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Equilibrium, Stationary Movement and Oscillation of a Free Solid

In certain situations, the motion of a free solid, which is only subject to known external efforts that can be expressed and quantified, but not to any links that would introduce onto the motion any unknown strains that would vary with it, can experience situations of equilibrium and oscillations around these. It can even be the motion sought for a particular mechanism. This is why the equations that govern these situations are of great interest to engineers. The approach we develop here is limited to the first order of magnitude (in the sense of limited mathematical developments) of the variations of the situation parameters of a solid; it leads to second order linear differential equations. Far more precise oscillatory motions would require going beyond this first order of magnitude in the limited development of equilibrium equations; this is not the objective of the following presentation, which instead indicates the procedure to do it.

1.1. Expression of the fundamental principle of dynamics for a free solid

In a Galilean frame $\langle g\rangle \equiv \left\langle \mathrm{O}_g \middle| \overrightarrow{x_g}\ \overrightarrow{y_g}\ \overrightarrow{z_g} \right\rangle$, the motion of the solid (S), to which is joined the frame $\langle S\rangle \equiv \left\langle \mathrm{O}_S \middle| \overrightarrow{x_S}\ \overrightarrow{y_S}\ \overrightarrow{z_S} \right\rangle$, obeys the fundamental principle of dynamics, the torsor expression of which is

$$\left\{\mathscr{A}_S^g\right\} = \left\{\overline{S} \to S\right\}$$

where $\left\{\overline{S} \to S\right\}$ refers to the external efforts torsor acting on (S).

At a point P of the solid, the dynamic torsor $\left\{\mathcal{A}_S^g\right\}$ has the following reduction elements:

$$\begin{cases} \vec{s}\left\{\mathcal{A}_S^g\right\} = \int_{\mathrm{M}\in S} \overrightarrow{J^{(g)}}(\mathrm{M})\,dm = m\,\overrightarrow{J^{(g)}}(\mathrm{G}) \\ \overrightarrow{\mathcal{M}_{\mathrm{P}}}\left\{\mathcal{A}_S^g\right\} = \int_{\mathrm{M}\in S} \overrightarrow{\mathrm{PM}} \wedge \overrightarrow{J^{(g)}}(\mathrm{M})\,dm \\ \quad " \quad " \qquad = m\,\overrightarrow{\mathrm{PG}} \wedge \overrightarrow{J^{(g)}}(\mathrm{P}) + I_{\mathrm{P}}(S)\,\overrightarrow{\omega_S^g}' + \overrightarrow{\omega_S^g} \wedge I_{\mathrm{P}}(S)\,\overrightarrow{\omega_S^g} \end{cases}$$

If the motion of the solid (S) is observed from a given frame $\langle\lambda\rangle$, the expression of the fundamental principle of dynamics involves two other torsors that correspond to the existence of the motion of $\langle\lambda\rangle$ in relation to the Galilean frame $\langle g\rangle$. This is the inertial drive torsor $\left\{\mathcal{A}_S^e\middle|{}^g_\lambda\right\}$ and the Coriolis drive torsor $\left\{\mathcal{A}_S^c\middle|{}^g_\lambda\right\}$, the reduction elements of which at a given point Q have the following generic expressions:

$$\begin{cases} \vec{s}\left\{\mathcal{A}_S^c\middle|{}^g_\lambda\right\} = 2\int_{\mathrm{M}\in S} \overrightarrow{\omega_\lambda^g} \wedge \overrightarrow{v^{(\lambda)}}(\mathrm{M})\,dm(\mathrm{M}) \\ \overrightarrow{\mathcal{M}_{\mathrm{Q}}}\left\{\mathcal{A}_S^c\middle|{}^g_\lambda\right\} = 2\int_{\mathrm{M}\in S} \overrightarrow{\mathrm{QM}} \wedge \left[\overrightarrow{\omega_\lambda^g} \wedge \overrightarrow{v^{(\lambda)}}(\mathrm{M})\right] dm(\mathrm{M}) \end{cases}$$

$$\begin{cases} \vec{s}\left\{\mathcal{A}_S^e\middle|{}^g_\lambda\right\} = \int_{\mathrm{M}\in S} \left[\overrightarrow{J^{(g)}}(\mathrm{O}_\lambda) + \overrightarrow{\omega_\lambda^g}' \wedge \overrightarrow{\mathrm{O}_\lambda\mathrm{M}} + \overrightarrow{\omega_\lambda^g} \wedge \left(\overrightarrow{\omega_\lambda^g} \wedge \overrightarrow{\mathrm{O}_\lambda\mathrm{M}}\right)\right] dm(\mathrm{M}) \\ \overrightarrow{\mathcal{M}_{\mathrm{Q}}}\left\{\mathcal{A}_S^e\middle|{}^g_\lambda\right\} = \ \ldots \\ \ldots \int_{\mathrm{M}\in S} \overrightarrow{\mathrm{QM}} \wedge \left[\overrightarrow{J^{(g)}}(\mathrm{O}_\lambda) + \overrightarrow{\omega_\lambda^g}' \wedge \overrightarrow{\mathrm{O}_\lambda\mathrm{M}} + \overrightarrow{\omega_\lambda^g} \wedge \left(\overrightarrow{\omega_\lambda^g} \wedge \overrightarrow{\mathrm{O}_\lambda\mathrm{M}}\right)\right] dm(\mathrm{M}) \end{cases}$$

Their development in the case of the solid (S) with an inertial center G and origin O_S, at a point P of this solid, among others O_S or G, is written as:

$$\begin{cases} \vec{s}\left\{\mathscr{A}_S^c \middle| {}^g_\lambda\right\} = 2m\, \overrightarrow{\omega_\lambda^g} \wedge \overrightarrow{v^{(\lambda)}}(\mathrm{G}) \\ \overrightarrow{\mathscr{M}_{\mathrm{P}\in(S)}}\left\{\mathscr{A}_S^c \middle| {}^g_\lambda\right\} = 2m\,\overrightarrow{\mathrm{PG}} \wedge \left[\overrightarrow{\omega_\lambda^g} \wedge \overrightarrow{v^{(\lambda)}}(\mathrm{P})\right] \; \ldots \\ \qquad \ldots \; + trace\, I_\mathrm{P}(S)\overrightarrow{\omega_\lambda^g} \wedge \overrightarrow{\omega_S^\lambda} + 2\overrightarrow{\omega_S^\lambda} \wedge I_\mathrm{P}(S)\overrightarrow{\omega_\lambda^g} \end{cases}$$

$$\begin{cases} \vec{s}\left\{\mathscr{A}_S^e \middle| {}^g_\lambda\right\} = m\left[\overrightarrow{J^{(g)}}(\mathrm{O}_\lambda) + \overrightarrow{\omega_\lambda^g}' \wedge \overrightarrow{\mathrm{O}_\lambda \mathrm{G}} + \overrightarrow{\omega_\lambda^g} \wedge \left(\overrightarrow{\omega_\lambda^g} \wedge \overrightarrow{\mathrm{O}_\lambda \mathrm{G}}\right)\right] \\ \overrightarrow{\mathscr{M}_{\mathrm{P}\in(S)}}\left\{\mathscr{A}_S^e \middle| {}^g_\lambda\right\} = \; \ldots \\ \quad \ldots \; m\,\overrightarrow{\mathrm{PG}} \wedge \left[\overrightarrow{J^{(g)}}(\mathrm{O}_\lambda) + \overrightarrow{\omega_\lambda^g}' \wedge \overrightarrow{\mathrm{O}_\lambda \mathrm{P}} + \overrightarrow{\omega_\lambda^g} \wedge \left(\overrightarrow{\omega_\lambda^g} \wedge \overrightarrow{\mathrm{O}_\lambda \mathrm{P}}\right)\right] \; \ldots \\ \qquad \ldots \; + I_\mathrm{P}(S)\overrightarrow{\omega_\lambda^g}' + \overrightarrow{\omega_\lambda^g} \wedge I_\mathrm{P}(S)\overrightarrow{\omega_\lambda^g} \end{cases}$$

In reference to the frame $\langle \lambda \rangle$, the expression of the fundamental principle becomes

$$\begin{aligned} \left\{\mathscr{A}_S^\lambda\right\} &= \left\{\mathscr{A}_S^g\right\} - \left\{\mathscr{A}_S^e \middle| {}^g_\lambda\right\} - \left\{\mathscr{A}_S^c \middle| {}^g_\lambda\right\} \\ \text{" "} \quad &= \left\{\overline{S} \to S\right\} - \left\{\mathscr{A}_S^e \middle| {}^g_\lambda\right\} - \left\{\mathscr{A}_S^c \middle| {}^g_\lambda\right\} \end{aligned}$$

where $\left\{\overline{S} \to S\right\}$ is the torsor of efforts given or calculable using the situation parameters of (S).

Besides, if the solid is free, meaning that no link is exerted on it, we get

$$\left\{\overline{S} \to S\right\} = \left\{\Delta \to S\right\}$$

1.2. Canonical form of the fundamental principle

The situation of the solid (S) in $\langle g \rangle$ is defined by six parameters Q_1, Q_2, Q_3, Q_4, Q_5 and Q_6. When considering the small movements and vibrations of a solid, it is actually the small variations on its situation parameters that we are examining. This is why it is essential to express clearly the fundamental principle and its scalar consequences depending on these parameters that need to vary in sufficiently small amounts to linearize relations and study them in this new configuration.

Thus, the preliminary work is to put scalar and vector expressions of the fundamental principle under canonical form, meaning under an explicit and generic form of situation parameters.

$$\overrightarrow{\mathrm{O}_g\mathrm{O}_S} = \vec{F}(Q_1, Q_2, Q_3)$$

$$\Rightarrow \begin{cases} \overrightarrow{v^{(g)}}(\mathrm{O}_S) = \dfrac{d^{(g)}}{dt}\overrightarrow{\mathrm{O}_g\mathrm{O}_S} = \displaystyle\sum_{i=1}^{3} \dfrac{\partial^{(g)}\vec{F}}{\partial Q_i} Q_i' \\ \overrightarrow{J^{(g)}}(\mathrm{O}_S) = \dfrac{d^{(g)^2}}{dt^2}\overrightarrow{\mathrm{O}_g\mathrm{O}_S} = \displaystyle\sum_{i=1}^{3} \dfrac{\partial^{(g)}\vec{F}}{\partial Q_i} Q_i'' + \sum_{i,j=1}^{3} \dfrac{\partial^{(g)^2}\vec{F}}{\partial Q_i \partial Q_j} Q_i' Q_j' \end{cases}$$

$$\overrightarrow{\omega_S^g} = \sum_{l=4}^{6} \overrightarrow{{}^{l}\delta}(Q_4, Q_5, Q_6) Q_l' = \sum_{l=4}^{6} \overrightarrow{{}^{l}\delta}(g, S) Q_l'$$

$$\Rightarrow \quad \overrightarrow{\omega_S^g}' = \frac{d^{(S)}}{dt} \sum_{l=4}^{6} \overrightarrow{{}^{l}\delta} Q_l' = \sum_{l=4}^{6} \overrightarrow{{}^{l}\delta} Q_l'' + \sum_{l,n=4}^{6} \frac{\partial^{(S)} \overrightarrow{{}^{l}\delta}}{\partial Q_n} Q_l' Q_n'$$

It is advisable to restate here a result on the derivative of the rotation rate vector $\overrightarrow{\omega_S^g}$:

$$\overrightarrow{\omega_S^g}' = \frac{d^{(g)}}{dt}\overrightarrow{\omega_S^g} = \frac{d^{(S)}}{dt}\overrightarrow{\omega_S^g}$$

1.2.1. *Dynamic resultant*

$$\overrightarrow{\omega_S^g}' \wedge \overrightarrow{\mathrm{O}_S\mathrm{G}} = \sum_{l=4}^{6}\left(\overrightarrow{{}^l\delta} \wedge \overrightarrow{\mathrm{O}_S\mathrm{G}}\right)Q_l'' + \sum_{l,n=4}^{6}\left(\frac{\partial^{(S)}\overrightarrow{{}^l\delta}}{\partial Q_n} \wedge \overrightarrow{\mathrm{O}_S\mathrm{G}}\right)Q_l'Q_n'$$

$$\overrightarrow{\omega_S^g} \wedge \left(\overrightarrow{\omega_S^g} \wedge \overrightarrow{\mathrm{O}_S\mathrm{G}}\right) = \sum_{l,n=4}^{6}\overrightarrow{{}^l\delta}Q_l' \wedge \left(\overrightarrow{{}^n\delta}Q_n' \wedge \overrightarrow{\mathrm{O}_S\mathrm{G}}\right)$$

$$\text{" "} \qquad = \sum_{l,n=4}^{6}\left[\overrightarrow{{}^l\delta} \wedge \left(\overrightarrow{{}^n\delta} \wedge \overrightarrow{\mathrm{O}_S\mathrm{G}}\right)\right]Q_l'Q_n'$$

$$\Rightarrow \quad \vec{s}\left\{\mathscr{A}_S^g\right\} = m\sum_{i=1}^{3}\frac{\partial^{(g)}\overrightarrow{F}}{\partial Q_i}Q_i'' + m\sum_{l=4}^{6}\left(\overrightarrow{{}^l\delta} \wedge \overrightarrow{\mathrm{O}_S\mathrm{G}}\right)Q_l'' \;\ldots$$

$$\ldots \; + m\sum_{i,j=1}^{3}\frac{\partial^{(g)^2}\overrightarrow{F}}{\partial Q_i \partial Q_j}Q_i'Q_j' \;\ldots$$

$$\ldots \; + m\sum_{l,n=4}^{6}\left[\frac{\partial^{(S)}\overrightarrow{{}^l\delta}}{\partial Q_n} \wedge \overrightarrow{\mathrm{O}_S\mathrm{G}} + \overrightarrow{{}^l\delta} \wedge \left(\overrightarrow{{}^n\delta} \wedge \overrightarrow{\mathrm{O}_S\mathrm{G}}\right)\right]Q_l'Q_n'$$

Synthetically, the dynamic resultant can be expressed as follows:

$$\vec{s}\left\{\mathscr{A}_S^g\right\} = \sum_{\alpha}\overrightarrow{s^\alpha}\left(Q_1,Q_2,\ldots,Q_6\right)Q_\alpha'' \;\ldots$$

$$\ldots + \sum_{\alpha,\beta}\overrightarrow{s^{\alpha\beta}}\left(Q_1,Q_2,\ldots,Q_6\right)Q_\alpha'Q_\beta'$$

1.2.2. *Dynamic moment at* O_S

$$\overrightarrow{\mathrm{O}_S\mathrm{G}} \wedge \overrightarrow{J^{(g)}}\left(\mathrm{O}_S\right) = \overrightarrow{\mathrm{O}_S\mathrm{G}} \wedge \sum_{i=1}^{3}\frac{\partial^{(g)}\overrightarrow{F}}{\partial Q_i}Q_i'' \;\ldots$$

$$\ldots \; + \overrightarrow{\mathrm{O}_S\mathrm{G}} \wedge \sum_{i,j=1}^{3}\frac{\partial^{(g)^2}\overrightarrow{F}}{\partial Q_i \partial Q_j}Q_i'Q_j'$$

$$I_{O_S}\,\overrightarrow{\omega_S^g}' = I_{O_S}\left(\sum_{l=4}^{6} \overrightarrow{{}^{l}\delta}\, Q_l'' + \sum_{l,n=4}^{6} \frac{\partial^{(S)}\overrightarrow{{}^{l}\delta}}{\partial Q_n} Q_l' Q_n'\right)$$

$$\text{" "} \quad = \sum_{l=4}^{6}\left(I_{O_S}\,\overrightarrow{{}^{l}\delta}\right) Q_l'' + \sum_{l,n=4}^{6}\left(I_{O_S}\,\frac{\partial^{(S)}\overrightarrow{{}^{l}\delta}}{\partial Q_n}\right) Q_l' Q_n'$$

$$\overrightarrow{\omega_S^g} \wedge I_{O_S}\,\overrightarrow{\omega_S^g} = \sum_{l=4}^{6} \overrightarrow{{}^{l}\delta}\, Q_l' \wedge \left(I_{O_S} \sum_{n=4}^{6} \overrightarrow{{}^{n}\delta}\, Q_n'\right)$$

$$\text{" "} \quad = \sum_{l,n=4}^{6}\left[\overrightarrow{{}^{l}\delta} \wedge \left(I_{O_S}\,\overrightarrow{{}^{n}\delta}\right)\right] Q_l' Q_n'$$

Thus, the canonical expression of the dynamic moment at O_S is written as:

$$\overrightarrow{\mathcal{M}_{O_S}} = \sum_{i=1}^{3} m\,\overrightarrow{O_S G} \wedge \frac{\partial^{(g)}\overrightarrow{F}}{\partial Q_i} Q_i'' + \sum_{l=4}^{6}\left(I_{O_S}\,\overrightarrow{{}^{l}\delta}\right) Q_l'' \quad \ldots$$

$$\ldots \quad + \sum_{i,j=1}^{3} m\,\overrightarrow{O_S G} \wedge \frac{\partial^{(g)^2}\overrightarrow{F}}{\partial Q_i \partial Q_j} Q_i' Q_j'$$

$$\ldots \quad + \sum_{l,n=4}^{6}\left(I_{O_S}\,\frac{\partial^{(S)}\overrightarrow{{}^{l}\delta}}{\partial Q_n} + \overrightarrow{{}^{l}\delta} \wedge \left(I_{O_S}\,\overrightarrow{{}^{n}\delta}\right)\right) Q_l' Q_n'$$

which can be written under the following synthetic form:

$$\overrightarrow{\mathcal{M}_{O_S}}\left\{\mathcal{A}_S^g\right\} = \sum_{\alpha} \overrightarrow{m_{O_S}^{\alpha}}\,(Q_1, Q_2, \ldots, Q_6)\, Q_\alpha''$$

$$+ \sum_{\alpha,\beta} \overrightarrow{m_{O_S}^{\alpha\beta}}\,(Q_1, Q_2, \ldots, Q_6)\, Q_\alpha' Q_\beta'$$

We thus see that the reduction elements of the dynamic torsor at a given point of the solid, in this case O_S, are each the sum of a linear form in Q_α'' and a quadratic form in $Q_\alpha' Q_\beta'$. The coefficients of these linear and quadratic forms depend only on parameters Q_α, excluding all temporal derivatives.

1.2.3. *Fundamental principle of dynamics*

$$\left\{\mathscr{A}_S^g\right\} = \left\{\overline{S} \to S\right\} = \{\Delta\}$$

The torsor representing the external efforts acting on (S) in its motion depends on:

– the situation of (S) in $\langle g\rangle$, meaning the parameters Q_α;

– the velocity of (S) throughout its motion in $\langle g\rangle$, i.e. the terms Q_α';

– the time (potentially).

The fundamental principle of dynamics applied to the motion of the free solid (S) in $\langle g\rangle$ results, under canonical form, in two vectorial equations:

$$\begin{cases} \sum\limits_{\alpha} \overrightarrow{s^{\alpha}} Q_\alpha'' + \sum\limits_{\alpha,\beta} \overrightarrow{s^{\alpha\beta}} Q_\alpha' Q_\beta' = \overrightarrow{s^{\Delta}} \left(Q_1,\ldots,Q_6 \middle| Q_1',\ldots,Q_6' \middle| t\right) \\ \sum\limits_{\alpha} \overrightarrow{m_{O_S}^{\alpha}} Q_\alpha'' + \sum\limits_{\alpha,\beta} \overrightarrow{m_{O_S}^{\alpha\beta}} Q_\alpha' Q_\beta' = \overrightarrow{\mathscr{M}_{O_S}^{\Delta}} \left(Q_1,\ldots,Q_6 \middle| Q_1',\ldots,Q_6' \middle| t\right) \end{cases}$$

1.3. Equilibrium of the free solid

1.3.1. *Equations of equilibrium*

The equilibrium of the solid (S) is defined in the following manner.

During the interval $\left(t_i, t_f\right)$, the solid (S) is at equilibrium in the frame $\langle g\rangle$ if its situation parameters maintain constant values, independent of time, meaning that

$$\forall t \in \left]t_i, t_f\right[, \; Q_\alpha = e_\alpha, \forall \alpha = 1, \ldots, 6,$$

$$\Rightarrow \; Q_\alpha' = 0, Q_\alpha'' = 0, \forall \alpha$$

On this time interval, the vector equalities that express the fundamental principle of dynamics are written as:

$$\begin{cases} \vec{0} = \overrightarrow{s^\Delta}\left(e_1, \ldots, e_6 \left| 0, \ldots, 0 \right| t\right) \\ \vec{0} = \overrightarrow{\mathcal{M}^\Delta_{\mathrm{O}_S}}\left(e_1, \ldots, e_6 \left| 0, \ldots, 0 \right| t\right) \end{cases}$$

One of the ways to satisfy these requests is that the two functions $\overrightarrow{s^\Delta}$ and $\overrightarrow{\mathcal{M}^\Delta_{\mathrm{O}_S}}$ be explicitly independent of time.

Another way to respond is that these two functions are divided as follows:

$$\overrightarrow{s^\Delta}\left(Q_1, \ldots, Q_6 \left| Q_1', \ldots, Q_6' \right| t\right) = \ldots$$

$$\ldots \; \overrightarrow{s^{\Delta *}}\left(Q_1, \ldots, Q_6 \left| Q_1', \ldots, Q_6' \right.\right) + \vec{\sigma}(t)$$

$$\overrightarrow{\mathcal{M}^\Delta_{\mathrm{O}_S}}\left(Q_1, \ldots, Q_6 \left| Q_1', \ldots, Q_6' \right| t\right) = \ldots$$

$$\ldots \; \overrightarrow{\mathcal{M}^{\Delta *}_{\mathrm{O}_S}}\left(Q_1, \ldots, Q_6 \left| Q_1', \ldots, Q_6' \right.\right) + \vec{\mu}(t)$$

with $\vec{\sigma}(t) = \vec{0}$ and $\overrightarrow{\mu_{\mathrm{O}_S}}(t) = \vec{0}$ for $t_i < t < t_f$.

In these conditions, the positions of equilibrium of (S) in $\langle g\rangle$ are given by two vector relations:

$$\begin{cases} \overrightarrow{s^{\Delta *}}\left(e_1,\ldots,e_6 \,|\, 0,\ldots,0\right)=\vec{0} \\ \overrightarrow{\mathscr{M}^{\Delta *}_{O_S}}\left(e_1,\ldots,e_6 \,|\, 0,\ldots,0\right)=\vec{0} \end{cases}$$

1.3.2. *Stability of equilibrium*

An equilibrium is said to be stable when, broken by a disturbance, it is restored with a succession of small movements, or oscillations, which maintain or bring the equilibrium back to its initial state.

1.4. General equations of small movements of a free solid

1.4.1. *Reminder of developments limited to the first order*

1.4.1.1. *Function of one variable*

We consider the function of one variable $f(x)$, a particular value a of this variable and its increase h around that particular value.

Let us state $x=a+h$.

$$f(x)=f(a+h)=f(a)+\frac{h}{1!}\frac{df}{dx}(a)+\frac{h^2}{2!}\frac{d^2 f}{dx^2}(a)+h^2\varepsilon(a,h)$$

with $\lim\limits_{h\to 0}\varepsilon(a,h)=0$.

We write:

$$D_1\left[f(a+h)\right]=f(a)+h\frac{df}{dx}(a)$$

the development limited to the first order of the function $f(x)$, around the particular value $x=a$ of its variable.

When the function is vectorial, $\vec{f}(x)$, it should be specified for any derivative of that function the basis (λ) where this derivative is established. Around that particular value $x=a$, we write:

$$D_1\left[\vec{f}(a+h)\right]=\vec{f}(a)+h\frac{d^{(\lambda)}\vec{f}}{dx}(a)$$

1.4.1.2. *Function of two variables*

We consider the function of two variables $f(x,y)$ and its variation around the couple of values $x=a$ and $y=b$. We state

$$x=a+h,\ y=b+k\ ,$$

$$f(a+h,b+k)=f(a,b)+\frac{h}{1!}\frac{\partial f}{\partial x}(a,b)+\frac{k}{1!}\frac{\partial f}{\partial y}(a,b)\ \ldots$$

$$\ldots\ +\frac{h^2}{2!}\frac{\partial^2 f}{\partial x^2}(a,b)+\frac{2hk}{2!}\frac{\partial^2 f}{\partial x\partial y}(a,b)+\frac{k^2}{2!}\frac{\partial^2 f}{\partial y^2}(a,b)+\ldots$$

as well as

$$D_1\left[f(a+h,b+k)\right]=f(a,b)+\frac{h}{1!}\frac{\partial f}{\partial x}(a,b)+\frac{k}{1!}\frac{\partial f}{\partial y}(a,b)$$

When the function of two variables is vectorial, $\vec{f}(x,y)$, around the couple of values (a,b), we state

$$D_1\left[\vec{f}(a+h,b+k)\right]=\vec{f}(a,b)+\frac{h}{1!}\frac{\partial^{(\lambda)}\vec{f}}{\partial x}(a,b)+\frac{k}{1!}\frac{\partial^{(\lambda)}\vec{f}}{\partial y}(a,b)$$

1.4.1.3. *Function of n variables*

If we now consider a function of n variables $f\left(x_1,\ldots,x_i,\ldots,x_n\right)$ and its variations neighboring the set of values $\left(a_1,\ldots,a_i,\ldots,a_n\right)$, where

$$x_i = a_i + h_i,\, i = 1,\ldots,n$$

we state

$$D_1\left[f\left(a_1+h_1,\ldots,a_n+h_n\right)\right] = f\left(a_1,\ldots,a_n\right) + h_i\frac{\partial f}{\partial x_i}\left(a_1,\ldots,a_n\right)$$

and for a vectorial function of n variables $\vec{f}\left(x_1,\ldots,x_n\right)$

$$D_1\left[\vec{f}\left(a_1+h_1,\ldots,a_n+h_n\right)\right] = \vec{f}\left(a_1,\ldots,a_n\right) + h_i\frac{\partial^{(\lambda)}\vec{f}}{\partial x_i}\left(a_1,\ldots,a_n\right)$$

1.4.1.4. *Variation of a product of functions*

Consider, to set the stage, two functions, $f\left(x\right)$ and $g\left(y\right)$, of one variable and the variation of their product $f\left(x\right)g\left(y\right)$, around the couple of values $x = a$ and $y = b$. We get

$$D_1\left[f\left(a+h\right)g\left(b+k\right)\right] = \;\ldots$$

$$\ldots\;\; f\left(a\right)g\left(b\right) + h\,g\left(b\right)\frac{df}{dx}\left(a\right) + k\,f\left(a\right)\frac{dg}{dy}\left(b\right)$$

and in the case of the product of a function $f\left(x\right)$ by k, infinitely small of the first order or more, neighboring the value $x = a$,

$$D_1\left[k\,f\left(a+h\right)\right] = D_1\left[k\left(f\left(a\right) + h\frac{df}{dx}\left(a\right)\right)\right] = k\,f\left(a\right)$$

since the product kh is infinitely small of at least order 2, thus negligible in a development limited to the first order.

1.4.2. *Equations of small movements of the free solid*

1.4.2.1. *Vector equations of small movements*

Perform the development limited to the first order of the dynamic torsor of the motion of the free solid (S) in the frame $\langle g \rangle$, around the position of equilibrium previously determined, characterized by the values $e_1, e_2, e_3, e_4, e_5, e_6$ of the situation parameters $Q_1, Q_2, Q_3, Q_4, Q_5, Q_6$ of this solid.

The small movements of the solid around its equilibrium position are expressed by

$$Q_1 = e_1 + \varepsilon_1, Q_2 = e_2 + \varepsilon_2, Q_3 = e_3 + \varepsilon_3$$
$$Q_4 = e_4 + \varepsilon_4, Q_5 = e_5 + \varepsilon_5, Q_6 = e_6 + \varepsilon_6$$

which we synthesize as

$$Q_\alpha = e_\alpha + \varepsilon_\alpha \quad \Rightarrow \quad Q_\alpha' = \varepsilon_\alpha', Q_\alpha'' = \varepsilon_\alpha''$$

We will admit, to simplify the reasoning without affecting the validity of the development, that the terms $\varepsilon_\alpha, \varepsilon_\alpha'$ and ε_α'' are infinitely small of the same order.

The two vector relations that express the fundamental principle are then written as:

$$\sum_\alpha \overrightarrow{s^\alpha}(Q_1,\ldots,Q_6)Q_\alpha'' + \sum_{\alpha\beta} \overrightarrow{s^{\alpha\beta}}(Q_1,\ldots,Q_6)Q_\alpha' Q_\beta' = \ \ldots$$
$$\ldots \ \overrightarrow{s^{\Delta *}}\left(Q_1,\ldots,Q_6 \middle| Q_1',\ldots,Q_6'\right) + \vec{\sigma}(t)$$

$$\sum_{\alpha} \overrightarrow{m_{O_S}^{\alpha}}(Q_1,\ldots,Q_6)Q_\alpha'' + \sum_{\alpha\beta} \overrightarrow{m_{O_S}^{\alpha\beta}}(Q_1,\ldots,Q_6)Q_\alpha' Q_\beta' = \ldots$$

$$\ldots \ \overrightarrow{\mathscr{M}_{O_S}^{\Delta *}}\left(Q_1,\ldots,Q_6 \middle| Q_1',\ldots,Q_6'\right) + \vec{\mu}(t)$$

expressions in which the development limited to the first order of their different terms gives

$$D_1\left[\sum_{\alpha} \overrightarrow{s^{\alpha}}(e_1+\varepsilon_1,\ldots,e_6+\varepsilon_6)\varepsilon_\alpha'' \ \ldots\right.$$

$$\left.\ldots \ + \sum_{\alpha\beta} \overrightarrow{s^{\alpha\beta}}(e_1+\varepsilon_1,\ldots,e_6+\varepsilon_6)\varepsilon_\alpha' \varepsilon_\beta'\right] = \sum_{\alpha} \overrightarrow{s^{\alpha}}(e_1,\ldots,e_6)\varepsilon_\alpha''$$

$$D_1\left[\overrightarrow{s^{\Delta *}}\left(e_1+\varepsilon_1,\ldots,e_6+\varepsilon_6 \middle| \varepsilon_1',\ldots,\varepsilon_6'\right) + \vec{\sigma}(t)\right] = \ \ldots$$

$$\ldots \ \overrightarrow{s^{\Delta *}}\left(e_1,\ldots,e_6 \middle| 0,\ldots,0\right) + \sum_{\alpha} \varepsilon_\alpha \frac{\partial \overrightarrow{s^{\Delta *}}}{\partial Q_\alpha}\left(e_1,\ldots,e_6 \middle| 0,\ldots,0\right) \ \ldots$$

$$\ldots \ + \sum_{\alpha} \varepsilon_\alpha' \frac{\partial \overrightarrow{s^{\Delta *}}}{\partial Q_\alpha'}\left(e_1,\ldots,e_6 \middle| 0,\ldots,0\right) + \vec{\sigma}(t)$$

where $\varepsilon_\alpha' \varepsilon_\beta'$ is infinitely small of the second order. At the equilibrium, we get

$$\overrightarrow{s^{\Delta *}}\left(e_1,\ldots,e_6 \middle| 0,\ldots,0\right) = \vec{0}$$

Thus, the first vector equation of small movements of the solid (S) around the position of equilibrium $(e) \equiv \left(e_1,\ldots,e_6 \middle| 0,\ldots,0\right)$

$$\sum_{\alpha} \overrightarrow{s^{\alpha}}(e)\varepsilon_\alpha'' = \sum_{\alpha} \varepsilon_\alpha \frac{\partial \overrightarrow{s^{\Delta *}}}{\partial Q_\alpha}(e) + \sum_{\alpha} \varepsilon_\alpha' \frac{\partial \overrightarrow{s^{\Delta *}}}{\partial Q_\alpha'}(e) + \vec{\sigma}(t)$$

We will obtain, in the same way, the development at the first order of the vector equation of the moment of small movements of (S) around the equilibrium position (e)

$$\sum_{\alpha} \overrightarrow{m^{\alpha}_{\mathrm{O}_S}}(e)\varepsilon_{\alpha}'' = \sum_{\alpha} \varepsilon_{\alpha} \frac{\partial \overrightarrow{\mathscr{M}^{\Delta *}_{\mathrm{O}_S}}}{\partial Q_{\alpha}}(e) + \sum_{\alpha} \varepsilon_{\alpha}' \frac{\partial \overrightarrow{\mathscr{M}^{\Delta *}_{\mathrm{O}_S}}}{\partial Q_{\alpha}'}(e) + \vec{\mu}(t)$$

1.4.2.2. *Applications*

1.4.2.2.1. Problem 1

The solid (S) to which is joined the frame $\langle S\rangle \equiv \left\langle \mathrm{O} \middle| \overrightarrow{x_S}\ \overrightarrow{y_S}\ \vec{z} \right\rangle$ is in a perfect rotoid link of axis $\left(\mathrm{O}\middle|\vec{z}\right)$ in relation to the Galilean frame $\langle g\rangle \equiv \left\langle \mathrm{O} \middle| \vec{x}\ \vec{y}\ \vec{z} \right\rangle$, where $\vec{x}$ is vertical descending. Its kinetic characteristics are as follows:

– mass m;

– center of inertia G defined by $\overrightarrow{\mathrm{OG}} = a\overrightarrow{x_s}$;

– inertial matrix $[I_{\mathrm{O}}]_{/(S)} \equiv \begin{bmatrix} A & -F & -E \\ -F & B & -D \\ -E & -D & C \end{bmatrix}$.

The spring (R), of stiffness k and free length ℓ_0, ties a fixed point of the solid A to a point A′ also fixed in the Galilean frame. It works under pulling up and down actions according to the law:

$$\vec{R} = k\left(1 - \frac{\ell_0}{\left\|\overrightarrow{\mathrm{AA}'}\right\|}\right)\overrightarrow{\mathrm{AA}'}$$

where $\overrightarrow{\mathrm{OA}} = c\,\overrightarrow{x_S}, \overrightarrow{\mathrm{OA}'} = b\,\vec{x}$.

g is the intensity of the local gravity field.

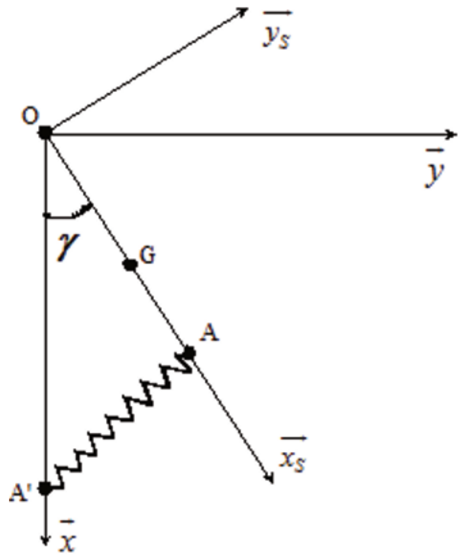

Figure 1.1. *Configuration of the oscillator*

Question 1: Determine the reduction elements at O of the torsor associated with the action exerted by the spring (R) on the solid (S).

$$\{\mathrm{R} \to S\}_{\mathrm{A}} = \left[\vec{R} \middle| \vec{0}\right] \quad , \quad \vec{R} = k\left(1 - \frac{\ell_0}{\left\|\overrightarrow{\mathrm{AA'}}\right\|}\right)\overrightarrow{\mathrm{AA'}}$$

with

$$\overrightarrow{\mathrm{AA'}} = \overrightarrow{\mathrm{OA'}} - \overrightarrow{\mathrm{OA}} = b\,\vec{x} - c\,\overrightarrow{x_S} = (b - c\cos\gamma)\vec{x} - c\sin\gamma\,\vec{y}$$

$$\left\|\overrightarrow{\mathrm{AA'}}\right\| = \sqrt{b^2 + c^2 - 2bc\cos\gamma}$$

$$\Rightarrow \quad \vec{s}\{\mathrm{R} \to S\} \quad \ldots$$

$$\ldots \quad = k\left(1 - \frac{\ell_0}{\sqrt{b^2 + c^2 - 2bc\cos\gamma}}\right)\left[(b - c\cos\gamma)\vec{x} - c\sin\gamma\,\vec{y}\right]$$

$$\overrightarrow{\mathscr{M}_{\mathrm{O}}}\{\mathrm{R} \to S\} = \overrightarrow{\mathscr{M}_{\mathrm{A}}}\{\mathrm{R} \to S\} + \overrightarrow{\mathrm{OA}} \wedge \vec{s}\{\mathrm{R} \to S\}$$

$$'' \quad '' \qquad = -kbc\sin\gamma\left(1 - \frac{\ell_0}{\sqrt{b^2 + c^2 - 2bc\cos\gamma}}\right)\vec{z}$$

Question 2: Write the differential equation governing the motion of (S) in relation to $\langle g \rangle$.

$$-\left\{\mathscr{A}_S^g\right\}_{\mathrm{O}} = \left[m\overrightarrow{J^{(g)}}(\mathrm{G}) \middle| I_{\mathrm{O}}\overrightarrow{\omega_S^g}' + \overrightarrow{\omega_S^g} \wedge I_{\mathrm{O}}\overrightarrow{\omega_S^g}\right]$$

$$\overrightarrow{\mathrm{OG}} = a\overrightarrow{x_S},\ \overrightarrow{\omega_S^g} = \gamma'\vec{z} = \gamma'\overrightarrow{z_S},\ \overrightarrow{\omega_S^g}' = \gamma''\vec{z}$$

$$\overrightarrow{v^{(g)}}(\mathrm{G}) = a\gamma'\overrightarrow{y_S},\ \overrightarrow{J^{(g)}}(\mathrm{G}) = a\left(-\gamma'^2\overrightarrow{x_S} + \gamma''\overrightarrow{y_S}\right)$$

$$I_{\mathrm{O}}\overrightarrow{\omega_S^g} = -E\gamma'\overrightarrow{x_S} - D\gamma'\overrightarrow{y_S} + C\gamma'\overrightarrow{z_S}$$

$$I_{\mathrm{O}}\overrightarrow{\omega_S^g}' = -E\gamma''\overrightarrow{x_S} - D\gamma''\overrightarrow{y_S} + C\gamma''\overrightarrow{z_S}$$

$$\left\{\mathscr{A}_S^g\right\}_{\mathrm{O}} = \left[ma\left(-\gamma'^2\overrightarrow{x_S} + \gamma'' y_S\right)\middle| \ldots\right.$$

$$\left.\ldots\left(D\gamma'^2 - E\gamma''\right)\overrightarrow{x_S} - \left(E\gamma'^2 + D\gamma''\right)\overrightarrow{y_S} + C\gamma''\overrightarrow{z_S}\right]$$

$$-\left\{\overline{S} \to S\right\} = \{\mathrm{R} \to S\} + \{\pi \to S\} + \left\{\mathscr{L}_{g \to S}\right\}$$

$$\{\pi \to S\}_{\mathrm{G}} = \left[mg\,\vec{x}\middle|\vec{0}\right]$$

$$\overrightarrow{\mathscr{M}_{\mathrm{O}}}\{\pi \to S\} = \overrightarrow{\mathscr{M}_{\mathrm{G}}}\{\pi \to S\} + \overrightarrow{\mathrm{OG}} \wedge \vec{S}\{\pi \to S\} = -mga\sin\gamma\,\vec{z}$$

As for the torsor associated with the action exerted by the link on the solid, we state

$$\left\{\mathscr{L}_{g \to S}\right\}_{\mathrm{O}} = \left[X\overrightarrow{x_S} + Y\overrightarrow{y_S} + Z\vec{z}\middle| L\overrightarrow{x_S} + M\overrightarrow{y_S} + N\vec{z}\right]$$

As the link is perfect in the motion of (S) in relation to $\langle g \rangle$, we write

$$\mathcal{P}^{(g)}\left\{\mathcal{L}_{g\to S}\right\}=\left\{{}^{g}_{S}\right\}\otimes\left\{\mathcal{L}_{g\to S}\right\}$$
$$\text{"}\quad\text{"}\quad=\overrightarrow{\omega_S^g}\cdot\overrightarrow{\mathcal{M}_{\mathrm{O}}}\left\{\mathcal{L}_{g\to S}\right\}+\overrightarrow{v^{(g)}}(\mathrm{O})\cdot\vec{s}\left\{\mathcal{L}_{g\to S}\right\}=N\gamma'=0$$

With this result being valid $\forall\gamma'$, the condition of perfect link becomes

$$N=0$$

These various elements allow us to write the scalar consequences of the fundamental principle applied to the motion of the solid (S) in relation to $\langle g \rangle$.

$$-ma\gamma'^2=X+mg\cos\gamma+k\left(1-\frac{\ell_0}{\sqrt{b^2+c^2-2bc\cos\gamma}}\right)(b\cos\gamma-c)$$
$$ma\gamma''=Y-mg\sin\gamma-k\left(1-\frac{\ell_0}{\sqrt{b^2+c^2-2bc\cos\gamma}}\right)\mathrm{c}\sin\gamma$$
$$0=Z$$

$$D\gamma'^2-E\gamma''=L$$
$$-E\gamma'^2-D\gamma''=M$$
$$C\gamma''=-mga\sin\gamma-kbc\sin\gamma\left(1-\frac{\ell_0}{\sqrt{b^2+c^2-2bc\cos\gamma}}\right)$$

We note that, with regard to these six equations, the first five contain link terms. They will be particularly appropriate for determining these links when the motion will be known. Only the last one is a differential equation in γ, this is therefore the sought movement equation.

$$C\gamma'' + \left[mga + kbc\left(1 - \frac{\ell_0}{\sqrt{b^2 + c^2 - 2bc\cos\gamma}}\right)\right]\sin\gamma = 0$$

Question 3: Write the equation that helps to determine the equilibrium position of (S) in relation to $\langle g \rangle$.

The values of the situation parameter of (S) in relation to $\langle g \rangle$, for which the solid is at equilibrium, are given as

$$\gamma = \gamma_e \quad \Rightarrow \quad \gamma_e' = 0, \gamma_e'' = 0$$

and verify the relation

$$\left[mga + kbc\left(1 - \frac{\ell_0}{\sqrt{b^2 + c^2 - 2bc\cos\gamma_e}}\right)\right]\sin\gamma_e = 0$$

Question 4: Deduce from the previous question what is the free length of the spring that allows (S) to reach the equilibrium position $\gamma_e = \dfrac{\pi}{3}$.

The equation of equilibrium leads to the following possibilities

– $\sin\gamma_e = 0 \quad \Rightarrow \quad \gamma_e = 0, \gamma_e = \pi$,

– $mga + kbc\left(1 - \dfrac{\ell_0}{\sqrt{b^2 + c^2 - 2bc\cos\gamma_e}}\right) = 0$

$$\Rightarrow \quad \ell_0 = \left(1 + \frac{mga}{kbc}\right)\sqrt{b^2 + c^2 - 2bc\cos\gamma_e}$$

For $\gamma_e = \dfrac{\pi}{3}, \ \cos\gamma_e = \dfrac{1}{2} \quad \Rightarrow \quad \ell_0 = \left(1 + \dfrac{mga}{kbc}\right)\sqrt{b^2 + c^2 - bc}$

Question 5: Determine the equation of the oscillator associated with (S), which governs its small movements in $\langle g \rangle$ around the equilibrium position $\gamma_e = \dfrac{\pi}{3}$. We will determine the natural frequency ϖ of these vibrations depending on k, a, b, c, mg, C.

We therefore consider a variation of the situation of (S) around the position of equilibrium γ_e by stating

$$\gamma = \gamma_e + \varepsilon \;\Rightarrow\; \gamma' = \varepsilon',\; \gamma'' = \varepsilon''$$

We apply these values in the movement equation that we then linearize to the first order in ε

$$C\varepsilon'' + D_1 \left\{ \left[mga + kbc \left(1 - \frac{\ell_0}{\sqrt{b^2 + c^2 - 2bc\cos(\gamma_e + \varepsilon)}} \right) \right] \sin(\gamma_e + \varepsilon) \right\} = 0$$

The development limited to the first order of the function $f(\gamma)$ in the neighborhood of the value $\gamma = \gamma_e$ has the following expression:

$$D_1\left[f(\gamma_e + \varepsilon) \right] = f(\gamma_e) + \frac{\varepsilon}{1!}\left(\frac{df}{d\gamma} \right)_{\gamma=\gamma_e} = \frac{\varepsilon}{1!}\left(\frac{df}{d\gamma} \right)_{\gamma=\gamma_e}$$

where $f(\gamma_e) = 0$

$$f(\gamma) = \left[mga + kbc \left(1 - \frac{\ell_0}{\sqrt{b^2 + c^2 - 2bc\cos\gamma}} \right) \right] \sin\gamma$$

$$\frac{df}{d\gamma}=\left[mga+kbc\left(1-\frac{\ell_0}{\sqrt{b^2+c^2-2bc\cos\gamma}}\right)\right]\cos\gamma\ \ldots$$

$$\ldots\ +\frac{kb^2c^2\ell_0\sin^2\gamma}{\left(b^2+c^2-2bc\cos\gamma\right)^{3/2}}$$

$$\Rightarrow\ \left(\frac{df}{d\gamma}\right)_{\gamma=\gamma_e}=\frac{kb^2c^2\ell_0\sin^2\gamma_e}{\left(b^2+c^2-2bc\cos\gamma_e\right)^{3/2}}$$

taking into account the relation between ℓ_0 and γ_e. For $\gamma_e=\dfrac{\pi}{3}$,

$$\left(\frac{df}{d\gamma}\right)_{\gamma=\frac{\pi}{3}}=\frac{3bc\left(kbc+mga\right)}{4\left(b^2+c^2-bc\right)}$$

thus the equation of the associated oscillator

$$C\varepsilon''+\frac{3bc\left(kbc+mga\right)}{4\left(b^2+c^2-bc\right)}\varepsilon=0$$

We notice that

$$b^2+c^2-bc=b^2+c^2-2bc+bc=\left(b-c\right)^2+bc>0$$

As the coefficient of ε in the equation of the associated oscillator is positive, we can write this equation under the following form:

$$\varepsilon''+\varpi^2\varepsilon=0$$

where the natural frequency of this oscillator has the following expression:

$$\varpi=\sqrt{\frac{3bc\left(kbc+mga\right)}{4C\left(b^2+c^2-bc\right)}}$$

1.4.2.2.2. Problem 2

We consider the solid (S), to which is joined the frame $\langle S\rangle \equiv \left\langle \mathrm{O}_S \middle| \overrightarrow{x_S}\ \overrightarrow{y_S}\ \overrightarrow{z_S} \right\rangle$, animated by a motion with fixed plane in the Galilean frame $\langle g\rangle \equiv \left\langle \mathrm{O}' \middle| \vec{x}\ \vec{y}\ \vec{z} \right\rangle$ where $\vec{x}$ is vertical descending.

g refers to the intensity of the local field of gravity.

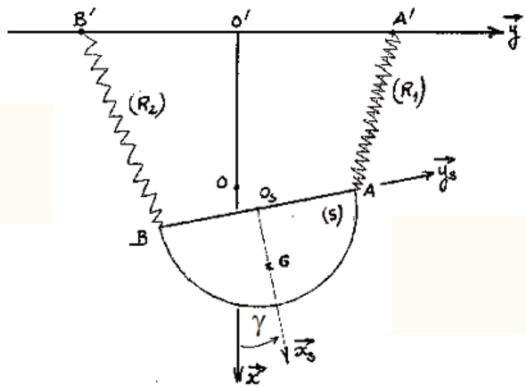

Figure 1.2. *Configuration of the oscillatory device*

The kinetic characteristics of the solid (S) are as follows:

– mass m;

– center of inertia G defined by $\overrightarrow{\mathrm{O}_s\mathrm{G}} = a\overrightarrow{x_s}$;

– inertial matrix $\left[I_{\mathrm{O}_S}\right]_{/(S)} \equiv \begin{bmatrix} A & -F & -E \\ -F & B & -D \\ -E & -D & C \end{bmatrix}$;

where the component $I_{\mathrm{O}_S}\vec{z}$ is orthogonal to the plane $\Pi\left(\overrightarrow{x_S}, \overrightarrow{y_S}\right)$

$$\Rightarrow \quad \overrightarrow{x_S} \cdot I_{\mathrm{O}_S}\vec{z} = \overrightarrow{y_S} \cdot I_{\mathrm{O}_S}\vec{z} = 0$$

The solid is hanged through two springs (R_1) and (R_2) attached, respectively, to the Galilean frame at points A' and B' defined as

$$\overrightarrow{\mathrm{B'O'}} = \overrightarrow{\mathrm{O'A'}} = c\,\vec{y}$$

The respective attachment points A and B of the springs on to the solid are fixed points on it and are defined as:

$$\overrightarrow{\mathrm{BO}_S} = \overrightarrow{\mathrm{O}_S\mathrm{A}} = b\,\overrightarrow{y_S}$$

The spring (R_1) is attached by its ends to points A and A', the spring (R_2), to points B and B'. They work under pulling up and down action, have the same stiffness k and the same free length ℓ_0.

A point O tied to $\langle g\rangle$, defined as $\overrightarrow{\mathrm{O'O}} = x_e\vec{x}$, is selected so that when (S) is unbalanced in relation to $\langle g\rangle$, the points O and O_S coincide and we get

$$\left(\overrightarrow{x_S}\right)_e = \vec{x} \text{ and } \left(\overrightarrow{y_S}\right)_e = \vec{y}$$

The situation of (S) in relation to $\langle g\rangle$ is defined as:

– $\overrightarrow{\mathrm{OO}_S} = x\vec{x} + y\vec{y}$;

– $\overrightarrow{x_S} = \cos\gamma\,\vec{x} + \sin\gamma\,\vec{y}$;

where x, y, γ are functions of time twice derivable.

Question 1: Determine, through their components in the basis (g) the reduction elements at O_S of the torsors associated with actions exerted, respectively, on (S) by the spring (R_1), the spring (R_2) and the field of gravity, depending on x, y and the trigonometric functions of γ.

$$-\ \{\mathrm{R}_1 \to S\}_{\mathrm{A}} = \left[k\left(1-\frac{\ell_0}{\left\|\overrightarrow{\mathrm{AA'}}\right\|}\right)\overrightarrow{\mathrm{AA'}} \middle| \vec{0} \right]$$

$$\overrightarrow{\mathrm{AA'}} = \overrightarrow{\mathrm{AO}_S} + \overrightarrow{\mathrm{O}_S\mathrm{O}} + \overrightarrow{\mathrm{OO'}} + \overrightarrow{\mathrm{O'A'}} \quad \ldots$$

$$\text{" "} = -(x + x_e - b\sin\gamma)\vec{x} - (y - c + b\cos\gamma)\vec{y}$$

$$\left\|\overrightarrow{\mathrm{AA'}}\right\| = \sqrt{\overrightarrow{\mathrm{AA'}}^2}$$

$$\text{" "} = \sqrt{(x+x_e)^2 + (c-y)^2 + b^2 - 2b(x+x_e)\sin\gamma - 2b(c-y)\cos\gamma}$$

$$\overrightarrow{\mathscr{M}_{\mathrm{O}_S}}\{\mathrm{R}_1 \to S\} = \overrightarrow{\mathscr{M}_{\mathrm{A}}}\{\mathrm{R}_1 \to S\} + \overrightarrow{\mathrm{O}_S\mathrm{A}} \wedge \vec{s}\{\mathrm{R}_1 \to S\}$$

$$\text{" "} \qquad = kb\left[(x+x_e)\cos\gamma + (y-c)\sin\gamma\right]\left(1-\frac{\ell_0}{\left\|\overrightarrow{\mathrm{AA'}}\right\|}\right)\vec{z}$$

$$\{\mathrm{R}_2 \to S\}_{\mathrm{B}} = \left[k\left(1-\frac{\ell_0}{\left\|\overrightarrow{\mathrm{BB'}}\right\|}\right)\overrightarrow{\mathrm{BB'}} \middle| \vec{0} \right]$$

$$\overrightarrow{\mathrm{BB'}} = \overrightarrow{\mathrm{BO}_S} + \overrightarrow{\mathrm{O}_S\mathrm{O}} + \overrightarrow{\mathrm{OO'}} + \overrightarrow{\mathrm{O'B'}} \quad \ldots$$

$$\text{" "} = -(x + x_e + b\sin\gamma)\vec{x} - (y + c - b\cos\gamma)\vec{y}$$

$$\left\|\overrightarrow{\mathrm{BB'}}\right\| = \sqrt{\overrightarrow{\mathrm{BB'}}^2}$$

$$\text{" "} = \sqrt{(x+x_e)^2 + (c+y)^2 + b^2 + 2b(x+x_e)\sin\gamma - 2b(c+y)\cos\gamma}$$

$$\overrightarrow{\mathscr{M}_{\mathrm{O}_S}}\{\mathrm{R}_2 \to S\} = \overrightarrow{\mathscr{M}_{\mathrm{B}}}\{\mathrm{R}_2 \to S\} + \overrightarrow{\mathrm{O}_S\mathrm{B}} \wedge \vec{s}\{\mathrm{R}_2 \to S\}$$

$$\text{" "} \qquad = -kb\left[(x+x_e)\cos\gamma + (y+c)\sin\gamma\right]\left(1-\frac{\ell_0}{\left\|\overrightarrow{\mathrm{BB'}}\right\|}\right)\vec{z}$$

$$-\ \{\pi \to S\}_{\mathrm{G}} = \left[mg\,\vec{x} \middle| \vec{0} \right]$$

$$\overrightarrow{\mathcal{M}_{\mathrm{O}_S}}\{\pi \to S\} = \overrightarrow{\mathcal{M}_{\mathrm{G}}}\{\pi \to S\} + \overrightarrow{\mathrm{O}_S\mathrm{G}} \wedge \vec{s}\{\pi \to S\} = -mga\sin\gamma\,\vec{z}$$

Question 2: Determine the relation that joins k, b, c, mg, ℓ_0, x_e at equilibrium. Since the situation at the equilibrium – as it is defined – is symmetrical, the two springs will then have the same length, written as ℓ.

The fundamental principle of dynamics applied to the motion of (S) in $\langle g \rangle$ is expressed by:

$$\left\{\mathcal{A}_S^g\right\} = \left\{\overline{S} \to S\right\} = \{R_1 \to S\} + \{R_2 \to S\} + \{\pi \to S\} + \left\{\mathcal{L}_{g\to S}\right\}$$

with

$$\left\{\mathcal{L}_{g\to S}\right\}_{\mathrm{O}_S} = \left[X\vec{x} + Y\vec{y} + Z\vec{z} \,\middle|\, L\vec{x} + M\vec{y} + N\vec{z} \right]$$

As this link is perfect in the motion of (S) in relation to $\langle g \rangle$, we get

$$\mathcal{P}^{(g)}\left\{\mathcal{L}_{g\to S}\right\} = \left\{{}^g_S\right\} \otimes \left\{\mathcal{L}_{g\to S}\right\} = 0$$

$$\text{"} \quad \text{"} \qquad = \overrightarrow{\omega_S^g} \cdot \overrightarrow{\mathcal{M}_{\mathrm{O}_S}}\left\{\mathcal{L}_{g\to S}\right\} + \overrightarrow{v^{(g)}}(\mathrm{O}_S) \cdot \vec{s}\left\{\mathcal{L}_{g\to S}\right\}$$

with $\overrightarrow{\omega_S^g} = \gamma'\vec{z}$, $\overrightarrow{\mathrm{OO}_S} = x\vec{x} + y\vec{y} \;\Rightarrow\; \overrightarrow{v^{(g)}}(\mathrm{O}_S) = x'\vec{x} + y'\vec{y}$

thus the condition of perfect link

$$\mathcal{P}^{(g)}\left\{\mathcal{L}_{g\to S}\right\} = N\gamma' + Xx' + Yy' = 0$$

Because this link is perfect throughout the entire motion, meaning $\forall\gamma'$, $\forall x'$ and $\forall y'$, this condition of perfect link ultimately is expressed by

$$X = 0, \quad Y = 0, \quad N = 0$$

Among six scalar consequences of the fundamental principle, there are three that contain no link term. As there are three situation parameters of the solid in $\langle g\rangle$: x, y, γ, we have three differential equations with three unknowns to study the motion of (S). They are the movement equations of (S) in relation to $\langle g\rangle$.

Those three equations are given as:

$$\vec{x}\cdot\vec{s}\left\{\mathscr{A}_S^g\right\}=\vec{x}\cdot\left[\vec{s}\{R_1\to S\}+\vec{s}\{R_2\to S\}+\vec{s}\{\pi\to S\}\right]$$

$$\vec{y}\cdot\vec{s}\left\{\mathscr{A}_S^g\right\}=\vec{y}\cdot\left[\vec{s}\{R_1\to S\}+\vec{s}\{R_2\to S\}+\vec{s}\{\pi\to S\}\right]$$

$$\vec{z}\cdot\overrightarrow{\mathscr{M}_{\mathrm{O}_S}}\left\{\mathscr{A}_S^g\right\}\ \ldots$$
$$\ldots\ =\vec{z}\cdot\left[\overrightarrow{\mathscr{M}_{\mathrm{O}_S}}\{R_1\to S\}+\overrightarrow{\mathscr{M}_{\mathrm{O}_S}}\{R_2\to S\}+\overrightarrow{\mathscr{M}_{\mathrm{O}_S}}\{\pi\to S\}\right]$$

At the equilibrium, the terms of the first member of each of the equations above are null since the dynamic torsor, which provides the three equations of equilibrium is null for the values at equilibrium:

$$\vec{x}\cdot\left[\vec{s}\{R_1\to S\}+\vec{s}\{R_2\to S\}+\vec{s}\{\pi\to S\}\right]=0$$
$$\vec{y}\cdot\left[\vec{s}\{R_1\to S\}+\vec{s}\{R_2\to S\}+\vec{s}\{\pi\to S\}\right]=0$$
$$\vec{z}\cdot\left[\overrightarrow{\mathscr{M}_{\mathrm{O}_S}}\{R_1\to S\}+\overrightarrow{\mathscr{M}_{\mathrm{O}_S}}\{R_2\to S\}+\overrightarrow{\mathscr{M}_{\mathrm{O}_S}}\{\pi\to S\}\right]=0$$

Since the positions of O and O_S at the equilibrium coincide, this equilibrium is expressed by these values

$$x_e=0,\quad y_e=0,\quad \gamma_e=0$$

which must satisfy the three equations above. Furthermore, we have

$$\overrightarrow{\mathrm{AA}'_e}^{\,2}=\overrightarrow{\mathrm{BB}'_e}^{\,2}=x_e^{\,2}+(b-c)^2$$

thus, the three equations of equilibrium

$$\begin{cases} -k\left(1-\dfrac{l_0}{\sqrt{x_e^{\,2}+(b-c)^2}}\right)x_e-k\left(1-\dfrac{l_0}{\sqrt{x_e^{\,2}+(b-c)^2}}\right)x_e+mg=0 \\ -k\left(1-\dfrac{l_0}{\sqrt{x_e^{\,2}+(b-c)^2}}\right)(b-c)+k\left(1-\dfrac{l_0}{\sqrt{x_e^{\,2}+(b-c)^2}}\right)(b-c)=0 \\ kb\left(1-\dfrac{l_0}{\sqrt{x_e^{\,2}+(b-c)^2}}\right)x_e-kb\left(1-\dfrac{l_0}{\sqrt{x_e^{\,2}+(b-c)^2}}\right)x_e=0 \end{cases}$$

which can be reduced to one single relation

$$mg=2k\left(1-\frac{l_0}{\sqrt{x_e^{\,2}+(b-c)^2}}\right)x_e,\ \text{where}\quad \sqrt{x_e^{\,2}+(b-c)^2}=\ell$$

Question 3: Determine the developments limited to the first order of the reduction elements at O_S of the dynamic torsor of (S) in its motion in relation to $\langle g\rangle$.

$$-\ \vec{s}\left\{\mathscr{A}_S^g\right\}=m\overrightarrow{J^{(g)}}(\mathrm{G}),$$

$$\overrightarrow{\mathrm{OG}}=\overrightarrow{\mathrm{OO}_S}+\overrightarrow{\mathrm{O}_S\mathrm{G}}=(x+a\cos\gamma)\vec{x}+(y+a\sin\gamma)\vec{y}$$

$$\overrightarrow{v^{(g)}}(\mathrm{G})=(x'-a\gamma'\sin\gamma)\vec{x}+(y'+a\gamma'\cos\gamma)\vec{y}$$

$$\overrightarrow{J^{(g)}}(\mathrm{G})=\left(x''-a\gamma''\sin\gamma-a\gamma'^2\cos\gamma\right)\vec{x}\ \ldots$$
$$\ldots+\left(y''+a\gamma''\cos\gamma-a\gamma'^2\sin\gamma\right)\vec{y}$$

$$-\ \overrightarrow{\mathscr{M}_{\mathrm{O}_S}}\left\{\mathscr{A}_S^g\right\}=m\overrightarrow{\mathrm{O}_S\mathrm{G}}\wedge\overrightarrow{J^{(g)}}(\mathrm{O}_S)+I_{\mathrm{O}_S}\overrightarrow{\omega_S^g}'+\overrightarrow{\omega_S^g}\wedge I_{\mathrm{O}_S}\overrightarrow{\omega_S^g}$$

Since, according to the statement,

$$\overrightarrow{x_S} \cdot I_{O_S} \vec{z} = \overrightarrow{x_S} \cdot \left(-E\,\overrightarrow{x_S} - D\,\overrightarrow{y_S} + C\vec{z}\right) = -E = 0$$
$$\overrightarrow{y_S} \cdot I_{O_S} \vec{z} = \overrightarrow{y_S} \cdot \left(-E\,\overrightarrow{x_S} - D\,\overrightarrow{y_S} + C\vec{z}\right) = -D = 0$$

$$I_{O_S} \overrightarrow{\omega_S^g}' = C\gamma''\vec{z} \quad , \quad \overrightarrow{\omega_S^g} \wedge I_{O_S} \overrightarrow{\omega_S^g} = \vec{0}$$

$$\overrightarrow{O_S G} \wedge \overrightarrow{J^{(g)}}(O_S) = a\left(-x''\sin\gamma + y''\cos\gamma\right)\vec{z}$$

we therefore have

$$\vec{s}\left\{\mathscr{A}_S^g\right\} = m\Big[\left(x'' - a\gamma''\sin\gamma - a\gamma'^2\cos\gamma\right)\vec{x} \quad \ldots$$
$$\ldots + \left(y'' + a\gamma''\cos\gamma - a\gamma'^2\sin\gamma\right)\vec{y}\Big]$$

$$\overrightarrow{\mathscr{M}_{O_S}}\left\{\mathscr{A}_S^g\right\} = \left[m\,a\left(-x''\sin\gamma + y''\cos\gamma\right) + C\gamma''\right]\vec{z}$$

In order to simplify the development limited to the first order of these two reduction elements of the dynamic torsor, we note that

– $\vec{s}\left\{\mathscr{A}_S^g\right\}$ has a linear part in Q_α'' and a quadratic part in $Q_\alpha' Q_\beta'$, under the form

$$\vec{s}\left\{\mathscr{A}_S^g\right\} = \sum_\alpha \overrightarrow{s^\alpha} Q_\alpha'' + \sum_{\alpha,\beta} \overrightarrow{s^{\alpha\beta}} Q_\alpha' Q_\beta'$$

– $\overrightarrow{\mathscr{M}_{O_S}}\left\{\mathscr{A}_S^g\right\}$ only has a linear part in Q_α'' and has the following form

$$\overrightarrow{\mathscr{M}_{O_S}}\left\{\mathscr{A}_S^g\right\} = \sum_\alpha \overrightarrow{m_{O_S}^\alpha} Q_\alpha''$$

The small movements of the solid (S) around its equilibrium position are characterized by

$$Q_\alpha = Q_{\alpha_e} + \varepsilon_\alpha \;\Rightarrow\; Q_\alpha' = \varepsilon_\alpha',\;\; Q_\alpha'' = \varepsilon_\alpha''$$

In the present problem

$$Q_1 = x_e + \varepsilon_1,\;\; Q_2 = y_e + \varepsilon_2,\;\; Q_3 = \gamma_e + \varepsilon_3$$

with $x_e = 0,\;\; y_e = 0,\;\; \gamma_e = 0$.

The study of small movements of the solid (S) around its equilibrium position (e) leads to linearizing to the first order the movement equations provided by the scalar consequences of the fundamental principle, i.e. the scalar consequences of the dynamic resultant projecting on $\vec{x}$ and $\vec{y}$ and that of the dynamic moment theorem projecting on $\vec{z}$

$$\vec{s}\left\{\mathscr{A}_S^g\right\} = \overrightarrow{s^\Delta}$$

$$D_1\left[\vec{s}\left\{\mathscr{A}_S^g\right\}\right] = \sum_\alpha \overrightarrow{s^\alpha}_{(e)}\varepsilon_\alpha'' = \;\ldots$$

$$\ldots\;\; \sum_\alpha \overrightarrow{s^\Delta}_{(e)} + \sum_\alpha \left(\frac{\partial \overrightarrow{s^\Delta}}{\partial Q_\alpha}\right)_{(e)} \varepsilon_\alpha + \sum_\alpha \left(\frac{\partial \overrightarrow{s^\Delta}}{\partial Q_\alpha'}\right)_{(e)} \varepsilon_\alpha'$$

$$\overrightarrow{\mathscr{M}_{O_S}}\left\{\mathscr{A}_S^g\right\} = \overrightarrow{M_{O_S}^\Delta}$$

$$D_1\left[\overrightarrow{\mathscr{M}_{O_S}}\left\{\mathscr{A}_S^g\right\}\right] = \sum_\alpha \overrightarrow{m_{O_S}^\alpha}_{(e)}\varepsilon_\alpha'' = \;\ldots$$

$$\ldots\;\; \sum_\alpha \overrightarrow{M_{O_S}^\Delta}_{(e)} + \sum_\alpha \left(\frac{\partial \overrightarrow{M_{O_S}^\Delta}}{\partial Q_\alpha}\right)_{(e)} \varepsilon_\alpha + \sum_\alpha \left(\frac{\partial \overrightarrow{M_{O_S}^\Delta}}{\partial Q_\alpha'}\right)_{(e)} \varepsilon_\alpha'$$

Identifying

$$\begin{cases} \overrightarrow{s^1} = m\vec{x} \;\Rightarrow\; \overrightarrow{s^1}_{(e)} = m\vec{x} \\ \overrightarrow{s^2} = m\vec{y} \;\Rightarrow\; \overrightarrow{s^2}_{(e)} = m\vec{y} \\ \overrightarrow{s^3} = ma\left(-\sin\gamma\vec{x} + \cos\gamma\vec{y}\right) \;\Rightarrow\; \overrightarrow{s^3}_{(e)} = ma\vec{y} \end{cases}$$

$$\begin{cases} \overrightarrow{m^1_{O_S}} = -ma\sin\gamma\vec{z} \;\Rightarrow\; \overrightarrow{m^1_{O_S\,(e)}} = \vec{0} \\ \overrightarrow{m^2_{O_S}} = ma\cos\gamma\vec{z} \;\Rightarrow\; \overrightarrow{m^2_{O_S\,(e)}} = ma\vec{z} \\ \overrightarrow{m^3_{O_S}} = C\vec{z} \;\Rightarrow\; \overrightarrow{m^3_{O_S\,(e)}} = C\vec{z} \end{cases}$$

we get

$$D_1\left[\vec{s}\left\{\mathscr{A}^g_S\right\}\right] = m\varepsilon_1''\vec{x} + m\left(\varepsilon_2'' + a\varepsilon_3''\right)\vec{y}$$

$$D_1\left[\overrightarrow{\mathscr{M}_{O_S}}\left\{\mathscr{A}^g_S\right\}\right] = \left(ma\varepsilon_2'' + C\varepsilon_3''\right)\vec{z}$$

Question 4: Write the equations of the oscillators associated with the motion of the solid (S) under the form $a_{\alpha\beta}\varepsilon_\beta'' + b_{\alpha\beta}\varepsilon_\beta = 0,\ \alpha = 1,2,3,\ \beta = 1,2,3$. We will explain the coefficients $a_{\alpha\beta}$ and $b_{\alpha\beta}$ depending on $m, a, b, c, C, k, \ell_0, \ell, g$.

When we examine the second members of the equations of dynamics, we note that the torsor $\left\{\overline{S} \to S\right\}$ is only the function of the parameters Q_α and not of their derivatives Q_α'. In the equations of the oscillators only the terms $\dfrac{\partial \overrightarrow{s^\Delta}}{\partial Q_\alpha}$ and $\dfrac{\partial \overrightarrow{M^\Delta_{O_S}}}{\partial Q_\alpha}$ will remain.

We set

$$\ell_1 = \left\|\overrightarrow{\mathrm{AA'}}\right\| = \sqrt{\left(x + x_e - b\sin\gamma\right)^2 + \left(y - c + b\cos\gamma\right)^2}$$

$$\ell_2 = \left\|\overrightarrow{\mathrm{BB'}}\right\| = \sqrt{\left(x + x_e + b\sin\gamma\right)^2 + \left(y + c - b\cos\gamma\right)^2}$$

$$\ell_{1(e)} = \ell_{2(e)} = \ell$$

$$\overrightarrow{s^{\Delta}} = k\left(1 - \frac{\ell_0}{\ell_1}\right)\left[-\left(x + x_e - b\sin\gamma\right)\vec{x} - \left(y - c + b\cos\gamma\right)\vec{y}\right] \;\ldots$$

$$\ldots\; + k\left(1 - \frac{\ell_0}{\ell_2}\right)\left[-\left(x + x_e + b\sin\gamma\right)\vec{x} - \left(y + c - b\cos\gamma\right)\vec{y}\right] \;\ldots$$

$$\ldots\; + mg\,\vec{x} + Z\,\vec{z}$$

$$\overrightarrow{M^{\Delta}_{O_S}} = kb\left(1 - \frac{\ell_0}{\ell_1}\right)\left[\left(x + x_e\right)\cos\gamma + \left(y - c\right)\sin\gamma\right]\vec{z} \;\ldots$$

$$\ldots\; - kb\left(1 - \frac{\ell_0}{\ell_2}\right)\left[\left(x + x_e\right)\cos\gamma + \left(y + c\right)\sin\gamma\right]\vec{z} \;\ldots$$

$$\ldots\; - mga\sin\gamma\,\vec{z} + L\,\vec{x} + M\,\vec{y}$$

We will not take the time here to develop the different derivatives of these two expressions depending on the three variables x, y, γ; we will leave it up to the readers to do so themselves. We finally obtain the three equations of the associated oscillators

$$m\varepsilon_1'' + 2k\left(1 - \frac{\ell_0}{\ell} + \frac{\ell_0}{\ell^3}x_e^{\,2}\right)\varepsilon_1 = 0$$

$$\begin{cases} m\varepsilon_2'' + ma\,\varepsilon_3'' \;\ldots \\ \ldots\; + 2k\left(1 - \dfrac{\ell_0}{\ell} + \dfrac{\ell_0}{\ell^3}\left(b - c\right)^2\right)\varepsilon_2 - 2kb\left(b - c\right)\dfrac{\ell_0}{\ell^3}x_e\varepsilon_3 = 0 \\ ma\,\varepsilon_2'' + C\,\varepsilon_3'' \;\ldots \\ \ldots\; - 2kb\left(b - c\right)\dfrac{\ell_0}{\ell^3}x_e\varepsilon_2 + \left[2kb\left(1 - \dfrac{\ell_0}{\ell} + \dfrac{\ell_0}{\ell^3}x_e^{\,2}\right) + mga\right]\varepsilon_3 = 0 \end{cases}$$

We observe, according to these results, that the vertical oscillation in x is decoupled from the two others. It is a sinusoidal motion with pulsation.

$$\varpi_1 = \sqrt{\frac{2k\left(1 - \frac{\ell_0}{\ell} + \frac{\ell_0}{\ell^3} x_e{}^2\right)}{m}}$$

The other two oscillators, in y and γ, are coupled and will need to be the objects of a separate analysis (see Chapter 3).

1.4.2.2.3. Problem 3

In the Galilean frame $\langle g \rangle \equiv \left\langle \mathrm{O}_g \middle| \overrightarrow{x_g}\, \overrightarrow{y_g}\, \overrightarrow{z_g} \right\rangle$, where $\overrightarrow{z_g}$ is vertical ascending, we consider a plate (S), to which we join the basis $(S) \equiv \left(\overrightarrow{x_S}\, \overrightarrow{y_S}\, \overrightarrow{z_S}\right)$. Its mechanical and kinetic characteristics are as follows:

– it is homogeneous, with a negligible thickness and a mass m;

– it is limited by an equilateral triangle $\mathrm{B_1B_2B_3}$, of height $3a$, pierced in its center with a circular hole;

– it has with a center of inertia G;

– it has with a moment of inertia of $C = \mathscr{I}_{\mathrm{G}|\overrightarrow{z_S}} = ma^2$. $\left(\mathrm{G}\middle|\overrightarrow{z_S}\right)$ is the principal inertial axis of the plate and subsequently the following products of inertia containing this axis, i.e. $D = \mathscr{P}_{G|\overrightarrow{z_S}\,\overrightarrow{x_S}}$ and $E = \mathscr{P}_{G|\overrightarrow{y_S}\,\overrightarrow{z_S}}$, are null. The inertial matrix of the solid in the basis (S) thus has the following form:

$$\left[I_{\mathrm{G}}(S)\right]_{/(S)} = \begin{bmatrix} A & -F & 0 \\ -F & B & 0 \\ 0 & 0 & ma^2 \end{bmatrix}$$

The basis (S) is defined by

$$\overrightarrow{x_S} = \frac{\overrightarrow{GB_1}}{\left\|\overrightarrow{GB_1}\right\|}, \quad \overrightarrow{z_S} \text{ orthogonal to the plane of } (S).$$

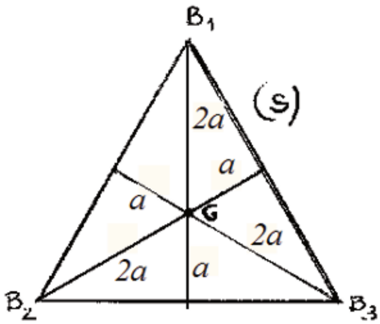

Figure 1.3. *Triangular plate*

The plate (S) is hanged in its three summits B_1,B_2,B_3 to summits A_1,A_2,A_3 of a fixed triangle defined as

$$\overrightarrow{O_gA_1} = 2a\,\overrightarrow{x_g}, \quad \overrightarrow{O_gA_2} = 2a\,\vec{u}\left(\frac{2\pi}{3}\right), \quad \overrightarrow{O_gA_3} = 2a\,\vec{u}\left(\frac{4\pi}{3}\right)$$

by three inextensible and perfectly flexible wires, with the same length ℓ and negligible thicknesses and masses, which we will consider taut and not crossed.

We also consider that during its motion, the plate will never hit the materialized plane $A_1A_2A_3$.

The plate is in a pivot sliding linkage of axis $\left(G\middle|\overrightarrow{z_S}\right)$ that coincides with the axis $\left(O_g\middle|\overrightarrow{z_g}\right)$.

We set $\overrightarrow{O_g G} = z\,\overrightarrow{z_g}$, $\widehat{\overrightarrow{x_g},\overrightarrow{x_S}} = \gamma$ measured on $\overrightarrow{z_g}$.

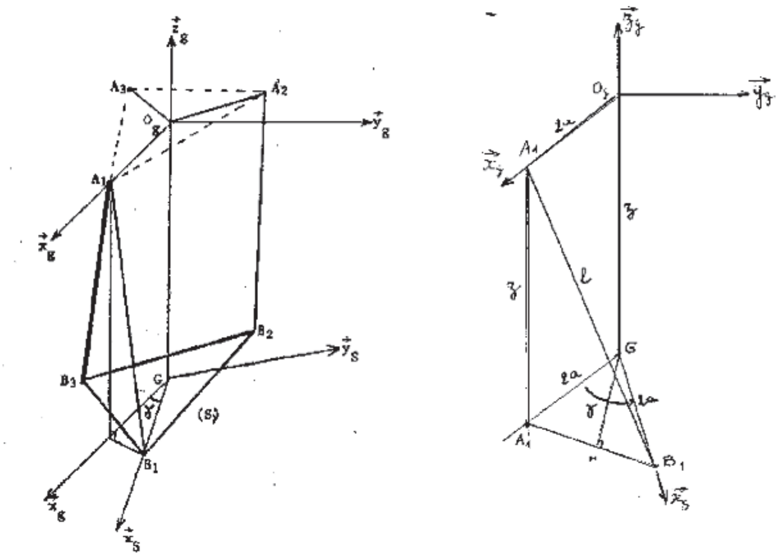

Figure 1.4. *Configuration of the studied device*

Question 1: Express z^2 according to γ.

Consider A_1' the orthogonal projection of A_1 on the plane of (S). We have

$$\overrightarrow{A_1A_1'} = z\,\overrightarrow{z_g} \quad \Rightarrow \quad \overrightarrow{A_1B_1} = \overrightarrow{A_1A_1'} + \overrightarrow{A_1'B_1}, \quad \overrightarrow{A_1A_1'} \perp \overrightarrow{A_1'B_1}$$

Consider the isosceles triangle $GA_1'B_1$, with an angle γ in G, and the orthogonal projection H of G on $A_1'B_1$ $\quad \Rightarrow \quad \overrightarrow{A_1'H} = \overrightarrow{HB_1}$.

In the triangle $GA_1'H$, we have

$$\frac{A_1'H}{GA_1'} = \sin\frac{\gamma}{2} \quad \Rightarrow \quad A_1'H = 2a\sin\frac{\gamma}{2} \quad \Rightarrow \quad \left\|\overrightarrow{A_1'B_1}\right\| = 4a\sin\frac{\gamma}{2}$$

According to Pythagoras's theorem, we have

$$\overrightarrow{\mathrm{A_1B_1}}^2 = \overrightarrow{\mathrm{A_1A_1'}}^2 + \overrightarrow{\mathrm{A_1'B_1}}^2$$

$$\ell^2 = z^2 + 16a^2\sin^2\frac{\gamma}{2} \;\Rightarrow\; z^2 = \ell^2 - 16a^2\sin^2\frac{\gamma}{2}$$

Question 2: Write the scalar equations deduced from the fundamental principle of dynamics, with the moments calculated in G, projected onto the basis (g).

$$\left\{\mathscr{A}_S^g\right\} = \left\{\overline{S} \to S\right\} = \{\pi \to S\} \;\ldots$$

$$\ldots\; + \{\mathrm{A_1B_1} \to S\} + \{\mathrm{A_2B_2} \to S\} + \{\mathrm{A_3B_3} \to S\} + \left\{\mathscr{L}_{g\to S}\right\}$$

$$-\left\{\mathscr{A}_S^g\right\}_{\mathrm{G}} = \left[m\overrightarrow{J^{(g)}}(\mathrm{G}) \middle| I_{\mathrm{G}}\overrightarrow{\omega_S^g}' + \overrightarrow{\omega_S^g} \wedge I_{\mathrm{G}}\overrightarrow{\omega_S^g} \right]$$

$$\overrightarrow{\mathrm{O}_g\mathrm{G}} = z\overrightarrow{z_g} \;\Rightarrow\; \overrightarrow{v^{(g)}}(\mathrm{G}) = z'\overrightarrow{z_g},\quad \overrightarrow{J^{(g)}}(\mathrm{G}) = z''\overrightarrow{z_g}$$

$$\overrightarrow{\omega_S^g} = \gamma'\overrightarrow{z_g},\quad \overrightarrow{\omega_S^g}' = \gamma''\overrightarrow{z_g}$$

$$\Rightarrow\; \left\{\mathscr{A}_S^g\right\}_{\mathrm{G}} = \left[m z''\overrightarrow{z_g} \middle| ma^2\gamma''\overrightarrow{z_g} \right]$$

$$-\{\pi \to S\}_{\mathrm{G}} = \left[-mg\,\overrightarrow{z_g} \middle| \vec{0} \right]$$

$$-\{\mathrm{A_1B_1} \to S\}_{\mathrm{B_1}} = \left[T_1\frac{\overrightarrow{\mathrm{B_1A_1}}}{\left\|\overrightarrow{\mathrm{B_1A_1}}\right\|} \middle| \vec{0} \right],$$ where T_1 is the tension of the wire.

$$\overrightarrow{\mathrm{B_1A_1}} = \overrightarrow{\mathrm{O}_g\mathrm{A_1}} - \left(\overrightarrow{\mathrm{O}_g\mathrm{G}} + \overrightarrow{\mathrm{GB_1}}\right) = 2a\overrightarrow{x_g} - \left(z\overrightarrow{z_g} + 2a\overrightarrow{x_S}\right)$$

$$\text{" "} \quad = 2a(1-\cos\gamma)\overrightarrow{x_g} - 2a\sin\gamma\,\overrightarrow{y_g} - z\overrightarrow{z_g}$$

We also have $\left\|\overrightarrow{\mathrm{B_1A_1}}\right\| = \left\|\overrightarrow{\mathrm{B_2A_2}}\right\| = \left\|\overrightarrow{\mathrm{B_3A_3}}\right\| = \ell$

$$\overrightarrow{\mathcal{M}_{\mathrm{G}}}\{\mathrm{A_1B_1} \to S\} = \overrightarrow{\mathcal{M}_{\mathrm{B_1}}}\{\mathrm{A_1B_1} \to S\} + \overrightarrow{\mathrm{GB_1}} \wedge \vec{s}\{\mathrm{A_1B_1} \to S\} \quad \ldots$$

$$\ldots \quad = 2a\,\overrightarrow{x_S} \wedge T_1 \frac{\overrightarrow{\mathrm{B_1A_1}}}{\ell} \quad \ldots$$

$$\ldots \quad = \left(-2az\sin\gamma\,\overrightarrow{x_g} + 2az\cos\gamma\,\overrightarrow{y_g} - 4a^2\sin\gamma\,\overrightarrow{z_g}\right)\frac{T_1}{\ell}$$

$$\{\mathrm{A_2B_2} \to S\}_{\mathrm{B_2}} = \left[\begin{array}{c|c} T_2 \dfrac{\overrightarrow{\mathrm{B_2A_2}}}{\left\|\overrightarrow{\mathrm{B_2A_2}}\right\|} & \vec{0} \end{array}\right]$$

$$\overrightarrow{\mathrm{B_2A_2}} = \overrightarrow{\mathrm{O}_g\mathrm{A_2}} - \left(\overrightarrow{\mathrm{O}_g\mathrm{G}} + \overrightarrow{\mathrm{GB_2}}\right) = 2a\,\vec{u}\left(\frac{2\pi}{3}\right) - \left[z\,\overrightarrow{z_g} + 2a\,\vec{u}\left(\gamma + \frac{2\pi}{3}\right)\right]$$

$$\text{""} \quad = -a\left(1 - \cos\gamma - \sqrt{3}\sin\gamma\right)\overrightarrow{x_g} + a\left(\sqrt{3} + \sin\gamma - \sqrt{3}\cos\gamma\right)\overrightarrow{y_g} - z\,\overrightarrow{z_g}$$

$$\overrightarrow{\mathcal{M}_{\mathrm{G}}}\{\mathrm{A_2B_2} \to S\} = \overrightarrow{\mathcal{M}_{\mathrm{B_2}}}\{\mathrm{A_2B_2} \to S\} + \overrightarrow{\mathrm{GB_1}} \wedge \vec{s}\{\mathrm{A_2B_2} \to S\} \quad \ldots$$

$$\ldots \quad = 2a\,\vec{u}\left(\gamma + \frac{2\pi}{3}\right) \wedge T_2 \frac{\overrightarrow{\mathrm{B_2A_2}}}{\ell} = \left[az\left(\sin\gamma - \sqrt{3}\cos\gamma\right)\overrightarrow{x_g} \quad \ldots\right.$$

$$\ldots \quad \left. -az\left(\cos\gamma + \sqrt{3}\sin\gamma\right)\overrightarrow{y_g} - 4a^2\sin\gamma\,\overrightarrow{z_g}\right]\frac{T_2}{\ell}$$

$$-\ \{\mathrm{A_3B_3} \to S\}_{\mathrm{B_3}} = \left[\begin{array}{c|c} T_3 \dfrac{\overrightarrow{\mathrm{B_3A_3}}}{\left\|\overrightarrow{\mathrm{B_3A_3}}\right\|} & \vec{0} \end{array}\right]$$

$$\overrightarrow{\mathrm{B_3A_3}} = \overrightarrow{\mathrm{O}_g\mathrm{A_3}} - \left(\overrightarrow{\mathrm{O}_g\mathrm{G}} + \overrightarrow{\mathrm{GB_3}}\right) = 2a\,\vec{u}\left(-\frac{2\pi}{3}\right) - \left[z\,\overrightarrow{z_g} + 2a\,\vec{u}\left(\gamma - \frac{2\pi}{3}\right)\right]$$

$$\text{""} \quad = -a\left(1 - \cos\gamma + \sqrt{3}\sin\gamma\right)\overrightarrow{x_g} - a\left(\sqrt{3} - \sin\gamma - \sqrt{3}\cos\gamma\right)\overrightarrow{y_g} - z\,\overrightarrow{z_g}$$

$$\overrightarrow{\mathscr{M}_{\mathrm{G}}}\left\{\mathrm{A}_3\mathrm{B}_3 \to S\right\} = \overrightarrow{\mathscr{M}_{\mathrm{B}_3}}\left\{\mathrm{A}_3\mathrm{B}_3 \to S\right\} + \overrightarrow{\mathrm{GB}_3} \wedge \vec{s}\left\{\mathrm{A}_3\mathrm{B}_3 \to S\right\} \quad \ldots$$

$$\ldots \quad = 2a\,\vec{u}\left(\gamma - \frac{2\pi}{3}\right) \wedge T_3 \frac{\overrightarrow{\mathrm{B}_3\mathrm{A}_3}}{\ell} = \left[az\left(\sin\gamma + \sqrt{3}\cos\gamma\right)\overrightarrow{x_g} \quad \ldots\right.$$

$$\left.\ldots \quad -az\left(\cos\gamma - \sqrt{3}\sin\gamma\right)\overrightarrow{y_g} - 4a^2 \sin\gamma\,\overrightarrow{z_g}\right]\frac{T_3}{\ell}$$

$$-\left\{\mathscr{L}_{g\to S}\right\}_{\mathrm{G}} = \left[X\,\overrightarrow{x_g} + Y\,\overrightarrow{y_g} + Z\,\overrightarrow{z_g} \middle| L\,\overrightarrow{x_g} + M\,\overrightarrow{y_g} + N\,\overrightarrow{z_g} \right] \quad .$$

The link is perfect in the motion of (S) in relation to $\langle g \rangle$:

$$\mathscr{P}^{(g)}\left\{\mathscr{L}_{g\to S}\right\} = \left\{{}^{g}_{S}\right\} \otimes \left\{\mathscr{L}_{g\to S}\right\} = 0 \quad \text{with} \quad \left\{{}^{g}_{S}\right\}_{\mathrm{G}} = \left[\gamma'\,\overrightarrow{z_g} \middle| z'\,\overrightarrow{z_g} \right]$$

$$\Rightarrow \quad Zz' + N\gamma' = 0 \qquad \forall z' \ , \ \forall \gamma' \quad \Rightarrow \quad Z = N = 0 \quad .$$

We therefore obtain six scalar consequences of the fundamental principle:

$$0 = 2a\frac{T_1}{\ell}(1-\cos\gamma) - a\frac{T_2}{\ell}\left(1-\cos\gamma - \sqrt{3}\sin\gamma\right) \quad \ldots$$

$$\ldots \quad -a\frac{T_3}{\ell}\left(1-\cos\gamma + \sqrt{3}\sin\gamma\right) + X$$

$$0 = -2a\frac{T_1}{\ell}\sin\gamma + a\frac{T_2}{\ell}\left(\sqrt{3} + \sin\gamma - \sqrt{3}\cos\gamma\right) \quad \ldots$$

$$\ldots \qquad -a\frac{T_3}{\ell}\left(1-\sin\gamma - \sqrt{3}\cos\gamma\right) + Y$$

$$m\,z'' = -mg - \frac{T_1+T_2+T_3}{\ell}z$$

$$0 = -2a\frac{T_1}{\ell}z\sin\gamma + a\frac{T_2}{\ell}z\left(\sin\gamma - \sqrt{3}\cos\gamma\right) \quad \ldots$$

$$\ldots \quad +a\frac{T_3}{\ell}z\left(\sin\gamma + \sqrt{3}\cos\gamma\right) + L$$

$$0 = 2a\frac{T_1}{\ell} z\cos\gamma - a\frac{T_2}{\ell} z\left(\cos\gamma + \sqrt{3}\sin\gamma\right) \ \ldots$$
$$\ldots \quad - a\frac{T_3}{\ell} z\left(\cos\gamma - \sqrt{3}\sin\gamma\right) + M$$

$$ma^2\gamma'' = -4a^2\,\frac{T_1 + T_2 + T_3}{\ell}\sin\gamma$$

Question 3: Express $T = T_1 + T_2 + T_3$, sum of the suspension tensions by the three wires, depending on $\gamma, \gamma', \gamma''$ and the data of the problem. Deduce a movement equation that only involves γ and its two first derivatives in time.

From the above scalar consequences, we extract the two following equations that contain no link terms:

$$\begin{cases} m\,z'' = -mg - \dfrac{T_1 + T_2 + T_3}{\ell}\,z \\ ma^2\gamma'' = -4a^2\,\dfrac{T_1 + T_2 + T_3}{\ell}\sin\gamma \end{cases}$$

$$\Rightarrow \quad ma^2\gamma'' = 4ma^2\,\frac{z'' + g}{z}\sin\gamma \quad \text{with} \quad z^2 = \ell^2 - 16a^2\sin^2\frac{\gamma}{2} \quad .$$

$$z\,z' = -4a^2\sin\gamma\,\gamma' \quad \Rightarrow \quad z'^2 = \frac{16a^4\sin^2\gamma}{\ell^2 - 16a^2\sin^2\dfrac{\gamma}{2}}\,\gamma'^2$$

$$z\,z'' + z'^2 = -4a^2\left(\sin\gamma\,\gamma'' + \cos\gamma\,\gamma'^2\right)$$

$$\Rightarrow \quad z\,z'' = -4a^2\left(\sin\gamma\,\gamma'' + \cos\gamma\,\gamma'^2\right) - \frac{16a^4\sin^2\gamma}{\ell^2 - 16a^2\sin^2\dfrac{\gamma}{2}}\,\gamma'^2$$

Since $\dfrac{z'' + g}{z} = \dfrac{z\,z''}{z^2} + \dfrac{g}{z}$ and $z = -\sqrt{\ell^2 - 16a^2\sin^2\dfrac{\gamma}{2}}$

we deduce the movement equation in γ

$$ma^2\gamma'' = -\frac{16ma^4\sin\gamma}{\ell^2 - 16a^2\sin^2\frac{\gamma}{2}}\left[\sin\gamma\gamma'' + \left(\cos\gamma + \frac{4a^2\sin^2\gamma}{\ell^2 - 16a^2\sin^2\frac{\gamma}{2}}\right)\gamma'^2\right]\ \ldots$$

$$\ldots\ -\frac{4mga^2\sin\gamma}{\sqrt{\ell^2 - 16a^2\sin^2\frac{\gamma}{2}}}$$

Question 4: Demonstrate that the position of equilibrium $\gamma = 0$ is a position of stable equilibrium. We will give the natural frequency of the oscillations around this equilibrium position. Express $\gamma(t)$ according to t, γ_0, γ_0' and determine the development limited to the first order of the global tension of the wires T .

At equilibrium, $\gamma = \gamma_e$, $\gamma_e' = 0$, $\gamma_e'' = 0$. With these equilibrium values, the above movement equation gives us:

$$4mga^2\sin\gamma_e = 0$$

We note that it is verified for $\gamma_e = 0$.

The position corresponding to $\gamma = 0$ is in fact a good equilibrium position.

Consider now a disturbance ε of the situation parameter γ around this equilibrium value. The small movements that ensue from this disturbance are expressed by

$$\gamma = \gamma_e + \varepsilon = \varepsilon \quad , \quad \gamma' = \varepsilon' \quad , \quad \gamma'' = \varepsilon''$$

and the development limited to the first order of the movement equation gives

$$ma^2\varepsilon'' + D_1\left[\frac{16ma^4\sin^2\gamma}{\ell^2 - 16a^2\sin^2\dfrac{\gamma}{2}}\gamma''\right]_{(e)} = D_1\left[\frac{-4mga^2\sin\gamma}{\sqrt{\ell^2 - 16a^2\sin^2\dfrac{\gamma}{2}}}\right]_{(e)}$$

because the term in γ'^2, of the second order, is negligible. Thus

$$D_1\left[\frac{16ma^4\sin^2\gamma}{\ell^2 - 16a^2\sin^2\dfrac{\gamma}{2}}\gamma''\right] = \ldots$$

$$\ldots \frac{16ma^4\sin^2\gamma_e}{\ell^2 - 16a^2\sin^2\dfrac{\gamma_e}{2}}\varepsilon'' + \varepsilon\varepsilon''\frac{d}{d\gamma}\left(\frac{16ma^4\sin^2\gamma}{\ell^2 - 16a^2\sin^2\dfrac{\gamma}{2}}\right)_{(e)} = 0$$

$$D_1\left[\frac{4mga^2\sin(\gamma_e+\varepsilon)}{\sqrt{\ell^2 - 16a^2\sin^2\dfrac{(\gamma_e+\varepsilon)}{2}}}\right] = \ldots$$

with,

$$\ldots \frac{4mga^2\sin\gamma_e}{\sqrt{\ell^2 - 16a^2\sin^2\dfrac{\gamma_e}{2}}} + \varepsilon\left(\frac{d}{d\gamma}\frac{4mga^2\sin\gamma}{\sqrt{\ell^2 - 16a^2\sin^2\dfrac{\gamma}{2}}}\right)_{(e)}$$

$$\left(\frac{d}{d\gamma}\frac{4mga^2\sin\gamma}{\sqrt{\ell^2 - 16a^2\sin^2\dfrac{\gamma}{2}}}\right)_{(e)} = \ldots$$

$$\ldots \left(\frac{4mga^2\sin\gamma\left(\ell^2 - 16a^2\sin^2\dfrac{\gamma}{2}\right) + 16mga^4\sin^2\gamma}{\left(\ell^2 - 16a^2\sin^2\dfrac{\gamma}{2}\right)^{3/2}}\right)_{(e)} = \frac{4mga^2}{\ell}$$

The equation of small movements around the position of equilibrium $\gamma_e = 0$ can therefore be presented as follows:

$$ma^2\varepsilon'' = -\frac{4mga^2}{\ell}\varepsilon$$

It is of the form $\varepsilon'' + \varpi^2\varepsilon = 0$ where $\varpi^2 = \dfrac{4g}{\ell}$. The solution to this equation that expresses the oscillations of the studied mechanical device is of the following form:

$$\varepsilon = \varepsilon_0 \cos\varpi t + \frac{\varepsilon_0'}{\varpi}\sin\varpi t$$

But as we can assimilate ε and γ for the small movements, we can write

$$\gamma = \gamma_0 \cos\varpi t + \frac{\gamma_0'}{\varpi}\sin\varpi t$$

According to the sixth scalar consequence of the fundamental principle:

$$m\ell\gamma'' + 4(T_1 + T_2 + T_3)\sin\gamma = 0$$

the first-order limited development of the above relation offers the following:

$$m\ell\varepsilon'' + D_1\left[4(T_1 + T_2 + T_3)\sin\gamma\right] = m\ell\varepsilon'' + \left[4(T_1 + T_2 + T_3)\sin\gamma\right]_{(e)} \quad \ldots$$

$$\ldots \quad + \varepsilon\left[4(T_1 + T_2 + T_3)\cos\gamma\right]_{(e)} + \varepsilon 4\sin\gamma_e\left[\frac{d}{d\gamma}(T_1 + T_2 + T_3)\right]_{(e)} = 0$$

or

$$m\ell\varepsilon'' + 4(T_1 + T_2 + T_3)_e\,\varepsilon = 0 \quad \text{with} \quad \varepsilon'' = -\varpi^2\varepsilon = -\frac{4g}{\ell}\varepsilon$$

$$\Rightarrow \quad (T_1 + T_2 + T_3)_e = mg\,.$$

This result confirms what we expected; at equilibrium, the tension in the wires maintains the plate subjected to its own weight.

1.4.2.3. *Torsor equation of small movements of a solid*

The dynamic torsor of small movements of a solid has the following reduction elements at O_S :

$$\begin{cases} \vec{s} = \sum_{\alpha} \overrightarrow{s^{\alpha}} Q_{\alpha}'' + \sum_{\alpha\beta} \overrightarrow{s^{\alpha\beta}} Q_{\alpha}' Q_{\beta}' \\ \overrightarrow{\mathcal{M}_{\mathrm{O}_S}} = \sum_{\alpha} \overrightarrow{m_{\mathrm{O}_S}^{\alpha}} Q_{\alpha}'' + \sum_{\alpha\beta} \overrightarrow{m_{\mathrm{O}_S}^{\alpha\beta}} Q_{\alpha}' Q_{\beta}' \end{cases}$$

The moment at Q of this torsor is given by

$$\overrightarrow{\mathcal{M}_{\mathrm{Q}}} = \overrightarrow{\mathcal{M}_{\mathrm{O}_S}} + \overrightarrow{\mathrm{QO}_S} \wedge \vec{s} = \sum_{\alpha} \overrightarrow{m_{\mathrm{O}_S}^{\alpha}} Q_{\alpha}'' + \sum_{\alpha\beta} \overrightarrow{m_{\mathrm{O}_S}^{\alpha\beta}} Q_{\alpha}' Q_{\beta}' \quad \ldots$$

$$\ldots \quad + \overrightarrow{\mathrm{QO}_S} \wedge \left(\sum_{\alpha} \overrightarrow{s^{\alpha}} Q_{\alpha}'' + \sum_{\alpha\beta} \overrightarrow{s^{\alpha\beta}} Q_{\alpha}' Q_{\beta}' \right)$$

$$\overrightarrow{\mathcal{M}_{\mathrm{Q}}} = \sum_{\alpha} \left(\overrightarrow{m_{\mathrm{O}_S}^{\alpha}} + \overrightarrow{\mathrm{QO}_S} \wedge \overrightarrow{s^{\alpha}} \right) Q_{\alpha}'' \quad \ldots$$

$$\ldots \quad + \sum_{\alpha\beta} \left(\overrightarrow{m_{\mathrm{O}_S}^{\alpha\beta}} + \overrightarrow{\mathrm{QO}_S} \wedge \overrightarrow{s^{\alpha\beta}} \right) Q_{\alpha}' Q_{\beta}'$$

By introducing the two following torsors:

$$\{\mathbf{A}_{\alpha}\}_{\mathrm{O}_S} = \left[\overrightarrow{s^{\alpha}} \middle| \overrightarrow{m_{\mathrm{O}_S}^{\alpha}} \right] \quad \text{and} \quad \{\mathbf{A}_{\alpha\beta}\}_{\mathrm{O}_S} = \left[\overrightarrow{s^{\alpha\beta}} \middle| \overrightarrow{m_{\mathrm{O}_S}^{\alpha\beta}} \right]$$

such as

$$\overrightarrow{\mathcal{M}_{\mathrm{Q}}} \{\mathbf{A}_{\alpha}\} = \overrightarrow{m_{\mathrm{O}_S}^{\alpha}} + \overrightarrow{\mathrm{QO}_S} \wedge \overrightarrow{s^{\alpha}}$$

$$\overrightarrow{\mathcal{M}_{\mathrm{Q}}} \{\mathbf{A}_{\alpha\beta}\} = \overrightarrow{m_{\mathrm{O}_S}^{\alpha\beta}} + \overrightarrow{\mathrm{QO}_S} \wedge \overrightarrow{s^{\alpha\beta}}$$

we can therefore write

$$\overrightarrow{\mathscr{M}_{\mathrm{Q}}} = \sum_{\alpha} \overrightarrow{\mathscr{M}_{\mathrm{Q}}}\left\{\mathbf{A}_{\alpha}\right\} Q_{\alpha}'' + \sum_{\alpha\beta} \overrightarrow{\mathscr{M}_{\mathrm{Q}}}\left\{\mathbf{A}_{\alpha\beta}\right\} Q_{\alpha}' Q_{\beta}'$$

and, lastly

$$\left\{\mathscr{A}_{S}^{g}\right\} = \sum_{\alpha}\left\{\mathbf{A}_{\alpha}\right\} Q_{\alpha}'' + \sum_{\alpha\beta}\left\{\mathbf{A}_{\alpha\beta}\right\} Q_{\alpha}' Q_{\beta}'$$

Accounting for the first-order limited development of the external efforts torsor, we established previously

$$\begin{cases} D_{1}\left[\overrightarrow{s^{\Delta}}\right] = \sum_{\alpha}\left[\varepsilon_{\alpha}\dfrac{\partial \overrightarrow{s^{\Delta *}}}{\partial Q_{\alpha}}(e) + \varepsilon_{\alpha}'\dfrac{\partial \overrightarrow{s^{\Delta *}}}{\partial Q_{\alpha}'}(e)\right] + \vec{\sigma}(t) \\ D_{1}\left[\overrightarrow{\mathscr{M}_{\mathrm{O}_S}^{\Delta}}\right] = \sum_{\alpha}\left[\varepsilon_{\alpha}\dfrac{\partial \overrightarrow{\mathscr{M}_{\mathrm{O}_S}^{\Delta *}}}{\partial Q_{\alpha}}(e) + \varepsilon_{\alpha}'\dfrac{\partial \overrightarrow{\mathscr{M}_{\mathrm{O}_S}^{\Delta *}}}{\partial Q_{\alpha}'}(e)\right] + \vec{\mu}(t) \end{cases}$$

and by setting

$$\begin{cases} \left\{\mathbf{C}_{\alpha}^{(e)}\right\}_{\mathrm{O}_S} = \left[\dfrac{\partial \overrightarrow{s^{\Delta *}}}{\partial Q_{\alpha}}(e) \middle| \dfrac{\partial \overrightarrow{\mathscr{M}_{\mathrm{O}_S}^{\Delta *}}}{\partial Q_{\alpha}}(e)\right] \\ \left\{\mathbf{B}_{\alpha}^{(e)}\right\}_{\mathrm{O}_S} = \left[\dfrac{\partial \overrightarrow{s^{\Delta *}}}{\partial Q_{\alpha}'}(e) \middle| \dfrac{\partial \overrightarrow{\mathscr{M}_{\mathrm{O}_S}^{\Delta *}}}{\partial Q_{\alpha}'}(e)\right] \end{cases} \quad \begin{cases} \left\{\mathbf{\Delta}^{t}\right\}_{\mathrm{O}_S} = \left[\vec{\sigma}(t) \middle| \vec{\mu}(t)\right] \\ \left\{\mathbf{A}_{\alpha}^{(e)}\right\}_{\mathrm{O}_S} = \left[\overrightarrow{s^{\alpha}}(e) \middle| \overrightarrow{m_{\mathrm{O}_S}^{\alpha}}(e)\right] \end{cases}$$

the first-order linearized torsor equation of small movements in $\langle g \rangle$ of the free solid (S), around its equilibrium position (e), is written as:

$$\sum_{\alpha}\left[\left\{\mathbf{A}_{\alpha}^{(e)}\right\}\varepsilon_{\alpha}'' + \left\{\mathbf{B}_{\alpha}^{(e)}\right\}\varepsilon_{\alpha}' + \left\{\mathbf{C}_{\alpha}^{(e)}\right\}\varepsilon_{\alpha}\right] = \left\{\mathbf{\Delta}^{t}\right\}$$

1.4.3. *Analytical mechanics of free solids*

1.4.3.1. *Canonical form of kinetic energy*

The position vector of the solid (S) in $\langle g\rangle$ has the following expression:

$$\overrightarrow{\mathrm{O}_g\mathrm{O}_S} = \vec{F}(Q_1,Q_2,Q_3)$$

and the rotation rate vector of (S) in relation to $\langle g\rangle$,

$$\overrightarrow{\omega_S^g} = \sum_{l=4}^{6} {}^{l}\vec{\delta}(Q_4,Q_5,Q_6)Q_l'$$

This leads us to

$$\overrightarrow{v^{(g)}}(\mathrm{O}_S) = \sum_{i=1}^{k} \frac{\partial^{(g)}\vec{F}}{\partial Q_i}(Q_1,Q_2,Q_3)Q_i'$$

The general expression of kinetic energy

$$2T^{(g)}(S) = \left\{{}^g_S\right\} \otimes \left\{p_S^g\right\} = m\left[\overrightarrow{v^{(g)}}(\mathrm{O}_S)\right]^2 \quad \ldots$$
$$\ldots \quad +2m\overrightarrow{v^{(g)}}(\mathrm{O}_S)\cdot\left(\overrightarrow{\omega_S^g}\wedge\overrightarrow{\mathrm{O}_S\mathrm{G}}\right)+\overrightarrow{\omega_S^g}\cdot\left(I_{\mathrm{O}_S}\overrightarrow{\omega_S^g}\right)$$

is then written under the canonical form

$$2T = \sum_{i=1}^{3}\sum_{j=1}^{3} m\frac{\partial^{(g)}\vec{F}}{\partial Q_i}\cdot\frac{\partial^{(g)}\vec{F}}{\partial Q_j}Q_i'Q_j' \quad \ldots$$
$$\ldots \quad +\sum_{i=1}^{3}\sum_{l=4}^{6} 2m\frac{\partial^{(g)}\vec{F}}{\partial Q_i}\cdot\left({}^{l}\vec{\delta}\wedge\overrightarrow{\mathrm{O}_S\mathrm{G}}\right)Q_i'Q_l' \quad \ldots$$
$$\ldots \quad +\sum_{l=4}^{6}\sum_{n=4}^{6} {}^{l}\vec{\delta}\cdot\left(I_{\mathrm{O}_S}\,{}^{n}\vec{\delta}\right)Q_l'Q_n'$$

In this formulation, three types of scalar functions appear

– $E^*_{ij} = m \dfrac{\partial^{(g)} \overrightarrow{F}}{\partial Q_i} \cdot \dfrac{\partial^{(g)} \overrightarrow{F}}{\partial Q_j}$

– $E^*_{il} = 2m \dfrac{\partial^{(g)} \overrightarrow{F}}{\partial Q_i} \cdot \left(\overrightarrow{{}^l\delta} \wedge \overrightarrow{\mathrm{O}_S \mathrm{G}} \right)$

– $E^*_{ln} = \overrightarrow{{}^l\delta} \cdot I_{\mathrm{O}_S} \overrightarrow{{}^n\delta}$

with $E^*_{li} = 2m \dfrac{\partial^{(g)} \overrightarrow{F}}{\partial Q_l} \cdot \left(\overrightarrow{{}^i\delta} \wedge \overrightarrow{\mathrm{O}_S \mathrm{G}} \right) = 0$ because $\dfrac{\partial^{(g)} \overrightarrow{F}}{\partial Q_l} = \vec{0}$ and $\overrightarrow{{}^i\delta} = \vec{0}$.

Setting

$$E_{\alpha\beta} = \frac{E^*_{\alpha\beta} + E^*_{\beta\alpha}}{2}$$

we express the kinetic energy in terms that are all symmetrical, under the following canonical form:

$$2T = \sum_{\alpha=1}^{6} \sum_{\beta=1}^{6} E_{\alpha\beta}\left(Q_1, \ldots, Q_6\right) Q_\alpha{}' Q_\beta{}'$$

1.4.3.2. *Equations of analytical mechanics*

The fundamental principle of dynamics applied to the motion of a free solid (S) in the Galilean frame $\langle g \rangle$ is written as:

$$\left\{ \mathscr{A}_S^g \right\} = \left\{ \Delta \right\}$$

REMARK.– *In the case where the frame $\langle \lambda \rangle$ from which the motion is observed would not be Galilean, the drive and Coriolis inertial torsors of this frame's motion in relation to a reference Galilean one would be incorporated in the term $\{\Delta\}$ of the above expression.*

The resulting Lagrange equation has the following expression:

$$(L_\alpha) \;\Rightarrow\; \left\{{}^{g}_{S,\alpha}\right\} \otimes \left\{\mathscr{A}^{g}_{S}\right\} = \left\{{}^{g}_{S,\alpha}\right\} \otimes \{\Delta\}$$

or

$$(L_\alpha) \;\Rightarrow\; \frac{d}{dt}\frac{\partial T}{\partial Q_\alpha{}'} - \frac{\partial T}{\partial Q_\alpha} = \left\{{}^{g}_{S,\alpha}\right\} \otimes \{\Delta\} = \Pi^{\Delta}_{\alpha}$$

Calculating these terms requires that we operate a double index identification of the expression of kinetic energy, meaning

$$2T = \sum_\alpha \sum_\beta E_{\alpha\beta}(Q_1,\ldots,Q_6)\, Q_\alpha{}' Q_\beta{}' = \sum_\beta \sum_\gamma E_{\beta\gamma}(Q_1,\ldots,Q_6)\, Q_\beta{}' Q_\gamma{}'$$

that leads to

$$\frac{\partial T}{\partial Q_\alpha{}'} = \sum_\beta E_{\alpha\beta} Q_\beta{}'$$

$$\Rightarrow\; \frac{d}{dt}\frac{\partial T}{\partial Q_\alpha{}'} = \sum_\beta E_{\alpha\beta} Q_\beta{}'' + \sum_\beta \sum_\gamma \frac{\partial E_{\alpha\beta}}{\partial Q_\gamma} Q_\beta{}' Q_\gamma{}'$$

$$\frac{\partial T}{\partial Q_\alpha} = \frac{1}{2} \sum_\beta \sum_\gamma \frac{\partial E_{\beta\gamma}}{\partial Q_\alpha} Q_\beta{}' Q_\gamma{}'$$

thus the canonical form of the first member of the Lagrange equation (L_α) is given as:

$$\frac{d}{dt}\frac{\partial T}{\partial Q_\alpha{}'} - \frac{\partial T}{\partial Q_\alpha} = \sum_\beta E_{\alpha\beta} Q_\beta{}'' + \sum_\beta \sum_\gamma \left(\frac{\partial E_{\alpha\beta}}{\partial Q_\gamma} - \frac{1}{2}\frac{\partial E_{\beta\gamma}}{\partial Q_\alpha} \right) Q_\beta{}' Q_\gamma{}'$$

The second member of the Lagrange equation (L_α) is provided by the partial power of known efforts $\{\Delta\}$

$$\Pi_\alpha^\Delta\left(Q_1,\ldots,Q_6\left|Q_1',\ldots,Q_6'\right|t\right)=\left\{{}_{S,\alpha}^{\;g}\right\}\otimes\{\Delta\}$$

with

$$-\ \text{for } i=1,2,3,\ \left\{{}_{S,i}^{\;g}\right\}_{O_S}=\left[\vec{0}\left|\frac{\partial^{(g)}\vec{F}}{\partial Q_i}\right.\right]\Rightarrow\ \Pi_i^\Delta=\frac{\partial^{(g)}\vec{F}}{\partial Q_i}\cdot\overrightarrow{s^\Delta}$$

$$-\ \text{for } l=4,5,6,\ \left\{{}_{S,1}^{\;g}\right\}_{O_S}=\left[{}^{l}\vec{\delta}\left|\vec{0}\right.\right]\Rightarrow\ \Pi_l^\Delta={}^{l}\vec{\delta}\cdot\overrightarrow{\mathscr{M}_{O_S}^\Delta}$$

1.4.3.3. *State of equilibrium of the solid*

The state of equilibrium of (S) throughout the interval of time $\left[t_i,t_f\right]$ is identified by the following values of situation parameters:

$$Q_\alpha=e_\alpha\quad\Rightarrow\quad Q_\alpha'=0,\quad Q_\alpha''=0$$

When the partial powers do not depend explicitly on time, $\forall t\in\left[t_i,t_f\right]$, they must be null in order to have equilibrium, so

$$\Pi_\alpha^\Delta\left(e_1,\ldots,e_6\left|0,\ldots,0\right.\right)=0$$

These relations are the equations that characterize the state of equilibrium of the solid.

When the partial powers depend explicitly on time, they must be under this form

$$\Pi_\alpha^\Delta\left(Q_1,\ldots,Q_6\left|Q_1',\ldots,Q_6'\right|t\right)=\Pi_\alpha^{\Delta*}\left(Q_1,\ldots,Q_6\left|Q_1',\ldots,Q_6'\right.\right)+\Pi_\alpha^{\Delta t}$$

and in order to have equilibrium in the time interval $\left[t_i, t_f\right]$ with

$$\Pi_\alpha^{\Delta *}\left(e_1,\ldots,e_6\middle|0,\ldots,0\right)=0 \quad \text{and} \quad \Pi_\alpha^{\Delta t}=0 \quad \forall t\in\left[t_i,t_f\right]$$

1.4.3.4. *Small movements of the solid around the state of equilibrium*

The small movements around the state of equilibrium identified by the values of situation parameters

$$Q_\beta = e_\beta, \quad \beta = 1,\ldots,6$$

are characterized by

$$Q_\beta = e_\beta + \varepsilon_\beta, \quad Q_\beta' = \varepsilon_\beta', \quad Q_\beta'' = \varepsilon_\beta''$$

where $\varepsilon_\beta, \varepsilon_\beta', \varepsilon_\beta''$ are infinitely small of the first order.

The developments limited to the first order of the two members of the Lagrange equations $\left(L_\alpha\right)$ are

$$-\ D_1\left[\frac{d}{dt}\frac{\partial T}{\partial Q_\alpha'}-\frac{\partial T}{\partial Q_\alpha}\right]=\sum_\beta E_{\alpha\beta}\left(e_1,\ldots,e_6\right)\varepsilon_\beta''=\sum_\beta E_{\alpha\beta}\left(e\right)\varepsilon_\beta''$$

$$-\ D_1\left[\Pi_\alpha^{\Delta *}+\Pi_\alpha^{\Delta t}\right]=D_1\left[\Pi_\alpha^{\Delta *}\right]+\Pi_\alpha^{\Delta t}$$

However

$$D_1\left[\Pi_\alpha^{\Delta *}\left(e_1+\varepsilon_1,\ldots,e_6+\varepsilon_6\middle|0+\varepsilon_1',\ldots,0+\varepsilon_6'\right)\right]= \ \ldots$$

$$\ldots \ \Pi_\alpha^{\Delta *}\left(e_1,\ldots,e_6\middle|0,\ldots,0\right)+\sum_\beta\frac{\partial\Pi_\alpha^{\Delta *}}{\partial Q_\beta}\left(e_1,\ldots,e_6\middle|0,\ldots,0\right)\varepsilon_\beta \ \ldots$$

$$\ldots \ +\sum_\beta\frac{\partial\Pi_\alpha^{\Delta *}}{\partial Q_\beta'}\left(e_1,\ldots,e_6\middle|0,\ldots,0\right)\varepsilon_\beta'$$

therefore $D_1\left[\Pi_\alpha^{\Delta *}\right]=\sum_\beta \frac{\partial \Pi_\alpha^{\Delta *}}{\partial Q_\beta}(e)\varepsilon_\beta+\sum_\beta \frac{\partial \Pi_\alpha^{\Delta *}}{\partial Q_\beta{}'}(e)\varepsilon_\beta{}'$.

The system of linear differential equations that govern the small movements of (S) around the position of equilibrium (e) is therefore

$$\sum_\beta E_{\alpha\beta}(e)\varepsilon_\beta{}''=\sum_\beta \frac{\partial \Pi_\alpha^{\Delta *}}{\partial Q_\beta}(e)\varepsilon_\beta+\sum_\beta \frac{\partial \Pi_\alpha^{\Delta *}}{\partial Q_\beta{}'}(e)\varepsilon_\beta{}'+\Pi_\alpha^{\Delta t}$$

By stating

$$a_{\alpha\beta}=E_{\alpha\beta}(e),\quad b_{\alpha\beta}=-\frac{\partial \Pi_\alpha^{\Delta *}}{\partial Q_\beta{}'},\quad c_{\alpha\beta}=-\frac{\partial \Pi_\alpha^{\Delta *}}{\partial Q_\beta},\quad d_\alpha(t)=\Pi_\alpha^{\Delta t}$$

we find the general form of the system governing the small movements of a free solid around a position of equilibrium

$$\sum_{\beta=1}^{6}\left(a_{\alpha\beta}\varepsilon_\beta{}''+b_{\alpha\beta}\varepsilon_\beta{}'+c_{\alpha\beta}\varepsilon_\beta\right)=d_\alpha(t)$$

1.4.3.5. *Application: problem 4*

The system $(\mathscr{D})$ is composed of four homogeneous rods, each with a same mass m and a same length ℓ, forming a jointed rhomb ABCD with a center of inertia G and diagonals AC and BD.

We will suppose that, for each rod, the moment of inertia in relation to its axis is null. But this moment in relation to an axis going through the center of inertia of this rod and orthogonal to its axis is equal to $\dfrac{m\ell^2}{12}$.

The links between these rods are perfect pivot linkages.

We study the motion of the rhomb in the Galilean frame $\langle g\rangle \equiv \langle O|\vec{x}\,\vec{y}\,\vec{z}\rangle$, where $\vec{x}$ is vertical descending.

We set $\overrightarrow{OG} = x\vec{x} + y\vec{y}$ and $\alpha = \widehat{\overrightarrow{AB},\overrightarrow{AG}}$, $\theta = \widehat{\vec{x},\overrightarrow{GA}}$, around $\vec{z}$.

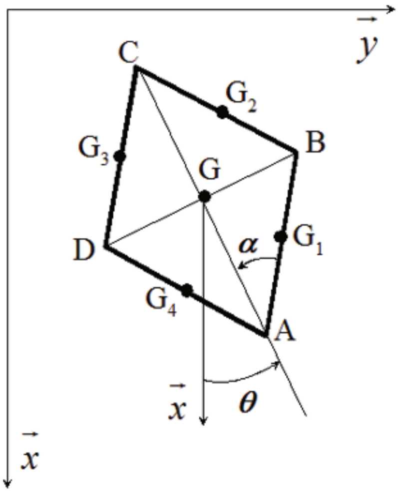

Figure 1.5. *Layout of the rhomb*

Question 1: Calculate the kinetic energy of each bar and that of the rhomb.

Consider G_1 the center of inertia of the bar AB.

$$\overrightarrow{OG_1} = \overrightarrow{OG} + \frac{\overrightarrow{GA} + \overrightarrow{GB}}{2}$$

with $\overrightarrow{GA} = \ell\cos\alpha\,\vec{u}(\theta)$, $\overrightarrow{GB} = \ell\cos\alpha\,\vec{u}\left(\theta + \dfrac{\pi}{2}\right)$

$$\Rightarrow \quad \overrightarrow{OG_1} = \left[x + \frac{\ell}{2}\cos(\alpha+\theta)\right]\vec{x} + \left[y + \frac{\ell}{2}\sin(\alpha+\theta)\right]\vec{y}$$

$$\overrightarrow{\mathrm{AB}} = \overrightarrow{\mathrm{GB}} - \overrightarrow{\mathrm{GA}} = -\ell\cos(\alpha-\theta)\vec{x} + \ell\sin(\alpha-\theta)\vec{y}$$
$$"" = \ell\cos\left[\pi-(\alpha-\theta)\right]\vec{x} + \ell\sin\left[\pi-(\alpha-\theta)\right]\vec{y}$$

$$\Rightarrow \quad \overrightarrow{\omega^{g}_{\mathrm{AB}}} = \left[\pi-(\alpha-\theta)\right]'\vec{z} = (\theta'-\alpha')\vec{z} = \overrightarrow{\omega_1}$$

We obtain similarly

$$\overrightarrow{\mathrm{OG}_2} = \left[x-\frac{\ell}{2}\cos(\alpha-\theta)\right]\vec{x} + \left[y+\frac{\ell}{2}\sin(\alpha-\theta)\right]\vec{y}$$
$$\overrightarrow{\mathrm{BC}} = \ell\cos\left[\pi+(\alpha+\theta)\right]\vec{x} + \ell\sin\left[\pi+(\alpha+\theta)\right]\vec{y}$$

$$\Rightarrow \quad \overrightarrow{\omega^{g}_{\mathrm{BC}}} = \left[\pi+(\alpha+\theta)\right]'\vec{z} = (\theta'+\alpha')\vec{z} = \overrightarrow{\omega_2}$$

$$\overrightarrow{\mathrm{OG}_3} = \left[x-\frac{\ell}{2}\cos(\alpha+\theta)\right]\vec{x} + \left[y-\frac{\ell}{2}\sin(\alpha+\theta)\right]\vec{y}$$
$$\overrightarrow{\mathrm{CD}} = \ell\cos(\theta-\alpha)\vec{x} + \ell\sin(\theta-\alpha)\vec{y}$$

$$\Rightarrow \quad \overrightarrow{\omega^{g}_{\mathrm{CD}}} = (\theta-\alpha)'\vec{z} = (\theta'-\alpha')\vec{z} = \overrightarrow{\omega_3}$$

$$\overrightarrow{\mathrm{OG}_4} = \left[x+\frac{\ell}{2}\cos(\alpha-\theta)\right]\vec{x} + \left[y-\frac{\ell}{2}\sin(\alpha-\theta)\right]\vec{y}$$
$$\overrightarrow{\mathrm{DA}} = \ell\cos(\theta+\alpha)\vec{x} + \ell\sin(\theta+\alpha)\vec{y}$$

$$\Rightarrow \quad \overrightarrow{\omega^{g}_{\mathrm{DA}}} = (\theta+\alpha)'\vec{z} = (\theta'+\alpha')\vec{z} = \overrightarrow{\omega_4}$$

$$-\ 2T^{(g)}(\mathrm{AB}) = 2T_1 = m\overrightarrow{v^{(g)}}(\mathrm{G}_1)^2 + \overrightarrow{\omega_1}\cdot I_{\mathrm{G}_1}(\mathrm{AB})\overrightarrow{\omega_1}$$

$$\overrightarrow{v^{(g)}}(\mathrm{G}_1) = \frac{d^{(g)}}{dt}\overrightarrow{\mathrm{OG}_1} = \left[x'-\frac{\ell}{2}\sin(\theta+\alpha)(\theta'+\alpha')\right]\vec{x} \ \ldots$$
$$\ldots \ + \left[y'+\frac{\ell}{2}\cos(\theta+\alpha)(\theta'+\alpha')\right]\vec{y}$$

$$\overrightarrow{\omega_1}\cdot I_{G_1}\overrightarrow{\omega_1}=\frac{m\ell^2}{12}(\theta'-\alpha')^2$$

$$2T_1=m\left[x'^2+y'^2+\frac{\ell^2}{3}\left(\theta'^2+\alpha'^2\right)-\ell x'(\theta'+\alpha')\sin(\theta+\alpha)\right.\ \ldots$$

$$\ldots\ \left.+\ell y'(\theta'+\alpha')\cos(\theta+\alpha)+\frac{\ell^2}{3}\theta'\alpha'\right]$$

We obtain similarly

$$2T_2=m\left[x'^2+y'^2+\frac{\ell^2}{3}\left(\theta'^2+\alpha'^2\right)+\ell x'(\theta'-\alpha')\sin(\theta-\alpha)\right.\ \ldots$$

$$\ldots\ \left.-\ell y'(\theta'-\alpha')\cos(\theta-\alpha)-\frac{\ell^2}{3}\theta'\alpha'\right]$$

$$2T_3=m\left[x'^2+y'^2+\frac{\ell^2}{3}\left(\theta'^2+\alpha'^2\right)+\ell x'(\theta'+\alpha')\sin(\theta+\alpha)\right.\ \ldots$$

$$\ldots\ \left.-\ell y'(\theta'+\alpha')\cos(\theta+\alpha)+\frac{\ell^2}{3}\theta'\alpha'\right]$$

$$2T_4=m\left[x'^2+y'^2+\frac{\ell^2}{3}\left(\theta'^2+\alpha'^2\right)-\ell x'(\theta'-\alpha')\sin(\theta-\alpha)\right.\ \ldots$$

$$\ldots\ \left.+\ell y'(\theta'-\alpha')\cos(\theta-\alpha)-\frac{\ell^2}{3}\theta'\alpha'\right]$$

As $(\mathscr{D})=(\mathrm{AB})\cup(\mathrm{BC})\cup(\mathrm{CD})\cup(\mathrm{DA})$ is a set of disjoint parts

$$2T=2T_1+2T_2+2T_3+2T_4=4m\left[x'^2+y'^2+\frac{\ell^2}{3}\left(\theta'^2+\alpha'^2\right)\right]$$

Question 2: We suppose that $(\mathscr{D})$ is only subject to efforts due to the links between the bars and those resulting from gravity. Calculate the Galilean power developed by these efforts and write the movement equations of $(\mathscr{D})$.

The links between the bars are internal to the system and are perfect. The set $(\mathscr{D})$ is then considered globally, the only power developed throughout its motion is that of the efforts resulting from gravity.

$$\mathscr{P}^{(g)}\left\{\overline{\mathscr{D}} \to \mathscr{D}\right\} = \mathscr{P}^{(g)}\{\pi \to \mathscr{D}\} = \left\{{}^{g}_{\mathscr{D}}\right\} \otimes \{\pi \to \mathscr{D}\}$$

$$\text{"} \quad \text{"} \qquad = \overrightarrow{\omega^{g}_{\mathscr{D}}} \cdot \vec{0} + 4mg\,\vec{x}\Big| \cdot \left(x'\vec{x} + y'\vec{y}\right) = 4mgx'$$

The four Lagrangian equations that result from the fundamental principle applied to the motion of $(\mathscr{D})$ are written as:

$$\frac{d}{dt}\frac{\partial T}{\partial x'} - \frac{\partial T}{\partial x} = \Pi^{(g)}_{x}\left\{\overline{\mathscr{D}} \to \mathscr{D}\right\} \;\Rightarrow\; 4m\,x'' - 0 = 4mg$$

$$\frac{d}{dt}\frac{\partial T}{\partial y'} - \frac{\partial T}{\partial y} = \Pi^{(g)}_{y}\left\{\overline{\mathscr{D}} \to \mathscr{D}\right\} \;\Rightarrow\; 4m\,y'' - 0 = 0$$

$$\frac{d}{dt}\frac{\partial T}{\partial \theta'} - \frac{\partial T}{\partial \theta} = \Pi^{(g)}_{\theta}\left\{\overline{\mathscr{D}} \to \mathscr{D}\right\} \;\Rightarrow\; 4m\frac{\ell^2}{3}\theta'' - 0 = 0$$

$$\frac{d}{dt}\frac{\partial T}{\partial \alpha'} - \frac{\partial T}{\partial \alpha} = \Pi^{(g)}_{\alpha}\left\{\overline{\mathscr{D}} \to \mathscr{D}\right\} \;\Rightarrow\; 4m\frac{\ell^2}{3}\alpha'' - 0 = 0$$

The set $(\mathscr{D})$ globally knows a uniform accelerated vertical falling motion along $\vec{x}$ under the effect of gravity, while its other parameters y, θ, α vary in a linear way throughout a lateral motion along $\vec{y}$, a global linear rotation θ and a variation in α of the opening of the rhomb.

Question 3: The plane of the rhomb coincides with $\Pi\left(\mathrm{O}\middle|\vec{x}\,\vec{y}\right)$ and $(\mathscr{D})$ is in perfect pivot link of axis $\left(\mathrm{O}\middle|\vec{z}\right) = \left(\mathrm{C}\middle|\vec{z}\right)$ with the Galilean. This condition remains valid for the following questions.

Express x and y and the kinetic energy of the rhomb according to α and θ.

The point C coinciding with O

$$\overrightarrow{\mathrm{OG}} = \overrightarrow{\mathrm{CG}} = \ell \cos\alpha \vec{u}(\theta) = x\vec{x} + y\vec{y}$$

$$\Rightarrow \begin{cases} x = \ell\cos\alpha\cos\theta, & x' = -\ell\alpha'\sin\alpha\cos\theta - \ell\theta'\cos\alpha\sin\theta \\ y = \ell\cos\alpha\sin\theta, & y' = -\ell\alpha'\sin\alpha\sin\theta + \ell\theta'\cos\alpha\cos\theta \end{cases}$$

$$2T = 4m\ell^2\left[\left(\cos^2\alpha + \frac{1}{3}\right)\theta'^2 + \left(\sin^2\alpha + \frac{1}{3}\right)\alpha'^2\right]$$

Question 4: A device outside of $(\mathscr{D})$ exerts on summit A a torsor of resultant vector $\vec{F} = k\overrightarrow{\mathrm{AC}}$ $(k > 0)$. Calculate the power developed by the efforts exerted over $(\mathscr{D})$.

Write the theorem of energy power and deduce a prime integral of motion, considering that the rhomb is dropped with no initial velocity, with A located on the axis $(\mathrm{O}|\vec{y})$ at a distance 2ℓ from O.

$$\left\{\overline{\mathscr{D}} \to \mathscr{D}\right\} = \{\pi \to \mathscr{D}\} + \left\{\mathscr{L}_{g\to\mathscr{D}}\right\} + \{F \to \mathscr{D}\}$$

where $\{F \to \mathscr{D}\}_{\mathrm{A}} = \left[k\overrightarrow{\mathrm{AC}}\middle|\vec{0}\right]$.

As the pivot linkage of $(\mathscr{D})$ with the Galilean frame is perfect, the power it develops is null. We therefore get

$$\mathscr{P}^{(g)}\left\{\overline{\mathscr{D}} \to \mathscr{D}\right\} = \left\{{}^{g}_{\mathscr{D}}\right\} \otimes \{\pi \to \mathscr{D}\} + \left\{{}^{g}_{\mathscr{D}}\right\} \otimes \{F \to \mathscr{D}\}$$

$$\overrightarrow{\mathscr{M}_{\mathrm{A}}}\left\{{}^{g}_{\mathscr{D}}\right\}=\overrightarrow{v^{(g)}}(\mathrm{A})=\frac{d^{(g)}}{dt}\overrightarrow{\mathrm{CA}}=\frac{d^{(g)}}{dt}\left(2\ell\cos\alpha\,\vec{u}(\theta)\right)$$

$$\text{" "}\quad=-2\ell\alpha'\sin\alpha\,\vec{u}(\theta)+2\ell\theta'\cos\alpha\,\vec{u}\left(\theta+\frac{\pi}{2}\right)$$

$$\left\{{}^{g}_{\mathscr{D}}\right\}\otimes\left\{F\to\mathscr{D}\right\}=\vec{s}\left\{{}^{g}_{\mathscr{D}}\right\}\cdot\vec{0}+\overrightarrow{\mathscr{M}_{\mathrm{A}}}\left\{{}^{g}_{\mathscr{D}}\right\}\cdot k\overrightarrow{\mathrm{AC}}$$

$$\text{" "}\quad=4k\ell^{2}\alpha'\sin\alpha\cos\alpha$$

$$\left\{{}^{g}_{\mathscr{D}}\right\}\otimes\left\{\pi\to\mathscr{D}\right\}=4mgx'=-4mg\ell\left(\alpha'\sin\alpha\cos\theta+\theta'\cos\alpha\sin\theta\right)$$

$$\mathscr{P}^{(g)}\left\{\overline{\mathscr{D}}\to\mathscr{D}\right\}=-4mg\ell\theta'\cos\alpha\sin\theta+4\ell\left(k\ell\cos\alpha-mg\cos\theta\right)\alpha'\sin\alpha$$

$$\text{" "}\quad=4mg\ell\frac{d}{dt}\left(\cos\alpha\cos\theta\right)-k\ell^{2}\frac{d}{dt}\cos2\alpha$$

The theorem of energy power can be expressed as follows:

$$\frac{dT}{dt}=\mathscr{P}^{(g)}\left\{\overline{\mathscr{D}}\to\mathscr{D}\right\}$$

Integrating this expression directly in relation to t, we find the following prime integral

$$2m\ell^{2}\left[\left(\cos^{2}\alpha+\frac{1}{3}\right)\theta'^{2}+\left(\sin^{2}\alpha+\frac{1}{3}\right)\alpha'^{2}\quad\ldots\right.$$

$$\left.\ldots\quad-\left(\cos^{2}\alpha_{0}+\frac{1}{3}\right)\theta_{0}'^{2}-\left(\sin^{2}\alpha_{0}+\frac{1}{3}\right)\alpha_{0}'^{2}\right]\quad\ldots$$

$$\ldots\quad=4mg\ell\left(\cos\alpha\cos\theta-\cos\alpha_{0}\cos\theta_{0}\right)-k\ell^{2}\left(\cos2\alpha-\cos2\alpha_{0}\right)$$

Yet, at the date $t=0$

$\alpha_{0}'=0,\quad\theta_{0}'=0$ null initial velocity,

$$\theta_0 = \frac{\pi}{2} \qquad \text{as} \quad \text{A is on the axis } \left(\mathrm{O}\middle|\vec{y}\right),$$

$$\alpha_0 = 0 \qquad \text{as} \quad \mathrm{CA} = 2\ell .$$

The prime integral then becomes

$$2m\ell\left[\left(\cos^2\alpha + \frac{1}{3}\right)\theta'^2 + \left(\sin^2\alpha + \frac{1}{3}\right)\alpha'^2\right] \ldots$$
$$\ldots = 4mg\cos\alpha\cos\theta + k\ell\left(1-\cos 2\alpha\right)$$

Question 5: We suppose that $0 \le \alpha < \frac{\pi}{2}$, $-\pi < \theta < \pi$. Determine the positions of equilibrium for $(\mathscr{D})$ and specify the ones that are stable. Calculate the periods of oscillation of both angular parameters α and θ around these stable positions.

As a preliminary remark, we note that, considering the area of variation of α we have

$$\sin\alpha \neq 1, \quad \cos\alpha \neq 0;$$

If we identify the position of equilibrium with $\alpha = \alpha_e, \quad \theta = \theta_e,$

$$\Rightarrow \quad \alpha_e' = 0, \ \alpha_e'' = 0; \quad \theta_e' = 0, \ \theta_e'' = 0$$

At the equilibrium, the first members of the Lagrange equations are null; the conditions of equilibrium α_e and θ_e verify the relations obtained by stating that the second members are also null, i.e.

$$\left(L_\alpha\right) \ \Rightarrow \ 4\ell\left(k\ell\cos\alpha_e - mg\cos\theta_e\right)\sin\alpha_e = 0$$
$$\left(L_\theta\right) \ \Rightarrow \ -4mg\ell\cos\alpha_e\sin\theta_e = 0$$

We then see two sets of solutions appear

Set 1: $\begin{cases} \sin\alpha_e = 0 \\ \sin\theta_e = 0 \end{cases} \Rightarrow \quad \alpha_e = 0, \quad \theta_e = 0$

Set 2: $\begin{cases} k\ell\cos\alpha_e = mg\cos\theta_e \quad (\cos\alpha_e \neq 0) \\ \cos\alpha_e \sin\theta_e = 0 \quad \Rightarrow \quad \sin\theta_e = 0, \quad \cos\theta_e = 1 \end{cases}$

thus $\quad \cos\alpha_e = \dfrac{mg}{k\ell}, \quad \theta_e = 0$.

In order to study the stability of these two equilibriums, we must write the Lagrangian equations relative to the two parameters α and θ and linearize them in the first order for small movements around these sets of equilibrium positions. Thus

$$(L_\alpha) \quad \Rightarrow \quad \frac{d}{dt}\frac{\partial T}{\partial \alpha'} - \frac{\partial T}{\partial \alpha} = \Pi_\alpha^{(g)}\left\{\overline{\mathscr{D}} \to \mathscr{D}\right\}$$

$$(L_\theta) \quad \Rightarrow \quad \frac{d}{dt}\frac{\partial T}{\partial \theta'} - \frac{\partial T}{\partial \theta} = \Pi_\theta^{(g)}\left\{\overline{\mathscr{D}} \to \mathscr{D}\right\}$$

or

$$(L_\alpha) \quad \Rightarrow \quad m\ell\alpha''\left(\sin^2\alpha + \frac{1}{3}\right) + m\ell\left(\alpha'^2 + \theta'^2\right)\sin\alpha\cos\alpha \quad \ldots$$
$$= \left(k\ell\cos\alpha - mg\cos\theta\right)\sin\alpha$$

$$(L_\theta) \quad \Rightarrow \quad \ell\theta''\left(\cos^2\alpha + \frac{1}{3}\right) - 2\ell\alpha'\theta'\sin\alpha\cos\alpha = -g\cos\alpha\sin\theta$$

To describe the small movements around the equilibrium positions, we state

$$\begin{cases} \alpha = \alpha_e + \varepsilon_1 \quad \Rightarrow \quad \alpha' = \varepsilon_1', \ \alpha'' = \varepsilon_1'' \\ \theta = \theta_e + \varepsilon_2 \quad \Rightarrow \quad \theta' = \varepsilon_2', \ \theta'' = \varepsilon_2'' \end{cases}$$

– For the stability of state 1: $\alpha_e = 0,\quad \theta_e = 0$

$$\begin{cases} \alpha = \varepsilon_1 \quad \Rightarrow \quad \alpha' = \varepsilon_1' \ , \ \alpha'' = \varepsilon_1'' \\ \theta = \varepsilon_2 \quad \Rightarrow \quad \theta' = \varepsilon_2' \ , \ \theta'' = \varepsilon_2'' \end{cases}$$

$$D_1[L_\alpha] \quad \Rightarrow \quad \frac{m\ell}{3}\varepsilon_1'' = \varepsilon_1 \left\{ \frac{\partial}{\partial \alpha}\left[(k\ell\cos\alpha - mg\cos\theta)\sin\alpha \right] \right\}_{(e)} \quad \ldots$$

$$\ldots \quad + \varepsilon_2 \left[\frac{\partial}{\partial \theta}(k\ell\cos\alpha - mg\cos\theta)\sin\alpha \right]_{(e)}$$

Thus

$$D_1[L_\alpha] \quad \Rightarrow \quad \varepsilon_1'' + \frac{3}{m\ell}(mg - k\ell)\varepsilon_1 = 0$$

Similarly, we obtain

$$D_1[L_\theta] \quad \Rightarrow \quad \varepsilon_2'' + \frac{3g}{4\ell}\varepsilon_2 = 0$$

We note that, considering these two equations of small movements:

- the second indicates that the equilibrium position $\theta_e = 0$ is stable, with its oscillation period

$$T_\theta = 2\pi\sqrt{\frac{4\ell}{3g}}$$

- the first indicates that the position of equilibrium $\alpha_e = 0$ is only stable if we have $mg - k\ell > 0$ and in this case, the oscillation period is

$$T_\alpha = 2\pi\sqrt{\frac{m\ell}{3(mg - k\ell)}}$$

– for the stability of set 2:

$$\theta_e = 0, \quad \cos\alpha_e = \frac{mg}{k\ell} \quad \Rightarrow \quad \sin^2\alpha_e = 1 - \left(\frac{mg}{k\ell}\right)^2$$

$$\begin{cases} \alpha = \alpha_e + \varepsilon_1 \quad \Rightarrow \quad \alpha' = \varepsilon_1', \ \alpha'' = \varepsilon_1'' \\ \theta = \varepsilon_2 \quad \Rightarrow \quad \theta' = \varepsilon_2', \ \theta'' = \varepsilon_2'' \end{cases}$$

In the same way as before, we obtain the developments limited to the first order of the two Lagrangian equations, which, with the above data, gives:

$$D_1[L_\alpha] \ \Rightarrow \ \varepsilon_1'' + \frac{3k}{m} \frac{(k\ell)^2 - (mg)^2}{4(k\ell)^2 - 3(mg)^2} \varepsilon_1 = 0$$

$$D_1[L_\theta] \ \Rightarrow \ \varepsilon_2'' + 3kg \frac{mg}{(k\ell)^2 + 3(mg)^2} \varepsilon_2 = 0$$

We note that

- the second equation indicates that the position of equilibrium $\theta_e = 0$ is stable, with its oscillation period

$$T_\theta = 2\pi \sqrt{\frac{(k\ell)^2 + 3(mg)^2}{3kmg^2}}$$

- the first equation indicates that the position of equilibrium defined by $\cos\alpha_e = \frac{mg}{k\ell} < 1$ (according to the properties of the function) is only stable if

$$\frac{(k\ell)^2 - (mg)^2}{4(k\ell)^2 - 3(mg)^2} > 0 \text{, which is verified since } \frac{mg}{k\ell} < 1.$$

1.5. Matrix expression of small movements of a free solid

To illustrate this form of representation of the equation of the small movements in $\langle g\rangle$ of a free solid (S) around the equilibrium position (e), it is preferable to use an example. Consider, therefore, in a Galilean frame $\langle g\rangle \equiv \left\langle \mathrm{O}_g \middle| \overrightarrow{x_g}\ \overrightarrow{y_g}\ \overrightarrow{z_g} \right\rangle$, the motion of free solid (S) to which is joined the frame $\langle S\rangle \equiv \left\langle \mathrm{O}_S \middle| \overrightarrow{x_S}\ \overrightarrow{y_S}\ \overrightarrow{z_S} \right\rangle$.

We consider that at equilibrium both frames coincide.

The situation parameters of the solid in $\langle g\rangle$ are defined by

$$\overrightarrow{\mathrm{O}_g\mathrm{O}_S} = \vec{F}(x,y,z) = x\,\overrightarrow{x_g} + y\,\overrightarrow{y_g} + z\,\overrightarrow{z_g}$$

$$\overrightarrow{y_g} \wedge \overrightarrow{z_S} = \vec{v}\sin Q_4, \quad \widehat{\overrightarrow{z_g},\vec{v}} = Q_5, \quad \widehat{\vec{v},\overrightarrow{x_S}} = Q_6 .$$

The position of equilibrium is characterized by

$$x_e = 0, \quad y_e = 0, \quad z_e = 0$$

$$Q_{4e} = \frac{\pi}{2}, \quad Q_{5e} = \frac{\pi}{2}, \quad Q_{6e} = 0$$

The small movements of the solid around this position of equilibrium are represented by the following values of the situation parameters:

$$\begin{cases} Q_1' = x' = \varepsilon_1', \quad Q_2' = y' = \varepsilon_2', \quad Q_3' = z' = \varepsilon_3' \\ Q_4' = \varepsilon_4', \quad Q_5' = \varepsilon_5', \quad Q_6' = \varepsilon_6' \end{cases}$$

and $Q_\alpha'' = \varepsilon_\alpha''$, $\alpha = 1,\ldots,6$.

1.5.1. *Using vector representation*

Accounting for the derivative values of the situation parameters around the position of equilibrium, the expression of dynamic resultant is composed of linear terms in Q_α'' and quadratic terms in $Q_\alpha' Q_\beta'$, which are of second order and subsequently negligible in a development limited to the first order. Under these conditions, the dynamic resultant is written as follows:

$$\vec{s}\left\{\mathscr{A}_S^g\right\} = m\sum_{i=1}^{3}\frac{\partial^{(g)}\overrightarrow{F}}{\partial Q_i}Q_i'' + m\sum_{l=4}^{6}\left(\overrightarrow{{}^l\delta}\wedge\overrightarrow{\mathrm{O}_S\mathrm{G}}\right)Q_l'' \ \ldots$$

$$\ldots\ + \text{ quadratic term}$$

thus its limited development D_1

$$\sum_\alpha \overrightarrow{s^\alpha}(e)\varepsilon_\alpha'' = m\sum_{i=1}^{3}\frac{\partial^{(g)}\overrightarrow{F}}{\partial Q_i}(e)\varepsilon_i'' + m\sum_{l=4}^{6}\left(\overrightarrow{{}^l\delta}\wedge\overrightarrow{\mathrm{O}_S\mathrm{G}}\right)(e)\varepsilon_l''$$

and taking account of the calculation of the development limited to the first order of the rotation rate of (S) in relation to $\langle g\rangle$ performed in section 1.6:

$$\frac{\partial^{(g)}\overrightarrow{F}}{\partial Q_1}(e) = \overrightarrow{x_g} = \vec{x} \quad , \quad \frac{\partial^{(g)}\overrightarrow{F}}{\partial Q_2}(e) = \overrightarrow{y_g} = \vec{y} \quad , \quad \frac{\partial^{(g)}\overrightarrow{F}}{\partial Q_3}(e) = \overrightarrow{z_g} = \vec{z}$$

$$\overrightarrow{{}^4\delta}(e) = \vec{x} \quad , \quad \overrightarrow{{}^5\delta}(e) = \vec{y} \quad , \quad \overrightarrow{{}^6\delta}(e) = \vec{z}$$

Let us also state that at the equilibrium

$$\overrightarrow{\mathrm{O}_S\mathrm{G}}(e) = a\vec{x} + b\vec{y} + c\vec{z}$$

$$\sum_\alpha \overrightarrow{s^\alpha}(e)\varepsilon_\alpha'' = m\left(\varepsilon_1''\vec{x} + \varepsilon_2''\vec{y} + \varepsilon_3''\vec{z}\right)\ \ldots$$

$$\ldots\ + m\left(\varepsilon_4''\vec{x} + \varepsilon_5''\vec{y} + \varepsilon_6''\vec{z}\right)\wedge\left(a\vec{x} + b\vec{y} + c\vec{z}\right)$$

Similarly, we have

$$\sum_{\alpha} \overrightarrow{m_{O_S}^{\alpha}}(e)\varepsilon_{\alpha}'' = m\overrightarrow{O_S G} \wedge \sum_{i=1}^{3} \frac{\partial^{(g)}\overrightarrow{F}}{\partial Q_i}(e)\varepsilon_i'' + \sum_{l=4}^{6} I_{O_S}\left(\overrightarrow{{}^{l}\delta\varepsilon_l''}\right)$$

ignoring the terms of order 2. With the following inertial operator

$$\left[I_{O_S}\right]_{/(\vec{x}\,\vec{y}\,\vec{z})} = \begin{bmatrix} A & -F & -E \\ -F & B & -D \\ -E & -D & C \end{bmatrix}$$

$$\begin{aligned} \Rightarrow \quad \sum_{\alpha} \overrightarrow{m_{O_S}^{\alpha}}(e)\varepsilon_{\alpha}'' &= \left[m\left(b\varepsilon_3'' - c\varepsilon_2''\right) + A\varepsilon_4'' - F\varepsilon_5'' - E\varepsilon_6''\right]\vec{x} \quad \ldots \\ \ldots \quad &+ \left[m\left(c\varepsilon_1'' - a\varepsilon_3''\right) - F\varepsilon_4'' + B\varepsilon_5'' - D\varepsilon_6''\right]\vec{y} \quad \ldots \quad . \\ \ldots \quad &+ \left[m\left(a\varepsilon_2'' - b\varepsilon_1''\right) - E\varepsilon_4'' - D\varepsilon_5'' + C\varepsilon_6''\right]\vec{z} \end{aligned}$$

To obtain a very synthetic matrix form of the equations of equilibrium, we state

$$\frac{\partial \overrightarrow{s^{\Delta *}}}{\partial Q_{\alpha}} = -\sum_{k} c_{k\alpha}\overrightarrow{x_k} \quad , \quad \frac{\partial \overrightarrow{s^{\Delta *}}}{\partial Q_{\alpha}'} = -\sum_{k} b_{k\alpha}\overrightarrow{x_k}$$

$$\frac{\partial \overrightarrow{\mathcal{M}_{O_S}^{\Delta *}}}{\partial Q_{\alpha}} = -\sum_{p} c_{p\alpha}\overrightarrow{x_p} \quad , \quad \frac{\partial \overrightarrow{\mathcal{M}_{O_S}^{\Delta *}}}{\partial Q_{\alpha}'} = -\sum_{p} b_{p\alpha}\overrightarrow{x_p}$$

$$\overrightarrow{s^{\alpha}}(e) = \sum_{k} a_{k\alpha}\overrightarrow{x_k} \quad , \quad \overrightarrow{m_{O_S}^{\alpha}}(e) = \sum_{p} a_{p\alpha}\overrightarrow{x_p}$$

$$\vec{\sigma}(t) = \sum_{k} \sigma_k \overrightarrow{x_k} = \sum_{k} d_k \overrightarrow{x_k} \quad , \quad \overrightarrow{\mu_{O_S}}(t) = \sum_{p} \mu_{O_S}^{p}\overrightarrow{x_p} = \sum_{p} d_p \overrightarrow{x_p}$$

with, at the equilibrium, in the studied problem

$$\overrightarrow{x_4} = \overrightarrow{x_1} = \vec{x}, \quad \overrightarrow{x_5} = \overrightarrow{x_2} = \vec{y}, \quad \overrightarrow{x_6} = \overrightarrow{x_3} = \vec{z}$$

We therefore get the following two index relations to express the considered equilibrium

$$\begin{cases} \sum\limits_{\alpha}\left(a_{k\alpha}\varepsilon_{\alpha}'' + b_{k\alpha}\varepsilon_{\alpha}' + c_{k\alpha}\varepsilon_{\alpha}\right) = d_k, & k = 1,2,3 \\ \sum\limits_{\alpha}\left(a_{p\alpha}\varepsilon_{\alpha}'' + b_{p\alpha}\varepsilon_{\alpha}' + c_{p\alpha}\varepsilon_{\alpha}\right) = d_p, & p = 4,5,6 \end{cases}$$

which we can group into one expression

$$\sum_{\alpha}\left(a_{\beta\alpha}\varepsilon_{\alpha}'' + b_{\beta\alpha}\varepsilon_{\alpha}' + c_{\beta\alpha}\varepsilon_{\alpha}\right) = d_{\beta}, \quad \beta = 1,\ldots,6$$

which then is suitable for the following matrix form

$$[a][\varepsilon''] + [b][\varepsilon'] + [c][\varepsilon] = [d]$$

where $[a],[b],[c]$ are (6×6) matrices and $[\varepsilon''],[\varepsilon'],[\varepsilon],[d]$ are (6×1) columns

In the case used here as an illustration, we have

$$[a] = \begin{bmatrix} m & 0 & 0 & 0 & mc & -mb \\ 0 & m & 0 & -mc & 0 & ma \\ 0 & 0 & m & mb & -ma & 0 \\ 0 & -mc & mb & A & -F & -E \\ mc & 0 & -ma & -F & B & -D \\ -mb & ma & 0 & -E & -D & C \end{bmatrix}$$

and $[b] = [c] = [0]$.

1.5.2. *Using analytical mechanics*

The kinetic energy $2T^{(g)}(S)$ can be expressed as a matrix under the following matrix form:

$$2T^{(g)}(S) = \bar{\mathtt{Q}}\,\mathtt{E}\,\mathtt{Q}$$

with

$$\mathtt{Q}(6,1) = \begin{bmatrix} Q_1' \\ \vdots \\ Q_6' \end{bmatrix} \quad , \quad \bar{\mathtt{Q}}(1,6) = \begin{bmatrix} Q_1' & \cdots & Q_6' \end{bmatrix}$$

$$\mathtt{E}(6,6) = \left[\begin{array}{cc|cc} E_{11} & \cdots & \cdots & E_{16} \\ \vdots & \ddots & \ddots & \vdots \\ \hline \vdots & \ddots & \ddots & \vdots \\ E_{61} & \cdots & \cdots & E_{66} \end{array}\right]$$

The different terms of the matrix $\mathtt{E}$ of dimensions $(6,6)$ have the following expressions, respectively

– from E_{11} to E_{33}: $\quad E_{\alpha\beta} = m\dfrac{\partial^{(g)}\overrightarrow{F}}{\partial Q_\alpha}\cdot\dfrac{\partial^{(g)}\overrightarrow{F}}{\partial Q_\beta}$,

– from E_{14} to E_{36}: $\quad E_{\alpha\beta} = m\dfrac{\partial^{(g)}\overrightarrow{F}}{\partial Q_\alpha}\cdot\left(\overrightarrow{^{\beta}\delta}\wedge\overrightarrow{\mathrm{O}_S\mathrm{G}}\right)$,

– from E_{41} to E_{63}: $\quad E_{\alpha\beta} = m\left(\overrightarrow{^{\alpha}\delta}\wedge\overrightarrow{\mathrm{O}_S\mathrm{G}}\right)\cdot\dfrac{\partial^{(g)}\overrightarrow{F}}{\partial Q_\beta}$,

– from E_{44} to E_{66}: $\quad E_{\alpha\beta} = \left(I_{\mathrm{O}_S}\,\overrightarrow{^{\alpha}\delta}\right)\cdot\overrightarrow{^{\beta}\delta}$.

In the example used, the characteristics of the state of equilibrium are

$$\begin{cases} Q_1' = x' = \varepsilon_1', \quad Q_2' = y' = \varepsilon_2', \quad Q_3' = z' = \varepsilon_3' \\ Q_4' = \varepsilon_4', \quad Q_5' = \varepsilon_5', \quad Q_6' = \varepsilon_6' \end{cases}$$

$$\overrightarrow{\mathrm{O}_S\mathrm{G}}(e) = a\vec{x} + b\vec{y} + c\vec{z}$$

$$\frac{\partial^{(g)}\vec{F}}{\partial Q_1}(e) = \overrightarrow{x_g} = \vec{x}, \quad \frac{\partial^{(g)}\vec{F}}{\partial Q_2}(e) = \overrightarrow{y_g} = \vec{y}, \quad \frac{\partial^{(g)}\vec{F}}{\partial Q_3}(e) = \overrightarrow{z_g} = \vec{z}$$

$${}^{4}\vec{\delta}(e) = \vec{x}, \quad {}^{5}\vec{\delta}(e) = \vec{y}, \quad {}^{6}\vec{\delta}(e) = \vec{z}$$

We get

$$\mathrm{E} = \begin{bmatrix} m & 0 & 0 & 0 & mc & -mb \\ 0 & m & 0 & -mc & 0 & ma \\ 0 & 0 & m & mb & -ma & 0 \\ 0 & -mc & mb & A & -F & -E \\ mc & 0 & -ma & -F & B & -D \\ -mb & ma & 0 & -E & -D & C \end{bmatrix}$$

The equations governing the small movements of (S) in $\langle g \rangle$ around the position of equilibrium (e) use the general matrix form

$$[a][\varepsilon''] + [b][\varepsilon'] + [c][\varepsilon] = [d]$$

which develops as follows:

$$m\varepsilon_1'' + mc\varepsilon_5'' - mb\varepsilon_6'' + b_{1\alpha}\varepsilon_\alpha' + c_{1\alpha}\varepsilon_\alpha = f_1(t)$$
$$m\varepsilon_2'' - mc\varepsilon_4'' + ma\varepsilon_6'' + b_{2\alpha}\varepsilon_\alpha' + c_{2\alpha}\varepsilon_\alpha = f_2(t)$$
$$m\varepsilon_3'' + mb\varepsilon_4'' - ma\varepsilon_5'' + b_{3\alpha}\varepsilon_\alpha' + c_{3\alpha}\varepsilon_\alpha = f_3(t)$$

$$-mc\varepsilon_2'' + mb\varepsilon_3'' + A\varepsilon_4'' - F\varepsilon_5'' - E\varepsilon_6'' + b_{4\alpha}\varepsilon_\varepsilon' + c_{4\alpha}\varepsilon_\alpha = f_4(t)$$

$$mc\varepsilon_1'' - ma\varepsilon_3'' - F\varepsilon_4'' + B\varepsilon_5'' - D\varepsilon_6'' + b_{5\alpha}\varepsilon_\varepsilon' + c_{5\alpha}\varepsilon_\alpha = f_5(t)$$

$$-mb\varepsilon_1'' + ma\varepsilon_2'' - E\varepsilon_4'' - D\varepsilon_5'' + C\varepsilon_6'' + b_{6\alpha}\varepsilon_\varepsilon' + c_{6\alpha}\varepsilon_\alpha = f_6(t)$$

1.5.3. *Relative situation of frames at the equilibrium*

We will continue to use the previous example to illustrate our ideas and consider the situation at equilibrium of the frame $\langle S\rangle$ in relation to the Galilean frame $\langle g\rangle$. The relative position of both frames is represented in Figure 1.6.

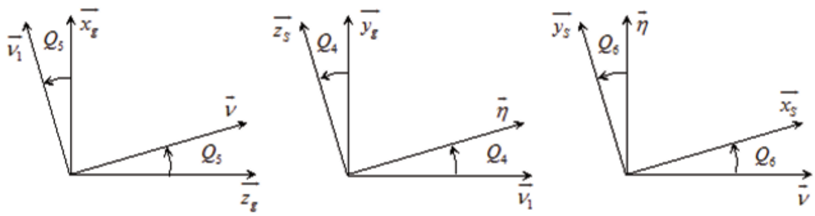

Figure 1.6. *Relative position of the bases*

The table of the change in bases appears

	$\overrightarrow{x_S}$	$\overrightarrow{y_S}$	$\overrightarrow{z_S}$
$\overrightarrow{x_g}$	$\cos Q_4 \cos Q_5 \sin Q_6 + \sin Q_5 \cos Q_6$	$\cos Q_4 \cos Q_5 \cos Q_6 - \sin Q_5 \sin Q_6$	$-\sin Q_4 \cos Q_5$
$\overrightarrow{y_g}$	$\sin Q_4 \sin Q_6$	$\sin Q_4 \cos Q_6$	$\cos Q_4$
$\overrightarrow{z_g}$	$-\cos Q_4 \sin Q_5 \sin Q_6 + \cos Q_5 \cos Q_6$	$-\cos Q_4 \sin Q_5 \cos Q_6 - \cos Q_5 \sin Q_6$	$\sin Q_4 \sin Q_5$

and becomes at the equilibrium position and then, at the first order around this position

$$\begin{array}{c|ccc} & \overrightarrow{x_S} & \overrightarrow{y_S} & \overrightarrow{z_S} \\ \hline \overrightarrow{x_g} & 1 & 0 & 0 \\ \overrightarrow{y_g} & 0 & 1 & 0 \\ \overrightarrow{z_g} & 0 & 0 & 1 \end{array}, \quad \begin{array}{c|ccc} & \overrightarrow{x_S} & \overrightarrow{y_S} & \overrightarrow{z_S} \\ \hline \overrightarrow{x_g} & 1 & -\varepsilon_6 & \varepsilon_5 \\ \overrightarrow{y_g} & \varepsilon_6 & 1 & -\varepsilon_4 \\ \overrightarrow{z_g} & -\varepsilon_5 & \varepsilon_4 & 1 \end{array}$$

Around this equilibrium position, the rotation rate, of general expression

$$\overrightarrow{\omega_S^g} = Q_4{}'\,\vec{v} + Q_5{}'\,\overrightarrow{y_g} + Q_6{}'\,\overrightarrow{z_S}$$

becomes upon first approach

$$D_1\left[\overrightarrow{\omega_S^g}\right] = D_1\left[\varepsilon_4{}'\,\vec{v}\right] + \varepsilon_5{}'\,\overrightarrow{y_g} + D_1\left[\varepsilon_6{}'\,\overrightarrow{z_S}\right]$$

since the vectors $\vec{v}$ and $\overrightarrow{z_S}$ are variable in relation to $\langle g \rangle$.

$$\vec{v} = \cos Q_5\,\overrightarrow{z_g} + \sin Q_5\,\overrightarrow{x_g} = \cos\left(\frac{\pi}{2} + \varepsilon_5\right)\overrightarrow{z_g} + \sin\left(\frac{\pi}{2} + \varepsilon_5\right)\overrightarrow{x_g}$$
$$"" = -\sin\varepsilon_5\,\overrightarrow{z_g} + \cos\varepsilon_5\,\overrightarrow{x_g}$$

$$\Rightarrow \quad D_1\left[\vec{v}\right] = \overrightarrow{x_g} - \varepsilon_5\,\overrightarrow{z_g}, \quad D_1\left[\varepsilon_4{}'\vec{v}\right] = \varepsilon_4{}'\,\overrightarrow{x_g}$$

$$\overrightarrow{z_S} = -\sin Q_4 \cos Q_5\,\overrightarrow{x_g} + \cos Q_4\,\overrightarrow{y_g} + \sin Q_4 \sin Q_5\,\overrightarrow{z_g}$$
$$"" = \cos\varepsilon_4 \sin\varepsilon_5\,\overrightarrow{x_g} - \sin\varepsilon_4\,\overrightarrow{y_g} + \cos\varepsilon_4 \cos\varepsilon_5\,\overrightarrow{z_g}$$

$$\Rightarrow \quad D_1\left[\overrightarrow{z_S}\right] = \varepsilon_5\,\overrightarrow{x_g} - \varepsilon_4\,\overrightarrow{y_g} + \overrightarrow{z_g}, \quad D_1\left[\varepsilon_6{}'\overrightarrow{z_S}\right] = \varepsilon_6{}'\,\overrightarrow{z_g}$$

The first-order limited development of the rotation rate of the basis (S) in relation to the basis (g) and that of its derivative in relation to time are therefore

$$D_1\left[\overrightarrow{\omega_S^g}\right]=\varepsilon_4'\,\overrightarrow{x_g}+\varepsilon_5'\,\overrightarrow{y_g}+\varepsilon_6'\,\overrightarrow{z_g}=\varepsilon_4'\,\overrightarrow{x_S}+\varepsilon_5'\,\overrightarrow{y_S}+\varepsilon_6'\,\overrightarrow{z_S}$$

$$\text{" "}\quad=\varepsilon_4'\,\vec{x}+\varepsilon_5'\,\vec{y}+\varepsilon_6'\,\vec{z}=\varepsilon_4'\,\overrightarrow{{}^4\delta}(e)+\varepsilon_5'\,\overrightarrow{{}^5\delta}(e)+\varepsilon_6'\,\overrightarrow{{}^6\delta}(e)$$

$$D_1\left[\overrightarrow{\omega_S^g}'\right]=\varepsilon_4''\,\vec{x}+\varepsilon_5''\,\vec{y}+\varepsilon_6''\,\vec{z}$$

1.6. Stationary movement

Among the solutions for equilibrium of a solid, there are some of them for which that equilibrium will have the following characteristics:

– for a part of the parameters each has a constant value;

– for the other part, each prime derivative in relation to time is also constant.

The solution for equilibrium is then a motion to which correspond fixed values of a few parameters and linear evolutions depending on time for the others. Such a movement is said to be *stationary*.

1.6.1. *Cyclic parameters*

In the case where the parameter Q_α possesses the following properties:

– the partial power $\Pi_\alpha^{(g)}$ is null;

– the kinetic energy T does not depend on $Q_\alpha \Rightarrow \dfrac{\partial T}{\partial Q_\alpha}=0$;

the Lagrangian equation $\left(L_\alpha\right)$ is then written as:

$$\frac{d}{dt}\frac{\partial T}{\partial Q_\alpha{}'}=0 \;\Rightarrow\; \frac{\partial T}{\partial Q_\alpha{}'}=C_\alpha$$

which leads to the prime integral

$$\sum_\beta E_{\alpha\beta}Q_\beta{}'=C_\alpha$$

The parameters Q_α that verify the two conditions above are called *cyclic parameters*.

The other parameters are said to be *non-cyclic*.

1.6.2. *Characterizing a stationary movement*

A solid $\left(S\right)$ is animated with a stationary movement in its frame of reference, starting from date $t=t_0$, if among its k ($k\le 6$) situation parameters Q_α, s some of them maintain a constant value while each of the other $k-s$ has its prime derivative in relation to time which has a constant value.

This is expressed for $t\ge t_0$, as

– $Q_i\left(t\right)=Q_i\left(t_0\right)=e_i,\quad i=1,\ldots,s$;

– $Q_j{}'\left(t\right)=Q_j{}'\left(t_0\right)=w_j,\quad j=s+1,\ldots,k$.

The k values $e_1,\ldots,e_s,w_{s+1},\ldots,w_k$ characterize the stationary movement of the solid $\left(S\right)$ in its frame of reference $\left\langle g\right\rangle$.

1.6.3. *Conditions of realization of a stationary movement*

Consider the case where, among the k degrees of freedom of the solid (S)

– the first s $(s<k)$ parameters Q_i are non-cyclic ($i=1,\ldots,s$);

– the last $k-s$ are cyclic, meaning that

$$\frac{\partial T}{\partial Q_j}=0,\quad \Pi_j^{(g)}=0 \quad \text{for} \quad j=s+1,\ldots,k$$

with $2T=\sum_{\beta}\sum_{\gamma}E_{\beta\gamma}\left(Q_1,\ldots,Q_s\right)Q_\beta{}'Q_\gamma{}' \;\Rightarrow\; \dfrac{\partial E_{\beta\gamma}}{\partial Q_j}=0$.

We then look for the conditions that allow the solid to experience a stationary movement characterized by

– for $i=1,\ldots,s$, $Q_i=e_i \;\Rightarrow\; Q_i{}'=0$, $Q_i{}''=0$;

– for $j=s+1,\ldots,k,\;\; Q_j=w_jt+v_j \;\Rightarrow\; Q_j{}'=w_j,\; Q_j{}''=0$.

For the first s values of Q_i, the Lagrange equations of rank i appear

$$\sum_{\beta=1}^{k}E_{i\beta}\,Q_\beta{}''-\frac{1}{2}\sum_{\beta=1}^{k}\sum_{\gamma=1}^{k}\frac{\partial E_{\beta\gamma}}{\partial Q_i}Q_\beta{}'Q_\gamma{}'+\sum_{\beta=1}^{k}\sum_{\gamma=1}^{s}\frac{\partial E_{i\beta}}{\partial Q_\gamma}Q_\beta{}'Q_\gamma{}'=\;\ldots$$

$$\ldots\;\Pi_i\left(Q_1,\ldots,Q_k\middle|Q_1{}',\ldots,Q_k{}'\middle|t\right)$$

since, for $\gamma>s$, $\dfrac{\partial E_{i\beta}}{\partial Q_\gamma}=0$ (cyclic variable).

In the situation of equilibrium that represents the considered stationary movement, considering that all terms Q'' are null and for β and $\gamma\le s$, $Q_\beta{}'=0$, $Q_\gamma{}'=0$, the above Lagrange equations can be written as:

$$-\frac{1}{2}\sum_{\beta=s+1}^{k}\sum_{\gamma=s+1}^{k}\frac{\partial E_{\beta\gamma}}{\partial Q_i}(e_1,\dots,e_s)\,w_\beta w_\gamma = \;\dots$$
$$\dots\;\;\Pi_i\left(e_1,\dots,e_s,w_{s+1}t+v_{s+1},\dots,w_k t+v_k\left|0,\dots,0,w_{s+1},\dots,w_k\right|t\right)$$

However, if the term of partial power Π_i is explicitly dependent on time t, the stationary movement will be impossible. Similarly, the parameters Q_j vary linearly with time; we will therefore limit ourselves to the case where the partial powers Π_i are independent of these parameters. For such an equilibrium to be possible, we must have

$$-\;\Pi_i\left(Q_1,\dots,Q_k\left|Q_1',\dots,Q_k'\right|t\right)=\Pi_i\left(Q_1,\dots,Q_k\left|Q_1',\dots,Q_k'\right.\right)+\mu(t)$$

with $\mu(t)=0$ for $t\geq t_0$.

In the previously defined conditions, the s relations that must satisfy the stationary movement of solid (S) in $\langle g\rangle$ surrounding parameters Q_α,

$$-\frac{1}{2}\sum_{\beta=s+1}^{k}\sum_{\gamma=s+1}^{k}\frac{\partial E_{\beta\gamma}}{\partial Q_i}(e_1,\dots,e_s)\,w_\beta w_\gamma = \;\dots$$
$$\dots\;\;\Pi_i\left(e_1,\dots,e_s\left|0,\dots,0,w_{s+1},\dots,w_k\right.\right).$$

For the $s-k$ cyclic parameters Q_j $(j=s+1,\dots,k)$, the corresponding Lagrange equations, of rank j, are written as:

$$\sum_{\beta=1}^{k}E_{j\beta}\,Q_\beta''+\sum_{\beta=1}^{k}\sum_{\gamma=1}^{s}\frac{\partial E_{j\beta}}{\partial Q_\gamma}Q_\beta' Q_\gamma'=\mu(t)$$

The studies developed in Chapter 4, which discuss gyroscopic motions, will provide applications for stationary movement and its stability.

1.6.4. *Neighboring motions and stability of a stationary movement*

We adopt the same conditions as before, i.e.

– s non-cyclic parameters Q_i, $i=1,...,s$;

– $s-k$ cyclic parameters Q_j, $j=s+1,\ldots,k$;

and a stationary movement defined by

$$Q_i = e_i,\ i=1,\ldots,s; \quad Q_j' = w_j,\ j=s+1,\ldots,k.$$

We now consider the respective variations of these parameters around their equilibrium values

$$Q_i = e_i + \varepsilon_i; \quad Q_j' = w_j + \varepsilon_j$$

The idea is then to linearize in the first order the aforementioned k Lagrange equations, keeping in mind that

$$Q_i'' = \varepsilon_i'' \ ,\ Q_i' = \varepsilon_i'; \quad Q_j'' = \varepsilon_j'$$

1.6.4.1. *Linearization of equations of rank j*

For equations at rank $j=s+1,\ldots,k$, the linearization to the first order gives, with $\mu(t)=0$ starting from time $t=t_0$, stating that $(e)\equiv(e_1,\ldots,e_s)$

$$\sum_{\beta=1}^{s} E_{j\beta}(e)\varepsilon_\beta'' + \sum_{\beta=s+1}^{k} E_{j\beta}(e)\varepsilon_\beta' + \sum_{\beta=s+1}^{k}\sum_{\gamma=1}^{s}\frac{\partial E_{j\beta}(e)}{\partial Q_\gamma}\varepsilon_\gamma' w_\beta = 0$$

1.6.4.2. *Linearization of equations of rank i*

The equations to linearize are the following:

$$\sum_{\beta=1}^{k} E_{i\beta}\, Q_\beta'' - \frac{1}{2}\sum_{\beta=1}^{k}\sum_{\gamma=1}^{k}\frac{\partial E_{\beta\gamma}}{\partial Q_i} Q_\beta' Q_\gamma' + \sum_{\beta=1}^{k}\sum_{\gamma=1}^{s}\frac{\partial E_{i\beta}}{\partial Q_\gamma} Q_\beta' Q_\gamma' = \ \ldots$$

$$\ldots\ \Pi_i\left(Q_1,\ldots,Q_k \middle| Q_1',\ldots,Q_k'\right) + \mu(t)$$

with $\mu(t)=0$ at the equilibrium. Then, we must linearize the terms of the two members.

$$D_1\left[\Pi_i\right] = \Pi_i\left(e_1,\ldots,e_s \middle| 0,\ldots,0,w_{s+1},\ldots,w_k\right)\ \ldots$$

$$\ldots\ -\sum_{\beta=1}^{s} c_{i\beta}\,\varepsilon_\beta - \sum_{\beta=1}^{s} b_{i\beta}\,\varepsilon_\beta' - \sum_{\beta=s+1}^{k} b_{i\beta}\,\varepsilon_\beta$$

where

$$\begin{cases} c_{i\beta} = -\dfrac{\partial \Pi_i}{\partial Q_\beta}\left(e_1,\ldots,e_s \middle| 0,\ldots,0,w_{s+1},\ldots,w_k\right), & \beta = 1,\ldots,s \\ b_{i\beta} = -\dfrac{\partial \Pi_i}{\partial Q_\beta'}\left(e_1,\ldots,e_s \middle| 0,\ldots,0,w_{s+1},\ldots,w_k\right), & \beta = 1,\ldots,k \end{cases}$$

$$-\sum_{\beta=1}^{k}\sum_{\gamma=1}^{k}\frac{\partial E_{\beta\gamma}}{\partial Q_i} Q_\beta' Q_\gamma' = \left[\sum_{\gamma=1}^{s}\varepsilon_\gamma' + \sum_{\gamma=s+1}^{k}\left(w_\gamma + \varepsilon_\gamma\right)\right]\sum_{\beta=1}^{k}\frac{\partial E_{\beta\gamma}}{\partial Q_i} Q_\beta'$$

with

$$-\,D_1\left[\sum_{\gamma=1}^{s}\varepsilon_\gamma'\sum_{\beta=1}^{k}\frac{\partial E_{\beta\gamma}}{\partial Q_i} Q_\beta'\right] = \sum_{\gamma=1}^{s}\varepsilon_\gamma'\sum_{\beta=s+1}^{k}\frac{\partial E_{\beta\gamma}}{\partial Q_i}(e)\, w_\beta$$

$$-\,D_1\left[\sum_{\gamma=s+1}^{k}\left(w_\gamma+\varepsilon_\gamma\right)\sum_{\beta=1}^{k}\frac{\partial E_{\beta\gamma}}{\partial Q_i} Q_\beta'\right] = \ \ldots$$

$$\ldots\ \sum_{\gamma=s+1}^{k} w_\gamma D_1\left[\sum_{\beta=1}^{k}\frac{\partial E_{\beta\gamma}}{\partial Q_i} Q_\beta'\right] + \sum_{\gamma=s+1}^{k}\varepsilon_\gamma D_1\left[\sum_{\beta=1}^{k}\frac{\partial E_{\beta\gamma}}{\partial Q_i} Q_\beta'\right]$$

where,

$$D_1\left[\frac{\partial E_{\beta\gamma}}{\partial Q_i}\right]=\frac{\partial E_{\beta\gamma}}{\partial Q_i}(e)+\sum_{j=1}^{s}\frac{\partial^2 E_{\beta\gamma}}{\partial Q_j \partial Q_i}\varepsilon_j$$

$$\Rightarrow \quad D_1\left[\sum_{\gamma=s+1}^{k}\left(w_\gamma+\varepsilon_\gamma\right)\sum_{\beta=1}^{k}\frac{\partial E_{\beta\gamma}}{\partial Q_i}Q_\beta{}'\right]= \ \ldots$$

$$\ldots \quad \sum_{\gamma=s+1}^{k} w_\gamma \sum_{\beta=1}^{k}\frac{\partial E_{\beta\gamma}}{\partial Q_i}(e)\varepsilon_\beta{}' + \sum_{\gamma=s+1}^{k}\varepsilon_\gamma \sum_{\beta=s+1}^{k}\frac{\partial E_{\beta\gamma}}{\partial Q_i}(e)w_\beta \quad \ldots$$

$$\ldots \ + \sum_{\gamma=s+1}^{k} w_\gamma \sum_{\beta=s+1}^{k}\left[\frac{\partial E_{\beta\gamma}}{\partial Q_i}(e)\left(w_\beta+\varepsilon_\beta\right)+w_\beta\sum_{j=1}^{s}\frac{\partial^2 E_{\beta\gamma}}{\partial Q_j \partial Q_i}\varepsilon_j\right]$$

To simplify this expression, we state

$$a_{\beta\gamma}=E_{\beta\gamma}(e), \qquad a_{\beta\gamma i}=\frac{\partial E_{\beta\gamma}}{\partial Q_i}(e), \qquad a_{\beta\gamma ij}=\frac{\partial^2 E_{\beta\gamma}}{\partial Q_i \partial Q_j}(e)$$

As we also have

$$-\frac{1}{2}\sum_{\beta=s+1}^{k}\sum_{\gamma=s+1}^{k}\frac{\partial E_{\beta\gamma}}{\partial Q_i}(e_1,\ldots,e_s)w_\beta w_\gamma = \ \ldots$$

$$\ldots \quad \Pi_i\left(e_1,\ldots,e_s \middle| 0,\ldots,0,w_{s+1},\ldots,w_k\right)$$

and we observe that via the set of indices and their ranges of variations,

$$\sum_{\gamma=1}^{s}\varepsilon_\gamma{}'\sum_{\beta=s+1}^{k}a_{\beta\gamma i}w_\beta=\sum_{\gamma=s+1}^{k}w_\beta\sum_{\beta=1}^{s}a_{\beta\gamma i}\varepsilon_\beta{}'$$

$$\sum_{\gamma=s+1}^{k}\sum_{\beta=s+1}^{k}a_{\beta\gamma i}w_\gamma\varepsilon_\beta=\sum_{\gamma=s+1}^{k}\sum_{\beta=1}^{s}a_{\beta\gamma i}w_\beta\varepsilon_\gamma$$

we obtain the s Lagrange equations (L_i), relative to the non-cyclic parameters Q_i $(i=1,\ldots,s)$, which complete the $k-s$ previously established equations, relative to the cyclic parameters Q_j $(j=s+1,\ldots,k)$

$$
\begin{aligned}
(L_i) \;\Rightarrow\; & \sum_{\beta=1}^{s} a_{i\beta}\boldsymbol{\varepsilon}_{\boldsymbol{\beta}}'' + \sum_{\beta=s+1}^{k} a_{i\beta}\boldsymbol{\varepsilon}_{\boldsymbol{\beta}}' + \sum_{\beta=1}^{s} b_{i\beta}\boldsymbol{\varepsilon}_{\boldsymbol{\beta}}' \;\ldots \\
& \ldots + \sum_{\gamma=1}^{s}\boldsymbol{\varepsilon}_{\boldsymbol{\gamma}}' \sum_{\beta=s+1}^{k} w_\beta \left(a_{i\beta\gamma} - a_{\beta\gamma i}\right) + \sum_{\beta=s+1}^{k} b_{i\beta}\boldsymbol{\varepsilon}_{\boldsymbol{\beta}} + \sum_{\beta=1}^{s} c_{i\beta}\boldsymbol{\varepsilon}_{\boldsymbol{\beta}} \;\ldots \\
& \ldots - \sum_{\beta=s+1}^{k}\sum_{\gamma=s+1}^{k} w_\gamma \left(a_{\beta\gamma i}\boldsymbol{\varepsilon}_{\boldsymbol{\beta}} + \frac{1}{2} w_\beta \sum_{j=1}^{s} a_{\beta\gamma ij}\boldsymbol{\varepsilon}_j \right) = 0
\end{aligned}
$$

$$
(L_j) \;\Rightarrow\; \sum_{\beta=1}^{s} a_{j\beta}\boldsymbol{\varepsilon}_{\boldsymbol{\beta}}'' + \sum_{\beta=s+1}^{k} a_{j\beta}\boldsymbol{\varepsilon}_{\boldsymbol{\beta}}' + \sum_{\gamma=1}^{s}\boldsymbol{\varepsilon}_{\boldsymbol{\gamma}}' \sum_{\beta=s+1}^{k} a_{j\beta\gamma} w_\beta = 0
$$

This system of k linear differential equations of the second order helps study variations in motion around the situation of equilibrium represented by the stationary movement of the studied device.

It can be put into the matrix form we encountered earlier

$$[a][\varepsilon''] + [b][\varepsilon'] + [c][\varepsilon] = [d]$$

and used as such with the methods that will be described in Chapter 2.

1.6.5. *Applications*

This chapter does not contain exercises or problems about stationary movement. The subject will be addressed further in Chapter 4, with the study of gyroscopic movement.