1

Set of Solids with Neither Loops Nor Branches

The direct application of the fundamental principle of dynamics to the movement of a chain of solids, as developed in the previous volumes of this series, assumes that each solid is considered as non-deformable. Yet, when the arrangements of these solids form loops or branches, during movement these are generally subject to structural deformation like tension, compression, bending, torsion, etc. The study of such movements requires recourse to theories on mechanical structures, reaching beyond the scope of this series.

1.1. Identifying a chain of solids with neither loops nor branches

In a Galilean frame $\langle g \rangle$, we consider a series of *n* solids (S_1), (S_2), …(S_i), …(S_n) linked to one another, forming a chain with neither loops nor branches if

(S_1) is linked to $\langle g \rangle$ and (S_2) only;

(S_2) is linked to (S_1) and (S_3) only;

...

(S_i) is linked to (S_{i-1}) and (S_{i+1}) only;

...

(S_n) is linked to (S_{n-1}) only.

Let:

– "upstream of (S_i)" refer to the subset of solids (S_j) such that $1 \leq j \leq i$;

– "downstream of (S_i)" refer to the subset of solids (S_j) such that $i \leq j \leq n$;

– "(S_{j-1})" refer to the previous solid (S_j);

– "(S_{j+1})" refer to the next solid (S_j).

The solid (S_1), with which the frame $\langle 1 \rangle$ is associated, is located with respect to the reference frame $\langle g \rangle$ using $k_1 (k_1 < 6)$ parameters $q_1^1, q_1^2, ..., q_1^{k_1}$ forming the group $(q_1) \equiv \left[q_1^1, q_1^2, ..., q_1^{k_1} \right]$.

Each of the solids in the chain is framed and located analogously with respect to that preceding it. Thus, the solid (S_i), with which the frame $\langle i \rangle$ is associated, is located with respect to solid (S_{i-1}), with which the frame $\langle i-1 \rangle$ is associated, using the $k_i (k_i < 6)$ parameters forming the group $(q_i) \equiv \left[q_i^1, q_i^2, ..., q_i^{k_1} \right]$.

1.2. Applying the fundamental principles of mechanics

The fundamental principles governing mechanics take on particular meaning and importance in the case of chains of solids and they are worth explaining and commenting on. The case of a set (D) composed of two solids (S_1) and (S_2) articulated to each other at a common point provides an appropriate illustration suitable for highlighting their application.

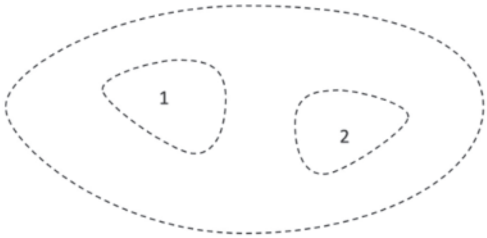

Figure 1.1. *Set of two disjoint solids*

Let us begin by recalling the two basic principles that are applied when studying chains of solids.

1.2.1. *Principle of effort generators*

Two effort generators $\sum'$ and $\sum''$, considered disjoint, i.e. such that $\sum' \cap \sum'' = \emptyset$, act jointly on the effort receiver $\mathbb{R}$ according to the diagram shown in Figure 1.2.

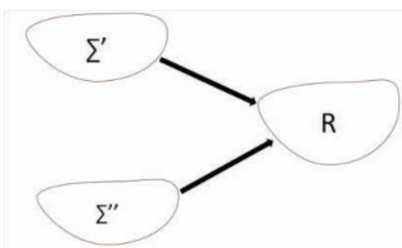

Figure 1.2. *Effort generation diagram*

The efforts exerted by $\sum'$ on $\mathbb{R}$ are represented by the torsor $\{\sum \to \mathbb{R}\}$, those exerted by $\sum''$ on $\mathbb{R}$ are represented by the torsor $\{\sum \to \mathbb{R}''\}$. and those exerted by uniting two effort generators $\sum'$ and $\sum''$ on the same effort receiver $\mathbb{R}$ are represented by the torsor $\{\sum' \cup \sum'' \to \mathbb{R}\}$ such that:

$$\left\{\sum{}' \cup \sum{}'' \to \mathbb{R}\right\} = \left\{\sum{}' \to \mathbb{R}\right\} + \left\{\sum{}'' \to \mathbb{R}\right\}$$

1.2.2. *Principle of effort receivers*

The effort generator acts simultaneously on the two effort receivers $\mathbb{R}'$ and $\mathbb{R}''$, disjoint ($\mathbb{R}' \cap \mathbb{R}'' = \emptyset$) according to the diagram shown in Figure 1.3.

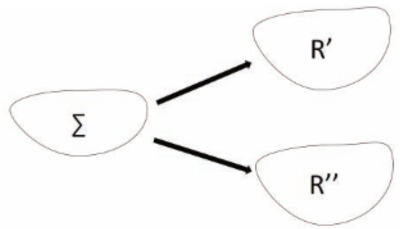

Figure 1.3. *Effort reception diagrams*

The efforts exerted by $\sum$ on $\mathbb{R}'$ are represented by the torsor $\{\sum \to \mathbb{R}'\}$, those exerted by $\sum$ on $\mathbb{R}''$ are represented by the torsor $\{\sum \to \mathbb{R}''\}$ and those that $\sum$ exerts jointly on the two effort receivers $\mathbb{R}'$ and $\mathbb{R}''$ are represented by the torsor $\{\sum \to \mathbb{R}' \cup \mathbb{R}''\}$ such that:

$$\{\sum \to \mathbb{R}' \cup \mathbb{R}''\} = \{\sum \to \mathbb{R}'\} + \{\sum \to \mathbb{R}''\}$$

1.2.3. *Applying the fundamental principle of dynamics*

In the Galilean frame $< g >$, consider the set $(D) = (S_1) \cup (S_2)$ To simplify the expression of the torsor equations that result from applying the fundamental principle of dynamics to the different moving bodies that may be considered, the two solids will be noted 1 and 2, respectively. These equations also involve the external environment of these bodies: $\left(\overline{D}\right)$ represents the external environment of $\left(D\right)$, that is, everything within its field of evolution, with the exception of those constituting $\left(D\right)$, strictly excluding (S_1) and (S_2), which are also disjoint.

We must also consider that the reduction elements of a dynamic torsor are summations achieved using the particles M of a solid, which makes it possible to write, insofar as the two solids are disjoint,

$$\{\mathcal{A}_D^g\} = \{\mathcal{A}_1^g\} + \{\mathcal{A}_2^g\}$$

1.2.3.1. *Applying the fundamental principle to solid 1*

As the frame $\langle g \rangle$ is Galilean, the fundamental principle of dynamics can be applied to the movement of solid 1 with the equation

$$\{\mathcal{A}_D^g\} = \{\overline{1} \rightarrow 1\}$$

Note that $\overline{1}$ is the outside of 1, composed of all that is not (S_1), and we have

$$\overline{1} = \overline{D} \cup 2$$

hence, according to the principle of effort generators

$$\{\overline{1} \rightarrow 1\} = \{\overline{D} \cup 2 \rightarrow 1\} = \{\overline{D} \rightarrow 1\} + \{2 \rightarrow 1\}$$

The fundamental principle of dynamics applied to the movement of (S_1) is therefore written as

$$\{\mathcal{A}_1^g\} = \{\overline{1} \rightarrow 1\} = \{\overline{D} \rightarrow 1\} + \{2 \rightarrow 1\}$$

By separating the actions expressed by each of the two torsors into known efforts and links, we obtain an exploitable expression of the fundamental principle of dynamics applied to the movement of (S_1)

$$\{\mathcal{A}_1^g\} = \{\Delta_{\overline{D}\rightarrow 1}\} + \{\Delta_{2\rightarrow 1}\} + \{\mathcal{L}_{\overline{D}\rightarrow 1}\} + \{\mathcal{L}_{2\rightarrow 1}\}$$

1.2.3.2. *Applying the fundamental principle to solid 2*

As the frame $\langle g \rangle$ is Galilean, the fundamental principle of dynamics can be applied to the movement of solid 2 with the equation

$$\{\mathcal{A}_2^g\} = \{\overline{2} \rightarrow 2\}$$

Note that $\overline{2}$, is the outside of 2, composed of all that is not (S_2), and we have $\overline{2} = \overline{D} \cup 1$

Hence, according to the principle of effort generators

$$\{\overline{2} \to 2\} = \{\overline{D} \cup 1 \to 2\} = \{\overline{D} \to 2\} + \{1 \to 2\}$$

The fundamental principle of dynamics applied to the movement of (S_2) is therefore written as

$$\{\mathcal{A}_2^g\} = \{\overline{2} \to 2\} = \{\overline{D} \to 2\} + \{1 \to 2\}$$

Thus, by separating each of the two torsors according to the known efforts and the links, we obtain:

$$\{\mathcal{A}_2^g\} = \{\Delta_{\overline{D} \to 2}\} + \{\Delta_{1 \to 2}\} + \{\mathcal{L}_{\overline{D} \to 2}\} + \{\mathcal{L}_{1 \to 2}\}$$

1.2.3.3. *Fundamental principle applied to the set (D)*

As it is assumed, in mechanics, that effort torsors may be separated according to the physical laws (gravitation, electromagnetism and contact forces), to which a mechanical set is subjected, the following expression expresses in short the application of the fundamental principle of dynamics to the movement of (D) in the Galilean frame $\langle g \rangle$

$$\{\mathcal{A}_D^g\} = \{\overline{D} \to D\},$$

where $D = 1 \cup 2$.

According to the principle of effort receivers

$$\{\overline{D} \to D\} = \{\overline{D} \to 1 \cup 2\} = \{\overline{D} \to 1\} + \{\overline{D} \to 2\}$$

$$\Longrightarrow \{\mathcal{A}_D^g\} = \{\overline{D} \to 1\} + \{\overline{D} \to 2\}$$

that is, after separating into known efforts and links:

$$\{\mathcal{A}_D^g\} = \{\Delta_{\overline{D} \to 1}\} + \{\Delta_{\overline{D} \to 2}\} + \{\mathcal{L}_{\overline{D} \to 1}\} + \{\mathcal{L}_{\overline{D} \to 2}\}$$

1.2.4. *Theorem of mutual actions*

The sum of the two equations obtained by application of the fundamental principle of dynamics to each solid (S_1) and (S_2), respectively, gives us:

$$\{\mathcal{A}_1^g\} + \{\mathcal{A}_2^g\} = \{\overline{D} \to 1\} + \{2 \to 1\} + \{\overline{D} \to 2\} + \{1 \to 2\}$$

Consequently

$$\{\mathcal{A}_D^g\} = \{\mathcal{A}_1^g\} + \{\mathcal{A}_2^g\} = \{\overline{D} \to 1\} + \{2 \to 1\} + \{\overline{D} \to 2\} + \{1 \to 2\}$$

With, in addition:

$$\{\mathcal{A}_D^g\} = \{\overline{D} \to 1\} + \{\overline{D} \to 2\}$$

The combination of these different equations results in the expression of the theorem of mutual actions that two solids exert on one another

$$\{2 \to 1\} + \{2 \to 1\} = \{0\}$$

If the separation into known efforts and links, conducted previously, is applied to this relation, this expression is then written as

$$\{\Delta_{2\to1}\} + \{\Delta_{1\to2}\} + \{\mathcal{L}_{2\to1}\} + \{\mathcal{L}_{1\to2}\} = \{0\}$$

In principle, whatever the physical law involved in the given or calculable actions exerted by solid (S_2) on solid (S_1), we can write:

$$\{\Delta_{2\to1}\} + \{\Delta_{1\to2}\} = \{0\}$$

We therefore deduce reciprocity in the link efforts that are exerted between the two solids

$$\{\mathcal{L}_{2\to1}\} + \{\mathcal{L}_{1\to2}\} = \{0\}$$

1.2.5. *Summary of equations obtained*

For the set $(D) = (S_1)\cup(S_2)$, to recap:

– two independent torsor equations, i.e. 12 scalar equations;

– three independent link torsors, unknown, $\{\mathcal{L}_{\bar{D}\to 1}\}$, $\{\mathcal{L}_{\bar{D}\to 2}\}$, $\{\mathcal{L}_{1\to 2}\}$, representing 18 scalar unknowns;

– p location parameters.

The 12 scalar equations identified above include 18+p unknowns. As a result, in a system corresponding to a set of two solids, there is a deficit of

18+p-12=6+p equations.

In the specific scenario where one of the three link torsors is null, the deficit is reduced to p equations.

1.3. Study of the movement of a chain of solids (case of three solids)

1.3.1. *Applying the fundamental principle of dynamics*

In order to illustrate the principles that apply to the study of the movement of several solids interconnected by means of an articulated device according to the rule presented above, we will explore further the case of three solids constituting a chain comprising neither loops nor branches, therefore meeting the following conditions:

$$(D) = (S_1)\cup(S_2)\cup(S_3) \text{ where } \begin{bmatrix} (S_1)\cap(S_2) = \emptyset \\ (S_2)\cap(S_3) = \emptyset \\ (S_1)\cap(S_3) = \emptyset \end{bmatrix}$$

The torsor equations translating the movement of each solid in the pseudo-Galilean frame $< \lambda >$ are written as

$$* \, (S_1) \Rightarrow \left\{\mathcal{A}_1^{\lambda}\right\} = \{\bar{1} \to 1\} = \{\bar{D} \to 1\} + \{2 \to 1\}$$

$$* (S_2) \Rightarrow \{\mathcal{A}_2^\lambda\} = \{\bar{2} \to 2\} = \{\bar{D} \to 2\} + \{1 \to 2\} + \{3 \to 2\}$$

$$* (S_3) \Rightarrow \{\mathcal{A}_3^\lambda\} = \{\bar{3} \to 3\} = \{\bar{D} \to 3\} + \{2 \to 3\}$$

The three solids are disjoint: given the composition of the chain and applying what was shown in section 1.2, according to the theorem of mutual actions, we can write that

$$\{2 \to 1\} + \{1 \to 2\} = \{0\}$$

$$\{3 \to 2\} + \{2 \to 3\} = \{0\}$$

We can furthermore deduce the torsor equation of the fundamental principle of dynamics by applying it to the mechanical set (D)

$$\{\mathcal{A}_D^\lambda\} = \{\mathcal{A}_1^\lambda\} + \{\mathcal{A}_2^\lambda\} + \{\mathcal{A}_3^\lambda\} = \{\bar{D} \to 1\} + \{\bar{D} \to 2\} + \{\bar{D} \to 3\}$$

The external and the link efforts acting on each solid come under the various laws of forces already mentioned, making it possible to separate the right-hand terms of the torsional equations according to these laws, applying the fundamental principle of dynamics to each solid, giving:

$$\{\mathcal{A}_1^\lambda\} = \{\Delta_{\bar{D}\to 1}\} + \{\mathcal{L}_{\lambda\to 1}\} - \{\Delta_{1\to 2}\} - \{\mathcal{L}_{1\to 2}\}$$

$$\{\mathcal{A}_2^\lambda\} = \{\Delta_{\bar{D}\to 2}\} + \{\Delta_{1\to 2}\} + \{\mathcal{L}_{1\to 2}\} - \{\Delta_{2\to 3}\} - \{\mathcal{L}_{2\to 3}\}$$

$$\{\mathcal{A}_3^\lambda\} = \{\Delta_{\bar{D}\to 3}\} + \{\Delta_{2\to 3}\} + \{\mathcal{L}_{2\to 3}\}$$

where the link of (S_1) with $(\bar{D})$ leads to the link with $< \lambda >$.

1.3.2. *Solidifying parameters*

In cases where links are independent of time, the velocity distributing torsors of the movement of (S_1) with respect to $< \lambda >$, of (S_2) with respect to $< 1 >$ and of (S_3) with respect to $< 2 >$ are expressed, respectively, as

$$\begin{Bmatrix}\lambda\\1\end{Bmatrix} = \begin{Bmatrix}\lambda\\2.\alpha_1\end{Bmatrix}(q_1^{\alpha_1}), \qquad \alpha_1 = 1, \dots, k_1 \in (q_1)$$

$$\begin{Bmatrix}1\\2\end{Bmatrix} = \begin{Bmatrix}1\\2.\alpha_2\end{Bmatrix}(q_2^{\alpha_2}), \qquad \alpha_2 = 1, \dots, k_2 \in (q_2)$$

$$\begin{Bmatrix}2\\3\end{Bmatrix} = \begin{Bmatrix}2\\3.\alpha_3\end{Bmatrix}(q_3^{\alpha_3}), \qquad \alpha_3 = 1, \dots, k_3 \in (q_3)$$

where $(q_1), (q_2)$ and (q_3) are without the parameter groups introduced in section 1.1 to identify the solids of a chain. It should be noted in particular that:

$$\begin{Bmatrix}\lambda\\2\end{Bmatrix} = \begin{Bmatrix}1\\2\end{Bmatrix} + \begin{Bmatrix}\lambda\\1\end{Bmatrix} = \begin{Bmatrix}\lambda\\1.\alpha_1\end{Bmatrix}(q_1^{\alpha_1})' + \begin{Bmatrix}1\\2.\alpha_2\end{Bmatrix}(q_2^{\alpha_2})'$$

As the parameters $q_2^{\alpha_2}$ are not involved in the movement of (S_1) with respect to $< \lambda >$, we have $\begin{Bmatrix}\lambda\\1.\alpha_2\end{Bmatrix} = \{0\}$

$$\Rightarrow \begin{Bmatrix}\lambda\\2.\alpha_2\end{Bmatrix} = \begin{Bmatrix}1\\2.\alpha_2\end{Bmatrix} + \begin{Bmatrix}\lambda\\1.\alpha_2\end{Bmatrix} = \begin{Bmatrix}1\\2.\alpha_2\end{Bmatrix}$$

Likewise, the parameters $q_1^{\alpha_1}$ are not involved in the movement of (S_2) with respect to $< 1 >$, and as a result $\begin{Bmatrix}1\\2.\alpha_1\end{Bmatrix} = \{0\}$

$$\Rightarrow \begin{Bmatrix}\lambda\\2.\alpha_1\end{Bmatrix} = \begin{Bmatrix}1\\2.\alpha_1\end{Bmatrix} + \begin{Bmatrix}\lambda\\1.\alpha_1\end{Bmatrix} = \begin{Bmatrix}\lambda\\1.\alpha_1\end{Bmatrix}$$

hence

$$\begin{Bmatrix}\lambda\\2\end{Bmatrix} = \begin{Bmatrix}\lambda\\2.\alpha_1\end{Bmatrix}(q_1^{\alpha_1})' + \begin{Bmatrix}\lambda\\2.\alpha_2\end{Bmatrix}(q_2^{\alpha_2})'$$

The same approach allows us to demonstrate that the relation

$$\begin{aligned}\begin{Bmatrix}\lambda\\3\end{Bmatrix} &= \begin{Bmatrix}2\\3\end{Bmatrix} + \begin{Bmatrix}1\\2\end{Bmatrix} + \begin{Bmatrix}\lambda\\1\end{Bmatrix}\\ &= \begin{Bmatrix}2\\3.\alpha_3\end{Bmatrix}(q_3^{\alpha_3})' + \begin{Bmatrix}1\\2.\alpha_2\end{Bmatrix}(q_2^{\alpha_2})' + \begin{Bmatrix}\lambda\\1.\alpha_1\end{Bmatrix}(q_1^{\alpha_1})'\end{aligned}$$

can be written more generally as we are faced with a chain of solids with neither loops nor branches.

$$\begin{Bmatrix}\lambda\\3\end{Bmatrix}=\begin{Bmatrix}\lambda\\3.\alpha_1\end{Bmatrix}(q_1^{\alpha_1})'+\begin{Bmatrix}\lambda\\3.\alpha_2\end{Bmatrix}(q_2^{\alpha_2})'+\begin{Bmatrix}\lambda\\3.\alpha_3\end{Bmatrix}(q_3^{\alpha_3})'$$

We can observe that the parameters $q_1^{\alpha_1}$ are involved in the movements of (S_1), (S_2) and (S_3) with respect to $<\lambda>$; indeed, if we block parameters $q_2^{\alpha_2}$ and $q_3^{\alpha_3}$ such that (S_2) and (S_3) are fixed with respect to (S_1), the only variation in the parameters $q_1^{\alpha_1}$ causes the location of the set (D) to evolve as if it were a single solid. We can therefore say that the parameters $q_1^{\alpha_1}$ are *solidifying* for the set

$$(D)=(S_1)\cup(S_2)\cup(S_3)$$

Likewise, the parameters $q_2^{\alpha_2}$ are involved in the movements of (S_2) and (S_3) with respect to $<\lambda>$. If we block parameters $q_1^{\alpha_1}$ as well as $q_3^{\alpha_3}$, only the variation in parameters $q_2^{\alpha_2}$ can be witnessed with respect to $<\lambda>$ or $<1>$, in the movement of set $(S_2)\cup(S_3)$, which behaves like a single solid. We can therefore say that the parameters $q_2^{\alpha_2}$ are solidifying for the set $(S_2)\cup(S_3)$.

1.3.3. *Movement equations*

Applying the fundamental principle of dynamics to the movements of the three solids, (S_1), (S_2) and (S_3) with respect to $<\lambda>$ results in $6\times 3=18$ equations, consequences of the three torsor relations introduced at the end of section 1.3.1.

Yet, these scalar relations contain link components in their right-hand terms. Moreover, they do not express the energy dimension of the movements, and as a result we cannot directly take advantage of this aspect of the movements studied such as, for example, the energy formulation of perfect links to obtain movement equations for the set (D).

We therefore turn to an analytic expression of this movement, and since we obtain as many analytical mechanics equations as there are situation parameters to express each of the three solids with respect to $< \lambda >$, that is to say, $k_1 + k_2 + k_3$ equations, they should be chosen carefully, given the solidifying nature of a number of these parameters.

1.3.3.1. *First set of equations*

The k_1 analytical mechanics equations relating to the k_1 location parameters $q_1^{\alpha_1}$ of the three solids (S_1), (S_2) and (S_3) result from the following three torsor relations:

$$(S_1): \begin{Bmatrix} \lambda \\ 1.\alpha_1 \end{Bmatrix} \{\mathcal{A}_1^\lambda\} = \cdots$$

$$\cdots \begin{Bmatrix} \lambda \\ 1.\alpha_1 \end{Bmatrix} \otimes [\{\Delta_{\bar{D}\to 1}\} + \{\mathcal{L}_{\lambda\to 1}\} - \{\Delta_{1\to 2}\} - \{\mathcal{L}_{1\to 2}\}]$$

$$(S_2): \begin{Bmatrix} \lambda \\ 2.\alpha_1 \end{Bmatrix} \{\mathcal{A}_2^\lambda\} = \cdots$$

$$\cdots \begin{Bmatrix} \lambda \\ 2.\alpha_1 \end{Bmatrix} \otimes [\{\Delta_{\bar{D}\to 2}\} + \{\Delta_{1\to 2}\} + \{\mathcal{L}_{1\to 2}\} - \{\Delta_{2\to 3}\} - \{\mathcal{L}_{2\to 3}\}]$$

$$(S_3): \begin{Bmatrix} \lambda \\ 3.\alpha_3 \end{Bmatrix} \{\mathcal{A}_3^\lambda\} = \cdots$$

$$\cdots \begin{Bmatrix} \lambda \\ 3.\alpha_1 \end{Bmatrix} \otimes [\{\Delta_{\bar{D}\to 3}\} + \{\Delta_{2\to 3}\} + \{\mathcal{L}_{2\to 3}\}]$$

And as these parameters are solidifying for the set

$(D) = (S_1) \cup (S_2) \cup (S_3)$ with respect to $< \lambda >$, we have

$$\begin{Bmatrix} \lambda \\ 1.\alpha_1 \end{Bmatrix} = \begin{Bmatrix} \lambda \\ 2.\alpha_1 \end{Bmatrix} = \begin{Bmatrix} \lambda \\ 3.\alpha_1 \end{Bmatrix}$$

$$\Rightarrow \begin{Bmatrix} \lambda \\ 1.\alpha_1 \end{Bmatrix} \otimes [\{\mathcal{A}_1^\lambda\} + \{\mathcal{A}_2^\lambda\} + \{\mathcal{A}_3^\lambda\}] = \begin{Bmatrix} \lambda \\ 1.\alpha_1 \end{Bmatrix} \{\mathcal{A}_D^\lambda\} = \cdots$$

$$\ldots \begin{Bmatrix} \lambda \\ 1.\alpha_1 \end{Bmatrix} \otimes [\{\Delta_{\bar{D}\to 1}\} + \{\mathcal{L}_{\lambda\to 1}\} + \{\Delta_{\bar{D}\to 2}\} + \{\Delta_{\bar{D}\to 3}\}]$$

relation that can be expressed in terms of the kinetic energy in the form

$$\left[\frac{d}{dt}\left(\frac{\partial T^\lambda(S_1)}{\partial\left(q_1^{\alpha_1}\right)'}\right) - \frac{\partial T^\lambda(S_1)}{\partial q_1^{\alpha_1}}\right] + \left[\frac{d}{dt}\left(\frac{\partial T^\lambda(S_2)}{\partial\left(q_1^{\alpha_1}\right)'}\right) - \frac{\partial T^\lambda(S_2)}{\partial q_1^{\alpha_1}}\right] \ldots$$

$$\ldots + \left[\frac{d}{dt}\left(\frac{\partial T^\lambda(S_3)}{\partial(q_1^{\alpha_1})\prime}\right) - \frac{\partial T^\lambda(S_3)}{\partial q_1^{\alpha_1}}\right] \ldots$$

$$\ldots = \prod\nolimits_{\alpha_1}^{\lambda}(\Delta_{\bar{D}\to 1}) + \prod\nolimits_{\alpha_1}^{\lambda}(\mathcal{L}_{\lambda\to 1}) + \prod\nolimits_{\alpha_1}^{\lambda}(\Delta_{\bar{D}\to 2}) + \prod\nolimits_{\alpha_1}^{\lambda}(\Delta_{\bar{D}\to 3})$$

As the three solids are disjoint and the kinetic energy is a summation for the set of particles of the field considered, we can also write

$$T^\lambda(D) = T^\lambda(S_1) + T^\lambda(S_2) + T^\lambda(S_3)$$

and for each parameter $q_1^{\alpha_1}$, we obtain one of the k_1 equations.

$$\frac{d}{dt}\left(\frac{\partial T^\lambda(D)}{\partial(q_1^{\alpha_1})\prime}\right) - \frac{\partial T^\lambda(D)}{\partial q_1^{\alpha_1}} = \prod\nolimits_{\alpha_1}^{\lambda}(\bar{D} \to D)$$

where the only link that appears is that of (S_1) with $< \lambda >$. If this link is perfect, that is, if $\prod_{\alpha_1}^{\lambda}(\mathcal{L}_{\lambda\to 1}) = 0$, the k_1 equations above constitute a first set of movement equations.

1.3.3.2. *Second set of equations*

The k_2 analytical mechanics equations relating to the k_2 location parameters $q_2^{\alpha_2}$ of the two solids (S_2) and (S_3) result from the following three torsor relations:

$$(S_1): \begin{Bmatrix} \lambda \\ 1.\alpha_2 \end{Bmatrix} \{\mathcal{A}_1^\lambda\} = \cdots$$

$$\ldots \begin{Bmatrix} \lambda \\ 1.\alpha_2 \end{Bmatrix} \otimes [\{\Delta_{\bar{D}\to 1}\} + \{\mathcal{L}_{\lambda\to 1}\} - \{\Delta_{1\to 2}\} - \{\mathcal{L}_{1\to 2}\}]$$

$$(\mathrm{S}_2)\colon \begin{Bmatrix} \lambda \\ 2.\alpha_2 \end{Bmatrix} \{\mathcal{A}_2^\lambda\} = \cdots$$

$$\ldots \begin{Bmatrix} \lambda \\ 2.\alpha_2 \end{Bmatrix} \otimes [\{\Delta_{\overline{\mathrm{D}}\to 2}\} + \{\Delta_{1\to 2}\} + \{\mathcal{L}_{1\to 2}\} - \{\Delta_{2\to 3}\} - \{\mathcal{L}_{2\to 3}\}]$$

$$(\mathrm{S}_3)\colon \begin{Bmatrix} \lambda \\ 3.\alpha_2 \end{Bmatrix} \{\mathcal{A}_3^\lambda\} = \cdots$$

$$\ldots \begin{Bmatrix} \lambda \\ 3.\alpha_2 \end{Bmatrix} \otimes [\{\Delta_{\overline{\mathrm{D}}\to 3}\} + \{\Delta_{2\to 3}\} + \{\mathcal{L}_{2\to 3}\}]$$

where

$$\begin{Bmatrix} \lambda \\ 1.\alpha_2 \end{Bmatrix} = \{0\} \text{ and } \begin{Bmatrix} \lambda \\ 2.\alpha_2 \end{Bmatrix} = \begin{Bmatrix} \lambda \\ 3.\alpha_2 \end{Bmatrix}$$

making it possible to write:

$$\begin{Bmatrix} \lambda \\ 2.\alpha_2 \end{Bmatrix} \otimes [\{\mathcal{A}_2^\lambda\} + \{\mathcal{A}_3^\lambda\}] = \begin{Bmatrix} \lambda \\ 2.\alpha_2 \end{Bmatrix} \{\mathcal{A}_{2\cup 3}^\lambda\} = \cdots$$

$$\ldots . \begin{Bmatrix} \lambda \\ 2.\alpha_2 \end{Bmatrix} \otimes [\{\Delta_{\overline{D}\to 2}\} + \{\Delta_{1\to 2}\} + \{\mathcal{L}_{1\to 2}\} + \{\Delta_{\overline{D}\to 3}\}]$$

or, in kinetic energy terms:

$$\left[\frac{d}{dt}\left(\frac{\partial T^\lambda(S_2)}{\partial (q_2^{\alpha_2})'}\right) - \frac{\partial T^\lambda(S_2)}{\partial q_2^{\alpha_2}}\right] + \left[\frac{d}{dt}\left(\frac{\partial T^\lambda(S_3)}{\partial (q_2^{\alpha_2})'}\right) - \frac{\partial T^\lambda(S_3)}{\partial q_2^{\alpha_2}}\right] \ldots$$

$$\ldots = \prod\nolimits_{\alpha_2}^{\lambda}(\Delta_{\overline{D}\to 2}) + \prod\nolimits_{\alpha_2}^{\lambda}(\Delta_{\overline{D}\to 3}) + \begin{Bmatrix} \lambda \\ 2.\alpha_2 \end{Bmatrix} \{\Delta_{1\to 2}\} + \begin{Bmatrix} \lambda \\ 2.\alpha_2 \end{Bmatrix} \{\mathcal{L}_{1\to 2}\}$$

By developing the latter relation, we obtain the following results:

1.3.3.2.1. Interpreting left-hand terms

As $T^\lambda(S_1)$ is only a function of the parameters $q_1^{\alpha_1}$, we therefore have

$$\frac{\partial T^{\lambda}(S_1)}{\partial (q_2^{\alpha_2})'} = 0 \text{ and } \frac{\partial T^{\lambda}(S_1)}{\partial q_2^{\alpha_2}} = 0$$

The left-hand term of the equation in question is thus written as

$$\frac{d}{dt}\left(\frac{\partial T^{\lambda}(D)}{\partial (q_2^{\alpha_2})'}\right) - \frac{\partial T^{\lambda}(D)}{\partial q_2^{\alpha_2}}$$

1.3.3.2.2. Interpreting right-hand terms

Consider the terms of the expression

$$\begin{Bmatrix} \lambda \\ 2.\alpha_2 \end{Bmatrix}\{\Delta_{1\to 2}\} + \begin{Bmatrix} \lambda \\ 2.\alpha_2 \end{Bmatrix}\{\mathcal{L}_{1\to 2}\}$$

The efforts expressed by torsors $\{\Delta_{1\to 2}\}$ and $\{\mathcal{L}_{1\to 2}\}$ are only exerted in the relative movements of the solids (S_1) and (S_2). They only put a power in play in these movements.

$$\mathcal{P}^{(1)}(\Delta_{1\to 2}) = \begin{Bmatrix} 1 \\ 2. \end{Bmatrix}\{\Delta_{1\to 2}\}$$
$$= \begin{Bmatrix} 1 \\ 2.\alpha_2 \end{Bmatrix}\{\Delta_{1\to 2}\}\left(q_2^{\alpha_2}\right)' = \prod\nolimits_{\alpha_2}^{1}(\Delta_{1\to 2})\left(q_2^{\alpha_2}\right)'$$

As $\begin{Bmatrix} \lambda \\ 1.\alpha_2 \end{Bmatrix} = \{0\}$ and $\begin{Bmatrix} 1 \\ 2.\alpha_2 \end{Bmatrix} = \begin{Bmatrix} \lambda \\ 2.\alpha_2 \end{Bmatrix}$

we obtain

$$\prod\nolimits_{\alpha_2}^{1}(\Delta_{1\to 2}) = \begin{Bmatrix} \lambda \\ 2.\alpha_2 \end{Bmatrix}\{\Delta_{1\to 2}\} = \prod\nolimits_{\alpha_2}^{\lambda}(\Delta_{1\to 2})$$

This result is only applicable because the chain of solids has neither loops nor branches. We obtain a similar result for the link torsor $\{\mathcal{L}_{1\to 2}\}$.

The second set of equations, relating to the parameters $q_2^{\alpha_2}$, is then written as

$$\left[\frac{d}{dt}\left(\frac{\partial T^{\lambda}(D)}{\partial (q_2^{\alpha_2})'}\right) - \frac{\partial T^{\lambda}(D)}{\partial q_2^{\alpha_2}}\right]\cdots$$

$$\ldots = \prod_{\alpha_2}^{\lambda}(\Delta_{\bar{D}\to 2}) + \prod_{\alpha_2}^{\lambda}(\Delta_{\bar{D}\to 3}) + \prod_{\alpha_2}^{\lambda}(\Delta_{1\to 2}) + \prod_{\alpha_2}^{\lambda}(\mathcal{L}_{1\to 2})$$

We thus have k_2 equations, in which the only link involved is that between (S_1) and (S_2). When this link is perfect, in other words when

$$\prod_{\alpha_2}^{1}(\mathcal{L}_{1\to 2}) = \prod_{\alpha_2}^{\lambda}(\mathcal{L}_{1\to 2}) = 0$$

we then obtain k_2 movement equations.

1.3.3.3. *Third set of equations*

The k_3 analytical mechanics equations relating to the k_3 location parameters $q_3^{\alpha_3}$ of the solid (S_3) result from the following three torsor relations:

$$(S_1): \begin{Bmatrix} \lambda \\ 1.\alpha_3 \end{Bmatrix} \{\mathcal{A}_1^{\lambda}\} = \cdots$$

$$\ldots \begin{Bmatrix} \lambda \\ 1.\alpha_3 \end{Bmatrix} \otimes [\{\Delta_{\bar{D}\to 1}\} + \{\mathcal{L}_{\lambda\to 1}\} - \{\Delta_{1\to 2}\} - \{\mathcal{L}_{1\to 2}\}]$$

$$(S_2): \begin{Bmatrix} \lambda \\ 2.\alpha_3 \end{Bmatrix} \{\mathcal{A}_2^{\lambda}\} = \cdots$$

$$\ldots \begin{Bmatrix} \lambda \\ 2.\alpha_3 \end{Bmatrix} \otimes [\{\Delta_{\bar{D}\to 2}\} + \{\Delta_{1\to 2}\} + \{\mathcal{L}_{1\to 2}\} - \{\Delta_{2\to 3}\} - \{\mathcal{L}_{2\to 3}\}]$$

$$(S_3): \begin{Bmatrix} \lambda \\ 3.\alpha_3 \end{Bmatrix} \{\mathcal{A}_3^{\lambda}\} = \cdots$$

$$\ldots \begin{Bmatrix} \lambda \\ 3.\alpha_2 \end{Bmatrix} \otimes [\{\Delta_{\bar{D}\to 3}\} + \{\Delta_{2\to 3}\} + \{\mathcal{L}_{2\to 3}\}]$$

where:

$$\begin{Bmatrix} \lambda \\ 1.\alpha_3 \end{Bmatrix} = \begin{Bmatrix} \lambda \\ 2.\alpha_3 \end{Bmatrix} = \{0\}$$

making it possible to write:

$$\left[\frac{d}{dt}\left(\frac{\partial T^{\lambda}(S_3)}{\partial (q_3^{\alpha_3})'}\right) - \frac{\partial T^{\lambda}(S_3)}{\partial q_3^{\alpha_3}}\right] \ldots$$

$$\ldots = \prod_{\alpha_3}^{\lambda}(\Delta_{\bar{D}\to 3}) + \begin{Bmatrix} \lambda \\ 3.\alpha_3 \end{Bmatrix}\{\Delta_{2\to 3}\} + \begin{Bmatrix} \lambda \\ 3.\alpha_3 \end{Bmatrix}\{\mathcal{L}_{2\to 3}\}$$

By developing the latter relation, we obtain the following results.

1.3.3.3.1. Interpreting the left-hand term

As $T^{\lambda}(S_1)$ and $T^{\lambda}(S_2)$ are not functions of the parameters $q_3^{\alpha_3}$, we can write

$$\left[\frac{d}{dt}\left(\frac{\partial T^{\lambda}(S_3)}{\partial\left(q_3^{\alpha_3}\right)'}\right) - \frac{\partial T^{\lambda}(S_3)}{\partial q_3^{\alpha_3}}\right] = \left[\frac{d}{dt}\left(\frac{\partial T^{\lambda}(D)}{\partial\left(q_3^{\alpha_3}\right)'}\right) - \frac{\partial T^{\lambda}(D)}{\partial q_3^{\alpha_3}}\right]$$

1.3.3.3.2. Interpreting right-hand terms

Consider the terms of the expression

$$\begin{Bmatrix} \lambda \\ 3.\alpha_3 \end{Bmatrix}\{\Delta_{2\to 3}\} + \begin{Bmatrix} \lambda \\ 3.\alpha_3 \end{Bmatrix}\{\mathcal{L}_{2\to 3}\}$$

The efforts expressed by torsors $\{\Delta_{2\to 3}\}$ and $\{\mathcal{L}_{2\to 3}\}$ are only exerted in the relative movements of solids (S_2) and (S_3). They only put in play a power in these movements.

$$\mathcal{P}^{(2)}(\Delta_{2\to 3}) = \begin{Bmatrix} 2 \\ 3 \end{Bmatrix}\{\Delta_{2\to 3}\} = \begin{Bmatrix} 2 \\ 3.\alpha_3 \end{Bmatrix}\{\Delta_{2\to 3}\}\left(q_3^{\alpha_3}\right)'$$

As $\begin{Bmatrix} 1 \\ 2.\alpha_3 \end{Bmatrix} = \{0\}$ and $\begin{Bmatrix} \lambda \\ 1.\alpha_3 \end{Bmatrix} = \{0\}$

$$\begin{Bmatrix} \lambda \\ 3.\alpha_3 \end{Bmatrix}\{\Delta_{2\to 3}\} = \begin{Bmatrix} 2 \\ 3.\alpha_3 \end{Bmatrix}\{\Delta_{2\to 3}\} = \prod_{\alpha_3}^{2}(\Delta_{2\to 3}) = \prod_{\alpha_3}^{\lambda}(\Delta_{2\to 3})$$

We obtain

$$\begin{Bmatrix} \lambda \\ 3.\alpha_3 \end{Bmatrix}\{\Delta_{2\to 3}\} = \begin{Bmatrix} 2 \\ 3.\alpha_3 \end{Bmatrix}\{\Delta_{2\to 3}\} = \prod_{\alpha_3}^{2}(\Delta_{2\to 3}) = \prod_{\alpha_3}^{\lambda}(\Delta_{2\to 3})$$

The term $\left\{\begin{matrix}\lambda \\ 3.\alpha_3\end{matrix}\right\}\{\Delta_{2\to3}\}$ is a power coefficient that has its place in an analytical mechanics equation, as well as $\left\{\begin{matrix}\lambda \\ 3.\alpha_3\end{matrix}\right\}\{\mathcal{L}_{2\to3}\}$.

We then obtain k_3 movement equations of the form

$$\left[\frac{d}{dt}\left(\frac{\partial T^{\lambda}(D)}{\partial\left(q_3^{\alpha_3}\right)'}\right)-\frac{\partial T^{\lambda}(D)}{\partial q_3^{\alpha_3}}\right]=\prod\nolimits_{\alpha_3}^{\lambda}(\overline{S_3}\to S_3)$$

with the only link terms that they contain being those relating to $\{\mathcal{L}_{2\to3}\}$. If the latter is perfect, that is if $\prod_{\alpha_3}^{2}(\mathcal{L}_{2\to3})=0$, these k_3 relations are counted among the movement equations.

1.3.4. *Determining the link unknowns*

In the case of all-perfect links, the $k_1+k_2+k_3$ analytical mechanics equations form $k_1+k_2+k_3$ movement equations to determine, as a function of time, the $(k_1+k_2+k_3)$ parameters $q_1^{\alpha_1}, q_2^{\alpha_2}, q_3^{\alpha_3}$.

The 18-$(k_1+k_2+k_3)$ equations provided by the direct application of the fundamental principle of dynamics associated with perfect link conditions enable the link unknowns to be determined.

1.4. Links between solids

Consider two solids (S_1), with surface contour Σ_1, to which the frame $<1>$ is attached, and (S_2), with surface contour Σ_2, to which the frame $<2>$ is attached.

1.4.1. *Link associated with the point contact of two solids*

The solid (S_2) is in intermittent contact with the single solid (S_1) at a point *M* of its surface contour Σ_2.

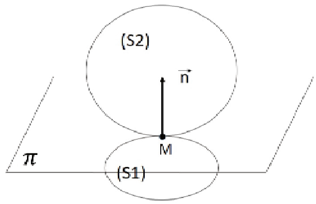

Figure 1.4. *Point contact of two solids*

$\vec{n}$ is the unitary vector normal to the instantaneous common plane Π at M tangent to the respective surface contours Σ_1 and Σ_2 of the two solids (S_1) and (S_2), directed from (S_1) to (S_2). The velocity distributing torsor of (S_2) with respect to (S_1) has the following reduction elements at M:

$$\left[\begin{array}{l} \overrightarrow{\omega_2^1} \\ \overrightarrow{\mathcal{M}_M}\begin{Bmatrix}1\\2\end{Bmatrix} \end{array}\right.$$

involving, by principle, six parameters.

During the relative displacement of the two solids onto one another, the point of contact is mobile on each of them. Under these conditions, if $\overrightarrow{\mathcal{M}_M}\begin{Bmatrix}1\\2\end{Bmatrix}$ has the dimensions of a velocity, this term only expresses the instantaneous speed with respect to (S_1) of the point of (S_2), which coincides, at this instant, with the point of contact; this vector therefore does not obey the velocity composition law applicable to a solid.

In the case where contact at M is persistent, which amounts to saying that at the moment considered, the component of $\overrightarrow{\mathcal{M}_M}\begin{Bmatrix}1\\2\end{Bmatrix}$ following the instantaneous normal $\vec{n}$ is null, the additional link relation provided by this condition is

$$\vec{n}.\overrightarrow{\mathcal{M}_M}\begin{Bmatrix}1\\2\end{Bmatrix} = 0$$

and reduces the number of degrees of freedom of (S_2).

The reduction elements at M of the link torsor $\{\mathcal{L}\}$ associated with the contact actions exerted by (S_1) on (S_2) are

$$\{\mathcal{L}\} = [\vec{s} \,\vdots \overrightarrow{M_Q}]$$

and introduce six scalar unknowns.

Let us examine the question of point contact in the specific case where the frame $< 1 >$ is Galilean. The torsor equation

$$\{\mathcal{A}_2^1\} = \{\Delta\} + \{\mathcal{L}\}$$

that results from the application of the fundamental principle of dynamics to the movement of (S_2) with respect to (S_1) gives six scalar equations.

With a scalar equation characterizing the link, we therefore have seven equations for 12 unknowns, hence a deficit of five scalar equations.

Moreover, if the link torsor has a null moment at M, a condition that results in three additional scalar equations, the deficit is then just two scalar equations.

Let us introduce a new condition to attempt to solve the problem, expressed by the laws of dry friction.

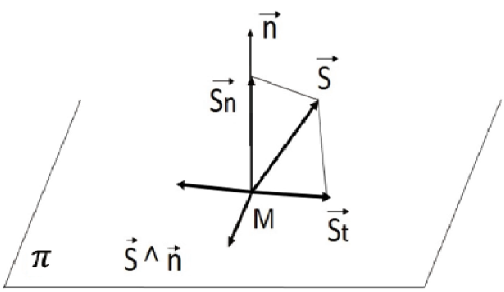

Figure 1.5. *Point contact at M*

The resultant $\vec{s}$ of the link efforts is projected in the form

$$\vec{s} = (\vec{n}.\vec{s})\vec{n} + \vec{n}\wedge(\vec{s}\wedge\vec{n})$$

where:

– $\vec{s}_n = (\vec{n}.\vec{s})\vec{n}$ defines the normal contact action;

– $\vec{s}_t = \vec{n}\wedge(\vec{s}\wedge\vec{n})$ defines the tangential contact action.

The power developed by the link efforts is given by:

$$\mathcal{P}_{\mathcal{L}}^{1} = \begin{Bmatrix}1\\2\end{Bmatrix}\otimes\{\mathcal{L}\} = \vec{s}.\overrightarrow{\mathcal{M}_M}\begin{Bmatrix}1\\2\end{Bmatrix}$$

$$\mathcal{P}_{\mathcal{L}}^{1} = [(\vec{n}.\vec{s})\vec{n} + \vec{n}\wedge(\vec{s}\wedge\vec{n})].\overrightarrow{\mathcal{M}_M}\begin{Bmatrix}1\\2\end{Bmatrix}$$

where $\vec{n}.\overrightarrow{\mathcal{M}_M}\begin{Bmatrix}1\\2\end{Bmatrix} = 0$ as we consider the contact to be persistent given that there is no displacement of point *M* along the normal, hence:

$$\mathcal{P}_{\mathcal{L}}^{1} = [\vec{n}\wedge(\vec{s}\wedge\vec{n})].\overrightarrow{\mathcal{M}_M}\begin{Bmatrix}1\\2\end{Bmatrix} = \vec{s_t}.\overrightarrow{\mathcal{M}_M}\begin{Bmatrix}1\\2\end{Bmatrix}$$

The configuration of this type of point contact can present several aspects, which are explained below.

1.4.1.1. *Configuration 1*

When the solid (S_2) is not bonded to (S_1), the relation

$$\vec{n}.\vec{s} > 0$$

indicates that, with the two vectors forming an acute angle, the normal contact action exerted by (S_1) on (S_2) is directed from (S_1) toward (S_2).

Cancellation of the normal contact action leads this contact at *M* to be interrupted by unbonding.

1.4.1.2. *Configuration 2*

When the sliding velocity of (S_2) on (S_1) at M is null, that is,

$$\overrightarrow{\mathcal{M}_M}\begin{Bmatrix}1\\2\end{Bmatrix} = \vec{0}$$

the tangential component of the link efforts is small compared to the normal component, which is reflected by the relation

$$\sqrt{\vec{s}_t^{\,2}} < f\sqrt{\vec{s}_n^{\,2}}$$

where:

$-f$ is the mutual-friction coefficient of (S_1) and (S_2), which depends on the nature of the materials, their surface condition and their temperature.

$-\sqrt{\vec{s}_n^{\,2}} = |\vec{s}.\vec{n}| = \vec{s}.\vec{n}$

$-\sqrt{\vec{s}_t^{\,2}} = \sqrt{[\vec{n}\Lambda(\vec{s}\Lambda\vec{n})]^2} = \sqrt{(\vec{s}\Lambda\vec{n})^2}$

$-\vec{s}.\vec{n} < f\sqrt{(\vec{s}\Lambda\vec{n})^2}$

If we examine the summary of equations available to address the question, introducing the relation $\overrightarrow{\mathcal{M}_M}\begin{Bmatrix}1\\2\end{Bmatrix} = \vec{0}$ provides the two missing scalar equations to reduce the deficit to zero.

As $\mathcal{P}_{\mathcal{L}}^1 = \vec{s}_t.\overrightarrow{\mathcal{M}_M}\begin{Bmatrix}1\\2\end{Bmatrix}$, the power developed with respect to the Galilean frame $< 1 >$ by the link associated with the contact actions that (S_1) exerts on (S_2) is null. This relation makes it possible to determine the contact action $\vec{s}$. This then involves verifying the two inequalities

$$\begin{bmatrix} \vec{n}.\vec{s} > 0 \\ \vec{n}.\vec{s} < f\sqrt{(\vec{s}\Lambda\vec{n})^2} \end{bmatrix}$$

and when the next equality is verified

$$\vec{n}.\vec{s} = f\sqrt{(\vec{s}\wedge\vec{n})^2}$$

There is a possibility of sliding at *M*.

1.4.1.3. *Configuration 3*

Let us now consider the case where

$$\overrightarrow{\mathcal{M}_M}\begin{Bmatrix}1\\2\end{Bmatrix} \neq \vec{0}$$

a condition that leads to the three relations

$$\left[\begin{array}{l} (1)\ \sqrt{\vec{s}_t^{\,2}} = f\sqrt{\vec{s}_n^{\,2}} \\ (2)\ \ \vec{s}_t\wedge\overrightarrow{\mathcal{M}_M}\begin{Bmatrix}1\\2\end{Bmatrix} = \vec{0} \\ (3)\overrightarrow{s}_t.\overrightarrow{\mathcal{M}_M}\begin{Bmatrix}1\\2\end{Bmatrix} < 0 \end{array}\right.$$

According to relation (2) above, the tangential contact action and the sliding velocity at *M* of (S_2) on (S_1) are colinear. This relation is written as

$$\overrightarrow{\mathcal{M}_M}\begin{Bmatrix}1\\2\end{Bmatrix}\wedge[\vec{n}\wedge(\vec{s}\wedge\vec{n})] = \vec{0}$$

$$\left[\overrightarrow{\mathcal{M}_M}\begin{Bmatrix}1\\2\end{Bmatrix}.(\vec{s}\wedge\vec{n})\right]\vec{n} - \left[\overrightarrow{\mathcal{M}_M}\begin{Bmatrix}1\\2\end{Bmatrix}.\vec{n}\right](\vec{s}\wedge\vec{n}) = \vec{0}$$

As we have the condition $\overrightarrow{\mathcal{M}_M}\begin{Bmatrix}1\\2\end{Bmatrix}.\vec{n} = 0$, we deduce from the preceding relation that

$$\overrightarrow{\mathcal{M}_M}\begin{Bmatrix}1\\2\end{Bmatrix}.(\vec{s}\wedge\vec{n}) = 0$$

According to relation (3), the tangential contact action and the sliding velocity at *M* of (S_2) on (S_1) are opposite. This relation is written as

$$\left[\vec{n}\Lambda(\vec{s}\Lambda\vec{n}).\overrightarrow{\mathcal{M}_M}\begin{Bmatrix}1\\2\end{Bmatrix}\right] < 0$$

$$\Rightarrow [\vec{s} - (\vec{n}.\vec{s})\vec{n}].\overrightarrow{\mathcal{M}_M}\begin{Bmatrix}1\\2\end{Bmatrix} < 0$$

With the same condition as before, we deduce that

$$\vec{s}.\overrightarrow{\mathcal{M}_M}\begin{Bmatrix}1\\2\end{Bmatrix} < 0$$

That is to say that the power developed with respect to the Galilean frame $< 1 >$ by the link torsor associated with the contact actions at M of (S_1) and (S_2) is negative.

Thus, when we have $\overrightarrow{\mathcal{M}_M}\begin{Bmatrix}1\\2\end{Bmatrix} \neq 0$, we have the three relations

$$\begin{bmatrix} \vec{s}.\vec{n} < f\ (\vec{s}\Lambda\vec{n})^2 \\ (\vec{s}\Lambda\vec{n}).\overrightarrow{\mathcal{M}_M}\begin{Bmatrix}1\\2\end{Bmatrix} = 0 \\ \vec{s}.\overrightarrow{\mathcal{M}_M}\begin{Bmatrix}1\\2\end{Bmatrix} < 0 \end{bmatrix}$$

The first two provide the two missing scalar equations to reduce the equation deficit to zero. Having then determined the contact action $\vec{s}$, it remains for the inequality $\vec{n}.\vec{s} > 0$ to be verified.

1.4.1.4. *Perfect links*

In the case of a perfect, frictionless link (f = 0), the tangential component of the contact action is null. This action is then expressed by

$$\vec{s} = s\vec{n} \quad \text{where } s > 0$$

The value 0 of the two components of the tangential contact action then provides the two equations enabling the deficit to be reduced to zero. Moreover, the power developed by the link is null.

1.4.2. *Link torsor associated with the line contact of two solids*

1.4.2.1. *Arbitrary line contact*

The contact between the two solids occurs along the curve's arc (C).

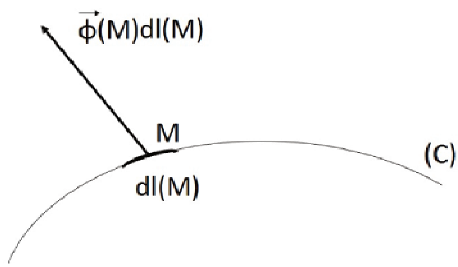

Figure 1.6. *Arbitrary line contact*

Associated with point M of the contact curve (C), whose vicinity on this curve (Figure 1.6) is defined by the elementary arc $d\ell(M)$, is the elementary force

$$\vec{\phi}(M)d\ell(M)$$

where $\vec{\phi}(M)$ is a linear force density.

The link torsor $\{\mathcal{L}\}$ between the two solids in contact has the following reduction elements at Q:

$$\left[\begin{array}{l} \vec{s}\{\mathcal{L}\} = \displaystyle\int_{M\in(C)} \vec{\phi}(M)d\ell(M) \\ \quad \overrightarrow{\mathcal{M}_Q}\{\mathcal{L}\} = \displaystyle\int_{M\in(C)} \overrightarrow{QM}\wedge\vec{\phi}(M)d\ell(M) \end{array}\right.$$

Six scalar components of the link torsor are, generally speaking, unknown.

1.4.2.2. *Straight line contact*

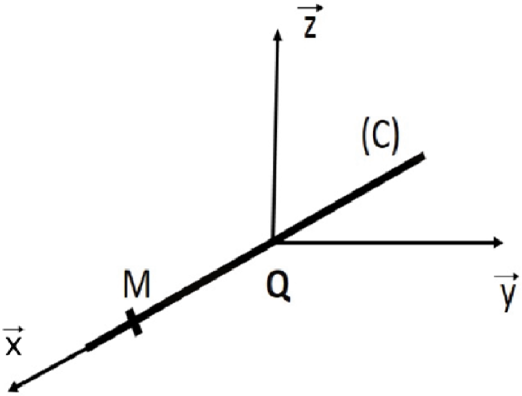

Figure 1.7. *Straight line contact*

The segment (C) of the line contact belongs to the common tangent plane $(Q|\vec{x}.\vec{y})$. The normal penetration velocity is null at any point on the contact segment. As a result, for the two points, M and Q, belonging to the contact segment, we can write

$$\vec{z}.\overrightarrow{\mathcal{M}_M}\begin{Bmatrix}1\\2\end{Bmatrix} = 0 \quad \Rightarrow \quad \vec{z}.\left[\overrightarrow{v^{(1)}}(O_2) + \overrightarrow{\omega_2^1}\Lambda\overrightarrow{O_2M}\right] = 0$$

$$\vec{z}.\overrightarrow{\mathcal{M}_Q}\begin{Bmatrix}1\\2\end{Bmatrix} = 0 \quad \Rightarrow \quad \vec{z}.\left[\overrightarrow{v^{(1)}}(O_2) + \overrightarrow{\omega_2^1}\Lambda\overrightarrow{O_2Q}\right] = 0$$

From the term-by-term difference for the last two expressions, we obtain

$$\vec{z}.\left(\overrightarrow{\omega_2^1}\Lambda\overrightarrow{QM}\right) = 0$$

By writing $\overrightarrow{QM} = x\vec{x}$, the relation

$$\vec{z}.\left(\overrightarrow{\omega_2^1}\Lambda x\vec{x}\right) = 0$$

is satisfied regardless of the point *M* located on (C) if

$$\vec{z}.\left(\overrightarrow{\omega_2^1}\wedge\vec{x}\right) = 0 \quad \Rightarrow \quad (\vec{x}\wedge\vec{z}).\overrightarrow{\omega_2^1} = 0 \quad \Rightarrow \quad \vec{y}\,.\overrightarrow{\omega_2^1} = 0$$

That is to say, the rotation rate vector of the solid (S_2), with respect to (S_1), is collinear to the vector $\vec{x}$ support of the contact line.

In the case of persistent straight line contact, the two relations

$$\vec{z}.\overrightarrow{\mathcal{M}_Q}\begin{Bmatrix}1\\2\end{Bmatrix} = 0$$

$$\vec{y}\,.\overrightarrow{\omega_2^1} = 0$$

accord four degrees of freedom to the solid (S_2).

The reduction elements at *Q* of the link torsor are given as

$$\left[\begin{array}{l} \vec{s}\{\mathcal{L}\} = \displaystyle\int_{M\in(C)} \vec{\phi}(x)dx \\ \overrightarrow{\mathcal{M}_Q}\{\mathcal{L}\} = \displaystyle\int_{M\in(C)} x\vec{x}\wedge\vec{\phi}(x)dx = \vec{x}\wedge\int_{M\in(C)} \vec{\phi}(x)xdx \end{array}\right.$$

As, according to the relation above, $\vec{x}.\overrightarrow{\mathcal{M}_Q}\{\mathcal{L}\} = 0$, the link torsor involves only five unknowns, its reduction elements at *Q* can be written as

$$\vec{s}\{\mathcal{L}\} = X\vec{x} + Y\vec{y} + Z\vec{z}$$

$$\overrightarrow{\mathcal{M}_Q}\{\mathcal{L}\} = M\vec{y} + N\vec{z}$$

With the two link equations and $L = 0$, we have a system of nine scalar equations for 12 unknowns, that is, a deficit of three equations.

In the case of a perfect link, we have:

$$\vec{\phi}(M) = \phi(x)\,\vec{z}$$

$$\vec{s}\{\mathcal{L}\} = \vec{z}\int_{M\in(C)} \phi(x)dx$$

$$\overrightarrow{\mathcal{M}_Q}\{\mathcal{L}\} = -\vec{y}\int_{M\in(C)} \phi(x)xdx$$

The link torsor involves only two unknowns and its reduction elements at Q are

$$\vec{s}\{\mathcal{L}\} = Z\vec{z}$$

$$\overrightarrow{\mathcal{M}_Q}\{\mathcal{L}\} = M\vec{y}$$

With the three additional equations

$$X = 0; \quad Y = 0; \quad N = 0$$

the deficit is reduced to zero.

1.4.3. *Link torsor associated with the surface contact of two solids*

1.4.3.1. *Arbitrary surface contact*

The contact is made along the surface (Σ)

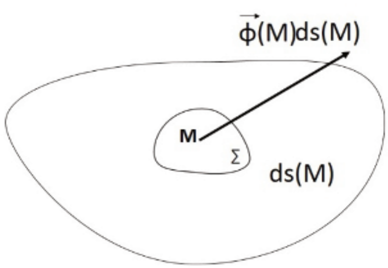

Figure 1.8. *Arbitrary surface contact*

Point M, belonging to the contact surface (Σ), is characterized on this surface by an elementary surface area $ds(M)$, which specifies its vicinity. At this point, we associate an elementary force with the efforts due to the contact action between the two solids

$$\vec{\phi}(M)ds(M)$$

where $\vec{\phi}(M)$ is a surface force density.

The reduction elements at Q of the link torsor are expressed by

$$\vec{s}\{\mathcal{L}\} = \int_{M\in(C)} \vec{\phi}(M)ds(M)$$

$$\overrightarrow{\mathcal{M}_Q}\{\mathcal{L}\} = \int_{M\in(C)} \overrightarrow{QM}\wedge\vec{\phi}(M)ds(M)$$

and their components are generally unknown.

1.4.3.2. *Plane surface contact*

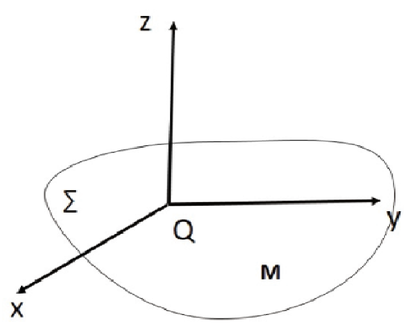

Figure 1.9. *Plane surface contact*

The plane surface (Σ) belongs to the tangent plane common to the two solids in contact.

With the normal penetration velocity being null at any point in the contact area, for the two points, M and Q, we have the two relations:

$$\vec{z}.\overrightarrow{\mathcal{M}_M}\begin{Bmatrix}1\\2\end{Bmatrix} = 0$$

$$\vec{z}.\overrightarrow{\mathcal{M}_Q}\begin{Bmatrix}1\\2\end{Bmatrix} = 0$$

which are summarized in a single equation:

$$\vec{z}.\left(\overrightarrow{\omega_2^1}\wedge\overrightarrow{QM}\right) = 0$$

By writing $\overrightarrow{QM} = x\vec{x} + y\vec{y}$, we obtain, successively

$$\vec{z}.\left(\overrightarrow{\omega_2^1}\wedge(x\vec{x} + y\vec{y})\right) = 0$$

$$\overrightarrow{\omega_2^1}.\left[(x\vec{x} + y\vec{y})\wedge\vec{z}\right] = 0$$

$$\overrightarrow{\omega_2^1}.(y\vec{x} - x\vec{y}) = 0$$

and the latter relation is satisfied, whatever the point *M* belonging to (Σ), if we simultaneously have

$$\begin{cases}\vec{x}.\overrightarrow{\omega_2^1} = 0\\ \vec{y}.\overrightarrow{\omega_2^1} = 0\end{cases}$$

In other words, the rolling rate vector of (S_2) on (S_1) is null.

In the case of persistent plane surface contact, the three relations

$$\vec{z}.\overrightarrow{\mathcal{M}_Q}\begin{Bmatrix}1\\2\end{Bmatrix} = 0$$

$$\vec{x}.\overrightarrow{\omega_2^1} = 0$$

$$\vec{y}.\overrightarrow{\omega_2^1} = 0$$

result in three degrees of freedom for the movement of (S_2).

The reduction elements of the link torsor at Q are

$$\vec{s}\{\mathcal{L}\} = \int_{M\in(\Sigma)} \vec{\phi}(x,y)dxdy$$

$$\overrightarrow{\mathcal{M}_Q}\{\mathcal{L}\} = \int_{M\in(\Sigma)} (x\vec{x} + y\vec{y})\Lambda\vec{\phi}(x,y)dxdy$$

Six components of this torsor are unknown.

With the three link equations, we attain a system of nine scalar equations for 12 unknowns, and consequently a deficit of three equations.

In the case where the link is perfect, we have:

$$\vec{\phi}(x,y) = \phi(x,y)\,\vec{z}$$

$$\vec{s}\{\mathcal{L}\} = \vec{z}\int_{M\in(\Sigma)} \phi(x,y)dx\,dy$$

$$\overrightarrow{\mathcal{M}_Q}\{\mathcal{L}\} = \vec{x}\int_{M\in(\Sigma)} \phi(x,y)ydx\,dy - \vec{y}\int_{M\in(\Sigma)} \phi(x,y)xdx\,dy$$

The link torsor involves only three scalar unknowns and its reduction elements at Q lead to:

$$\vec{s}\{\mathcal{L}\} = Z\vec{z}$$

$$\overrightarrow{\mathcal{M}_Q}\{\mathcal{L}\} = L\vec{x} + M\vec{y}$$

With the three additional equations

$$X = 0; \quad Y = 0; \quad N = 0$$

the deficit is reduced to zero.

1.4.4. *Fundamental links between two solids in contact*

The reduction elements at Q of the link torsor $\{\mathcal{L}\}$ associated with the actions of (S_1) on (S_2) have the following general expressions:

$$\vec{s}\{\mathcal{L}\} = X\vec{x} + Y\vec{y} + Z\vec{z}$$

$$\overrightarrow{\mathcal{M}_Q}\{\mathcal{L}\} = L\vec{x} + M\vec{y} + N\vec{z}$$

1.4.4.1. *Pivot linkage with axis* $(\boldsymbol{Q}|\vec{\boldsymbol{z}})$

This link only allows one degree of freedom. The velocity distributor

$$\begin{Bmatrix}1\\2\end{Bmatrix}_Q = [\gamma'\vec{z} \vdots \vec{0}]$$

is accompanied by the partial distributor

$$\begin{Bmatrix}1\\2.\gamma\end{Bmatrix}_Q = [\vec{z} \vdots \vec{0}]$$

If the link is perfect, we have

$$N = 0$$

and the link torsor displays five unknowns.

1.4.4.2. *Sliding linkage with axis* $(\boldsymbol{Q}|\vec{\boldsymbol{z}})$

This link only allows one degree of freedom. The velocity distributor

$$\begin{Bmatrix}1\\2\end{Bmatrix} = [\vec{0} \vdots z'\vec{z}]$$

is accompanied by the partial distributor

$$\begin{Bmatrix}1\\2.z\end{Bmatrix}_Q = [\vec{0} \vdots \vec{z}]$$

If the link is perfect, we have

$$Z = 0$$

and the link torsor displays five unknowns.

1.4.4.3. *Bolt linkage (sliding-pivot) with axis* $(\boldsymbol{Q}\,|\,\vec{\boldsymbol{z}})$

This link only allows two degrees of freedom. The velocity distributor

$$\begin{Bmatrix} 1 \\ 2 \end{Bmatrix}_Q = [\gamma'\vec{z} \,\vdots\, z'\vec{z}]$$

is accompanied by the two partial distributors

$$\begin{Bmatrix} 1 \\ 2.\gamma \end{Bmatrix}_Q = [\vec{z} \,\vdots\, \vec{0}] \quad ; \quad \begin{Bmatrix} 1 \\ 2.z \end{Bmatrix} = [\vec{0} \,\vdots\, \vec{z}]$$

If the link is perfect, we have

$$N = 0 \quad \text{and} \quad Z = 0$$

and the link torsor then displays four unknowns.

1.4.4.4. *Helical linkage with axis* $(\boldsymbol{Q}|\vec{\boldsymbol{z}})$

This link only allows one degree of freedom. The velocity distributor

$$\begin{Bmatrix} 1 \\ 2 \end{Bmatrix}_Q = \left[\gamma'\vec{z} \,\vdots\, \frac{p}{2\pi}\gamma'\vec{z}\right]$$

with p the helix pitch only admits a single partial distributor

$$\begin{Bmatrix} 1 \\ 2.\gamma \end{Bmatrix}_Q = \left[\vec{z} \,\vdots\, \frac{p}{2\pi}\vec{z}\right] \quad ;$$

If the link is perfect, we have

$$N + \frac{p}{2\pi} Z = 0$$

and the link torsor depends on five unknowns.

1.4.4.5. *Ball-joint linkage of center Q*

This link only allows three degrees of freedom. The velocity distributor

$$\begin{Bmatrix}1\\2\end{Bmatrix}_Q = \left[\psi'\vec{z} + \theta'\vec{n} + \varphi'\vec{Z} : \vec{0}\right]$$

admits the three partial distributors

$$\begin{Bmatrix}1\\2.\psi\end{Bmatrix}_Q = [\vec{z} : \vec{0}]; \quad \begin{Bmatrix}1\\2.\theta\end{Bmatrix}_Q = [\vec{n} : \vec{0}]; \quad \begin{Bmatrix}1\\2.\varphi\end{Bmatrix}_Q = [\vec{Z} : \vec{0}]$$

If the link is perfect, we have

$$L = M = N = 0$$

and the link torsor depends on three unknowns.

1.4.4.6. *Fixed-plane linkage with the plane* $\Pi(Q\,|\,\vec{x}.\vec{y})$

This link only allows three degrees of freedom. The velocity distributor

$$\begin{Bmatrix}1\\2\end{Bmatrix}_Q = [\gamma'\vec{z} + \vdots\ x'\vec{x} + y'\vec{y}]$$

is accompanied by the three partial distributors

$$\begin{Bmatrix}1\\2.\gamma\end{Bmatrix}_Q = [\vec{z} : \vec{0}]\,; \quad \begin{Bmatrix}1\\2.x\end{Bmatrix}_Q = [\vec{O} : \vec{x}]; \quad \begin{Bmatrix}1\\2.y\end{Bmatrix}_Q = [\vec{0} : \vec{y}]$$

If the link is perfect, we have

$$N = 0 \qquad \text{and} \qquad X = Y = 0$$

and the link torsor then depends on three unknowns.