i=1}^{m}\sum_{j=1}^{n}\lambda_i\mu_j\left(\vec{a_i}\cdot\vec{b_j}\right).$$

1.3.2. *Scalar square – unit vector*

As $\cos\left(\widehat{\vec{a},\vec{a}}\right)=1$, we obtain the *scalar square of* $\vec{a}$ by the operation:

$$\vec{a}\cdot\vec{a} = \vec{a}^{\,2} = \left\|\vec{a}\right\|^2 \quad \text{so} \quad \left\|\vec{a}\right\| = \sqrt{\vec{a}^{\,2}}\,.$$

If we consider a group of three linearly independent vectors $\left(\vec{U_1},\vec{U_2},\vec{U_3}\right)$ of the affine space, we can define the three vectors:

$$\vec{u_1} = \frac{\overrightarrow{U_1}}{\left\|\overrightarrow{U_1}\right\|} \quad , \quad \vec{u_2} = \frac{\overrightarrow{U_2}}{\left\|\overrightarrow{U_2}\right\|} \quad , \quad \vec{u_3} = \frac{\overrightarrow{U_3}}{\left\|\overrightarrow{U_3}\right\|} \quad ,$$

which are vectors whose norms are equal to one, called *unit vectors* and that can be the basis (u) of the vector affine space considered. This basis (u) is more usefully represented by the unit vectors $\left(\vec{u_1}, \vec{u_2}, \vec{u_3}\right)$, because they offer a same measurement reference in all directions. This basis will be denoted as:

$$(u) \equiv \left(\vec{u_1}\ \vec{u_2}\ \vec{u_3}\right).$$

1.3.3. *Geometric interpretation of the scalar product*

Consider two vectors $\vec{a}$ and $\vec{b}$, and project orthogonally the vector $\vec{b}$ to the straight support of vector $\vec{a}$ whose unit vector is $\vec{u} = \dfrac{\vec{a}}{\left\|\vec{a}\right\|}$.

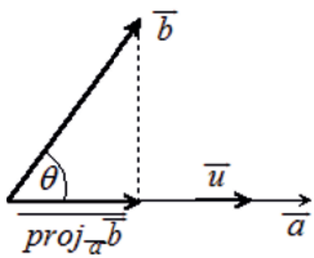

Figure 1.3. *Orthogonal projection of a vector on an axis*

The algebraic measurement of vector $\overrightarrow{proj_{\vec{a}}\vec{b}}$, denoted as $\left|proj_{\vec{a}}\vec{b}\right|$, which is the orthogonal projection of the vector $\vec{b}$ on the direction of the unit vector $\vec{u}$, is given by:

$$\left|proj_{\vec{a}}\vec{b}\right| = \left\|\vec{b}\right\|\cos\theta = \vec{u}\cdot\vec{b} = \frac{\vec{a}\cdot\vec{b}}{\left\|\vec{a}\right\|}.$$

The projection vector $\overrightarrow{proj_{\vec{a}}\vec{b}}$ is therefore written as:

$$\overrightarrow{proj_{\vec{a}}\vec{b}} = \left(\vec{u}\cdot\vec{b}\right)\vec{u} = \frac{\left(\vec{a}\cdot\vec{b}\right)\vec{a}}{\vec{a}^{\,2}}.$$

Under these conditions, we can also write:

$$\vec{a}\cdot\vec{b} = \left\|\vec{a}\right\|\times\left|proj_{\vec{a}}\vec{b}\right| = \vec{a}\cdot\overrightarrow{proj_{\vec{a}}\vec{b}}.$$

We consider the projection $\vec{b}$ on $-\vec{a}$ according to Figure 1.4.

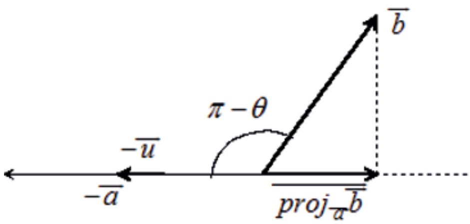

Figure 1.4. *Orthogonal projection of a vector on an axis*

This vector is expressed as:

$$\overrightarrow{proj_{-\vec{a}}\vec{b}} = \left(-\vec{u}\cdot\vec{b}\right)\left(-\vec{u}\right) = \left(\vec{u}\cdot\vec{b}\right)\vec{u} = \frac{\left(\vec{a}\cdot\vec{b}\right)\vec{a}}{\left\|\vec{a}\right\|^2} = \overrightarrow{proj_{\vec{a}}\vec{b}},$$

and its algebraic measure has the value:

$$\left|proj_{-\vec{a}}\vec{b}\right| = \left\|\vec{b}\right\|\cos(\pi-\theta) = -\left\|\vec{b}\right\|\cos\theta = -\vec{u}\cdot\vec{b} = -\frac{\vec{a}\cdot\vec{b}}{\left\|\vec{a}\right\|} = -\left|proj_{\vec{a}}\vec{b}\right|$$

1.3.4. *Solving the equation* $\vec{a} \cdot \vec{x} = 0$

Consider a given vector $\vec{a}$ and the equation $\vec{a} \cdot \vec{x} = 0$. It accepts two types of solution:

$$\begin{cases} \vec{x} = \vec{0} \\ \vec{x} \perp \vec{a} \quad \text{ so } \widehat{\vec{a}, \vec{x}} = \dfrac{\pi}{2} \text{ and } \cos\dfrac{\pi}{2} = 0 \end{cases}.$$

Therefore, the scalar product of two vectors is zero if one of the two vectors, at least, is zero, or if the two vectors are orthogonal.

1.4. Vector product $\vec{a} \wedge \vec{b}$

1.4.1. *Definition*

The vector product of the two vectors $\vec{a}$ and $\vec{b}$ of affine space E_3 of dimension 3 is represented by the operation $\vec{a} \wedge \vec{b}$. Its result is the vector $\vec{c} = \vec{a} \wedge \vec{b}$, which has the following properties:

– it is orthogonal to the plane formed by the vectors $\vec{a}$ and $\vec{b}$: $\vec{c} \perp \Pi\left(\vec{a}, \vec{b}\right)$;

– it is oriented so that the trihedron $\left(\vec{a}, \vec{b}, \vec{c}\right)$ is direct (see below);

– $\left\|\vec{c}\right\| = \left\|\vec{a}\right\| \times \left\|\vec{b}\right\| \times \left\|\sin\theta\right\|$ with $-\pi \le \theta \le \pi$.

NOTE.– According to the rule of the corkscrew by Maxwell, a corkscrew planted perpendicular to the plane $\Pi\left(\vec{a}, \vec{b}\right)$ progresses, when rotated from $\vec{a}$ toward $\vec{b}$, in the direction of the vector $\vec{c}$.

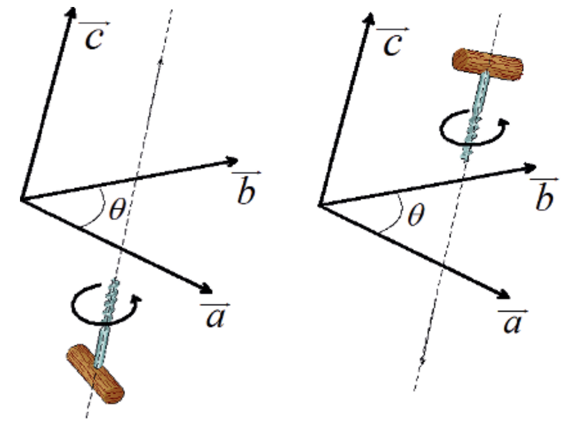

Figure 1.5. *Maxwell's corkscrew rule*

1.4.2. *Geometric interpretation of the vector product*

We consider the triangle OAB built on the two vectors $\vec{a}$ and $\vec{b}$ according to Figure 1.6.

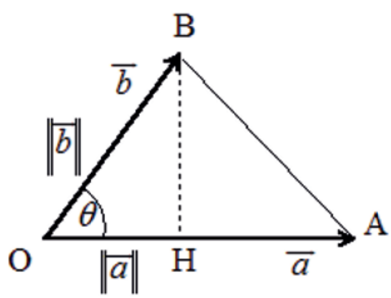

Figure 1.6. *Geometric interpretation of the vector product*

As such, the area of its surface is equal to $\frac{1}{2} \times \mathrm{OA} \times \mathrm{BH}$, so:

$$\mathscr{Aire}(\mathrm{OAB}) = \frac{1}{2} \times \|\vec{a}\| \times \|\vec{b}\| \sin\theta = \frac{1}{2}\|\vec{a} \wedge \vec{b}\| \quad .$$

1.4.3. *Properties of vector product*

The vector product has properties that are either its own or combined with the other operations of vectors:

– the vector product is anticommutative, that is to say:

$$\vec{b} \wedge \vec{a} = -\vec{a} \wedge \vec{b}.$$

This property results from the fact that the trihedrons $\left(\vec{a}, \vec{b}, \vec{a} \wedge \vec{b}\right)$ and $\left(\vec{b}, \vec{a}, \vec{b} \wedge \vec{a}\right)$ must be both direct;

– multiplication by a scalar obeys the rule:

$$\left(\lambda\vec{a}\right) \wedge \vec{b} = \lambda\left(\vec{a} \wedge \vec{b}\right) = \vec{a} \wedge \left(\lambda\vec{b}\right)$$

– the vector product is distributive with respect to the addition of vectors:

$$\vec{a} \wedge \left(\overrightarrow{b_1} + \overrightarrow{b_2}\right) = \vec{a} \wedge \overrightarrow{b_1} + \vec{a} \wedge \overrightarrow{b_2}$$

$$\left(\overrightarrow{a_1} + \overrightarrow{a_2}\right) \wedge \vec{b} = \overrightarrow{a_1} \wedge \vec{b} + \overrightarrow{a_2} \wedge \vec{b}$$

– the vector product of two linear combinations of vectors is developed:

$$\left(\sum_{i=1}^{m} \lambda_i \overrightarrow{a_i}\right) \wedge \left(\sum_{j=1}^{n} \mu_j \overrightarrow{b_j}\right) = \sum_{i=1}^{m} \sum_{j=1}^{n} \lambda_i \mu_j \left(\overrightarrow{a_i} \wedge \overrightarrow{b_j}\right).$$

1.4.4. *Solving the equation* $\vec{a} \wedge \vec{x} = \vec{0}$

Consider a given vector $\vec{a}$ and equation $\vec{a} \wedge \vec{x} = 0$. It accepts two types of solution:

$$\begin{cases} \vec{x} = \vec{0} \\ \vec{x} = \lambda\vec{a} \qquad \text{and, as a result,} \ \widehat{\vec{a}, \vec{x}} = 0 \ \text{ and } \ \sin 0 = 0 \end{cases}.$$

Therefore, the vector product of two vectors is zero if one of the two vectors, at least, is zero, or if the two vectors are co-linear.

1.5. Mixed product $\left(\vec{a},\vec{b},\vec{c}\right)$

1.5.1. *Definition*

The mixed product of three vectors is the scalar product denoted as $\left(\vec{a},\vec{b},\vec{c}\right)$, which represents the operation:

$$\left(\vec{a},\vec{b},\vec{c}\right)=\vec{a}\cdot\left(\vec{b}\wedge\vec{c}\right).$$

1.5.2. *Geometric interpretation of the mixed product*

Consider the trihedron formed by the three vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$.

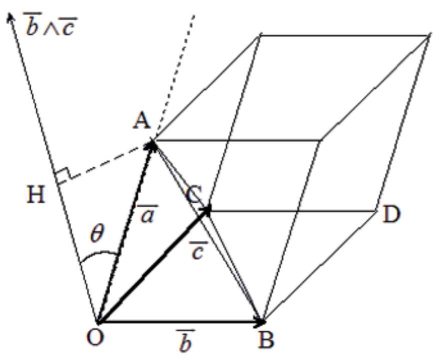

Figure 1.7. *Geometric interpretation of the mixed product*

Consider the vector product $\vec{b}\wedge\vec{c}$; according to the geometric interpretation of this operation, we write:

$$\text{Area (OBDC)} = 2\times\text{Area (OBC)} = \left\|\vec{b}\wedge\vec{c}\right\|.$$

The segment OH, orthogonal projection of $\vec{a}$ on the vector $\vec{b} \wedge \vec{c}$ which is orthogonal to the plane $\Pi\left(\vec{b}, \vec{c}\right)$, is the height of the parallelepiped constructed on the three vectors $\vec{a}$, $\vec{b}$, $\vec{c}$. The volume of the parallelepiped is equal to:

$$\mathscr{Aire}(\text{OBDC}) \times \text{OH}, \text{ with } \text{OH} = \left\|\vec{a}\right\| \times \cos\theta .$$

We deduce that

$$\begin{aligned} \mathscr{Vol}\left[\text{parallelepiped}\left(\vec{a}, \vec{b}, \vec{c}\right)\right] &= 6 \times \mathscr{Vol}\left[\text{tetrahedron}(\text{OABC})\right] \\ \text{"} \quad \text{"} \quad &= \left\|\vec{a} \cdot \left(\vec{b} \wedge \vec{c}\right)\right\| = \left\|\left(\vec{a}, \vec{b}, \vec{c}\right)\right\| \end{aligned} .$$

If we express this result as:

$$\left|\mathscr{Vol}\left[\text{parallelepiped}\left(\vec{a}\,\vec{b}\,\vec{c}\right)\right]\right| = \left(\vec{a}, \vec{b}, \vec{c}\right),$$

this algebraic volume is positive when the trihedron $\left(\vec{a}, \vec{b}, \vec{c}\right)$ is direct, negative when it is indirect.

1.5.3. *Properties of the mixed product*

1.5.3.1. *Cases of nullity of the mixed product*

The relationship $\left(\vec{a}, \vec{b}, \vec{c}\right) = 0$ corresponds to the following cases:

– $\vec{a} = \vec{0}$;

– $\vec{b} \wedge \vec{c} = \vec{0}$, which means one of these $\begin{cases} \vec{b} = \vec{0} \\ \vec{c} = \vec{0} \\ \exists \lambda \in \mathbb{R} \text{ such as } \vec{b} = \lambda \vec{c} \end{cases}$;

– $\vec{a} \perp \vec{b} \wedge \vec{c}$, which means $\exists\, \lambda, \mu \in \mathbb{R}$ such as $\vec{a} = \lambda \vec{b} + \mu \vec{c}$.

The mixed product of three vectors is zero when one of the vectors is zero or when two of them are at least linked.

1.5.3.2. *Circular permutation of terms*

If we consider the parallelepiped constructed on the three vectors $\vec{a}$, $\vec{b}$, $\vec{c}$, the calculation of its volume does not depend on the order in which one considers them. We can thus calculate at first the vector product of $\vec{a}$ and $\vec{b}$, then project $\vec{c}$ onto the vector $\vec{a} \wedge \vec{b}$, or the vector product $\vec{c} \wedge \vec{a}$ and project $\vec{b}$ onto this vector; the main thing is to conserve the order of vectors, since the algebraic sign of the volume depends on the direct or indirect order of the three vectors. We can either write the equalities:

$$\vec{a} \cdot \left(\vec{b} \wedge \vec{c} \right) = \vec{b} \cdot \left(\vec{c} \wedge \vec{a} \right) = \vec{c} \cdot \left(\vec{a} \wedge \vec{b} \right),$$

that is to say we can invert the operations, such that:

$$\vec{a} \cdot \left(\vec{b} \wedge \vec{c} \right) = \left(\vec{a} \wedge \vec{b} \right) \cdot \vec{c}.$$

Symbolically therefore we write:

$$\left(\vec{a}, \vec{b}, \vec{c} \right) = \left(\vec{b}, \vec{c}, \vec{a} \right) = \left(\vec{c}, \vec{a}, \vec{b} \right).$$

1.5.3.3. *Permutation of two terms*

For the same reasons related to the direct or indirect nature of the trihedron formed by the three vectors of a mixed product, in the order they are listed, the permutation of two vectors changes the sign of the result of the operation. So

$$\left(\vec{a}, \vec{c}, \vec{b} \right) = -\left(\vec{a}, \vec{b}, \vec{c} \right).$$

1.5.3.4. *Multiplication by a scalar*

Multiplying a mixed product by a scalar amounts to multiplying one of the vectors by this scalar:

$$\lambda \times \left(\vec{a}, \vec{b}, \vec{c} \right) = \left(\lambda \vec{a}, \vec{b}, \vec{c} \right) = \left(\vec{a}, \lambda \vec{b}, \vec{c} \right) = \left(\vec{a}, \vec{b}, \lambda \vec{c} \right).$$

1.5.3.5. *Distributivity*

The operation is distributive with respect to the addition of vectors on each member of the mixed product, as, for example:

$$\left(\overrightarrow{a_1}+\overrightarrow{a_2},\vec{b},\vec{c}\right)=\left(\overrightarrow{a_1},\vec{b},\vec{c}\right)+\left(\overrightarrow{a_2},\vec{b},\vec{c}\right).$$

1.5.3.6. *Mixed product of a combination of vectors*

$$\left(\sum_{i=1}^{l}\lambda_i\overrightarrow{a_i},\sum_{j=1}^{m}\mu_j\overrightarrow{b_j},\sum_{k=1}^{n}\nu_k\overrightarrow{c_k}\right)=\sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n}\lambda_i\mu_j\nu_k\left(\overrightarrow{a_i},\overrightarrow{b_j},\overrightarrow{c_k}\right).$$

1.6. Vector calculus in the affine space of dimension 3

1.6.1. *Orthonormal basis*

The orthogonal projection of vectors applies practically trigonometric functions. A trirectangular trihedron vector corresponds to linearly independent vectors since the orthogonal projection of one of them, on the plane formed by the two others, is always zero. The mechanical formalism is thereby greatly simplified. So it is worthwhile to resort, if another need does not justify it, to the orthogonal bases and, moreover, normed.

The vectors that constitute the orthonormal basis $(u)=\left(\overrightarrow{u_1}\,\overrightarrow{u_2}\,\overrightarrow{u_3}\right)$ therefore verify all the relations:

$$\begin{array}{lll}\overrightarrow{u_1}^2=1 & \overrightarrow{u_2}^2=1 & \overrightarrow{u_3}^2=1\\ \overrightarrow{u_1}\cdot\overrightarrow{u_2}=0 & \overrightarrow{u_2}\cdot\overrightarrow{u_3}=0 & \overrightarrow{u_3}\cdot\overrightarrow{u_1}=0\\ \overrightarrow{u_1}\wedge\overrightarrow{u_2}=\overrightarrow{u_3} & \overrightarrow{u_2}\wedge\overrightarrow{u_3}=\overrightarrow{u_1} & \overrightarrow{u_3}\wedge\overrightarrow{u_1}=\overrightarrow{u_2}\end{array}\quad.$$

1.6.2. *Analytical expression of the scalar product*

Consider the orthogonal basis $(u) = \left(\overrightarrow{u_1}\,\overrightarrow{u_2}\,\overrightarrow{u_3}\right)$, the two vectors $\vec{a}$ and $\vec{b}$, and their respective components (a_1, a_2, a_3) and (b_1, b_2, b_3) on it. These vectors are thus written:

$$\vec{a} = a_1\,\overrightarrow{u_1} + a_2\,\overrightarrow{u_2} + a_3\,\overrightarrow{u_3} \quad \text{and} \quad \vec{b} = b_1\,\overrightarrow{u_1} + b_2\,\overrightarrow{u_2} + b_3\,\overrightarrow{u_3}\,.$$

The scalar product:

$$\vec{a}\cdot\vec{b} = \left(a_1\,\overrightarrow{u_1} + a_2\,\overrightarrow{u_2} + a_3\,\overrightarrow{u_3}\right)\cdot\left(b_1\,\overrightarrow{u_1} + b_2\,\overrightarrow{u_2} + b_3\,\overrightarrow{u_3}\right),$$

thus is developed, taking into account the properties of the unit vectors of the orthonormal basis:

$$\vec{a}\cdot\vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3\,.$$

If the basis was not orthonormal, we should take it into account, in the development, the expression of the different scalar products $\overrightarrow{u_i}\cdot\overrightarrow{u_j}$ that are involved.

In the analytical form, the scalar square of the vector $\vec{a}$ is written as:

$$\vec{a}^{\,2} = \vec{a}\cdot\vec{a} = a_1^{\,2} + a_2^{\,2} + a_3^{\,2}\,.$$

1.6.3. *Analytical expression of the vector product*

Similarly, the vector product of the two vectors:

$$\vec{a} = a_1\,\overrightarrow{u_1} + a_2\,\overrightarrow{u_2} + a_3\,\overrightarrow{u_3} \quad \text{and} \quad \vec{b} = b_1\,\overrightarrow{u_1} + b_2\,\overrightarrow{u_2} + b_3\,\overrightarrow{u_3}$$

is written, taking into account the properties of the orthonormal basis:

$$\begin{aligned}\vec{a} \wedge \vec{b} &= \left(a_1 \overrightarrow{u_1} + a_2 \overrightarrow{u_2} + a_3 \overrightarrow{u_3} \right) \wedge \left(b_1 \overrightarrow{u_1} + b_2 \overrightarrow{u_2} + b_3 \overrightarrow{u_3} \right) \\ " \quad &= (a_2 b_3 - a_3 b_2) \overrightarrow{u_1} + (a_3 b_1 - a_1 b_3) \overrightarrow{u_2} + (a_1 b_2 - a_2 b_1) \overrightarrow{u_3}\end{aligned}.$$

We can calculate, using a practical method, this vector product by using the following determinant that is effectively developed with respect to the first line:

$$\begin{vmatrix} \overrightarrow{u_1} & \overrightarrow{u_2} & \overrightarrow{u_3} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (a_2 b_3 - a_3 b_2) \overrightarrow{u_1} - (a_1 b_3 - a_3 b_1) \overrightarrow{u_2} + (a_1 b_2 - a_2 b_1) \overrightarrow{u_3}$$

in order to obtain the expression.

1.6.4. *Analytical expression of the mixed product*

Consider the three vectors:

$$\begin{aligned}&\vec{a} = a_1 \overrightarrow{u_1} + a_2 \overrightarrow{u_2} + a_3 \overrightarrow{u_3}, \quad \vec{b} = b_1 \overrightarrow{u_1} + b_2 \overrightarrow{u_2} + b_3 \overrightarrow{u_3}, \\ &\vec{c} = c_1 \overrightarrow{u_1} + c_2 \overrightarrow{u_2} + c_3 \overrightarrow{u_3},\end{aligned}$$

and their mixed product $\left(\vec{a}, \vec{b}, \vec{c} \right)$:

$$\begin{aligned}\left(\vec{a}, \vec{b}, \vec{c} \right) &= \vec{a} \cdot \left(\vec{b} \wedge \vec{c} \right) = \left(a_1 \overrightarrow{u_1} + a_2 \overrightarrow{u_2} + a_3 \overrightarrow{u_3} \right) \ldots \\ &\qquad \ldots \quad \cdot \left[(b_2 c_3 - b_3 c_2) \overrightarrow{u_1} + (b_3 c_1 - b_1 c_3) \overrightarrow{u_2} + (b_1 c_2 - b_2 c_1) \overrightarrow{u_3} \right], \\ " \quad &= a_1 (b_2 c_3 - b_3 c_2) + a_2 (b_3 c_1 - b_1 c_3) + a_3 (b_1 c_2 - b_2 c_1) \\ \Rightarrow \quad &\left(\vec{a}, \vec{b}, \vec{c} \right) = a_1 (b_2 c_3 - b_3 c_2) + a_2 (b_3 c_1 - b_1 c_3) + a_3 (b_1 c_2 - b_2 c_1)\end{aligned}$$

results that may also be calculated by using the determinant:

$$\left|\vec{a},\vec{b},\vec{c}\right| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

1.7. Applications of vector calculus

1.7.1. *Double vector product*

1.7.1.1. *Definition*

Consider the three vectors $\vec{a}$, $\vec{b}$, $\vec{c}$, and the operation $\vec{a} \wedge \left(\vec{b} \wedge \vec{c}\right)$. The vector $\vec{b} \wedge \vec{c}$ is orthogonal to the plane $\Pi\left(\vec{b},\vec{c}\right)$; therefore, the vector $\vec{d} = \vec{a} \wedge \left(\vec{b} \wedge c\right)$ that is orthogonal to $\vec{b} \wedge \vec{c}$ belongs to this plane according to the following diagram.

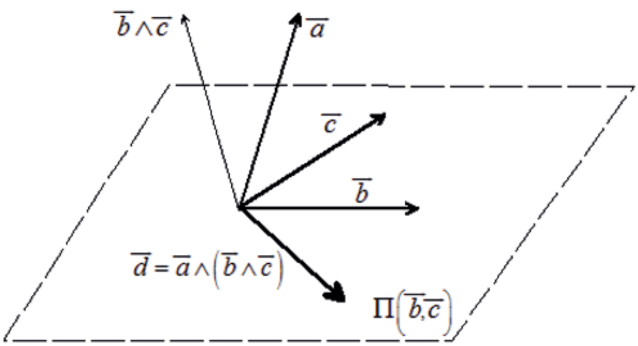

Figure 1.8. *Illustration of the double vector product*

So we can write $\exists \lambda, \mu \in \mathbb{R}$ such as $\vec{d} = \lambda\vec{b} + \mu\vec{c}$.

In the direct orthonormal basis $(u) = \left(\overrightarrow{u_1}\,\overrightarrow{u_2}\,\overrightarrow{u_3}\right)$, consider the three vectors:

$$\begin{cases} \vec{a} = a_1\overrightarrow{u_1} + a_2\overrightarrow{u_2} + a_3\overrightarrow{u_3} \\ \vec{b} = b_1\overrightarrow{u_1} + b_2\overrightarrow{u_2} + b_3\overrightarrow{u_3} \\ \vec{c} = c_1\overrightarrow{u_1} + c_2\overrightarrow{u_2} + c_3\overrightarrow{u_3} \end{cases},$$

where $\vec{b} \wedge \vec{c} = (b_2c_3 - b_3c_2)\overrightarrow{u_1} + (b_3c_1 - b_1c_3)\overrightarrow{u_2} + (b_1c_2 - b_2c_1)\overrightarrow{u_3}$.

If we consider the component $\overrightarrow{u_1}$ of the vector $\vec{d} = \vec{a} \wedge \left(\vec{b} \wedge \vec{c}\right)$, which is of the form $\vec{d} = \lambda\vec{b} + \mu\vec{c}$, we can write:

$$\begin{aligned} \lambda b_1 + \mu c_1 &= a_2(b_1c_2 - b_2c_1) - a_3(b_3c_1 - b_1c_3) \\ \text{"} \quad &= (a_1c_1 + a_2c_2 + a_3c_3)b_1 - (a_1b_1 + a_2b_2 + a_3b_3)c_1 . \\ \text{"} \quad &= \left(\vec{a}\cdot\vec{c}\right)b_1 - \left(\vec{a}\cdot\vec{b}\right)c_1 \end{aligned}$$

In the same way, we obtain:

$$\lambda b_2 + \mu c_2 = \left(\vec{a}\cdot\vec{c}\right)b_2 - \left(\vec{a}\cdot\vec{b}\right)c_2, \quad \lambda b_3 + \mu c_3 = \left(\vec{a}\cdot\vec{c}\right)b_3 - \left(\vec{a}\cdot\vec{b}\right)c_3,$$

and we can identify:

$$\lambda = \vec{a}\cdot\vec{c}, \quad \mu = \vec{a}\cdot\vec{b}.$$

This gives the development of the double vector product

$$\vec{a} \wedge \left(\vec{b} \wedge \vec{c}\right) = \left(\vec{a}\cdot\vec{c}\right)\vec{b} - \left(\vec{a}\cdot\vec{b}\right)\vec{c}.$$

1.7.1.2. *Particular characteristics of the double vector product*

$$\left(\vec{a} \wedge \vec{b}\right) \wedge \vec{c} = \vec{c} \wedge \left(\vec{b} \wedge \vec{a}\right) = \left(\vec{c}\cdot\vec{a}\right)\vec{b} - \left(\vec{c}\cdot\vec{b}\right)\vec{a}.$$

The difference, memberwise, of this relationship and the previous yields:

$$\vec{a} \wedge \left(\vec{b} \wedge \vec{c}\right) = \left(\vec{a} \wedge \vec{b}\right) \wedge \vec{c} + \left(\vec{c}\cdot\vec{b}\right)\vec{a} - \left(\vec{a}\cdot\vec{b}\right)\vec{c}.$$

In particular, when $\vec{c} = \vec{a}$, we have:

$$\vec{a} \wedge \left(\vec{b} \wedge \vec{a}\right) = \left(\vec{a} \wedge \vec{b}\right) \wedge \vec{a},$$

and if $\exists \lambda \in \mathbb{R}$ such as $\vec{c} = \lambda \vec{a}$:

$$\vec{a} \wedge \left(\vec{b} \wedge \lambda \vec{a}\right) - \left(\vec{a} \wedge \vec{b}\right) \wedge \lambda \vec{a} = \left(\lambda \vec{a} \cdot \vec{b}\right)\vec{a} - \left(\vec{a} \cdot \vec{b}\right)\lambda \vec{a} = \vec{0},$$

with the result that if $\vec{a}$ and $\vec{c}$ are collinear:

$$\vec{a} \wedge \left(\vec{b} \wedge \vec{c}\right) = \left(\vec{a} \wedge \vec{b}\right) \wedge \vec{c}.$$

1.7.1.3. *The case for nullity with the double vector product*

Consider the relationship $\vec{a} \wedge \left(\vec{b} \wedge \vec{c}\right) = \vec{0}$. It is checked if:

– at least one of the three vectors, $\vec{a}$, $\vec{b}$ or $\vec{c} = \vec{0}$;

– the vector $\vec{b} \wedge \vec{c} = \vec{0} \quad \Rightarrow \quad \exists \lambda \in \mathbb{R}$ such as $\vec{b} = \lambda \vec{c}$;

– the vector $\vec{a}$ is collinear with $\vec{b} \wedge \vec{c}$;

$\Rightarrow \quad \exists \lambda \in \mathbb{R}$ such as $\vec{a} = \lambda\left(\vec{b} \wedge \vec{c}\right)$ or else $\vec{a} \perp \Pi\left(\vec{b}, \vec{c}\right)$.

1.7.1.4. *Projection of a vector on a plane*

Consider the plane $\Pi\left(\vec{a}\right)$ whose normal vector is $\vec{a}$, and the unit vector $\vec{u} = \dfrac{\vec{a}}{\left\|\vec{a}\right\|}$; consider the vector $\vec{b}$ and its orthogonal projection on the plane (parallel to $\vec{a}$).

The vector $\vec{b}$ is expressed in terms of its two projections:

$$\vec{b} = \overrightarrow{proj_{\vec{a}} \vec{b}} + \overrightarrow{proj_{\Pi\left(\vec{a}\right)} \vec{b}} \quad \text{with} \quad \overrightarrow{proj_{\vec{a}} \vec{b}} = \left(\vec{u} \cdot \vec{b}\right)\vec{u} = \frac{\left(\vec{a} \cdot \vec{b}\right)}{\vec{a}^{\,2}}\vec{a}.$$

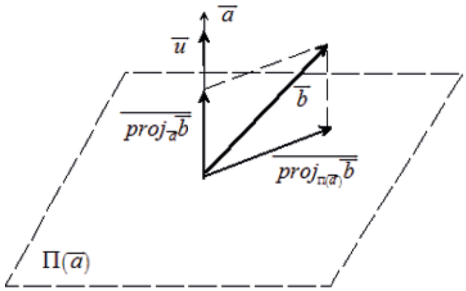

Figure 1.9. *Projection of a vector on a plane*

We deduce that:

$$\overrightarrow{proj_{\Pi(a)}b} = \vec{b} - \overrightarrow{proj_{\overline{a}}b} = \vec{b} - \frac{\left(\vec{a}\cdot\vec{b}\right)}{\vec{a}^{\,2}}\vec{a}\,.$$

From the formula of the double vector product:

$$\vec{u} \wedge \left(\vec{b} \wedge \vec{u}\right) = \vec{u}^{\,2}\vec{b} - \left(\vec{u}\cdot\vec{b}\right)\vec{u}\,,$$

we obtain the expression of the three-dimensional projection of a vector on an axis of unit vector $\vec{u}$ and on the plane orthogonal to this axis:

$$\vec{b} = \left(\vec{u}\cdot\vec{b}\right)\vec{u} + \vec{u} \wedge \left(\vec{b} \wedge \vec{u}\right)\ .$$

Therefore, the projection of a vector $\vec{b}$ on an orthogonal plane, in the direction defined by vector $\vec{a}$, and the unit vector $\vec{u} = \dfrac{\vec{a}}{\left\|\vec{a}\right\|}$, is given by:

$$\overrightarrow{proj_{\Pi(a)}b} = \vec{u} \wedge \left(\vec{b} \wedge \vec{u}\right) = \frac{\vec{a} \wedge \left(\vec{b} \wedge \vec{a}\right)}{\vec{a}^{\,2}}\,.$$

1.7.2. *Resolving the equation* $\vec{a} \cdot \vec{x} = b$

Consider two data, a vector $\vec{a} = a_1\overrightarrow{u_1} + a_2\overrightarrow{u_2} + a_3\overrightarrow{u_3}$ and a scalar b.

This is about determining the vector $\vec{x} = x_1\overrightarrow{u_1} + x_2\overrightarrow{u_2} + x_3\overrightarrow{u_3}$, satisfying the equation $\vec{a} \cdot \vec{x} = b$, that is to say the relationship $a_1x_1 + a_2x_2 + a_3x_3 = b$ that admits, as an equation, a double infinity of solutions.

Indeed, if there are infinitely many solutions to choose x_1, for example, this value taken, there are still an infinite number of values to choose for x_2; x_3 then depends on these two choices.

Suppose that we know a particular solution $\overrightarrow{x_0}$ that satisfies the equation $\vec{a} \cdot \overrightarrow{x_0} = b$; it also satisfies the relationship:

$$\vec{a} \cdot \left(\vec{x} - \overrightarrow{x_0} \right) = 0,$$

that is to say that the vector $\vec{x} - \overrightarrow{x_0}$ is orthogonal to $\vec{a}$.

We only needs to know one of these particular solutions in order to then express the general form of the equation's solution.

Choosing one, among all the possible solutions, which is collinear with $\vec{a}$, so the form $\overrightarrow{x_0} = \lambda\vec{a}$.

$$\vec{a} \cdot \overrightarrow{x_0} = \vec{a} \cdot \lambda\vec{a} = \lambda\vec{a}^{\,2} = b \quad \Rightarrow \quad \lambda = \frac{b}{\vec{a}^{\,2}}.$$

By stating:

$$\overrightarrow{\mathrm{OA}} = \vec{a}, \ \overrightarrow{\mathrm{OM}_0} = \overrightarrow{x_0} \quad \text{and} \quad \overrightarrow{\mathrm{OM}} = \vec{x},$$

the extreme M of the vector position $\overrightarrow{OM}$ is situated in the plane $\Pi\left(M_0|\vec{a}\right)$, passing through M_0 and orthogonal to $\overrightarrow{OA}$.

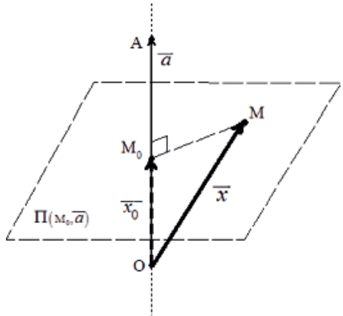

Figure 1.10. *Resolving the equation* $\vec{a}\cdot\vec{x} = b$

1.7.3. *Resolving the equation* $\vec{a}\wedge\vec{x} = \vec{b}$

Such an equation only has a solution if $\vec{a}$ and $\vec{b}$ are orthogonal, that is to say if:

$$\vec{a}\cdot\vec{b} = \vec{0} \text{ so } a_1b_1 + a_2b_2 + a_3b_3 = 0 .$$

If we know a particular solution $\overrightarrow{x_0}$ satisfying $\vec{a}\wedge\overrightarrow{x_0} = \vec{b}$, we can write:

$$\vec{a}\wedge\left(\vec{x} - \overrightarrow{x_0}\right) = \vec{0} ,$$

that is to say that the vector $\vec{x} - \overrightarrow{x_0}$ is collinear to $\vec{a}$, and is:

$$\vec{x} - \overrightarrow{x_0} = \lambda\vec{a} \quad \text{so} \quad \vec{x} = \overrightarrow{x_0} + \lambda\vec{a} .$$

Among all the possible vector solutions, we choose, if there is, $\overrightarrow{x_0}$ orthogonal to $\vec{a}$, that is to say:

$$\vec{a}\cdot\overrightarrow{x_0} = 0 \quad \text{with} \quad \vec{a}\wedge\overrightarrow{x_0} = \vec{b} .$$

Multiplying vectorially left by $\vec{a}$ the vector product above, we obtain:

$$\vec{a} \wedge \left(\vec{a} \wedge \overrightarrow{x_0}\right) = \left(\vec{a} \cdot \overrightarrow{x_0}\right)\vec{a} - \vec{a}^{\,2}\,\overrightarrow{x_0} = -\vec{a}^{\,2}\,\overrightarrow{x_0} = \vec{a} \wedge \vec{b} \quad \Rightarrow \quad \overrightarrow{x_0} = \frac{\vec{b} \wedge \vec{a}}{\vec{a}^{\,2}} .$$

The general solution is thus in the form of:

$$\vec{x} = \frac{\vec{b} \wedge \vec{a}}{\vec{a}^{\,2}} + \lambda \vec{a} .$$

By stating, as per the diagram below:

$$\overrightarrow{\mathrm{OA}} = \vec{a}\,,\quad \overrightarrow{\mathrm{OB}} = \vec{b}\,,\quad \overrightarrow{\mathrm{OM}_0} = \overrightarrow{x_0} \quad \text{and} \quad \overrightarrow{\mathrm{OM}} = \vec{x}\,,$$

the extremity M of the vector $\overrightarrow{\mathrm{OM}}$ is located on the straight line passing through M_0 and parallel to $\overrightarrow{\mathrm{OA}}$.

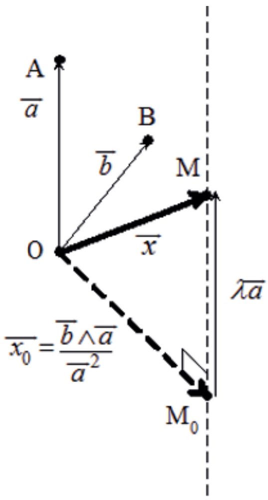

Figure 1.11. *Resolving the equation* $\vec{a} \wedge \vec{x} = \vec{b}$

1.7.4. *Equality of Lagrange*

Consider two vectors $\vec{a}$ and $\vec{b}$, and perform the scalar square of the two following terms, the one, the scalar $\vec{a}\cdot\vec{b}$, the other, the vector $\vec{a}\wedge\vec{b}$.

$$\begin{aligned}&\left(\vec{a}\cdot\vec{b}\right)^2 = (\,a_1b_1 + a_2b_2 + a_3b_3\,)^2\\ &\left(\vec{a}\wedge\vec{b}\right)^2 = (\,a_2b_3 - a_3b_2\,)^2 + (\,a_3b_1 - a_1b_3\,)^2 + (\,a_1b_2 - a_2b_1\,)^2\end{aligned}.$$

The sum memberwise of these two expressions yields, after development and simplification:

$$\left(\vec{a}\wedge\vec{b}\right)^2 + \left(\vec{a}\cdot\vec{b}\right)^2 = \left(a_1^2 + a_2^2 + a_3^2\right)\left(b_1^2 + b_2^2 + b_3^2\right) = \left(\vec{a}^{\,2}\right)\left(\vec{b}^{\,2}\right).$$

This equality of Lagrange is also written as:

$$\left(\frac{\vec{a}\wedge\vec{b}}{\|\vec{a}\|\|\vec{b}\|}\right)^2 + \left(\frac{\vec{a}\cdot\vec{b}}{\|\vec{a}\|\|\vec{b}\|}\right)^2 = 1,$$

and reduces, in fact, to the trigonometric expression $\sin^2\theta + \cos^2\theta = 1$.

1.7.5. *Equations of planes*

1.7.5.1. *Plane normal to a vector and passing through a point*

In the affine space that is associated with the frame $\left\langle \mathrm{O}\middle|\overrightarrow{u_1}\,\overrightarrow{u_2}\,\overrightarrow{u_3}\right\rangle$, we consider a vector $\vec{a} = a_1\overrightarrow{u_1} + a_2\overrightarrow{u_2} + a_3\overrightarrow{u_3}$ represented by the vector position $\overrightarrow{\mathrm{OA}}$ and a unit vector $\vec{v} = v_1\overrightarrow{u_1} + v_2\overrightarrow{u_2} + v_3\overrightarrow{u_3}$.

The question is to express the equation of the plane $\Pi\left(\mathrm{A}\middle|\vec{v}\right)$ passing through the point A and orthogonal to $\vec{v}$.

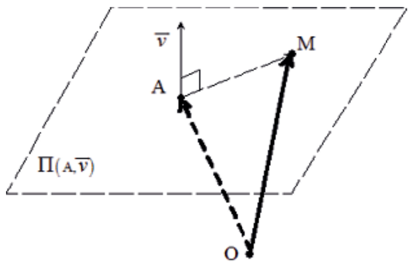

Figure 1.12. *Plane normal to a vector and passing through a point*

Consider the point M in this plane:

$$\overrightarrow{OM} = x_1\overrightarrow{u_1} + x_2\overrightarrow{u_2} + x_3\overrightarrow{u_3} .$$

This point M is such that:

$$\overrightarrow{AM} \cdot \vec{v} = \left(\overrightarrow{OM} - \overrightarrow{OA}\right) \cdot \vec{v} = 0 .$$

The plane $\Pi\left(A\middle|\vec{v}\right)$ is the locus of points M that satisfy the equation:

$$\overrightarrow{OM} \cdot \vec{v} = \overrightarrow{OA} \cdot \vec{v} ,$$

and has the Cartesian equation:

$$v_1x_1 + v_2x_2 + v_3x_3 = v_1a_1 + v_2a_2 + v_3a_3 .$$

1.7.5.2. *Plane defined by two vectors and passing through a point*

Consider the two vectors $\vec{b}$ and $\vec{c}$, the plane containing them, passing through the point A of vector position $\overrightarrow{OA}$. Any point M of the plane $\Pi\left(A\middle|\vec{b},\vec{c}\right)$ is such that:

$$\overrightarrow{AM} \in \Pi\left(A\middle|\vec{b},\vec{c}\right) \Rightarrow \exists \lambda, \mu \in \mathbb{R} \text{ such that } \overrightarrow{AM} = \lambda\vec{b} + \mu\vec{c} .$$

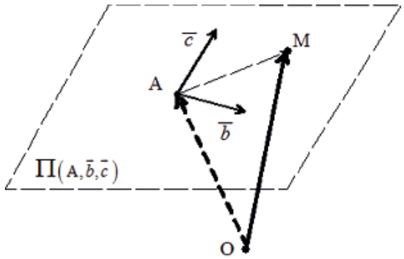

Figure 1.13. *Plane defined by two vectors and through a point*

According to Figure 1.13, we have:

$$\overrightarrow{OM} = \overrightarrow{OA} + \overrightarrow{AM} = \overrightarrow{OA} + \lambda\vec{b} + \mu\vec{c}\,.$$

In considering the scalar product of this expression by $\vec{b} \wedge \vec{c}$, which defines the orthogonal direction to the plane, one is brought back to the previous problem.

The equation of the plane is given by the equation:

$$\left(\vec{b} \wedge \vec{c}\right) \cdot \overrightarrow{OM} = \left(\vec{b} \wedge \vec{c}\right) \cdot \overrightarrow{OA}\,.$$

1.7.6. *Relations within the triangle*

Note the triangle represented in the figure below.

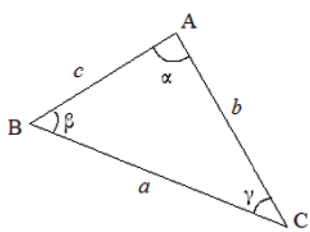

Figure 1.14. *Configuration of a triangle*

Consider the relation of the Chasles $\overrightarrow{\mathrm{BC}} = \overrightarrow{\mathrm{BA}} + \overrightarrow{\mathrm{AC}}$ and multiply its two members, scalarly, by $\overrightarrow{\mathrm{BC}}$. We obtain

$$\begin{array}{l} \overrightarrow{\mathrm{BC}}^{2} = \overrightarrow{\mathrm{BC}} \cdot \overrightarrow{\mathrm{BA}} + \overrightarrow{\mathrm{BC}} \cdot \overrightarrow{\mathrm{AC}} = \overrightarrow{\mathrm{BC}} \cdot \overrightarrow{\mathrm{BA}} + \overrightarrow{\mathrm{CB}} \cdot \overrightarrow{\mathrm{CA}} \\ \quad \Rightarrow \qquad a^2 = ac\cos\beta + ab\cos\gamma \end{array},$$

therefore $a = c\cos\beta + b\cos\gamma$.

Considering now the scalar square of the Chasles relation:

$$\begin{aligned} \overrightarrow{\mathrm{BC}}^{2} &= \left(\overrightarrow{\mathrm{BA}} + \overrightarrow{\mathrm{AC}}\right)^{2} = \overrightarrow{\mathrm{BA}}^{2} + \overrightarrow{\mathrm{AC}}^{2} + 2 \times \overrightarrow{\mathrm{BA}} \cdot \overrightarrow{\mathrm{AC}} \\ &= \overrightarrow{\mathrm{AC}}^{2} + \overrightarrow{\mathrm{AB}}^{2} - 2 \times \overrightarrow{\mathrm{AC}} \cdot \overrightarrow{\mathrm{AB}}, \end{aligned}$$

so $a^2 = b^2 + c^2 - 2bc\cos\alpha$.

According to the definition of the vector product, the area $\mathscr{A}$ of the triangle is given by:

$$2\mathscr{A} = bc\sin\alpha = ca\sin\beta = ab\sin\gamma,$$

and, by dividing each term of this expression by the product abc, we obtain a fundamental relationship of the triangle:

$$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} = \frac{\sin\gamma}{c}.$$

1.8. Vectors and basis changes

1.8.1. *Einstein's convention*

Given the size of mathematical expressions that will be developed in this work and formulas to be used therein, it quickly becomes essential to have a condensed notation system that facilitates writing and exploitation. The *Einstein notation convention* is a response to that expectation. To illustrate the principle, consider the example of a system of linear equations of n variables x_i for $i=1,\ldots,n$

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2 \\ \ldots \\ a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n = b_n \end{cases}.$$

Because of the Einstein notation convention, this system can be written in the following condensed form

$$a_{ij}x_j = b_i \ , \ \forall i, j = 1,\ldots,n,$$

and is interpreted as follows:

– the index i *that appears only once* in each member of the equality indicates that there are as many relations as values of this index, so here n; it is identified as *free index*. These are the n equations of the above system mixed into one relationship;

– the index j that appears only twice in the left section of the equality means that in each of the n equations of the given system, this member consists of the sum of n monomials obtained by giving to j its n values. This index is called *summation index*. The summation index only has meaning if the index *appears only twice* in a monomial convention.

NOTE.– Widely used in digital programming, this agreement is particularly interesting if the various indices used have the same range of variation; as will often be the case in the situations that will be discussed in this book.

For complete information on using the Einstein convention, take the example of the expression index:

$$a_{ij} = b_{i\alpha}c_{\alpha\beta}d_{\beta j}$$

– indices i and j appear only once in each of the two members. They are free indices. When their values are chosen, the expression of the relevant term a_{ij} outcomes;

– the indices α and β appear, however, twice in the right side of the index relationship; so these are the summation indices.

Consider, for the four indices, the values $1,2,3$. We obtain, for example, for $i=1$, $j=2$:

$$\begin{aligned} a_{12} = {} & b_{11}c_{11}d_{12} + b_{11}c_{12}d_{22} + b_{11}c_{13}d_{32} \ldots \\ & \ldots + b_{12}c_{21}d_{12} + b_{12}c_{22}b_{22} + b_{12}c_{23}d_{32} \ldots . \\ & \ldots + b_{13}c_{31}d_{12} + b_{13}c_{32}d_{22} + b_{13}c_{33}d_{32} \end{aligned}$$

1.8.2. *Transition table from basis (e) to basis (E)*

Consider two direct orthonormal bases:

$$(e) = \left(\overrightarrow{x_1}\,\overrightarrow{x_2}\,\overrightarrow{x_3}\right), \quad (E) = \left(\overrightarrow{X_1}\,\overrightarrow{X_2}\,\overrightarrow{X_3}\right),$$

and a vector $\vec{f}$ whose projection is expressed in the two bases.

$$\begin{cases} \vec{f} = \left(\vec{f}\cdot\overrightarrow{x_1}\right)\overrightarrow{x_1} + \left(\vec{f}\cdot\overrightarrow{x_2}\right)\overrightarrow{x_2} + \left(\vec{f}\cdot\overrightarrow{x_3}\right)\overrightarrow{x_3} = \left(\vec{f}\cdot\overrightarrow{x_\alpha}\right)\overrightarrow{x_\alpha} = f_\alpha \overrightarrow{x_\alpha} \\ \vec{f} = \left(\vec{f}\cdot\overrightarrow{X_1}\right)\overrightarrow{X_1} + \left(\vec{f}\cdot\overrightarrow{X_2}\right)\overrightarrow{X_2} + \left(\vec{f}\cdot\overrightarrow{X_3}\right)\overrightarrow{X_3} = \left(\vec{f}\cdot\overrightarrow{X_\beta}\right)\overrightarrow{X_\beta} = F_\beta \overrightarrow{X_\beta} \end{cases}$$

The relationship between the two bases is expressed by the following expressions:

$$p_{\alpha\beta} = \overrightarrow{x_\alpha}\cdot\overrightarrow{X_\beta}; \quad \overrightarrow{x_\alpha} = p_{\alpha\beta}\overrightarrow{X_\beta}; \quad \overrightarrow{X_\beta} = p_{\alpha\beta}\overrightarrow{x_\alpha},$$

which can be summarized by the transition table $p(e,E)$ from the basis (e) to basis (E) where the reading is done by line to express the vectors of basis (e) according to those of basis (E) or by column to determine those (E) based on those (e).

$$\begin{array}{c|ccc} \overline{(e)}\|\underline{(E)} & \overrightarrow{X_1} & \overrightarrow{X_2} & \overrightarrow{X_3} \\ \hline \overrightarrow{x_1} & p_{11} & p_{12} & p_{13} \\ \overrightarrow{x_2} & p_{21} & p_{22} & p_{23} \\ \overrightarrow{x_3} & p_{31} & p_{32} & p_{33} \end{array}.$$

This table is also used to express the components of a vector of a basis in the other.

$$\begin{aligned} f_\alpha &= \vec{f}\cdot\overrightarrow{x_\alpha} = \vec{f}\cdot\left(p_{\alpha\beta}\overrightarrow{X_\beta}\right) = p_{\alpha\beta}\left(\vec{f}\cdot\overrightarrow{X_\beta}\right) = p_{\alpha\beta}F_\beta \\ F_\beta &= \vec{f}\cdot\overrightarrow{X_\beta} = \vec{f}\cdot\left(p_{\alpha\beta}\overrightarrow{x_\alpha}\right) = p_{\alpha\beta}\left(\vec{f}\cdot\overrightarrow{x_\alpha}\right) = p_{\alpha\beta}f_\alpha \end{aligned}.$$

If we use a matrix representation of vectors and basis change operation, we can write:

$$\begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} = \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{bmatrix} \begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix} = \begin{bmatrix} p_{11} & p_{21} & p_{31} \\ p_{12} & p_{22} & p_{32} \\ p_{13} & p_{23} & p_{33} \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}.$$

From $[p(e,E)]$, the matrix directly transcribed from the transition table from basis (e) to basis (E), we have:

$$[p(e,E)] = \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{bmatrix},$$

and we see that the matrix that corresponds to a vector from the basis (E) to a vector of basis (e) is the transpose of the previous one, which can be expressed by:

$$[p(E,e)] = \overline{[p(e,E)]}.$$

The basis change of the vector $\overrightarrow{f}$ between (e) and (E) is therefore written as a matrix:

$$[f]_{/(e)} = [p(e,E)][f]_{/(E)}$$
$$[f]_{/(E)} = [p(E,e)][f]_{/(e)} = \overline{[p(e,E)]}[f]_{/(e)}\ .$$

1.8.3. *Characterization of the transition table*

The fact that the transition table takes place between two direct orthonormal bases means that its nine elements $p_{\alpha\beta}$ are not independent of each other. First note that the bases are orthonormal:

$$\overrightarrow{x_\alpha}\cdot\overrightarrow{x_\beta} = \delta_{\alpha\beta} \quad \text{and} \quad \overrightarrow{X_\mu}\cdot\overrightarrow{X_\nu} = \delta_{\mu\nu}\ ,$$

$$\Rightarrow \quad \overrightarrow{x_\alpha}\cdot\overrightarrow{x_\beta} = p_{\alpha\mu}\overrightarrow{X_\mu}\cdot p_{\beta\nu}\overrightarrow{X_\nu} = p_{\alpha\mu}p_{\beta\nu}\overrightarrow{X_\mu}\cdot\overrightarrow{X_\nu}$$
$$= p_{\alpha\mu}p_{\beta\nu}\delta_{\mu\nu} = p_{\alpha i}p_{\beta i} = \delta_{\alpha\beta},$$

which gives six relations that should verify the elements of the transition table between two bases.

We get the same six relations between the elements of the transition table by writing:

$$\overrightarrow{X_\mu}\cdot\overrightarrow{X_\nu} = p_{\alpha\mu}\overrightarrow{x_\alpha}\cdot p_{\beta\nu}\overrightarrow{x_\beta} = p_{\alpha\mu}p_{\beta\nu}\overrightarrow{x_\alpha}\cdot\overrightarrow{x_\beta} = p_{\alpha\mu}p_{\beta\nu}\delta_{\alpha\beta} = p_{i\mu}p_{i\nu} = \delta_{\mu\nu}\ .$$

In addition, the orientation of the bases introduces new relationships, whether they are direct or indirect. If the basis (e) is direct, as it is orthonormal, the relationship below is enough to express its orientation.

$$\overrightarrow{x_1} = \overrightarrow{x_2}\wedge\overrightarrow{x_3}\ ,$$

since the three vectors are orthogonal and this relationship enlightens unequivocally on the basis orientation. In the change of basis, the relation is written as:

$$p_{1\alpha}\overrightarrow{X_\alpha} = p_{2\beta}\overrightarrow{X_\beta} \wedge p_{3\gamma}\overrightarrow{X_\gamma} = p_{2\beta}p_{3\gamma}\overrightarrow{X_\beta} \wedge \overrightarrow{X_\gamma} .$$

The terms of this expression exist only if the values of the three indices α,β,γ come within a circular permutation $1,2,3$ or $1,3,2$, which gives another three relationships that should cross-check with the terms of the table. Since these are six distinct terms, it is clear that these relations are redundant.

When a transition table or a matrix change in basis has been established, we should check that:

– non-diagonal terms are anti-symmetric;

– vectors are unitary, that is to say, for each row or each column, the sum of squares of the three elements is equal to 1.

– vectors are orthogonal, that is to say the sum of the products of the relevant terms of two lines or two columns is zero, in checking it in the three combinations of lines or columns (1+2, 1+3, 2+3).

This control is required when drawing up transition table.