Chapter 1

Finite Elements and Shape Functions

There is a wide range of existing literature on finite elements, both on theoretical aspects (for example [Oden, Reddy-1977], [Ciarlet-1978], [Hughes-1987], [Ciarlet-1991]) and on practical aspects ([Zienkiewicz, Taylor-1989], [Bathe-1996], [Dhatt *et al.* 2007]). The purpose of this chapter, therefore, is not to provide *yet* another description of this method, but rather to introduce a point of view that is strongly oriented toward the underlying geometric aspects. Indeed, classically, these are all the aspects of approximations of functions (polynomial space or others, convergence, convergence rate, etc.) that are examined. We will, thus, only review basic definitions related to finite elements[1], as well as their shape functions. The classic case of finite elements whose degrees of freedom are nodal values of the considered functions (in other words, like Lagrange elements) is described for complete elements, reduced elements as well as rational elements. The less common finite elements such as Hermite elements, for example, where nodal or other derivatives are involved are not explicitly considered[2].

1.1. Basic concepts

The finite elements method allows us to calculate an approximate solution to a problem formulated in terms of a system of partial derivatives over a continuum Ω across two related approximations: a spatial approximation and an approximation for calculated solutions. The physical problem under study is modeled by a system of partial derivatives equations that constitutes a continuous problem with its operators, parameters, data and boundary conditions. The finite element method consists of searching for solutions in a particular space of functions (a Sobolev space) that is built on a discretization or a mesh of the domain. The continuous formulation is replaced by a weak formulation via the Galerkin or Ritz methods, which generally leads to

1. In order to specify the manner in which we study them and in order to establish the notations used.

2. These enriched elements may, nonetheless, be interpreted from the geometric point of view as Lagrange elements of a certain degree.

a matrix system (leading to a linear system or a problem of eigenvalues) of finite dimension. Thus, the first approximation, spatial approximation, is related to the fact that the continuous domain Ω is replaced by a discrete domain, denoted by Ω_h, which is composed of the union of simple geometric elements (triangle, quadrilateral, tetrahedron, etc.) denoted by K. We have $\Omega_h = \cup_{K \in \mathcal{T}_h} K$, where $\mathcal{T}_h$ designates the mesh corresponding to a parameter of size h which, therefore, refers to the size of the elements K and, therefore, to the finesse of the mesh. Other than the case where the domain Ω is polygonal (polyhedral), $\Omega \neq \Omega_h$ for any h. The second approximation is related to the construction of the space of solution functions based on the first spatial approximation, in other words, on the mesh. In the case of Lagrange finite elements, the restriction of this space to each element of the mesh is a linear combination of polynomials (or rational fractions), with each of these being a Lagrange interpolant of nodal values (where the solutions are, therefore, only one or several instantiations of these nodal values, also known as *degrees of freedom*). Thus, each element is associated with a list of nodes (comprising its vertices) which makes it possible to define the Lagrange interpolant. A first-order polynomial interpolant, in particular, is defined based on three nodes for a triangular element, where these nodes are the vertices of the triangle. Generally speaking, one node may support several degrees of freedom (a value, a derivative, etc.). As indicated, the solution function is known based on its value for degrees of freedom for the entire mesh and, thus, only at the nodes of the mesh[3]. If the functional solution space is known, the values at each point are only an evaluation of the Lagrange interpolants at these points. In summary, a finite element is characterized by the triplet $\{K, \Sigma_K, P_K\}$, where:

– K denotes the geometric element (triangle, etc.);

– Σ_K denotes the list of nodes of K;

– P_K denotes the space of chosen functions, here the polynomial Lagrange interpolants of Σ_K.

The geometry of an element K as well as the polynomial interpolants are determined based on the nodes of Σ_K. Notably, the degree of these interpolants is directly related to the number of nodes of Σ_K. In order to be able to define this geometry and these interpolants, we consider a space of reference (or reference space)[4], in which the reference element denoted by $\hat{K}$ is defined, with a fixed (uniform) distribution of nodes, said to be nodes of reference. The real or physical element of K is the image of $\hat{K}$ by an application, denoted by F_K, mapping the reference nodes on to the physical nodes. Thus, to sweep the points M of K, we sweep the points $\hat{M}$ of $\hat{K}$ and we have $M = F_K(\hat{M})$. More precisely, by designating the nodes of the element of reference[5] $\hat{K}$ by $\hat{A}_i$, $1 \leq i \leq n$, and the Lagrange interpolants of these nodes by p_i we have:

$$M = \sum_{i=1}^{n} p_i(\hat{M}) A_i, \qquad [1.1]$$

3. A problem is thus obtained, where the size is the number of degrees of freedom at all the nodes in the mesh. The continuous problem has therefore been replaced by a discrete problem

4. This can be regarded as a space of parameters and we thus interpret K as a geometric patch, defined over the space of parameters and the application F_K relative to the space P_K.

5. Or the nodal sequence.

where $A_i = F_K(\hat{A}_i)$ represents the nodes of K. The Lagrange interpolants p_i verify the following two fundamental properties:

$$p_i(\hat{A}_j) = \delta_{ij} \quad \text{and} \quad \sum_{i=1}^{n} p_i = 1,$$

where $\delta_{ii} = 1$ and $\delta_{ij} = 0$ for $j \neq i$. The first property characterizes the Lagrange interpolants while the second represents the partition of unity over each element. Thus, the restriction of the solution s_K of P_K to the element K is written as:

$$s_K(M) = \sum_{i=1}^{n} p_i(\hat{M}) s_K(A_i), \tag{1.2}$$

where $s_K(A_i)$ represents the nodal values for the solution restricted at K.

This definition of the physical or real elements via the space of reference (space $\hat{K}$) is *fundamental* for three reasons: the geometric definition of the elements, their geometric validity and the definition of the interpolation space.

The geometric definition of these elements is based on their nature and the distribution of their nodes and will be the focus of Chapter 2. Let us mention here that the position of any point M of an element K is indirectly defined via the space of reference; indeed, it is difficult to know whether a given point is in a given element[6] without making use of the space of reference. Even when using this, the localization of the point, that is locating its antecedent $\hat{M}$, would require the resolution of the nonlinear equations $M = F_K(\hat{M})$ in the general case. This is resolved using a combination of dichotomies and Newton methods. The geometric validity of a physical or real element can be established relatively simply only by analyzing the application F_K, especially the sign of its Jacobian, $\mathcal{J}(F_K)$, as we will see in the following chapters. This problem, which already exists for classic first-order elements, is all the more present in the cases of higher order elements, especially for isoparametric elements (with curved boundaries of the same degree as the interpolants). These are often used to reliably approach curved geometry. Moreover, if dK (respectively, $d\hat{K}$) represents an element of surface integration or volume integration in the physical or real space (respectively, space of reference), we have $dK = \mathcal{J}(F_K) d\hat{K}$ and as the integration over the element K is carried out via the space of reference (thus, over $\hat{K}$) to make it simple, this integration is valid if and only if $\mathcal{J}(F_K) > 0$ (the validity of the element thus guarantees the validity of the calculations carried out via the space of reference). Finally, the interpolation space is naturally constructed via the space of reference in which the polynomials p_i are defined.

As concerns relation [1.1], it must be noted that for a straight finite element (the edges are straight segments) with a degree other than one, with a uniform distribution of nodes, as we will see further on, this relation can be written more simply as $\displaystyle\sum_{i=1}^{m} q_i(\hat{M}) A_i$ where only the vertex

6. Except for first-degree simplices.

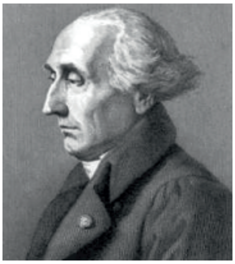

Figure 1.1.
Joseph Louis Lagrange (1736–1813)

nodes A_i, of number m, are taken into consideration and the shape function q_i are those of the finite element with a degree of one[7] of the element of the same geometry as the element being considered.

To conclude this brief presentation of the finite element method, we state in advance that we will be very sensitive, beyond the problem of the geometric validity of the elements, to their quality in shape (or regularity) – concepts that will be elaborated on in due time.

1.2. Shape functions, complete elements

We will construct the shape functions of common finite elements (Lagrange) that are complete (seen as patches) based on two generic functions. We will then consider the case of reduced elements (thus, in the case of quadrilaterals and hexahedra, the serendipity elements may be generalized). Finally, we will observe the case of rational elements with a brief allusion to the elements that may be constructed using B-splines or Nurbs functions.

1.2.1. *Generic expression of shape functions*

Shape functions were initially introduced by engineers to resolve elasticity problems using the finite element methods. In [Dhatt *et al.* 2007], we find a very comprehensive overview of shape functions of classic finite elements. In this chapter, we will review those functions that are expressed with spaces of reference that may be different (to place ourselves closest to the spaces made up of the patches seen in CAD).

The shape functions of the *isoparametric*[8] Lagrange elements of degree d are expressed in a generic manner. We will then adopt a purely geometric point of view (let us recall that the

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7. If the distribution of nodes is not uniform, the simplified formula remains true, but the q_i are no longer those of the first degree and, for example, even for a straight edge, the image of the midpoint of the edge of the space of reference cannot be the midpoint of the physical edge.

8. Let us repeat here that the shape functions serve to both define the transformation of the reference element toward the current element, a purely geometric aspect, as well as the polynomials of the finite element interpolation.

elements are seen as patches) and the notations used are those of this community. The reference element $\hat{K}$ is seen as a space of parameters and these are denoted by u, v, ..., rather than by $\hat{x}, \hat{y}$, ..., with classic correspondence between these two sets for the systems expressed as Barycentric coordinates ($u = 1 - \hat{x} - \hat{y}, v = \hat{x}$ and $w = \hat{y}$, in two dimensions, for a triangle).

For *simplices*, we work within a barycentric coordinate system and based on a generic function of degree d, which will be denoted by $\phi_i^d(u)$ or simply $\phi_i(u)$ when there is no ambiguity:

$$\boxed{\phi_i(u) = \frac{1}{i!}\prod_{l=0}^{d-1}(du - l)} \quad \text{for} \quad i \neq 0 \quad \text{and} \quad \phi_0(u) = 1, \qquad [1.3]$$

we construct the shape function of the index ijk with $i + j + k = d$ using:

$$p_{ijk}(u, v, w) = \phi_i(u)\phi_j(v)\phi_k(w), \qquad [1.4]$$

and the element, a triangle, is written as a function of the shape functions and its nodes:

$$\sum_{ijk} p_{ijk}(u, v, w)A_{ijk}. \qquad [1.5]$$

It can be immediately verified that $p_{ij}(u_k, v_l) = \delta_{ij,kl}$, even though the generic functions do not possess this property. Indeed, if we write $u_l = \frac{l}{d}$, ..., we then have $\phi_i(u_i) = \frac{1}{i!}\prod_{l=0}^{d-1}(i - l) = 1$ but $\phi_i(u_l) = 0$ uniquely for $l = 0, ..., i - 1$ - see the below schema (corresponding to the degree $d = 6$) where we show the isolines in u, v and w together with the value 0 or 1 for the three generic functions at the nodes of the reference element.

```
            .                               0
          .   .                           0   0
        .   .   .                       0   0   0
      .   .   .   .                   .   0   0   0
    .  ijk  .   .   .               .   1   0   0   0
  .   .   .   .   .   .           .   .   .   0   0   0
.   .   .   .   .   .   .       .   .   .   .   0   0   0

                              u   1 5/6 4/6 3/6 2/6 1/6  0
```

```
                0                            1           .
              0   .                        5/6         .   .
            0   .   .                    4/6         .   .   .
          0   .   .   .                3/6         .   .   .   .
        0   1   .   .   .            2/6         .   1   .   .   .
      0   .   .   .   .   .        1/6     0   0   0   0   0   0
    0   .   .   .   .   .   .      0     0   0   0   0   0   0   0

v   0 1/6 2/6 3/6 4/6 5/6  1         w
```

We have identical expressions in three dimensions, namely, for a tetrahedron:

$$p_{ijkl}(u,v,w,t) = \phi_i(u)\phi_j(v)\phi_k(w)\phi_l(t) \quad \text{and} \quad \sum_{ijkl} p_{ijkl}(u,v,w,t)A_{ijkl}. \qquad [1.6]$$

To familiarize ourselves with the indices, we give the example of a triangle of degree $d = 3$ that has 10 nodes, shown on the right with the classical numbering (vertices first, the nodes of the edges and then internal nodes) and on the left with the natural indices of the barycentric systems. The indexing of the shape functions is, quite obviously, identical.

```
              003                                  3

          102      012                          8     7

      201      111     021                   9    10     6

  300     210     120     030             1     4     5     2
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For the *tensor elements*, we use the natural coordinate system and use the generic function with degree d relative to the nodes $u_l = \frac{l}{d}$ for $l = 0, ..., d$, written as $\phi_i^d(u)$, if necessary, otherwise simply $\phi_i(u)$:

$$\boxed{\phi_i(u) = \frac{(-1)^i}{i!(d-i)!} \prod_{\substack{l=0 \\ l \neq i}}^{l=d} (l - du)}, \qquad [1.7]$$

we construct the shape function of the index ij by:

$$p_{ij}(u,v) = \phi_i(u)\phi_j(v). \qquad [1.8]$$

It must be noted that, of course $p_{ij}(u_k, v_l) = \delta_{ij,kl}$, but this is also the case for the generic function, $\phi_i(u_l) = \delta_{il}$, contrary to the earlier case.

The element, a quadrilateral[9] is written as given above based on the shape functions and its nodes:

$$\sum_{ij} p_{ij}(u,v)A_{ij}. \qquad [1.9]$$

In three dimensions, for a hexahedron this gives:

$$p_{ijk}(u,v,w) = \phi_i(u)\phi_j(v)\phi_k(w) \quad \text{and} \quad \sum_{ijk} p_{ijk}(u,v,w)A_{ijk}. \qquad [1.10]$$

To see what the indices are, we give the example of a quadrilateral of degree $d = 3$ that has 16 nodes with, on the right, classical numbering (vertices first, nodes of the edges and then internal nodes, observing for the latter nodes, that several conventions may be possible) and on the left with the natural indices (where, therefore, everything is natural).

03	13	23	33		4	10	9	3
02	12	22	32		11	15	16	8
01	11	21	31		12	13	14	7
00	10	20	30		1	5	6	2

Pentahedra or prisms are defined at u, v, w via functions as in [1.3] for the triangular faces and at t via the function [1.7].

Pyramidal elements (which are also pentahedra) are useful to ensure a consistent transition between hexahedral elements and tetrahedral elements. They are, however, difficult to define. We propose defining them as first-degree *pyramids*, which is not necessarily common, like complete but degenerated first-degree hexahedra, by precisely identifying the quadrilateral face, said to be the *base face*. This gives a very simple definition for these elements[10]. For the degrees 2 and 3, we propose the same approach, *but* starting from reduced hexahedra[11], which will be seen later on. Thus, these pyramids will only have, as nodes, their vertices and 1 or 2 nodes per edge. Consequently, we will have 13 second-degree nodes and 21 third-degree nodes. For higher degrees, there does not seem to be a plausible obvious definition.

9. In the literature, we sometimes see the term "quadrangle". This is incorrect even though everyone understands it.

10. Indeed, we can find more complicated definitions, which bring in shape functions of rational fractions. These definitions have the same fault, a singularity at the apex, and are certainly more costly.

11. Starting from complete elements makes it possible to have a reasonable geometric construction but will give finite elements which are surprising, to say the least. In particular, at a degree of 2, a triangular face will have an internal node.

1.2.2. *Explicit expression for degrees 1–3*

We only explicit the *typical* function or functions, for the common elements with a degree of 1–3. Functions [1.3] and [1.4] or [1.7] and [1.8] are used depending on the case under consideration.

Triangles of degrees 1–3

We use relations [1.3] and [1.4]. It is enough to calculate the desired $\phi_i(u)$ and then to express the *typical* $p_{ijk}(u,v,w)$ based on which the other shape functions can be easily deduced.

• Degree 1: we have $\phi_0(u) = 1$ and $\phi_1(u) = u$. There is only one typical function, $p_{100}(u,v,w)$, that is:

$$p_{100}(u,v,w) = \phi_1(u)\phi_0(v)\phi_0(w) = u.$$

• Degree 2: we have $\phi_0(u) = 1, \phi_1(u) = 2u$ and $\phi_2(u) = \frac{1}{2}2u(2u-1)$. There are two typical functions, $p_{200}(u,v,w)$ and $p_{110}(u,v,w)$, that is:

$$p_{200}(u,v,w) = \phi_2(u)\phi_0(v)\phi_0(w) = u(2u-1),$$

$$\text{and} \quad p_{110}(u,v,w) = \phi_1(u)\phi_1(v)\phi_0(w) = 4uv.$$

• Degree 3: we have $\phi_0(u) = 1, \phi_1(u) = 3u$, $\phi_2(u) = \frac{1}{2}3u(3u-1)$ and $\phi_3(u) = \frac{1}{6}3u(3u-1)(3u-2)$. There are three typical functions (see the schema given above for the indices) $p_{300}(u,v,w), p_{210}(u,v,w)$ and $p_{111}(u,v,w)$, and we find:

$$p_{300}(u,v,w) = \phi_3(u)\phi_0(v)\phi_0(w) = \frac{1}{2}u(3u-1)(3u-2),$$

$$\text{and} \quad p_{210}(u,v,w) = \phi_2(u)\phi_1(v)\phi_0(w) = \frac{1}{2}3u(3u-1)3v = \frac{9}{2}u(3u-1)v,$$

$$\text{then} \quad p_{111}(u,v,w) = \phi_1(u)\phi_1(v)\phi_1(w) = 27uvw.$$

Where $u = 1 - \hat{x} - \hat{y}, v = \hat{x}$ and $w = \hat{y}$, we find these same expressions in the usual variables. Through symmetry or rotation, we find all the shape functions of the elements, that is:

• Degree 1: $p_{100}(\hat{x},\hat{y}) = 1 - \hat{x} - \hat{y}$.

• Degree 2: $p_{200}(\hat{x},\hat{y}) = (1 - \hat{x} - \hat{y})(1 - 2\hat{x} - 2\hat{y})$ and $p_{110}(\hat{x},\hat{y}) = 4\hat{x}(1 - \hat{x} - \hat{y})$.

• Degree 3: $p_{300}(\hat{x},\hat{y}) = \frac{1}{2}(1 - \hat{x} - \hat{y})(2 - 3\hat{x} - 3\hat{y})(1 - 3\hat{x} - 3\hat{y})$ and $p_{210}(\hat{x},\hat{y}) = \frac{9}{2}\hat{x}(1 - \hat{x} - \hat{y})(2 - 3\hat{x} - 3\hat{y})$ and, finally, $p_{111}(\hat{x},\hat{y}) = 27\hat{x}\hat{y}(1 - \hat{x} - \hat{y})$.

Tetrahedra of degrees 1–3

We can simply extend the case of triangles and, in the barycentric coordinate system, the shape functions of tetrahedra are identical to those of triangles. Thus, we have:

• Degree 1: A single typical function, $p_{1000}(u,v,w,t)$, and we find: $p_{1000}(u,v,w,t) = u$.

• Degree 2: Two typical functions $p_{2000}(u, v, w, t)$ and $p_{1100}(u, v, w, t)$, that is: $p_{2000}(u, v, w, t) = u(2u - 1)$ and $p_{1100}(u, v, w, t) = 4uv$.

• Degree 3: Three typical functions (see the above schema for the indices) $p_{3000}(u, v, w, t), p_{2100}(u, v, w, t)$ and $p_{1110}(u, v, w, t)$, that is: $p_{3000}(u, v, w, t) = \frac{1}{2}u(3u - 1)(3u - 2)$, $p_{2100}(u, v, w, t) = \frac{9}{2}u(3u - 1)v$ and $p_{1110}(u, v, w, t) = 27uvw$.

Where $u = 1 - \hat{x} - \hat{y} - \hat{z}, v = \hat{x}, w = \hat{y}$ and $t = \hat{z}$, we find these same expressions in the usual variables. Through symmetry or rotation, we find all the shape functions.

Quadrilaterals of degrees 1–3

We use relations [1.7] and [1.8]. It is enough to calculate the desired $\phi_i(u)$ and then to express the *typical* $p_{ij}(u, v)$ based on which the other shape functions will be deduced. We open relation [1.7] for $i = 0$ and $i = 1$. We see:

$$\phi_0(u) = \frac{1}{d!}(1 - du)(2 - du)...(d - du) \quad \text{and}$$

$$\phi_1(u) = \frac{-1}{(d-1)!}(-du)(2-du)(3-du)...(d-du) = \frac{1}{(d-1)!}du(2-du)(3-du)...(d-du),$$

and then use these expressions, the only ones which can be used (by symmetry) for the degrees 1–3.

• Degree 1: we have $\phi_0(u) = 1 - u$. There is only one typical function, $p_{00}(u, v)$, and we find:

$$p_{00}(u, v) = \phi_0(u)\phi_0(v) = (1 - u)(1 - v).$$

• Degree 2: we have $\phi_0(u) = \frac{1}{2}(1-2u)(2-2u) = (1-2u)(1-u)$ and $\phi_1(u) = 2u(2-2u) = 4u(1 - u)$. There are three typical functions, $p_{00}(u, v), p_{10}(u, v)$ and $p_{11}(u, v)$, that is:

$$p_{00}(u, v) = \phi_0(u)\phi_0(v) = (1 - 2u)(1 - u)(1 - 2v)(1 - v),$$

$$p_{10}(u, v) = \phi_1(u)\phi_0(v) = 4u(1 - u)(1 - 2v)(1 - v),$$

$$p_{11}(u, v) = \phi_1(u)\phi_1(v) = 16u(1 - u)v(1 - v).$$

• Degree 3: we have $\phi_0(u) = \frac{1}{6}(1 - 3u)(2 - 3u)(3 - 3u) = \frac{1}{2}(1 - 3u)(2 - 3u)(1 - u)$ and $\phi_1(u) = \frac{1}{2}3u(2 - 3u)(3 - 3u) = \frac{9}{2}u(2 - 3u)(1 - u)$. There are three typical functions, see the above scheme for the indices $p_{00}(u, v), p_{10}(u, v)$ and $p_{11}(u, v)$, and we find:

$$p_{00}(u, v) = \phi_0(u)\phi_0(v) = \frac{1}{4}(1 - 3u)(2 - 3u)(1 - u)(1 - 3v)(2 - 3v)(1 - v),$$

$$p_{10}(u, v) = \phi_1(u)\phi_0(v) = \frac{9}{4}u(2 - 3u)(1 - u)(1 - 3v)(2 - 3v)(1 - v),$$

$$p_{11}(u, v) = \phi_1(u)\phi_1(v) = \frac{81}{4}u(2 - 3u)(1 - u)v(2 - 3v)(1 - v).$$

With $u = \hat{x}$ and $v = \hat{y}$, we find these same expressions in the usual variables. Through symmetry or rotation, we find all the shape functions of the elements. As an exercise, let us express $p_{20}(u, v)$ for a third-degree quadrilateral. It is enough to carry out a symmetry at u, thus $p_{20}(u, v) = p_{10}(1 - u, v)$, or:

$$p_{20}(u, v) = -\frac{9}{4}u(1 - 3u)(1 - u)(1 - 3v)(2 - 3v)(1 - v).$$

Hexahedra of degrees 1–3

We go back to the previously obtained functions for quadrilaterals and extend them mechanically.

• Degree 1: A single typical function, $p_{000}(u, v, w)$, thus we find:

$$p_{000}(u, v, w) = (1 - u)(1 - v)(1 - w).$$

• Degree 2: Four typical functions, $p_{000}(u, v, w), p_{100}(u, v, w), p_{110}(u, v, w)$ and $p_{111}(u, v, w)$, thus we find:

$$p_{000}(u, v, w) = (1 - 2u)(1 - u)(1 - 2v)(1 - v)(1 - 2w)(1 - w),$$

$$p_{100}(u, v, w) = 4u(1 - u)(1 - 2v)(1 - v)(1 - 2w)(1 - w),$$

$$p_{110}(u, v, w) = 16u(1 - u)v(1 - v)(1 - 2w)(1 - w),$$

$$p_{111}(u, v, w) = 64u(1 - u)v(1 - v)w(1 - w).$$

• Degree 3: Four typical functions, $p_{000}(u, v, w), p_{100}(u, v, w), p_{110}(u, v, w)$ and $p_{111}(u, v, w)$, we thus find:

$$p_{000}(u, v, w) = \frac{1}{8}(1 - 3u)(2 - 3u)(1 - u)(1 - 3v)(2 - 3v)(1 - v)(1 - 3w)(2 - 3w)(1 - w),$$

$$p_{100}(u, v, w) = \frac{9}{8}u(2 - 3u)(1 - u)(1 - 3v)(2 - 3v)(1 - v)(1 - 3w)(2 - 3w)(1 - w),$$

$$p_{110}(u, v, w) = \frac{81}{8}u(2 - 3u)(1 - u)v(2 - 3v)(1 - v)(1 - 3w)(2 - 3w)(1 - w),$$

$$p_{111}(u, v, w) = \frac{729}{8}u(2 - 3u)(1 - u)v(2 - 3v)(1 - v)w(2 - 3w)(1 - w).$$

With $u = \hat{x}, v = \hat{y}$ and $w = \hat{z}$ in notation $\hat{}$.

Pentahedra of degrees 1–3

We go back to the $\phi_i()$ functions of a triangle and those of a quadrilateral in the third direction. The coordinates are, thus, barycentric in the plane and natural in the third direction.

• Degree 1: A single typical function, $p_{1000}(u, v, w, t)$, and we find:

$$p_{1000}(u, v, w, t) = u(1 - t).$$

• Degree 2: Four typical functions, $p_{2000}(u, v, w, t), p_{1100}(u, v, w, t), p_{2001}(u, v, w, t)$ and $p_{1101}(u, v, w, t)$ that correspond to the vertices, the nodes of edges shared by a triangle or two quadrilateral faces, and to the nodes on the quadrilateral faces, that is:

$$p_{2000}(u, v, w, t) = u(2u - 1)(1 - 2t)(1 - t),$$

$$p_{1100}(u, v, w, t) = 4uv(1 - 2t)(1 - t),$$

$$p_{2001}(u, v, w, t) = 4u(2u - 1)t(1 - t),$$

$$p_{1101}(u, v, w, t) = 16uvt(1 - t).$$

• Degree 3: Five typical functions, $p_{3000}(u, v, w, t), p_{2100}(u, v, w, t), p_{3001}(u, v, w, t)$, $p_{1110}(u, v, w, t)$ and $p_{2101}(u.v, w, t)$ that correspond to the vertices, the nodes on the edges shared by a triangle or two quadrilateral faces and to the nodes on the triangular and quadrilateral faces, that is:

$$p_{3000}(u, v, w, t) = \frac{1}{4}u(3u - 1)(3u - 2)(1 - 3t)(2 - 3t)(1 - t),$$

$$p_{2100}(u, v, w, t) = \frac{9}{4}u(3u - 1)v(1 - 3t)(2 - 3t)(1 - t),$$

$$p_{3001}(u, v, w, t) = \frac{9}{4}u(3u - 1)(3u - 2)t(2 - 3t)(1 - t),$$

$$p_{1110}(u, v, w, t) = \frac{27}{2}uvw(1 - 3t)(2 - 3t)(1 - t),$$

$$p_{2101}(u, v, w, t) = \frac{81}{4}u(3u - 1)vt(2 - 3t)(1 - t).$$

Pyramids of degrees 1–3

For a degree of 1, we go back to the definition of a complete hexahedron, with the formulae [1.10], that is $p_{ijk}(u, v, w) = \phi_i(u)\phi_j(v)\phi_k(w)$ and $\sum_{ijk} p_{ijk}(u, v)A_{ijk}$, by positing $A_{ijd} = A_{00d}$ for all the (i, j) couples. We can deduce from this that the shape functions $p_{ijk}(u, v, w)$ are those of the hexahedron for the index $k \neq d$. On the other hand, the missing function, $p_{ood}(u, v, w)$, is obtained by taking the sum of $p_{ijd}(u, v, w)$. Thus, $p_{00d}(u, v, w) =$

$\left\{\sum_i \sum_j \phi_i(u)\phi_j(v)\right\} \phi_d(w)$. Consequently (classic property of the $\phi_i(u)\phi_j(v)$ which give a sum of 1), we have $p_{00d}(u,v,w) = \phi_d(w) = \dfrac{(-1)^d}{d!} \prod_{l=0}^{l=d-1} (l - dw)$.

As $d = 1$, we find:

- Degree 1: $p_{001}(u,v,w) = w$.

For the degrees 2 and 3, we use the same shortcut, but based on reduced elements (see further on). We thus find that:

- Degree 2: $p_{002}(u,v,w) = -w(1-2w)$.
- Degree 3: $p_{003}(u,v,w) = \frac{1}{2}w(1-3w)(2-3w)$.

1.3. Shape functions, reduced elements

The idea behind reduced elements is to bring down the number of internal nodes while retaining an acceptable level of precision, that is a sufficiently rich polynomial space. The nodes, thus, necessarily include the border nodes of the complete element, but we try to do away with all or part[12] of the internal nodes.

Who came up with these elements? It was most probably mechanical engineers, especially, at the beginning, for the second-degree Lagrange quadrilateral with eight nodes. This element is a serendipity element, from the legend of the three princes of Serendip (in the photo). Indeed, one construction of this element (and beyond those of other degrees) consists of imposing, as a polynomial space, a space that includes as basis all the monomials of a maximum degree of 2 in all the variables (the classical space P^2 of triangles). On doing this, it is seen that the two monomials u^2v and uv^2 are covered "for free", without really having been the goal of the operation. An unexpected gift and hence the term *serendipity*, which qualifies quadrilaterals and hexahedra of this nature. The existence of reduced elements for other geometries is more problematic. However, we know of the third-degree triangle with nine nodes (and thus the hexahedron with 16 nodes) but not of reduced simplices that are of interest for higher degrees. For other element types, for degrees that are not too high, it is possible to delete certain nodes (in particular, the case of those with faces comes to mind) but the resulting polynomial space is not very clear.

While literature is hardly prolix on the subject, other than for a degree of 2, we find several methods to try and construct reduced elements. We can separate the methods that work at the level of the matrices of rigidity (resulting from weak formulation) themselves to eliminate the nodes, typically by condensation. We find two categories of methods: one based on

12. And this is strictly necessary when the degree increases.

Figure 1.2.
The three princes of Serendip

Taylor expansions truncated to the desired order [Ciarlet-1978], [Bernardi *et al.* 2004], or again, [George, Borouchaki-2017], and the other based on the direct search for the polynomial space by imposing its basis, as developed by [Arnold, Awanou-2011] and [Floater, Gillette-2014]. For quadrilaterals and hexahedra, we will propose another approach based on a generalization of the transfinite interpolation, a transformation that is also known for a degree of 2 (for instance, Coons' patches) but is also valid for a degree of 3 (and we will also have a Coons' patch). For higher degrees, the method does not yield acceptable results and hence the idea of seeking a generalization – called *generalized transfinite interpolation* [George, Borouchaki-2015]. It must be noted that this approach is very close to a concept discussed in [Floater, Gillette-2014].

1.3.1. *Simplices, triangles and tetrahedra*

We know of the reduced, third-degree triangle with nine nodes (the complete triangle has 10 nodes). The first question is to understand its construction. Another question is to know whether such reduced triangles exist for higher degrees.

We can find an answer to the first question in [Bernardi *et al.* 2004]. However, there is no answer to the second question in the literature.

The polynomial space of the triangle is the usual space P^3 in which we add a condition that stipulates that the value of a polynomial evaluated at the barycenter (of the reference element, thus the node $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ in barycentric coordinates) is expressed as a linear combination of the values of this polynomial evaluated at the nodes on the edge of the reference element. If q designates a polynomial, we have the following relation:

$$12q(A_{111}) + 2\sum_{ijk\in\mathcal{S}} q(A_{ijk}) - 3\sum_{ijk\in\mathcal{A}} q(A_{ijk}) = 0,$$

where $\mathcal{S}$ designates the indices for the vertices and $\mathcal{A}$ designates the indices for the edge nodes. In [Bernardi *et al.* 2004], it is always shown that consequently, the polynomial space that is thereby reduced contains the space P^2. This is the definition that we will retain to define reduced triangles.

- Triangle of degree 3. Let us recall the system used for indices:

```
            003
        102     012
     201    111     021
 300    210     120     030
```

The concept is to begin with the Taylor expansion of a function q at the vertices and the nodes expressed using the central node; we then introduce the vectors $\overrightarrow{v_{ijk}} = \overrightarrow{A_{111}A_{ijk}}$ and the successive derivatives of q, seen in an abstract manner as linear operators, bilinear operators, etc. They are denoted by $D^1.(\overrightarrow{u}), D^2.(\overrightarrow{u}, \overrightarrow{u})$, etc., the coefficient (the binomial coefficient) being included in the operator. Thus, as we are looking for the space P^2, the development stops at the second derivative and is written as, at A_{300}:

$$q(A_{300}) = q(A_{111}) + D^1.(\overrightarrow{v_{300}}) + D^2.(\overrightarrow{v_{300}}, \overrightarrow{v_{300}}).$$

Upon taking the sum of the three vertices, we find:

$$\sum_{ijk \in \mathcal{S}} q(A_{ijk}) = 3q(A_{111}) + \sum_{ijk \in \mathcal{S}} D^1.(\overrightarrow{v_{ijk}}) + \sum_{ijk \in \mathcal{S}} D^2.(\overrightarrow{v_{ijk}}, \overrightarrow{v_{ijk}}),$$

and as the operator $D^1.()$ is linear and A_{111} is the barycenter, this is reduced to:

$$\sum_{ijk \in \mathcal{S}} q(A_{ijk}) = 3q(A_{111}) + \sum_{ijk \in \mathcal{S}} D^2.(\overrightarrow{v_{ijk}}, \overrightarrow{v_{ijk}}). \qquad [1.11]$$

This same development is expressed for the edge nodes. For example:

$$q(A_{210}) = q(A_{111}) + D^1.(\overrightarrow{v_{210}}) + D^2.(\overrightarrow{v_{210}}, \overrightarrow{v_{210}}),$$

and on summing, we obtain:

$$\sum_{ijk \in \mathcal{A}} q(A_{ijk}) = 6q(A_{111}) + \sum_{ijk \in \mathcal{A}} D^1.(\overrightarrow{v_{ijk}}) + \sum_{ijk \in \mathcal{A}} D^2.(\overrightarrow{v_{ijk}}, \overrightarrow{v_{ijk}}),$$

of which only the following remains (the terms related to $D^1.()$ cancel each other out two by two):

$$\sum_{ijk \in \mathcal{A}} q(A_{ijk}) = 6q(A_{111}) + \sum_{ijk \in \mathcal{A}} D^2.(\overrightarrow{v_{ijk}}, \overrightarrow{v_{ijk}}). \qquad [1.12]$$

As $A_{210} = \frac{2A_{300}+A_{030}}{3}$, etc., we can express the different $\overrightarrow{v_{ijk}}$ vectors uniquely as a function of three vectors related to the vertices. Thus, for example: $\overrightarrow{v_{210}} = \dfrac{2\overrightarrow{v_{300}} + \overrightarrow{v_{030}}}{3}$.

We now calculate $D^2.(\overrightarrow{v_{210}},\overrightarrow{v_{210}}) + D^2.(\overrightarrow{v_{120}},\overrightarrow{v_{120}})$, and find:

$$\frac{4}{9}D^2.(\overrightarrow{v_{300}},\overrightarrow{v_{300}}) + \frac{4}{9}D^2.(\overrightarrow{v_{300}},\overrightarrow{v_{030}}) + \frac{1}{9}D^2.(\overrightarrow{v_{030}},\overrightarrow{v_{030}})$$

$$+\frac{1}{9}D^2.(\overrightarrow{v_{300}},\overrightarrow{v_{300}}) + \frac{4}{9}D^2.(\overrightarrow{v_{300}},\overrightarrow{v_{030}}) + \frac{4}{9}D^2.(\overrightarrow{v_{030}},\overrightarrow{v_{030}}),$$

that is, in total $\frac{5}{9}D^2.(\overrightarrow{v_{300}},\overrightarrow{v_{300}}) + \frac{8}{9}D^2.(\overrightarrow{v_{300}},\overrightarrow{v_{030}}) + \frac{5}{9}D^2.(\overrightarrow{v_{030}},\overrightarrow{v_{030}})$.

By taking the sum of the three edges, we find:

$$\frac{10}{9}\sum_{ijk\in\mathcal{S}} D^2.(\overrightarrow{v_{ijk}},\overrightarrow{v_{ijk}}) + \frac{8}{9}(D^2.(\overrightarrow{v_{300}},\overrightarrow{v_{030}}) + D^2.(\overrightarrow{v_{030}},\overrightarrow{v_{003}}) + D^2.(\overrightarrow{v_{003}},\overrightarrow{v_{300}})).$$

We will express the crossed terms in functions of terms related only to the vertices, but as $D^2.(\overrightarrow{v_{300}}+\overrightarrow{v_{030}}+\overrightarrow{v_{003}},\overrightarrow{v_{300}}+\overrightarrow{v_{030}}+\overrightarrow{v_{003}}) = 0$, we have:

$$0 = \sum_{ijk\in\mathcal{S}} D^2.(\overrightarrow{v_{ijk}},\overrightarrow{v_{ijk}}) + 2(D^2.(\overrightarrow{v_{300}},\overrightarrow{v_{030}}) + D^2.(\overrightarrow{v_{030}},\overrightarrow{v_{003}}) + D^2.(\overrightarrow{v_{003}},\overrightarrow{v_{300}})),$$

or, in other terms:

$$(D^2.(\overrightarrow{v_{300}},\overrightarrow{v_{030}}) + D^2.(\overrightarrow{v_{030}},\overrightarrow{v_{003}}) + D^2.(\overrightarrow{v_{003}},\overrightarrow{v_{300}}) = -\frac{1}{2}\sum_{ijk\in\mathcal{S}} D^2.(\overrightarrow{v_{ijk}},\overrightarrow{v_{ijk}}),$$

thus, the above sum is reduced to $\frac{2}{3}\sum_{ijk\in\mathcal{S}} D^2.(\overrightarrow{v_{ijk}},\overrightarrow{v_{ijk}})$ and relation [1.12] is written as:

$$\sum_{ijk\in\mathcal{A}} q(A_{ijk}) = 6q(A_{111}) + \frac{2}{3}\sum_{ijk\in\mathcal{S}} D^2.(\overrightarrow{v_{ijk}},\overrightarrow{v_{ijk}}),$$

to finish, we identify this last sum (second derivatives) in the two relations, which gives the following combination:

$$\sum_{ijk\in\mathcal{S}} q(A_{ijk}) - 3q(A_{111}) = \frac{3}{2}\sum_{ijk\in\mathcal{A}} q(A_{ijk}) - 6q(A_{111}))$$

and, finally, we have found the defining property[13] of the desired polynomial space:

$$\boxed{12q(A_{111}) + 2\sum_{ijk\in\mathcal{S}} q(A_{ijk}) - 3\sum_{ijk\in\mathcal{A}} q(A_{ijk}) = 0}. \qquad [1.13]$$

The function of the space must verify this relation, especially the desired reduced shape functions, the p^s_{ijk}. By symmetry, we look for these functions in the form:

$$p^s_{ijk}(u,v,w) = p_{ijk}(u,v,w) + \alpha p_{111}(u,v,w),\ ijk \in \mathcal{S},$$

13. The equivalent of a serendipity relation.

$$p^s_{ijk}(u,v,w) = p_{ijk}(u,v,w) + \beta p_{111}(u,v,w),\ ijk \in \mathcal{A}.$$

In other words, there are only two parameters to be found and relation [1.13] gives the solution:

$$\boxed{\alpha = -\frac{1}{6}} \quad \text{and} \quad \boxed{\beta = \frac{1}{4}}.$$

Starting from these two values, we can find the expression for the two typical functions:

$$p^s_{300}(u,v,w) = \frac{1}{2}u(2u^2 + 2v^2 + 2w^2 - 5uv - 5vw - 5uw),$$

$$p^s_{210}(u,v,w) = \frac{9}{4}uv(4u - 2v + w),$$

which gives:

$$p^s_{300}(x,y) = \frac{9}{2}(1-x-y)(\frac{2}{9} - x - y + xy + x^2 + y^2),$$

$$p^s_{210}(x,y) = \frac{9}{4}x(1-x-y)(4-6x-3y).$$

To conclude the discussion on the reduced third-degree triangle, what remains is to show its polynomial space. By construction, P^2 is covered. The question is to find the other three monomials that are covered here. To do this, we begin with the relations (in the complete space):

$$x^l y^m = \sum_{ijk} \omega^{lq}_{ijk} p_{ijk}(x,y),$$

and the coefficients ω^{lq}_{ijk} are found through instantiations (for example, $\omega^{00}_{ijk} = 1$) for any ijk. Next, we begin with the relation $p^s_{ijk}(x,y) = p_{ijk}(x,y) + \alpha p_{111}(x,y)$, for example, and we write its opposite, that is $p_{ijk}(x,y) = p^s_{ijk}(x,y) - \alpha p_{111}(x,y)$. It is then sufficient to write it into an expression of $x^l y^m$ to verify whether the coefficient on p_{111} is null or not. These calculations were carried out in [George *et al.* 2014] and the result is illustrated in the following diagram:

$$\begin{array}{ccc} & 1 & \\ & x \quad y & \\ x^2 & xy & y^2 \\ x^3 + 2x^2y & x^2y - xy^2 & y^3 + 2xy^2, \end{array}$$

and the unexpected gift is the last line of the diagram (though we have lost the monomials x^3, x^2y, xy^2 and y^3 present in the complete space P^3).

- Triangle of degree 4

Construction by Taylor expansion, by enforcing the presence of the space P^3 in the reduced fourth-degree triangle leads to an impasse [George *et al.* 2014]. We cannot even find the space P^2. On the other hand, by enforcing the presence of P^2 alone, we find a solution (a total of 12 monomials or combinations), thus an element of 12 nodes, but this element seems inadequate with respect to any calculation.

Higher degrees present the same syndrome

- Tetrahedron of degree 3

Following the discussion on triangles, we think that only the reduced third-degree tetrahedron is of interest. The typical reduced shape functions are given in [George *et al.* 2014], and we find:

$$p^s_{3000}(u,v,w,t) = p_{3000}(u,v,w,t) - \frac{1}{6}p_{1110}(u,v,w,t) - \frac{1}{6}p_{1101}(u,v,w,t) - \frac{1}{6}p_{1011}(u,v,w,t)$$

which gives, at u, v, w and t:

$$p^s_{3000}(u,v,w,t) = \frac{1}{2}u(2u^2 - 5uv - 5uw - 5ut + 2v^2 - 5vw - 5vt + 2w^2 - 5wt + 2t^2),$$

and, at x, y and z:

$$p^s_{3000}(x,y,z) = \frac{1}{2}(1-x-y-z)(2 - 9x - 9y - 9z + 9x^2 + 9xy + 9xz + 9y^2 + 9yz + 9z^2).$$

For $p^s_{2100}(u,v,w,t)$, the dependency is at $p_{2100}(u,v,w,t)$ and the internal functions of the two incident faces, that is:

$$p^s_{2100}(u,v,w,t) = p_{2100}(u,v,w,t) + \frac{1}{4}p_{1110}(u,v,w,t) + \frac{1}{4}p_{1101}(u,v,w,t)$$

$$= \frac{9}{4}uv(4u - 2v + w + t),$$

and at x, y and z we have : $p^s_{2100}(x,y,z) = \frac{9}{4}(1-x-y-z)x(4 - 6x - 3y - 3z)$.

1.3.2. *Tensor elements, quadrilateral and hexahedral elements*

A serendipity tensor element of degree d for each variable is, in theory, defined starting from information on the boundary nodes (incomplete elements) and the specific shape functions that are based on these nodes. As all of these nodes are incomplete, there exists (except in exceptional cases) an infinity of shape functions related to these nodes. To establish a unique and adequately rich solution, we enforce that the space resulting from these shape functions contains the polynomial space of degree d at all the variables, the classic space P^d.

For small degrees, this definition also implies the presence of two monomials (in the case of quadrilaterals), $u^d v$ and uv^d in the resulting space, hence the term “serendipity”. In general, in two dimensions, the serendipity space is thus defined including the space P^d and the monomials of degree d for one of the variables and a degree of 1 for the others. Thus, in two dimensions, this space has the dimension $\frac{(d+1)(d+2)}{2} + 2$. Consequently, the unique information on the border nodes does not make it possible to cover the case of degrees higher than 3. Indeed, for a degree of 4, an additional node is required; for a degree of 5, three additional nodes are required and so on. The number of internal nodes is, thus, equal to $\frac{(d+1)(d+2)}{2} + 2 - 4d = \frac{d(d-5)}{2} + 3$. These internal nodes are arranged starting from the center of the element. From the degree of 5, the nodes of the corresponding element cannot be positioned symmetrically with respect to the center of the element (thus be independent from the local numbering of the nodes).

In three dimensions, the serendipity space comprises the space P^d and the monomials of a degree s higher than d having at least $s-d$ linear terms (variables) [Arnold, Awanou-2011]. For example, a serendipity hexahedron of degree 2, with 20 nodes includes the polynomial space P^2 with three variables: the third-degree monomials, namely u^2v, uv^2, u^2w, uw^2, v^2w, vw^2, uvw, as well as the fourth-degree monomials, namely u^2vw, uv^2w, uvw^2, thus, in total, 20 monomials.

To find a method of construction for serendipity elements of an arbitrary degree, we first consider the case of a degree of 2, which requires no internal node and where the construction method is none other than transfinite interpolation. In an analogous manner, we then propose a generalization of this method, called generalized transfinite interpolation, resulting, for any degree, in *symmetric* elements.

In order to phase the indices following the two variables in a transfinite interpolation of different degrees following these variables, we introduce two functions $k(i)$ and $l(i)$ where i is an index, defined for a given degree d, by:

$$k(i) = i \times d, \qquad l(0) = 0 \quad \text{and} \quad l(i) = i+1, i = 1, d-3 \quad \text{and finally} \quad l(d-2) = d.$$

• Quadrilateral of degree 2. A Taylor expansion [Bernardi *et al.* 2004] may be used to construct this element. However, we know that this quadrilateral can also be expressed via a classic transfinite interpolation [Gordon, Hall-1973] and is therefore written using $\phi_i(.)$ and the nodes, as:

$$\sum_{i=0}^{1}\sum_{j=0}^{2}\phi_i^1(u)\phi_j^2(v)A_{k(i),j} + \sum_{i=0}^{2}\sum_{j=0}^{1}\phi_i^2(u)\phi_j^1(v)A_{i,k(j)} - \sum_{i=0}^{1}\sum_{j=0}^{1}\phi_i^1(u)\phi_j^1(v)A_{k(i),k(j)}, \tag{1.14}$$

shortened to:

$$\sigma_{12}(u,v) + \sigma_{21}(u,v) - \sigma_{11}(u,v) \quad \text{or even more simply} \quad \boxed{\sigma_{12} + \sigma_{21} - \sigma_{11}}\,.$$

To obtain the second-degree tensor expression for each variable, we will rewrite the ϕ_*^1, the index $_*$ for $_i$ or $_j$, for a degree of 2. We thus consider two $\phi_*^1(u)$ functions and we express them on the basis of the $\phi_*^2(u)$ of degree 2 (the coefficients being unknown):

$$\alpha_0\phi_0^2(u) + \alpha_1\phi_1^2(u) + \alpha_2\phi_2^2(u) = \phi_0^1(u)$$
$$\text{and} \quad \beta_0\phi_0^2(u) + \beta_1\phi_1^2(u) + \beta_2\phi_2^2(u) = \phi_1^1(u),$$

we then instantiate u for three values $0, \frac{1}{2}$ and 1. The property $\phi_i^2(u_j) = \delta_{ij}$ (where the u_j are the nodes of the reference element (hence the three values above)) makes it possible to find the 6 coefficients. We have (a result that is also generalizable for any degree):

$$\frac{2}{2}\phi_0^2(u) + \frac{1}{2}\phi_1^2(u) + \frac{0}{2}\phi_2^2(u) = \phi_0^1(u) \quad \text{and} \quad \frac{0}{2}\phi_0^2(u) + \frac{1}{2}\phi_1^2(u) + \frac{2}{2}\phi_2^2(u) = \phi_1^1(u),$$

$$\text{thus:} \quad \sum_{i=0}^{1} \phi_i^1(u) A_{k(i)} = \sum_{i=0}^{2} \phi_i^2(u) Q_i,$$

$$\text{where} \quad Q_0 = A_0 \quad , \quad Q_1 = \frac{A_0 + A_2}{2} = \sum_{i+i_1=1} \frac{A_{k(i)}}{2} \quad \text{and} \quad Q_2 = A_2.$$

It must be noted that to go from a degree of 1–2, a simple elevation of degree is carried out here (this will generally be true to go from a degree of 1 to any degree d, see below) in Lagrange formalism with a method that is not the same as that used to elevate the degree in the case of Bézier shapes (see Chapter 3). Coming back to the element makes it possible to find the classic form, $\sigma_{12} = \sum_{I=0}^{2} \sum_{j=0}^{2} \phi_I^2(u) \phi_j^2(v) Q_{Ij}$, by defining the nodes $Q_{*,j}, j = 0, ..., 2$, by:

$$Q_{0j} = A_{k(0),j} = A_{0j}, \quad Q_{1j} = \sum_{m=0}^{1} \frac{1}{2} A_{k(m),j} = \frac{A_{0j} + A_{2j}}{2}, \quad Q_{2j} = A_{k(1),j} = A_{2j}. \qquad [1.15]$$

The second term in the sum, σ_{21}, is treated in the same manner by defining an analogous sequence, with relation [1.15] becoming, for $i = 0, .., 2$:

$$Q_{i0} = A_{i,k(0)} = A_{i0}, \quad Q_{i1} = \sum_{m=0}^{1} \frac{1}{2} A_{i,k(m)} = \frac{A_{i0} + A_{i2}}{2}, \quad Q_{i2} = A_{i,k(1)} = A_{i2}. \qquad [1.16]$$

Similarly, for the third term, σ_{11}, we construct the node sequence:

$$Q_{0j}, Q_{2j}, Q_{i0}, Q_{i2} \quad \text{as above and} \quad Q_{11} = \sum_{i=0}^{1} \sum_{j=0}^{1} \frac{A_{k(i),k(j)}}{4}. \qquad [1.17]$$

We write the initial second-degree patch by introducing the nodes Q_{ij} (defined above) in each of the three terms in its definition and then expressing these nodes as functions of the initial A_{ij}. We thus obtain a complete patch by inventing the central node. This node (see the above schema) is naturally denoted by A_{11}, which is the node that comes up with respect to the term $\phi_1^2(u)\phi_1^2(v)$. Consequently, we have three contributions. The first is $\frac{A_{01} + A_{21}}{2}$, the second is $\frac{A_{10} + A_{12}}{2}$, the last term is $\frac{A_{20} + A_{22} + A_{00} + A_{02}}{4}$ and upon summing we have:

$$A_{11} = \frac{A_{01} + A_{21} + A_{10} + A_{12}}{2} - \frac{A_{00} + A_{20} + A_{22} + A_{02}}{4},$$

which is summarized in the following schema:

```
02    12    22            -1/4   1/2   -1/4

01   [11]   21             1/2   [11]   1/2

00    10    20            -1/4   1/2   -1/4
```

In conclusion, with the specific, made-up node A_{11} and the initial nodes, the element is written as a complete patch, that is $\sum_{i=0}^{2}\sum_{j=0}^{2}\phi_i^2(u)\phi_j^2(v)A_{ij} = \sum_{i=0}^{2}\sum_{j=0}^{2}p_{ij}^2(u,v)A_{ij}$.

Following this construction, we can explicitly calculate the reduced shape functions. Let us recall that $\phi_0(u) = (1-2u)(1-u)$ and that $\phi_1(u) = 4u(1-u)$. The first shape function, index 00, is the polynomial resulting with respect to A_{00}, thus via A_{00}, weight 1, directly and via A_{11}, weight $-\frac{1}{4}$, or:

$$p_{00}^s(u,v) = p_{00}(u,v) - \frac{1}{4}p_{11}(u,v) = (1-u)(1-v)(1-2u-2v).$$

The other typical function is that of index 10, or, similarly:

$$p_{10}^s(u,v) = p_{10}(u,v) + \frac{1}{2}p_{11}(u,v) = 4u(1-u)(1-v).$$

We establish the expression for all the reduced shape functions via symmetry.

It must be noted that the initial relation gives the result directly. That is:

$$p_{00}^s(u,v) = \phi_0^1(u)\phi_0^2(v) + \phi_0^2(u)\phi_0^1(v) - \phi_0^1(u)\phi_0^1(v).$$

This was not evident in the above form, $p_{00}(u,v) - \frac{1}{4}p_{11}(u,v)$, and for the other typical function, index 10, we directly have:

$$p_{10}^s(u,v) = \phi_1^2(u)\phi_0^1(v).$$

However, this apparent simplicity is only true when $d = 2$ (without internal node) and, as we will see, when $d = 3$ (also without internal node).

- Quadrilateral of degree d

The idea, here as well, is to avoid using Taylor expansions and instead find an approach based on transfinite interpolation (as for degree 2) or, more precisely, to see how to generalize this transformation to meet our needs.

A symmetrical serendipity element includes, as nodes, all the nodes of the complete element except for the nodes of the *first ring*. This collection of nodes is formed by the internal nodes of the complete element that are immediate neighbors of the border nodes:

$$A_{1j} \text{ and } A_{d-1,j} \quad \text{for} \quad 1 \leq j \leq d-1 \quad \text{and} \quad A_{i1} \text{ and } A_{i,d-1} \quad \text{for} \quad 1 \leq i \leq d-1.$$

Such an element is identical, up to a degree of 4, to a classic serendipity element, and beyond this degree is richer than the classic serendipity element [Arnold, Awanou-2011], [Floater, Gillette-2014], but it has the advantage of being symmetrical. By considering this collection of nodes, we construct the element via the following relation called generalized transfinite interpolation:

$$\sum_{i=0}^{1}\sum_{j=0}^{d}\psi_i^1(u)\phi_j^d(v)A_{k(i),j} + \sum_{i=0}^{d}\sum_{j=0}^{1}\phi_i^d(u)\psi_j^1(v)A_{i,k(j)} + \sum_{i=0}^{d-2}\sum_{j=0}^{d-2}\psi_i^{d-2}(u)\psi_j^{d-2}(v)A_{l(i),l(j)}$$

$$-\sum_{i=0}^{1}\sum_{j=0}^{d-2}\psi_i^1(u)\psi_j^{d-2}(v)A_{k(i),l(j)}-\sum_{i=0}^{d-2}\sum_{j=0}^{1}\psi_i^{d-2}(u)\psi_j^1(v)A_{l(i),k(j)}, \qquad [1.18]$$

where ψ_*^1 is a first-degree shape function relative to the nodes with index 0 and d, ψ_*^{d-2} is a shape function of degree $d-2$ relative to all the nodes except those of index 1 and $d-1$. By definition, the ψ_*^1 function coincides with the classic function ϕ_*^1, and this is the same for a degree of 2, while for other degrees the ψ_*^{d-2} function is different [14] from ϕ_*^{d-2}. The ψ_*^{d-2} functions will be made explicit later in relation [1.19].

In shortened form, this interpolation can be written as:

$$\boxed{\sigma_{1d}+\sigma_{d1}+\sigma_{d-2,d-2}-\sigma_{1,d-2}-\sigma_{d-2,1}}\,.$$

This interpolation is a generalization of the transfinite interpolation in the presence of internal nodes. We will establish that this interpolation has the right properties, that is it covers the serendipity space. We first show that all the monomials $u^k v$ or uv^k are present. To do this, we write the shape functions for the reduced element, denoted by $\phi_{*,*}^{s,d}(u,v)$, in terms of the ϕ and the ψ of different degrees. To conclude, we express the shape functions uniquely as functions of ϕ.

The four kinds of shape functions

The $\phi_{*,*}^{s,d}(u,v)$ are of four types, corresponding to corners, index $_{00}$ and similar indices, to the "first" nodes, index $_{10}$ and similar indices, to the edge nodes with the other index between 2 and $d-2$, like $_{20,30,\ldots}$, to the internal nodes, the two indices between 2 and $d-2$, like $_{22,23,\ldots}$. The shape functions are expressed as a function of the classic shape functions, the ϕ functions, and the (non-classic) shape functions, the ψ functions. These are written as classic functions, but are cancelled out at the nodes of a different distribution (non-uniform). See the example of $\psi_0^3(u)$. The $\phi_{00}^{s,d}(u,v)$ function is that which comes up with regard to A_{00}, thus:

$$\phi_{00}^{s,d}(u,v)=\psi_0^1(u)\phi_0^d(v)+\phi_0^d(u)\psi_0^1(v)+\psi_0^{d-2}(u)\psi_0^{d-2}(v)-\psi_0^1(u)\psi_0^{d-2}(v)-\psi_0^{d-2}(u)\psi_0^1(v),$$

where, as we have seen, $\psi_*^1=\phi_*^1$, thus:

$$\phi_{00}^{s,d}(u,v)=\phi_0^1(u)\phi_0^d(v)+\phi_0^d(u)\phi_0^1(v)+\psi_0^{d-2}(u)\psi_0^{d-2}(v)-\phi_0^1(u)\psi_0^{d-2}(v)-\psi_0^{d-2}(u)\phi_0^1(v).$$

From this, we deduce the three other analogous functions:

$$\phi_{d0}^{s,d}(u,v)=\phi_1^1(u)\phi_0^d(v)+\phi_d^d(u)\phi_0^1(v)+\psi_{d-2}^{d-2}(u)\psi_0^{d-2}(v)-\phi_1^1(u)\psi_0^{d-2}(v)-\psi_{d-2}^{d-2}(u)\phi_0^1(v),$$

$$\phi_{0d}^{s,d}(u,v)=\phi_0^1(u)\phi_d^d(v)+\phi_0^d(u)\phi_1^1(v)+\psi_0^{d-2}(u)\psi_{d-2}^{d-2}(v)-\phi_0^1(u)\psi_{d-2}^{d-2}(v)-\psi_0^{d-2}(u)\phi_1^1(v),$$

14. Indeed, for example, for a degree $d=5$ and for the index 0, we have $\psi_0^3(u)=\frac{1}{6}(2-5u)(3-5u)(1-u)$ and this function has a value of 1 for $u=0$ and is cancelled out at $\frac{2}{5}$, $\frac{3}{5}$ and 1, while $\phi_0^3(u)=\frac{1}{2}(1-3u)(2-3u)(1-u)$ which has a value of 1 for u and is cancelled out at $\frac{1}{3}$, $\frac{2}{3}$ and 1.

$$\phi_{dd}^{s,d}(u,v)=\phi_1^1(u)\phi_d^d(v)+\phi_d^d(u)\phi_1^1(v)+\psi_{d-2}^{d-2}(u)\psi_{d-2}^{d-2}(v)-\phi_1^1(u)\psi_{d-2}^{d-2}(v)-\psi_{d-2}^{d-2}(u)\phi_1^1(v).$$

The function $\phi_{10}^{s,d}(u,v)$ is the function that is seen with respect to A_{10}, that is $\phi_{10}^{s,d}(u,v) = \phi_1^d(u)\phi_0^1(v)$, from this we also deduce the three analogous functions:

$\phi_{d-1,0}^{s,d}(u,v) = \phi_{d-1}^d(u)\phi_0^1(v)$, $\phi_{1d}^{s,d}(u,v) = \phi_1^d(u)\phi_1^1(v)$ and $\phi_{d-1,d}^{s,d}(u,v) = \phi_{d-1}^d(u)\phi_1^1(v)$

and, by symmetry, the similar relations obtained starting from $\phi_{01}^{s,d}(u,v) = \phi_0^1(u)\phi_1^d(v)$, that is:

$\phi_{0,d-1}^{s,d}(u,v) = \phi_0^1(u)\phi_{d-1}^d(v)$, $\phi_{d1}^{s,d}(u,v) = \phi_1^1(u)\phi_1^d(v)$ and $\phi_{d,d-1}^{s,d}(u,v) = \phi_1^1(u)\phi_{d-1}^d(v)$.

The $\phi_{20}^{s,d}(u,v)$ function is that which is seen with regard to A_{20}, thus, simply:

$$\phi_{20}^{s,d}(u,v) = \phi_{l(1),0}^{s,d}(u,v) = \phi_2^d(u)\phi_0^1(v) + \psi_1^{d-2}(u)\psi_0^{d-2}(v) - \psi_1^{d-2}\phi_0^1(v),$$

from which we deduce:

$$\phi_{30}^{s,d}(u,v) = \phi_{l(2),0}^{s,d}(u,v) = \phi_3^d(u)\phi_0^1(v) + \psi_2^{d-2}(u)\psi_0^{d-2}(v) - \psi_2^{d-2}\phi_0^1(v),$$

etc., up to the index $d-2$. Similarly for $\phi_{2d}^{s,d}(u,v)$, ..., $\phi_{02}^{s,d}(u,v)$, ... $\phi_{d2}^{s,d}(u,v)$, ...

The central functions come uniquely from the middle term in the general definition. That is:

$$\phi_{l(i),l(j)}^{s,d}(u,v) = \psi_i^{d-2}(u)\psi_j^{d-2}(v).$$

The serendipity space is covered

By construction, relation [1.18], the $\phi_{*,*}^{s,d}(u,v)$ cover the monomials in the serendipity space. Indeed, it is enough to show the typical monomials as we see in diagram [1.3.2] for a degree of 5, via the following interpretation:

– two diagonals covering the monomials u^k and $u^k v$ as well as their two counterparts for v^k and uv^k, for $k = 0, ..., d$,

– the central diamond covering the monomials $u^k v^l$ where $k = 0, ..., d-2$ and $l = 0, ..., d-2$,

noting that these regions overlap and that we can also restrict the second region to the diamond described by $u^k v^l$ but with $k = 2, ..., d-2$ and $l = 2, ..., d-2$. To show that these monomials of the serendipity space are covered, we will show that the first region may be expressed by essentially considering the first two terms of relation [1.18] while the second region essentially concerns the third term of this same relation.

It is obvious that the monomials u^k and $u^k v$, with $k = 0, ..., d$, may be expressed as linear combinations of the only $\phi_i^d(u)\psi_j^1(v)$ of the second term of the relation, as the $\phi_i^d(u)$ form a polynomial basis with a degree lower than or equal to d and the $\psi_j^1(v)$ form a basis of polynomials with a degree lower than or equal to 1. We will show that we can choose a specific combination of other monomials (of other terms) of the complete relation that ensure that the total combination remains the initial combination with the $\phi_i^d(u)\psi_j^1(v)$. First, we fix a combination of monomials

$$\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & u & & v & & & & \\
 & & & u^2 & & uv & & v^2 & & & \\
 & & u^3 & & u^2v & & uv^2 & & v^3 & & \\
 & u^4 & & u^3v & & u^2v^2 & & uv^3 & & v^4 & \\
u^5 & & u^4v & & u^3v^2 & & u^2v^3 & & uv^4 & & v^5 \\
 & u^5v & & & & u^3v^3 & & & & uv^5 &
\end{array}$$

Table 1.1. *The diagram of the basis monomials for a degree of 5. We find the monomials up to the degree of 5 complemented (last line) by some monomials of degree 6, whence the monomial u^3v^3 (which is added to the classic monomials of the serendipity elements due to the symmetry imposed in our definition)*

$\psi_i^1(u)\phi_1^d(v)$ and $\psi_i^1(u)\phi_{d-1}^d(v)$ such that the first term is identical to the fourth term (so that the $\phi_j^d(v)$ degenerate to $\psi_j^{d-2}(v)$). We then consider a combination of monomials $\psi_j^{d-2}(v)$ for $2 \leq j \leq d-2$ such that $\psi_j^{d-2}(v)$ degenerates to $\psi_j^1(v)$ ensuring the equality of the third and fourth terms. Thus, because of these combinations, only the second term yielding the desired combination remains. By symmetry, we obtain an analogous result for the monomials v^k and uv^k.

As concerns the central diamond, it is obvious that the monomials u^kv^l with $k = 0, ..., d-2$ and $l = 0, ..., d-2$, may be expressed as linear combinations of only the monomials of the third term of the complete relationship, that is, $\psi_i^{d-2}(u)\psi_j^{d-2}(v)$, as these monomials form a polynomial basis with a degree lower than or equal to $d-2$. Similarly, we establish a combination of monomials $\phi_1^d(u)$ and $\phi_{d-1}^d(u)$ so that $\phi_i^d(u)$ coincides with $\psi_i^{d-2}(u)$ (idem at v) such that the first (respectively, second) term and the fourth (respectively, fifth) term are identical and the total relation is reduced to the third term, which yields the desired combination.

The polynomial space is, thus, exactly the classic serendipity space for $d \leq 4$, and this same space, enriched, for $d \geq 5$ by the monomial(s) u^kv^l where $k = 0, ..., d-2$, $l = 0, ..., d-2$ and $k + l \geq d + 1$. As an example, for degree 5, we saw the additional monomial u^3v^3, for degree 6, we will find the monomials u^4v^3, u^3v^4 and u^4v^4 as the additional monomials.

Expression for $\phi_{*,*}^{s,d}(u,v)$ as a function of the classic function ϕ

In the expression for the reduced shape functions, we have the terms in ϕ and in ψ that we will rewrite using the ϕ functions, of degree d. There are two cases to consider: the treatment of the first-degree ψ and the $(d-2)$-degree ψ.

We saw that $\psi_*^1(u) = \phi_*^1(u)$ and, as we will see later, it is enough to increase the degree from 1 to d. However, the treatment of the ψ with a degree of $d-2$ is more technical. Each ψ_i^{d-2} can be written as the following linear combination:

$$\psi_i^{d-2}(u) = \phi_{l(i)}^d(u) + \alpha_{l(i)}\phi_1^d(u) + \beta_{l(i)}\phi_{d-1}^d(u), \qquad [1.19]$$

where the coefficients α_i and β_i are calculated such that the degree of this combination is $d-2$. Each ψ_i^{d-2} (thus, the above combination) is also a particular Lagrange interpolant (the property on δ_{ij} is verified for $j = l(i)$). To calculate the coefficients α_i and β_i, we go back to the expression for the complete shape functions, that is[15]:

$$\phi_i^d(u) = (-1)^i \frac{d^d C_i^d}{d!} \prod_{\substack{l=0 \\ l \neq i}}^{l=d} (\frac{l}{d} - u),$$

an expression in which the index i is mute and will, in due course, have the value $l(i)$. In this expression, the coefficient of the term in u^d is:

$$(-1)^i \frac{d^d C_i^d}{d!} (-1)^d = (-1)^{d-i} \frac{d^d C_i^d}{d!},$$

and that in u^{d-1} is:

$$(-1)^i \frac{d^d C_i^d}{d!} (-1)^{d-1} \sum_{\substack{l=0 \\ l \neq i}}^{d} \frac{l}{d} = (-1)^{d+i-1} \frac{d^d C_i^d}{d!} \frac{1}{d} \left\{ \frac{d(d+1)}{2} - i \right\}.$$

We also enforce that the terms at u^d and u^{d-1} of the combination are null. We then obtain:

$$(-1)^{d-i} C_i^d + \alpha_i (-1)^{d-1} C_1^d + \beta_i (-1)^1 C_{d-1}^d = 0.$$

This gives a first equation, that is:

$$(E_1) \qquad (-1)^i \frac{C_i^d}{d} - \alpha_i + (-1)^{d-1} \beta_i = 0.$$

The second equation is written as:

$$(-1)^{d+i-1} C_i^d \left\{ \frac{d(d+1)}{2} - i \right\} + (-1)^d C_1^d \left\{ \frac{d(d+1)}{2} - 1 \right\} \alpha_i + C_{d-1}^d \left\{ \frac{d(d+1)}{2} - (d-1) \right\} \beta_i = 0.$$

Upon multiplying by $(-1)^{1-d}$, we obtain:

$$(-1)^i C_i^d \left\{ \frac{d(d+1)}{2} - i \right\} - C_1^d \left\{ \frac{d(d+1)}{2} - 1 \right\} \alpha_i + (-1)^{d-1} C_{d-1}^d \left\{ \frac{d(d+1)}{2} - (d-1) \right\} \beta_i = 0,$$

an equation that contains the preceding one and is thus reduced to:

$$(-1)^i i C_i^d - C_1^d \alpha_i + (-1)^{d-1} C_{d-1}^d (d-1) \beta_i = 0,$$

and finally, this second equation can be reduced to:

$$(E_2) \qquad (-1)^i i \frac{C_i^d}{d} - \alpha_i + (-1)^{d-1} (d-1) \beta_i = 0.$$

15. The coefficient of the binomial is denoted by C_i^d with a degree d as exponent. This is in order to be homogenous during our writing of Bernstein polynomials where d is also an exponent.

Thus, the system to be solved is:

$$\begin{cases} (E_1) & (-1)^i \frac{C_i^d}{d} - \alpha_i + (-1)^{d-1}\beta_i = 0 \\ (E_2) & (-1)^i i \frac{C_i^d}{d} - \alpha_i + (-1)^{d-1}(d-1)\beta_i = 0, \end{cases}$$

which gives the solution, for $i = 0, d$ and different from 1 (for α_i) and from $d-1$ (for β_i; $(E_2) - (E_1)$ gives β_i, we then calculate α_i either by carrying over or by symmetry. We find (for the indices $l(i)$):

$$\boxed{\alpha_{l(i)} = (-1)^{l(i)} C_{l(i)}^d \frac{d-1-l(i)}{d(d-2)}} \quad \text{and} \quad \boxed{\beta_{l(i)} = (-1)^{d-l(i)} C_{l(i)}^d \frac{l(i)-1}{d(d-2)}}, \qquad [1.20]$$

noting that we can also define these coefficients for the values 1 and $d-1$ of the indices (values not covered by $l(i)$ when i varies from 0 to $d-2$) and that, therefore, $\alpha_{d-1} = 0$, $\beta_1 = 0$ and $\alpha_1 = \beta_{d-1} = -1$. We can thus write the formulae [1.20] with the index i, for $0 \le i \le d$. Let us note that the coefficients are symmetrical, $\alpha_0 = \beta_d$, ..., $\alpha_d = \beta_0$.

We use this strategy to treat the ψ^{d-2} functions. We can then express the typical shape functions of the reduced element. Thus, for the "corner" functions, here with an index $_{00}$, we have:

$$\phi_{00}^{s,d}(u,v) \;=\; \phi_0^1(u)\phi_0^d(v) + \phi_0^d(u)\phi_0^1(v) + \psi_0^{d-2}(u)\psi_0^{d-2}(v) - \phi_0^1(u)\psi_0^{d-2}(v) - \psi_0^{d-2}(u)\phi_0^1(v),$$

which is expressed as:

$$\begin{aligned} \boxed{\phi_{00}^{s,d}(u,v)} \;&=\; \phi_0^1(u)\phi_0^d(v) + \phi_0^d(u)\phi_0^1(v) \\ &+ \left\{\phi_0^d(u) + \alpha_0\phi_1^d(u) + \beta_0\phi_{d-1}^d(u)\right\}\left\{\phi_0^d(v) + \alpha_0\phi_1^d(v) + \beta_0\phi_{d-1}^d(v)\right\} \\ &- \phi_0^1(u)\left\{\phi_0^d(v) + \alpha_0\phi_1^d(v) + \beta_0\phi_{d-1}^d(v)\right\} - \left\{\phi_0^d(u) + \alpha_0\phi_1^d(u) + \beta_0\phi_{d-1}^d(u)\right\}\phi_0^1(v). \end{aligned}$$

The other functions of this type can be directly obtained. For example, using symmetry we have, for u:

$$\begin{aligned} \phi_{d0}^{s,d}(u,v) \;&=\; \phi_{00}^{s,d}(1-u,v) \;=\; \phi_0^1(1-u)\phi_0^d(v) + \phi_0^d(1-u)\phi_0^1(v) \\ &+ \left\{\phi_0^d(1-u) + \alpha_0\phi_1^d(1-u) + \beta_0\phi_{d-1}^d(1-u)\right\}\left\{\phi_0^d(v) + \alpha_0\phi_1^d(v) + \beta_0\phi_{d-1}^d(v)\right\} \\ &- \phi_0^1(1-u)\left\{\phi_0^d(v) + \alpha_0\phi_1^d(v) + \beta_0\phi_{d-1}^d(v)\right\} - \left\{\phi_0^d(1-u) + \alpha_0\phi_1^d(1-u) + \beta_0\phi_{d-1}^d(1-u)\right\}\phi_0^1(v), \end{aligned}$$

$$\begin{aligned} \text{therefore} \quad \phi_{d0}^{s,d}(u,v) \;&=\; \phi_1^1(u)\phi_0^d(v) + \phi_d^d(u)\phi_0^1(v) \\ &+ \left\{\phi_d^d(u) + \alpha_0\phi_{d-1}^d(u) + \beta_0\phi_1^d(u)\right\}\left\{\phi_0^d(v) + \alpha_0\phi_1^d(v) + \beta_0\phi_{d-1}^d(v)\right\} \\ &- \phi_1^1(u)\left\{\phi_0^d(v) + \alpha_0\phi_1^d(v) + \beta_0\phi_{d-1}^d(v)\right\} - \left\{\phi_d^d(u) + \alpha_0\phi_{d-1}^d(u) + \beta_0\phi_1^d(u)\right\}\phi_0^1(v). \end{aligned}$$

The second typical function is that of the index $_{10}$, whose expression is particularly simple:

$$\boxed{\phi_{10}^{s,d}(u,v)} \;=\; \phi_1^d(u)\psi_0^1(v) \;=\; \phi_1^d(u)\phi_0^1(v).$$

The third type of function is that of the functions associated with the edge nodes between the index 2 and the index $d-2$. For example, for the index $_{20}$. we find:

$$\boxed{\phi_{20}^{s,d}(u,v)} \;=\; \phi_{l(1),0}^{s,d}(u,v) \;=\; \phi_2^d(u)\psi_0^1(v) + \psi_1^{d-2}(u)\psi_0^{d-2}(v) - \psi_1^{d-2}(u)\phi_0^1(v).$$

$$\text{Therefore} \quad \phi_{20}^{s,d}(u,v) \;=\; \phi_2^d(u)\phi_0^1(v)$$
$$+ \left\{\phi_2^d(u) + \alpha_2\phi_1^d(u) + \beta_2\phi_{d-1}^d(u)\right\} \left\{\phi_0^d(v) + \alpha_0\phi_1^d(v) + \beta_0\phi_{d-1}^d(v)\right\}$$
$$- \left\{\phi_2^d(u) + \alpha_2\phi_1^d(u) + \beta_2\phi_{d-1}^d(u)\right\} \phi_0^1(v).$$

Finally, the functions associated with the internal nodes are written as:

$$\boxed{\phi_{l(i),l(j)}^{s,d}(u,v)} \;=\; \psi_i^{d-2}(u)\psi_j^{d-2}(v)$$
$$= \left\{\phi_{l(i)}^d(u) + \alpha_{l(i)}\phi_1^d(u) + \beta_{l(i)}\phi_{d-1}^d(u)\right\} \left\{\phi_{l(j)}^d(v) + \alpha_{l(j)}\phi_1^d(v) + \beta_{l(j)}\phi_{d-1}^d(v)\right\}.$$

As with $d = 3$, the general formula is similar[16] to a second-degree formula. This case can be worked through either via the above formulae or directly.

- Third-degree quadrilateral. The element is written with $\phi_i(.)$ and the nodes as:

$$\sum_{i=0}^{1}\sum_{j=0}^{3}\phi_i^1(u)\phi_j^3(v)A_{k(i),j} + \sum_{i=0}^{3}\sum_{j=0}^{1}\phi_i^3(u)\phi_j^1(v)A_{i,k(j)} - \sum_{i=0}^{1}\sum_{j=0}^{1}\phi_i^1(u)\phi_j^1(v)A_{k(i),k(j)}.$$

We mechanically repeat the construction of the second degree. In particular, we make use of the fact that:

$$\frac{3}{3}\phi_0^3(u) + \frac{2}{3}\phi_1^3(u) + \frac{1}{3}\phi_2^3(u) + \frac{0}{3}\phi_3^3(u) = \phi_0^1(u)$$
$$\text{and that} \quad \frac{0}{3}\phi_0^3(u) + \frac{1}{3}\phi_1^3(u) + \frac{2}{3}\phi_2^3(u) + \frac{3}{3}\phi_3^3(u) = \phi_1^1(u),$$

and from this we deduce that, for example, the first term of the expression, σ_{13}, is formulated as:

$$\sigma_{13} = \sum_{I=0}^{3}\sum_{j=0}^{3}\phi_I^3(u)\phi_j^3(v)Q_{Ij},$$

$$\text{where} \quad Q_{0j} = A_{k(0),j} = A_{0j}, Q_{Ij} = \frac{(3-I)A_{0j} + I\,A_{3j}}{3},\; I = 1,2 \text{ and } Q_{3j} = A_{k(1),j} = A_{3j}.$$

The second term is treated in the same manner as the last term. By rewriting the initial formula, term by term, to the degree of 3 using Q_{ij} and then replacing these based on the initial nodes, we find a complete statement to a degree of 3 by constructing the four missing nodes. For example, the construction of the node that is naturally denoted by A_{11} corresponds to the following schema:

```
03     13     23    33            -2/9     1/3    0   -1/9

02                  32              0                   0

01    [11]          31             2/3    [11]         1/3

00     10     20    30            -4/9     2/3    0   -2/9
```

16. The functions ϕ and ψ are identical.

With this particular node, A_{11}, as well as A_{21}, A_{22} and A_{12}, the four made-up nodes, and the initial nodes, the element can be written as a complete patch:

$$\sum_{i=0}^{3}\sum_{j=0}^{3}\phi_i^3(u)\phi_j^3(v)A_{ij} = \sum_{i=0}^{3}\sum_{j=0}^{3}p_{ij}^3(u,v)A_{ij}.$$

This construction makes it possible to explicitly calculate the reduced shape forms. Let us recall that $\phi_0(u) = \frac{1}{2}(1-3u)(2-3u)(1-u)$, $\phi_1(u) = \frac{9}{2}u(2-3u)(1-u)$ and that $\phi_2(u) = -\frac{9}{2}u(1-3u)(1-u)$. The first shape function, index 00, is the polynomial that comes up with respect to A_{00}, and thus via A_{00} itself, with the weight 1, and via A_{11}, A_{21}, A_{12} and A_{22}, with the weights $-\frac{4}{9}, -\frac{2}{9}, -\frac{2}{9}$ and $-\frac{1}{9}$, that is:

$$p_{00}^{s,3}(u,v) = p_{00}(u,v) - \frac{4}{9}p_{11}(u,v) - \frac{2}{9}p_{21}(u,v) - \frac{2}{9}p_{12}(u,v) - \frac{1}{9}p_{22}(u,v),$$

$$= \frac{1}{4}(1-3u)(2-3u)(1-u)(1-3v)(2-3v)(1-v) - \frac{4}{9}\frac{9}{2}u(2-3u)(1-u)\frac{9}{2}v(2-3v)(1-v)$$

$$+\frac{2}{9}\frac{9}{2}u(1-3u)(1-u)\frac{9}{2}v(2-3v)(1-v) + \frac{2}{9}\frac{9}{2}u(2-3u)(1-u)\frac{9}{2}v(1-3v)(1-v)$$

$$-\frac{1}{9}\frac{9}{2}u(1-3u)(1-u)\frac{9}{2}v(1-3v)(1-v)$$

$$= \frac{9}{2}(1-u)(1-v)\left\{\frac{2}{9}\frac{1}{4}(1-3u)(2-3u)(1-3v)(2-3v) - 2u(2-3u)v(2-3v)\right.$$

$$\left. +u(1-3u)v(2-3v) + u(2-3u)v(1-3v) - \frac{1}{2}u(1-3u)v(1-3v)\right\}$$

$$= \frac{9}{2}(1-u)(1-v)(\frac{2}{9} - u - v + u^2 + v^2).$$

The other typical function is that of index 10, which is in a similar manner obtained via A_{10}, A_{11} and A_{12}, therefore:

$$p_{10}^{s,3}(u,v) = p_{10}(u,v) + \frac{2}{3}p_{11}(u,v) + \frac{1}{3}p_{12}(u,v),$$

$$= \phi_1^3(u)\left\{\phi_0^3(v) + \frac{2}{3}\phi_1^3(v) + \frac{1}{3}\phi_2^3(v)\right\} = \phi_1^3(u)\phi_0^1(v) = \frac{9}{2}u(2-3u)(1-u)(1-v).$$

We establish the expression for the reduced shape functions using symmetry.

It must be noted that the initial relation directly gives the result, that is:

$$p_{00}^{s,3}(u,v) = \phi_0^1(u)\phi_0^3(v) + \phi_0^3(u)\phi_0^1(v) - \phi_0^1(u)\phi_0^1(v)$$

$$= \frac{1}{2}(1-u)(1-3v)(2-3v)(1-v) + \frac{1}{2}(1-3u)(2-3u)(1-u)(1-v) - (1-u)(1-v)$$

$$= \frac{1}{2}(1-u)(1-v)\left\{(1-3v)(2-3v)+(1-3u)(2-3u)-2\right\}$$

$$= \frac{1}{2}(1-u)(1-v)\left\{2-9u-9v+9u^2+9v^2\right\},$$

as above. And for the index 10, we directly find that:

$$p_{10}^{s,3}(u,v) = \phi_1^3(u)\phi_0^1(v),$$

as for degree 2.

• Quadrilateral of degree $d \geq 4$. The element is written with $\phi_i(.)$, $\psi_i(.)$ and nodes via formula [1.18].

As seen already, the formula involves first-degree functions and functions with a degree of $d-2$.

Rewriting the ψ_i functions using the classic ϕ_i functions also makes it possible to reconstruct a complete element equivalent to the reduced element by inventing the missing nodes, the other nodes being identical (to the initial nodes). These missing nodes are linear combinations of the nodes of the reduced element and are obtained by the same process of moving from ψ_i to ϕ_i from degree 1 or $d-2$ to the degree d. Further on, we will return to this process to establish these combinations.

***Degree of* 1 *to* d**

From a degree of 1, we can write an expression for a degree d. These formulae have already been seen:

$$\boxed{\sum_{k=0}^{d} \frac{d-k}{d}\phi_k^d(u) = \phi_0^1(u)} \quad \text{and} \quad \boxed{\sum_{k=0}^{d} \frac{k}{d}\phi_k^d(u) = \phi_1^1(u)}.$$

Thus:

$$\sum_{i=0}^{1} \phi_i^1(u)A_{k(i)} = \sum_{i=0}^{d} \phi_i^d(u)\frac{(d-i)A_0 + iA_d}{d} = \sum_{i=0}^{d} \phi_i^d(u)Q_i,$$

where $Q_i = \frac{(d-i)A_0+iA_d}{d}$. This rewritten form will be used to work on the terms at u or at v for σ_{1d}, σ_{d1}, $\sigma_{1,d-2}$ and $\sigma_{d-2,1}$ in relation [1.18].

***Degree* $d-2$ *to* d**

To work with the degree $d-2$, we once again consider, at a degree of d and with i different from 1 and from $d-1$, the linear combination $\psi_i^{d-2}(u) = \phi_{l(i)}^d(u) + \alpha_{l(i)}\phi_1^d(u) + \beta_{l(i)}\phi_{d-1}^d(u)$, with the coefficients α_i and β_i calculated such that the degree falls from 2. Thus, as calculated above, relations [1.20]:

$$\boxed{\alpha_{l(i)} = (-1)^{l(i)} C_{l(i)}^d \frac{d-1-l(i)}{d(d-2)}} \quad \text{and} \quad \boxed{\beta_{l(i)} = (-1)^{d-l(i)} C_{l(i)}^d \frac{l(i)-1}{d(d-2)}}.$$

Let us take a curve of degree d, which is written as $\sum_{i=0}^{d} \phi_i^d(u) A_i$ where the nodes other than A_1 and A_{d-1} are assumed to be known and we construct the nodes A_1 and A_{d-1} using the formulae:

$$A_1 = \sum_{i \neq 1, i \neq d-1} \alpha_i A_i \quad \text{and} \quad A_{d-1} = \sum_{i \neq 1, i \neq d-1} \beta_i A_i. \qquad [1.21]$$

We then have:

$$\sum_{i=0}^{d} \phi_i^d(u) A_i = \phi_o^d(u) A_0 + \sum_{\substack{i \neq 1 \\ i \neq d-1}} \alpha_i \phi_1^d(u) A_i + \sum_{i=2,d-2} \phi_i^d(u) A_i + \sum_{\substack{i \neq 1 \\ i \neq d-1}} \beta_i \phi_{d-1}^d(u) A_i + \phi_d^d(u) A_d,$$

which we group at $A_0, A_2, ..., A_{d-2}$ and A_d and we find:

$$\sum_{i=0}^{d} \phi_i^d(u) A_i = \sum_{i=0}^{d-2} \psi_i^{d-2}(u) A_{l(i)},$$

which is, therefore, a curve of degree $d-2$. This shortcut will make it possible to work on the degrees $d-2$ in formula [1.18] by inventing the missing nodes via formulae [1.21], making it possible to write an expression for a degree d.

When applied to a curve, this result gives the relation:

$$\sum_{i=0}^{d-2} \psi_i^{d-2}(u) A_{l(i)} = \sum_{i=0}^{d} \phi_i^d(u) Q_i,$$

where:

$$Q_0 = A_0, Q_1 = \sum_{\substack{i \neq 1 \\ i \neq d-1}} \alpha_i A_i,\ Q_i = A_i, i = 2, d-2,\ Q_{d-1} = \sum_{\substack{i \neq 1 \\ i \neq d-1}} \beta_i A_i \quad \text{and} \quad Q_d = A_d.$$

The tensor nature of the elements makes it possible to use this established result for curves in order to determine the missing nodes of the complete equivalent element[17]. Thus, we first complete the isocurves constant in v to obtain $Q_{i,1}$ and $Q_{i,d-1}$ for $i \neq 1$ and $i \neq d-1$. We then complete the isocurves constant in u to obtain the $Q_{1,j}$ and the $Q_{d-1,j}$ for all j (which fills all the missing nodes in the first phase).

As an illustration, we explain the case of fourth- and fifth-degree quadrilaterals and, for degree 4, we graphically explain relation [1.18], which gives the following diagram:

17. The complete equivalent element, see Chapter 3, will be used to study the geometric validity of the reduced element.

```
04  .   .   .  44       04  14  24  34  44       04  .   24  .   44

03             43       .               .        .               .

02             42   +   .               .    +   02      22      42

01             41       .               .        .               .

00  .   .   .  40       00  10  20  30  40       00   .  20  .   40
```

which corresponds to σ_{14}, σ_{41} and σ_{22}, terms to which we add two correction terms that correspond to σ_{12} and σ_{21}, that is:

```
      04  .   .   .  44       04  .   24  .   44

      .              .        .               .

  -   02             42   -   .               .

      .              .        .               .

      00  .   .   .  40       00  .   20  .   40
```

The degree 4

This element has 17 nodes of which one is internal (against 25, for the complete element). The coefficients α_i have a value of $\alpha_0 = \frac{3}{8}, \alpha_2 = \frac{6}{8}, \alpha_3 = 0$ and $\alpha_4 = -\frac{1}{8}$. Using these values, we have the following distribution schema for A_{11}:

```
04    14    24    34    44      -3/64   1/4   -18/64    0     5/64

03                      43        0                            0

02          22          42     -18/64          36/64        -18/64

01   [11]               41       3/4   [11]                   1/4

00    10    20    30    40     -27/64   3/4   -18/64    0    -3/64
```

and, by symmetry, we can find A_{13}, A_{31} and A_{33}. To give the example of a non-corner missing node, let us consider A_{21}. We thus find the distribution shown in the schema:

04	14	24	34	44	1/16	0	-1/8	0	1/16
03				43	0				0
02		22		42	-6/16		6/8		-6/16
01		[21]		41	1/2		[21]		1/2
00	10	20	30	40	-3/16	0	3/8	0	-3/16

and, by symmetry, we find A_{12}, A_{32} and A_{23}.

We can now make explicit the typical functions $\phi_{00}^{s,4}(u,v)$, $\phi_{10}^{s,4}(u,v)$, $\phi_{20}^{s,4}(u,v)$ and $\phi_{22}^{s,4}(u,v)$, either via the general formula or via the weights that are thus found.

For $\phi_{00}^{s,4}(u,v)$, via the general formula (we must calculate α_0 and β_0) we obtain:

$$\alpha_0 = \frac{3}{8} \quad \text{and} \quad \beta_0 = -\frac{1}{8}.$$

We than have the expression:

$$\begin{aligned}
\phi_{00}^{s,4}(u,v) \;=&\; \phi_0^1(u)\phi_0^4(v) + \phi_0^4(u)\phi_0^1(v) \\
&+ \left\{\phi_0^4(u) + \frac{3}{8}\phi_1^4(u) - \frac{1}{8}\phi_3^4(u)\right\}\left\{\phi_0^4(v) + \frac{3}{8}\phi_1^4(v) - \frac{1}{8}\phi_3^4(v)\right\} \\
&-\phi_0^1(u)\left\{\phi_0^4(v) + \frac{3}{8}\phi_1^4(v) - \frac{1}{8}\phi_3^4(v)\right\} - \left\{\phi_0^4(u) + \frac{3}{8}\phi_1^4(u) - \frac{1}{8}\phi_3^4(u)\right\}\phi_0^1(v) \\
=&\; \left\{\phi_0^4(u) + \frac{3}{8}\phi_1^4(u) - \frac{1}{8}\phi_3^4(u)\right\}\left\{\phi_0^4(v) + \frac{3}{8}\phi_1^4(v) - \frac{1}{8}\phi_3^4(v)\right\} \\
&-\phi_0^1(u)\left\{\frac{3}{8}\phi_1^4(v) - \frac{1}{8}\phi_3^4(v)\right\} - \left\{\frac{3}{8}\phi_1^4(u) - \frac{1}{8}\phi_3^4(u)\right\}\phi_0^1(v),
\end{aligned}$$

as:

$$\phi_0^1(u) = \sum_{k=0}^{4} \frac{4-k}{4}\phi_k^4(u) = \phi_0^4(u) + \frac{3}{4}\phi_1^4(u) + \frac{2}{4}\phi_2^4(u) + \frac{1}{4}\phi_3^4(u),$$

we can express everything using classic functions of a degree of 4 and we find:

$$\text{at} \quad \phi_0^4(u)\phi_0^4(v) : 1,$$

$$\text{at} \quad \phi_1^4(u)\phi_1^4(v) : \frac{9}{64} - \frac{3}{4}\frac{3}{8} - \frac{3}{8}\frac{3}{4} = -\frac{27}{64},$$

$$\text{at} \quad \phi_2^4(u)\phi_1^4(v) : -\frac{2}{4}\frac{3}{8} = -\frac{3}{16},$$

etc. We can also, as is obvious, directly find the weights of the two preceding diagrams.

At the end of the explanation, we start from:

$$\phi_0^1(u) = 1 - u,$$

$$\phi_0^4(u) = \frac{1}{6}(1-4u)(2-4u)(3-4u)(1-u) = \frac{1}{3}(1-4u)(1-2u)(3-4u)(1-u),$$

$$\phi_1^4(u) = -\frac{1}{6}(-4u)(2-4u)(3-4u)(4-4u) = \frac{16}{3}u(1-2u)(3-4u)(1-u),$$

$$\phi_3^4(u) = -\frac{1}{6}(-4u)(1-4u)(2-4u)(4-4u) = \frac{16}{3}u(1-4u)(1-2u)(1-u),$$

and $(1-u)(1-v)$ is factorized. We thus obtain, via a formal system of calculation, the expression (for details, see [George *et al.* 2014a]):

$$\phi_{00}^{s,4}(u,v) \;=\; \frac{1}{3}(1-u)(1-v)(3-22u-22v+48u^2+12uv+48v^2-32u^3-32v^3).$$

Similarly, from the same reference, we have:

$$\phi_{10}^{s,4}(u,v) \;=\; \frac{16}{3}u(1-u)(1-2u)(3-4u)(1-v),$$

$$\phi_{20}^{s,4}(u,v) \;=\; 4u(1-u)(1-v)(-3+16u-2v-16u^2),$$

$$\phi_{22}^{s,4}(u,v) \;=\; 16u(1-u)v(1-v).$$

The degree 5

This element has 24 nodes of which four are internal (as opposed to 36 for the complete element) and is different from the classic element [Arnold, Awanou-2011], [Floater, Gillette-2014], which has only 23 nodes, and thus a non-symmetry. This makes it a non-symmetrical element when the degrees of freedom are of a nodal value(s) type. The α_i sequence is $\alpha_0 = \frac{4}{15}, \alpha_2 = \frac{20}{15}, \alpha_3 = -\frac{10}{15}, \alpha_4 = 0$ and $\alpha_5 = \frac{1}{15}$, which gives the following schemas for A_{11}, with the factor $\frac{1}{45}$:

05	15	25	35	45	55		-4	9	-8	4	0	-1
04					54		0					0
03		23	33		53		16		-40	20		4
02		22	32		52		-32		80	-40		-8
01					51		36	[11]				9
00	10	20	30	40	50		-16	36	-32	16	0	-4

and for A_{21}, with the factor $\frac{1}{75}$:

05	15	25	35	45	55		-3	0	5	0	0	-2
04					54		0					0
03		23	33		53		30		-50	0		20
02		22	32		52		-60		100	0		-40
01					51		45		[21]			30
00	10	20	30	40	50		-12	0	20	0	0	-8

Starting from A_{11}, we can easily find A_{41}, A_{14} and A_{44}. From A_{21}, we can find $A_{31}, A_{42}, A_{43}, A_{34}, A_{33}, A_{12}$ and A_{13}.

Shape functions

These distribution schemas make it possible to find the explicit expression for the reduced shape functions. We once again see that the general formula also contains the distribution coefficients.

There are four types of functions: the corner functions, such as $\phi_{00}^{s,5}(u,v)$, those analogous to $\phi_{10}^{s,5}(u,v)$, those analogous to $\phi_{20}^{s,5}(u,v)$ and, finally, the four "central" functions, analogous to $\phi_{22}^{s,5}(u,v)$. These expressions are given in [George *et al.* 2014a], that is:

$$\phi_{00}^{s,5}(u,v) = \frac{1}{72}(v-1)(u-1)\left\{72 - 750u - 750v + 2625u^2 + 1250uv + 2625v^2\right.$$

$$\left.-3750u^3 - 1250u^2v - 1250uv^2 - 3750v^3 + 1875u^4 + 1250u^2v^2 + 1875v^4\right\},$$

$$\phi_{10}^{s,5}(u,v) = \frac{25}{24}u(2-5u)(3-5u)(4-5u)(1-u)(1-v),$$

$$\phi_{20}^{s,5}(u,v) = \frac{25}{36}u(5u-3)(u-1)(v-1)(12 - 75u + 25v + 75u^2 - 25v^2),$$

$$\phi_{22}^{s,5}(u,v) = q_{22}(u,v) = \frac{625}{36}uv(5v-3)(v-1)(5u-3)(u-1).$$

The case of hexahedra

A hexahedron is seen as the tensor product of a quadrilateral in the third direction. Consequently, the formulae seen for two dimensions can be extrapolated directly, which gives:

$$\boxed{\theta_{112} + \theta_{121} + \theta_{211} - 2\,\theta_{111}},$$

for $d = 2$ and, otherwise:

$$\boxed{\theta_{11d} + \theta_{1d1} + \theta_{d11} + \theta_{d-2,d-2,d-2} - \theta_{1,1,d-2} - \theta_{1,d-2,1} - \theta_{d-2,1,1}},$$

as the $\sigma_{..}(u,v)$ are replaced by the $\theta_{...}(u,v,w)$. It must be noted that the general formula, for $d = 3$ again gives a formula similar to that for degree 2 as in two dimensions.

The techniques developed can be directly applied, whether this is to increase a degree of 1 to a degree d or to interpret a curve with a degree of $d-2$ as a curve with a degree of d via the construction of the two nodes of indices 1 and $d-1$.

We conclude by giving the typical reduced form functions for the case $d = 2$ and the case $d = 3$.

The second-degree serendipity hexahedron has 20 nodes, the vertices and two nodes per edge. These two typical functions are:

$$p^s_{000}(u,v,w) = (1-u)(1-v)(1-w)(1-2u-2v-2w),$$

$$p^s_{100}(u,v,w) = 4u(1-u)(1-v)(1-w).$$

The third-degree serendipity hexahedron has 32 nodes, the vertices and two nodes per edge. These two typical functions are:

$$p^s_{000}(u,v,w) = \frac{9}{2}(1-u)(1-v)(1-w)(\frac{2}{9} - u - v - w + u^2 + v^2 + w^2),$$

$$p^s_{100}(u,v,w) = \frac{9}{2}u(2-3u)(1-u)(1-v)(1-w).$$

1.3.3. *Other elements, prisms and pyramids*

As prisms have a triangular base, only the cases where $d = 2$ and $d = 3$ are relevant. The second- and third-degree pyramids, seen as degenerate hexahedra, are, by nature, already reduced elements.

1.4. Shape functions, rational elements

Lagrange elements with a degree at least equal to 2 make it possible to have a better approximation of curved boundaries (better than straight elements, even of smaller size). On the contrary, approaching a circle (which is, nonetheless, of a degree of 2) is done using pieces of parabolas, from which comes the idea of using rational shape functions. This idea, already an old one (dating back to the 1980s at least) was combined with a modern trend, *isogeometric analysis*. Refer to [Cottrell *et al*. 2009] for example.

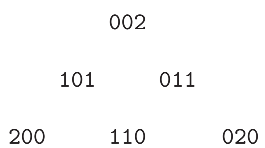
1.4.1. *Rational triangle with a degree of 2 or arbitrary degree*

The second-degree rational triangle is commonly seen as a patch and is described by a rational Bézier based on its control points. In an analogous manner, we will consider the following definition for this triangle:

$$\sigma(u,v,w) = \frac{\displaystyle\sum_{i+j+k=2} \omega_{ijk} p_{ijk}(u,v,w) A_{ijk}}{\displaystyle\sum_{i+j+k=2} \omega_{ijk} p_{ijk}(u,v,w)},$$

where the ω_{ijk} are positive or null weights (but not all null weights). By denoting the denominator by $D(u,v,w)$, that is: $D(u,v,w) = \displaystyle\sum_{i+j+k=2} \omega_{ijk} p_{ijk}(u,v,w)$, the expression becomes:

$$\sigma(u,v,w) = \frac{1}{D(u,v,w)} \sum_{i+j+k=2} \omega_{ijk} p_{ijk}(u,v,w) A_{ijk}. \qquad [1.22]$$



The shape functions, denoted by an exponent r (for "rational"), are direct. For example:

$$p^r_{200}(u,v,w) = \frac{\omega_{200}}{D(u,v,w)} p_{200}(u,v,w) \quad \text{and} \quad p^r_{110}(u,v,w) = \frac{\omega_{110}}{D(u,v,w)} p_{110}(u,v,w),$$

with the p_{ijk} seen earlier, $p_{200}(u,v,w) = u(2u-1)$ and $p_{110}(u,v,w) = 4uv$. Let us note that if the weights are equal, we find $p^r_{ijk}(u,v,w) = p_{ijk}(u,v,w)$.

We have the same shape for all degrees, that is: $p^r_{ijk}(u,v,w) = \dfrac{\omega_{ijk}}{D(u,v,w)} p_{ijk}(u,v,w)$.

We will return to these triangular elements in Chapter 3, to find a rational Bézier formulation and, thus, control the sign of their Jacobians.

1.4.2. *Rational quadrilateral of an arbitrary degree*

In a manner similar to the patches, we introduce the same definition to describe quadrilaterals of an arbitrary degree, that is:

$$\sigma(u,v) = \frac{\displaystyle\sum_{i=0}^{d}\sum_{j=0}^{d} \omega_{ij} p_{ij}(u,v) A_{ij}}{\displaystyle\sum_{i=0}^{d}\sum_{j=0}^{d} \omega_{ij} p_{ij}(u,v)}.$$

where the ω_{ijk} are positive or null weights (but not all null weights). By denoting the denominator by $D(u,v)$, or: $D(u,v) = \sum_{i=0}^{d}\sum_{j=0}^{d} \omega_{ij} p_{ij}(u,v)$, the expression becomes:

$$\sigma(u,v) = \frac{1}{D(u,v)} \sum_{i=0}^{d}\sum_{j=0}^{d} \omega_{ij} p_{ij}(u,v,w) A_{ij}. \qquad [1.23]$$

From which we deduce the expression for the shape functions: $p_{ij}^r(u,v) = \frac{\omega_{ij}}{D(u,v)} p_{ij}(u,v)$.

We will return to these quadrilateral elements in Chapter 3 as well, to find a rational Bézier formulation and, thus, control their Jacobian signs.

1.4.3. *General case, B-splines or Nurbs elements*

Rational tetrahedra and hexahedra may be expressed in exactly the same manner. For other elements, prisms and pyramids, this is less common.

There is nothing to prevent formally writing the B-splines or Nurbs elements with an identical definition. However, geometric validation (Chapter 3) of these type of elements is shown to be technically complicated, despite the fact that B-splines can be decomposed in Bézier shapes and Nurbs can be decomposed in rational Bézier curves (which, in turn, are expressed in classic Bézier curves but in an additional dimension of space). It must be noted that the literature on these elements does not mention this issue of validity, implicitly assuming that the question does not arise.

*

* *

Our description of finite elements and, more precisely, shape functions has been deliberately oriented by a very geometric view of elements, which are, therefore, considered to be patches. As such, this point of view naturally leads us to review Lagrangian formalism by transforming it into Bézier formalism. Beacuse of the underlying properties, this makes it possible to approach, in a relatively easy manner, the delicate problem of geometric validity of the elements seen either as finite elements or as geometric patches. Let us indicate the existence of several INRIA research reports that discuss in detail the subjects studied here, especially those that concern details on technical calculations that are not fully explained here.

This choice has resulted in certain important subjects being neglected. For example, the Hermite finite elements where, among the degrees of freedom, we also find derivatives. These are subjects that may be studied by themselves. These elements are richer in their interpolant aspects but, geometrically, they remain identical to Lagrange elements (classically, only those with a degree of 1) and the geometric validation is the same as for Lagrangian elements.

To conclude, even though we have discussed the case of elements of any order, in practice only first- or second-order elements (complete or reduced) are presently commonly used in concrete numerical simulations. This is especially true at the industrial level, while third-order (complete or reduced) elements are used more intensively in geometric modeling. It may be tempting to go up in order (or degree), however this greatly complicates the geometric processes involved.