PART 1

# Learning to use Pontryagin's Maximum Principle

# 1

# The Classical Calculus of Variations

## 1.1. Introduction: notations

The theory of optimal control has its roots in the calculus of variations.

The calculus of variations is an ancient branch of mathematics and one of its earliest famous problems was that of Queen Dido. According to legend, in the 9th Century BC, Queen Dido marked the borders of the town of Carthage using the skin of a single bull. She is said to have cut this skin into fine strips to form a very long leather cord with which she formed a semicircle, the two ends of which touched the coast.

Since the 18th Century, using the calculus of variations and the Euler–Lagrange equations, it has been known that if the coast is straight, then the semicircle with a fixed length is indeed the curve that maximizes an integral criterion (the area bordered by the curve and the side that is assumed to be straight, in the Dido problem), with an integral constraint (the length of the curve, in the Dido problem) and whose ends belong to a given set (the coast, assumed to be straight, in the Dido problem) or are fixed.

The term "calculus of variations" refers to a mathematical technique that considers local modifications of a solution curve and that characterizes its (local) optimality if every *admissible* variation (i.e. such that the disturbed solution passes through the same points $A$ and $B$ and verifies the same integral constraints, if any) has a criterion that is not as good. This is not exactly a "control" problem, but this very technique of "variations" is used to obtain the necessary conditions of optimality in the theory of optimal control, and would go on to give rise to the maximum principle, stated in the 1950s.

In the 19th Century, Hamilton laid the foundations of mechanics through which it was possible to calculate the trajectories, over time, of mechanical systems with $n$

degrees of freedom. This was based on the principle of least action, which is similar to a problem of calculus of variations and which was reflected in the formalism that Hamilton introduced, i.e. the minimization of the Hamiltonian. This approach proved to be very powerful as it made it possible to characterize the movements of mechanical systems using a system with $n$ first-order differential equations; the Hamiltonian was interpreted to be the total energy of the system.

The Hamiltonian considered in the maximum principle may be seen as a generalization of the Hamiltonian for mechanics for more general dynamic systems, which do not necessarily belong to mechanics and for which we also seek to characterize, over time, the trajectories that would minimize a criterion.

The aim of this chapter is to show how optimal control problems and the conditions required for the optimality of the maximum principle follow the calculus of variations (especially the Euler–Lagrange equation, which is also a necessary condition for optimality) and in what way they are more general and, therefore, more "difficult".

In order to do this, we will begin by going back to the well-known concept of the derivative of a function to revise in detail how the existence of a minimum can be related to the nullity of partial derivatives *when the minimum is in the domain of interest*, and what happens to this condition when the minimum is attained on a boundary point. We will then see that the classic Euler–Lagrange conditions may be seen as the necessary conditions for optimality *when the optimum is attained at a point in the interior of the domain of interest*, while the maximum principle (which will be briefly explained) corresponds to the case where the optimum is attained on a boundary point of the domain. We will use usual writing conventions. The reader can refer to section A1.1 (Appendix 1) for more information on these.

## 1.2. Minimizing a function

### 1.2.1. *Minimum of a function of one variable*

Let $f : \mathbb{R} \mapsto \mathbb{R}$ be a differentiable function whose minimum is attained at a point $x_0$. We then know that the derivative of $f$ is null at this point.

$$\left\{ f(x_0) = \min_{x \in \mathbb{R}} f(x) \right\} \Longrightarrow \left( \frac{df}{dx} \right)_{x=x_0} = 0, \qquad [1.1]$$

where $\left(\frac{df}{dx}\right)_{x=x_0}$ denotes the derivative of the function $f$, evaluated at the point $x_0$. Let us review the demonstration of this fact. According to the definition of the derivative at $x_0$, we have:

$$\left(\frac{df}{dx}\right)_{x=x_0} = \lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h} \qquad [1.2]$$

Let us write:

$$\varepsilon(h) = \frac{f(x_0+h)-f(x_0)}{h} - \left(\frac{df}{dx}\right)_{x=x_0}; \qquad \lim_{h\to 0}\varepsilon(h) = 0 \qquad [1.3]$$

Upon multiplying by $h$ we arrive at the relationship:

$$f(x_0+h)-f(x_0) = h\left(\left(\frac{df}{dx}\right)_{x=x_0} + \varepsilon(h)\right); \quad \lim_{h\to 0}\varepsilon(h) = 0 \qquad [1.4]$$

The left-hand side is either a positive value or 0 as $x_0$ realizes the minimum of $f$ and thus:

$$h > 0 \Longrightarrow \left(\left(\frac{df}{dx}\right)_{x=x_0} + \varepsilon(h)\right) \geq 0$$
$$h < 0 \Longrightarrow \left(\left(\frac{df}{dx}\right)_{x=x_0} + \varepsilon(h)\right) \leq 0$$

Consequently, since $\varepsilon(h)$ tends to 0, $\left(\frac{df}{dx}\right)_{x=x_0} = 0$.

*In this demonstration, we must recall that it is essential that $h$ can take positive or negative values.* This is clearly possible as the function is defined over all integers, $\mathbb{R}$. This would be possible even if $f$ were only defined on an open interval.

Let us assume that $f$ is always defined and differentiable over $\mathbb{R}$, but we are only interested in a closed interval $[a,b]$. We assume that:

$$f(x_0) = \min_{x\in[a,b]} f(x) \qquad [1.5]$$

The condition then becomes:

$$\begin{aligned}\left\{f(x_0) = \min_{x\in[a,b]} f(x) \text{ et } x_0 \in ]a,b[ \right\} &\Longrightarrow \left(\frac{df}{dx}\right)_{x=x_0} = 0\\ \left\{f(a) = \min_{x\in[a,b]} f(x) \right\} &\Longrightarrow \left(\frac{df}{dx}\right)_{x=a} \geq 0\\ \left\{f(b) = \min_{x\in[a,b]} f(x) \right\} &\Longrightarrow \left(\frac{df}{dx}\right)_{x=b} \leq 0\end{aligned} \qquad [1.6]$$

In effect, for the interval limits the increase, $h$, can no longer be given values that are both positive and negative. Thus, there remains only one sign condition in the derivative (see Figure 1.1). Therefore, the necessary condition for optimality takes a different form depending on whether the minimum is attained at a point in the interior or on the boundary of the domain of definition.

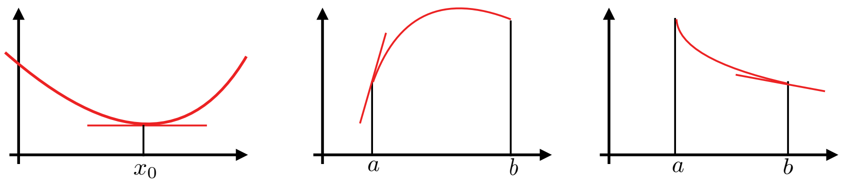


**Figure 1.1.** *Minimum of a function. For a color version of this figure, see www.iste.co.uk/harmand/bioprocesses.zip*

One final remark: what we just saw remains true if $x_0$ is a local minimum instead of a global minimum.

### 1.2.2. *Minimum of a function of two variables*

Let $D$ be a domain of $\mathbb{R}^2$ and $(x,y) \mapsto f(x,y)$ be a function of class $\mathcal{C}^1$ over $D$. Let:

$$\nabla f(x) = \begin{bmatrix} \partial_x f(x,y) \\ \partial_y f(x,y) \end{bmatrix}$$

be its gradient vector. The solutions of the differential equation:

$$\begin{bmatrix} \dfrac{dx}{dt} \\ \dfrac{dy}{dt} \end{bmatrix} = -\nabla f(x, y) \qquad [1.7]$$

are the curves of $f$ *with the steepest descent*. A contour line of $f$

$$t \mapsto (\xi(t), \eta(t))$$

is a curve such that $f(\xi(t), \eta(t))$ is constant, and thus has a null derivative. If we derive this, we arrive at:

$$\frac{d}{dt}(f(\xi(t), \eta(t)) = \partial_x f(\xi(t), \eta(t)) \frac{d\xi(t)}{dt} + \partial_y f(\xi(t), \eta(t)) \frac{d\eta(t)}{dt} = 0 \qquad [1.8]$$

As the vector:

$$\begin{bmatrix} \dfrac{d\xi}{dt} \\ \dfrac{d\eta}{dt} \end{bmatrix}$$

is the vector that is tangent to the contour line, we can read from [1.8] that the level curves are orthogonal to the gradient vectors and, thus, to the lines with the steepest descent curves.

An IGN map represents the "level curves" or the "contour lines" for the function "altitude above the point". In Figure 1.2, we have represented, on a portion of the map at 1:25,000, four lines with the greatest gradient. We see here that other than the point quoted "1397", which is a local peak, at all other points the gradient is non-null and, locally, the contour lines appear as a network of lines that are practically parallel (all along the stream in blue, the contour lines appear to no longer be differentiable, but perhaps upon enlargement, what seems angular may become smooth again). If we wish to map a two-variable function in the neighborhood of a point $(x_0, y_0)$ where the gradient is non-null, we trace a network of near-parallel lines and draw a vector perpendicular to the contour line that passes through $(x_0, y_0)$ and goes in the direction in which $f$ increases – the steepest upward line – as indicated in Figure 1.3.

If $(x_0, y_0)$ is a point in the interior of the domain, $\mathcal{D}$, where $f$ attains a local minimum, then the two partial derivatives $\partial x f(x_0, y_0)$ and $\partial y f(x_0, y_0)$ are 0: *the gradient vector at* $(x_0, y_0)$ *is 0*. This is obvious, because if $(x_0, y_0)$ is a local minimum of $f$ it is the same as $x_0$ for the function of one variable $x \mapsto f(x, y_0)$ and as $y_0$ for the single-variable function $y \mapsto f(x_0, y)$.

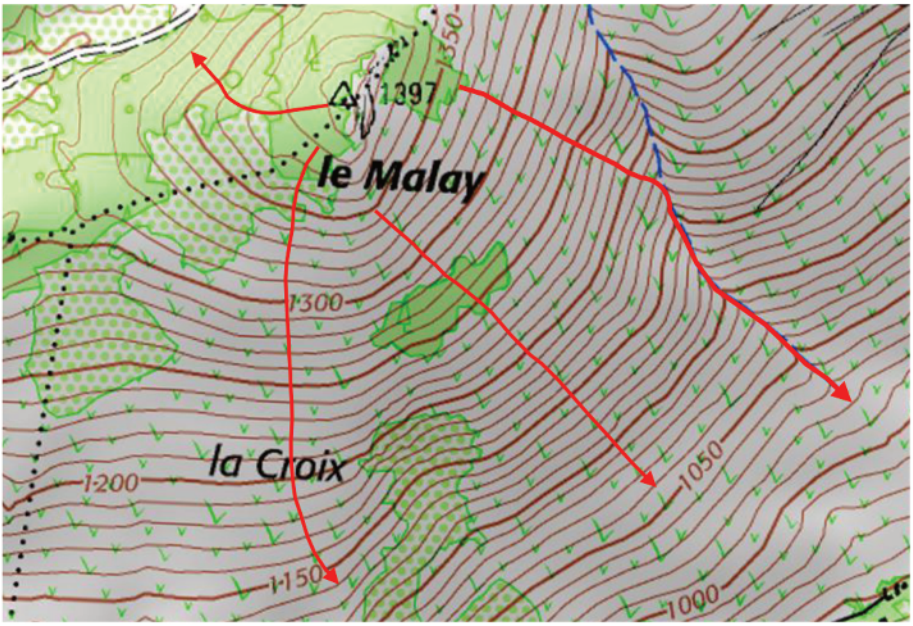


**Figure 1.2.** *Map scale of 1:25,000. For a color version of this figure, see www.iste.co.uk/harmand/bioprocesses.zip*

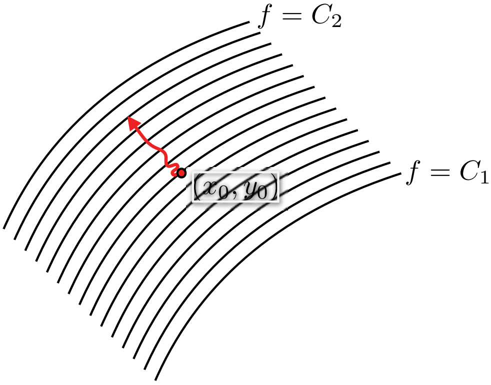


**Figure 1.3.** *Gradient vector: $C_2 > C_1$*

In the case of a single variable, we have seen that if the minimum is attained at the boundary of the set that we try to minimize, it is no longer necessary for the derivative to be 0, but it must simply be positive or 0 on the left, or negative or 0 on the right. A similar condition exists for two (or more) variables and we will state that condition below.

PROPOSITION 1.1.– Let $f$ be a function of two variables defined over $\mathbb{R}^2$. Let $\mathcal{D}$ be a domain and let us assume that $(x_0, y_0)$ is a local minimum of $f$ over $\mathcal{D}$. There exists $\varepsilon > 0$ such that if we write that $B_\varepsilon(x_0, y_0) = \{(x, y) \,:\, ||(x - x_0, y - y_0)|| \leq \varepsilon\}$ :

$$f(x_0, y_0) = \min_{(x,y)\in\mathcal{D}\cap B_\varepsilon(x_0,y_0)} f(x, y) \qquad [1.9]$$

Let us assume that $(x_0, y_0)$ is situated at a point on the boundary, $\partial\mathcal{D}$ of $\mathcal{D}$ and let us assume that at this point the boundary is a differentiable curve $t \mapsto (\alpha(t), \beta(t))$ with $\alpha(0) = x_0$ and $\beta(0) = y_0$. Then the gradient of $f$ at the point $(x_0, y_0)$ is either null or orthogonal to the boundary $\partial\mathcal{D}$ and goes in the direction of the interior of $\mathcal{D}$.

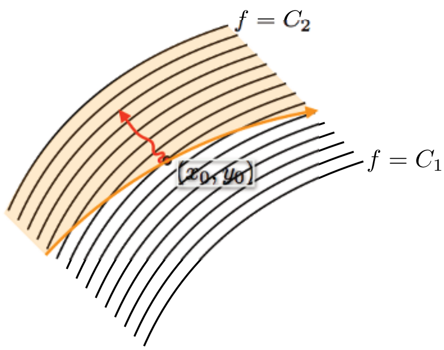


**Figure 1.4.** *Minimum of $f$ at the boundary of the domain of definition. For a color version of this figure, see www.iste.co.uk/harmand/bioprocesses.zip*

Indeed, let us assume that the curve that makes up the boundary $\mathcal{D}$ is not orthogonal to the gradient of $f$ at the point $(x_0, y_0)$, as indicated in Figure 1.4, where the boundary of $\mathcal{D}$ is the yellow curve that marks out the boundary of the yellow domain (situated above). We can see that if we start from the point $(x_0, y_0)$ and follow the yellow curve toward the right (in the direction of the arrow), the function $f$ diminishes – we still assume that $C_1 < C_2$, which contradicts the hypothesis.

REMARK 1.1.– The conclusion of proposition 1.1 can be reformulated in the following manner:

*Let $\psi_{(x_0,y_0)}$ denote a normal that enters the domain $\mathcal{D}$ in crossing the boundary at $(x_0, y_0)$. There exists a constant, $\lambda \geq 0$, such that:*

$$\nabla f(x_0, y_0) = \lambda \psi_{(x_0,y_0)} \tag{1.10}$$

## 1.3. Minimization of a functional: Euler–Lagrange equations

### 1.3.1. *The problem*

In classic mechanics, in the "Lagrangian" form we posit the principle that the trajectory of a material point whose coordinates are $x_i(t);\ i = 1, 2, 3$ going from $a$ to $b$ must minimize the total energy of the system, that is the integral:

$$\int_0^T \frac{m}{2} \sum_{i=1}^{3} \dot{x}_i(t)^2 - V(x_1(t), x_2(t), x_3(t))\, dt$$

where the first term is kinetic energy and the function, $V$, is potential energy. At time $t$, the integrated quantity depends both on the position $x(t)$ and the speed $\dot{x}(t)$. This explains our interest in the following problem:

Consider the space $D$ of functions of class $\mathcal{C}^1 : t \mapsto \gamma(t)$ from $[0, T]$ in $\mathbb{R}^n$ such that $\gamma(0) = \gamma_0$ and $\gamma(T) = \gamma_1$. We speak of *arcs* or *paths* that connect $\gamma_0$ to $\gamma_1$. We take a function of class $\mathcal{C}^1$ from $\mathbb{R}^n \times \mathbb{R}^n$ in $\mathbb{R}$:

$$(x_1, x_2) \mapsto l(x_1, x_2)$$

For any function $\gamma(\cdot)$, we consider the integral:

$$J = \int_0^T l(\gamma(t), \dot{\gamma}(t))\, dt \tag{1.11}$$

In doing so, we have defined an application of $D$ in $\mathbb{R}$, which we denote by:

$$\gamma(\cdot) \mapsto J(\gamma(\cdot)) = \int_0^T l(\gamma(t), \dot{\gamma}(t))\, dt \tag{1.12}$$

The application, $J$, defined by [1.12] is called a *functional* (to emphasize the fact that the variables in the application are not numbers or vectors, but are functions). Then, the minimization problem is as follows:

DEFINITION 1.1.– *Problem P0. Let $T > 0$ and $D$ be the set of functions of class $\mathcal{C}^1$ $t \mapsto \gamma(t)$ defined on $[0\,,\,T]$ such that $\gamma(0) = \gamma_0$ and $\gamma(T) = \gamma_1$. We examine the optimization problem*

$$\min_{\gamma(.)\in D} J(\gamma(\cdot)) = \int_0^T l(\gamma(t), \dot{\gamma}(t))\, dt$$

### 1.3.2. *The differential of $J$*

To simplify the writing, we assume in this section that $n = 1$; the function $l$ is, therefore, a function with *two* real variables: $x_1$ and $x_2$.

Our functional, $J$, is thus defined by the relationship [1.12]. We assume that $\gamma^\star(\cdot)$ is a local minimum of $J$. A function of class $\mathcal{C}^1$ with a value in $\mathbb{R}^n$, such that $\delta(0) = \delta(T) = 0$, is denoted by:

$$t \mapsto \delta(t)$$

and we call this a disturbance in $\gamma^\star(\cdot)$. It is clear that for any real number $\varepsilon$, the function:

$$\gamma^\star(\cdot) + \varepsilon\delta(\cdot)$$

belongs to $D$. We consider the function:

$$\varepsilon \mapsto J_\delta(\varepsilon) = J(\gamma^\star(\cdot) + \varepsilon\delta(\cdot)) \qquad [1.13]$$

It is clear that for any $\delta$, the function $J_\delta(\cdot)$ attains a local minimum for $\varepsilon = 0$, thus for any $\delta$, the derivative of $J_\delta(\cdot)$ is 0 at 0. We have:

$$J_\delta(\varepsilon) = \int_0^T l\big(\gamma^\star(t) + \varepsilon\delta(t), \dot{\gamma^\star}(t) + \varepsilon\dot{\delta}(t)\big)\, dt$$

We know from the theorem on differentiation under the integral sign that:

$$\frac{dJ_\delta}{d\varepsilon}(\varepsilon) = \int_0^T \frac{\partial l\big(\gamma^\star(t) + \varepsilon\delta(t), \dot{\gamma^\star}(t) + \varepsilon\dot{\delta}(t)\big)}{\partial\varepsilon}\, dt$$

that is:

$$\begin{aligned}\frac{dJ_\delta}{d\varepsilon}(\varepsilon) = \int_0^T \Big(&\partial_1 l\big(\gamma^\star(t) + \varepsilon\delta(t), \dot{\gamma^\star}(t) + \varepsilon\dot{\delta}(t)\big)\delta(t)\\ &+\partial_2 l\big(\gamma^\star(t) + \varepsilon\delta(t), \dot{\gamma^\star}(t) + \varepsilon\dot{\delta}(t)\big)\dot{\delta}(t)\Big)\, dt\end{aligned}$$

Thus, for any disturbance $\delta(\cdot)$:

$$\int_0^T \Big(\partial_1 l\big(\gamma^\star(t), \dot{\gamma}^\star(t)\big)\delta(t) + \partial_2 l\big(\gamma^\star(t), \dot{\gamma}^\star(t)\big)\dot{\delta}(t)\Big)\, dt = 0 \qquad [1.14]$$

We write:

$$f(t) = \partial_1 l(\gamma^\star(t), \dot{\gamma}^\star(t)) \qquad g(t) = \partial_2 l(\gamma^\star(t), \dot{\gamma}^\star(t))$$

We will demonstrate the following lemma given by Dubois-Raymond (1879):

LEMMA 1.1.– Let $f$ and $g$ be two continuous functions such that for any differentiable function $\delta$ such that $\delta(0) = \delta(T) = 0$, we have:

$$\int_0^T f(t)\delta(t) + g(t)\dot{\delta}(t)\, dt = 0$$

Thus, the function $g$ is differentiable and verifies:

$$\frac{dg}{dt}(t) = f(t), \quad t \in [0, T]$$

PROOF.– Let $F$ be a primitive of $f$. Through integration by parts, we have:

$$\int_0^T f(t)\delta(t)\, dt = \Big[F(t)\delta(t)\Big]_0^T - \int_0^T F(t)\dot{\delta}(t)\, dt$$

and as $\delta(0) = \delta(T) = 0$, it remains $\int_0^T f(t)\delta(t)\, dt = -\int_0^T F(t)\dot{\delta}(t)\, dt$.

Thus, for any differentiable function $\delta$ such that $\delta(0) = \delta(T) = 0$, we have:

$$\int_0^T f(t)\delta(t) + g(t)\dot{\delta}(t)\, dt = \int_0^T (g(t) - F(t))\dot{\delta}(t)\, dt = 0$$

Let us write that $h = g - F$ and consider:

$$\delta(t) = \int_0^t (h(\tau) - c)\ d\tau \quad \text{avec} \quad c = \frac{1}{T}\int_0^T h(\tau)\, d\tau$$

We thus have:

$$\begin{aligned}\int_0^T (h(t) - c)^2\ dt &= \int_0^T (h(t) - c)\dot{\delta}(t)\, dt \\ &= \int_0^T h(t)\dot{\delta}(t)\, dt - c\int_0^T \dot{\delta}(t)\, dt = 0\end{aligned}$$

As $t \mapsto (h(t) - c)^2$ is a positive function, it can be deduced that $h(t) - c = 0$ for any $t \in [0, T]$, or, in an equivalent manner, $h' = 0$, that is $g' = f$.

□

This lemma, applied to [1.14], demonstrates:

PROPOSITION 1.2.– Given a function of class $\mathcal{C}^1$:

$$(x_1, x_2) \mapsto l(x_1, x_2)$$

defined over $\mathbb{R} \times \mathbb{R}$. If the arc $\gamma^\star(\cdot)$ is differentiable and realizes the local minimum of the functional:

$$\phi(\gamma(\cdot)) = \int_0^T l(\gamma(t), \dot{\gamma}(t))\, dt$$

over the set of differentiable arcs such that $\gamma(0) = \gamma_0$ and $\gamma(T) = \gamma_1$, then $\gamma^\star(\cdot)$ satisfies the (Euler–Lagrange) equation:

$$\frac{d}{dt}\partial_2 l(\gamma^\star(t), \dot{\gamma}^\star(t)) = \partial_1 l(\gamma^\star(t), \dot{\gamma}^\star(t))$$

We leave it to the reader to extend this proposition to the case where $n$ is arbitrary.

THEOREM 1.1.– (Euler–Lagrange Equations) Given a function of class $\mathcal{C}^1$:

$$(x_1, x_2) \mapsto l(x_1, x_2)$$

defined over $\mathbb{R}^n \times \mathbb{R}^n$. If the arc $\gamma^\star(\cdot)$ is differentiable and realizes the local minimum of the functional:

$$\phi(\gamma(\cdot)) = \int_0^T l(\gamma(t), \dot{\gamma}(t))\, dt$$

over the set of differentiable arcs such that $\gamma(0) = \gamma_0$ and $\gamma(T) = \gamma_1$, then $\gamma^\star(\cdot)$ is twice-differentiable and satisfies the (Euler–Lagrange) equation:

$$\frac{d}{dt}\frac{\partial l}{\partial x_2}(\gamma^\star(t), \dot{\gamma}^\star(t)) = \frac{\partial l}{\partial x_1}(\gamma^\star(t), \dot{\gamma}^\star(t))$$

In classical mechanics, we use the more concise form:

$$\frac{d}{dt}\frac{\partial l}{\partial \dot{\gamma}}(\gamma, \dot{\gamma}) = \frac{\partial l}{\partial \gamma}(\gamma, \dot{\gamma})$$

and the following mnemonic: if we formally replace $\dot{\gamma}$ by $\frac{d\gamma}{dt}$ on the left we simplify the equation and find the member on the right.

### 1.3.3. *Examples*

EXAMPLE 1.1.– The problem to be solved is:

$$\min_{x(.)\in D} \int_0^1 (\dot{x}(t))^2 dt$$

$$D = \{x(\cdot) : x(0) = 0 \; ; \; x(1) = 1\}$$

We can easily resolve this problem directly, without using the Euler–Lagrange equations, by working as given below. Given two functions $x(\cdot)$ and $y(\cdot)$, defined over the interval $[0, 1]$, we have the following fundamental inequality (Schwartz's inequality):

$$\left(\int_0^1 x(t)y(t)\,dt\right)^2 \leq \int_0^1 x^2(t)\,dt \int_0^1 y^2(t)\,dt \tag{1.15}$$

In effect, for any real $\lambda$, the polynomial in $\lambda$:

$$\int_0^1 (\lambda x(t) + y(t))^2\,dt = \lambda^2 \int_0^1 x^2(t)\,dt + 2\lambda \int_0^1 x(t)y(t)\,dt + \int_0^1 y^2(t)\,dt \tag{1.16}$$

is positive or 0 and hence, its discriminant:

$$\Delta' = \left(\int_0^1 x(t)y(t)\,dt\right)^2 - \int_0^1 y^2(t)\,dt \int_0^1 x^2(t)\,dt \tag{1.17}$$

must be negative or 0, that is the inequality [1.15]. When this inequality is applied to:

$$\int_0^1 (\dot{x}(t))^2\,dt = \int_0^1 (\dot{x}(t))^2 (1)^2\,dt \tag{1.18}$$

It results in:

$$\int_0^1 (\dot{x}(t))^2\,dt \geq \left(\int_0^1 (\dot{x}(t)) \times 1\right)^2\,dt = (x(1) - x(0))^2 = 1 \tag{1.19}$$

On the one hand, the quantity to minimize is always greater than or equal to 1. On the other hand, if $x(t) = t$, one has $\dot{x} = 1$ whose integral of the square on $[0, 1]$ is 1. One solution to our problem is: $x^\star(t) = t$.

What do the Euler–Lagrange equations tell us? We have $l(x_1, x_2) = x_2^2$ and thus:

$$\frac{\partial l}{\partial x_1} = 0, \quad \frac{\partial l}{\partial x_2} = 2x_2$$

According to the Euler–Lagrange equations, because $x^\star(\cdot)$ is the optimum, we must have:

$$\frac{d}{dt}(2\dot{x}^\star(t)) = 0$$

that is: $\ddot{x}^\star(t) = 0$. This results in $\dot{x}^\star(t) = C$ with $C$ a constant, or $x^\star(t) = 0 + Ct$ and as we must have $x^\star(1) = 1$, $C = 1$.

EXAMPLE 1.2.– Let the problem to resolve be:

$$\min_{x(.)\in D} \int_0^1 (x(t))^2 \, dt$$
$$D = \{x(\cdot) : x(0) = 0 \; ; \; x(1) = 1\}$$

We have $l(x_1, x_2) = x_1^2$ and therefore:

$$\frac{\partial l}{\partial x_1} = 2x_1, \quad \frac{\partial l}{\partial x_2} = 0$$

The Euler–Lagrange equation is:

$$\frac{d}{dt}(0) = 2x(t)$$

that is $x(t) \equiv 0$. The solutions to the Euler–Lagrange equations are reduced to a single function, which cannot satisfy the boundary conditions. This corresponds to the fact that this problem has no optimal solution, as we can easily see. Let us take, for example, the family of functions:

$$x_\varepsilon(t) = \varepsilon \frac{t}{1 + \varepsilon - t}$$

The integral of $x_\varepsilon^2(t)$ over $[0, 1]$ tends to $0$ when $\varepsilon$ tends to $0$ but there is no *continuous function* whose value is $0$ at $0$ and $1$ at $1$ for which the integral of its square is 0.

This example shows that when we write:

$$\min_{x(\cdot)\in D} \int_0^1 (x(t))^2 \, dt$$

$$D = \{x(\cdot) : x(0) = 0 \; ; \; x(1) = 1\}$$

we must *specify the regularity* of the functions for which we are searching for a minimum. As we have stated, *without specifying the type of function we are considering* there is no answer to our problem. Indeed, we have just seen that if:

$$D = \{x(\cdot) \in \mathcal{C}^0 : x(0) = 0 \; ; \; x(1) = 1\}$$

then the problem has no solution. But we could also have considered, for $D$, the set of continuous piecewise functions on $[0, 1]$. For these functions, the integral of the square is well defined and the minimum is attained for the function $x(t)$, which is 0 over $[0, 1[$ and equal to 1 for $t = 1$.

Hence, from this point onwards, we will take care to clearly specify the class of functions over which we seek an optimum.

EXAMPLE 1.3.– Profile of a road.

We consider the following problem. We wish to construct a straight road that connects two cities, A and B, separated by a certain distance $L$. Between these two points, the altitude of the ground is a certain function $a(t)$ such that $a(0)$ is the altitude of city A, while $a(L)$ is the altitude of city B. $x(t)$ denotes the profile (i.e. the altitude of the road as a function of its distance to $A$). The cost of construction is given by:

$$J(x(\cdot)) = \int_0^L (x(t) - a(t))^2 \, dt = \int_0^L l(x_1, x_2) \, dt$$

which reveals that as long as we follow the profile exactly, the cost is 0 (negligible). However, when a tunnel must be dug out (the portion $p_1, p_2$ in Figure 1.5) or when a viaduct must be constructed (the portion $p_2, p_3$), the cost depends on the distance between the road and the profile. For such an application, it is essential that the profile $x(t)$ be a continuous and differentiable function (a road is not a staircase). For mathematical simplicity, we will assume that both cities are at the altitude 0 but are separated by a plateau whose altitude is constant and equal to $h$. We thus begin by examining the following problems:

*Problem: Road*-1

$$\min_{x(\cdot) \in D} \int_0^L (x(t) - h)^2 \, dt$$

$$D = \{x(\cdot) \in \mathcal{C}^1 : x(0) = 0 \; ; \; x(L) = 0\}$$

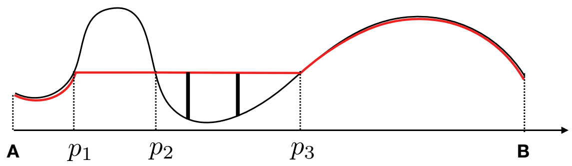


**Figure 1.5.** *Optimal road profile. For a color version of this figure, see www.iste.co.uk/harmand/bioprocesses.zip*

This problem has no solution. A road that attains an altitude $h$ at the distance $\delta$, remains at the altitude $h$ up to the point $L - \delta$ and then comes down to the altitude 0 has a cost lower than $2\delta h^2$ (see Figure 1.6). However, if $\delta$ is very small, then a road of this kind is not acceptable as the slope needed to reach the altitude $h$ would be, on average, $h/\delta$, and would be excessive if $\delta$ is too small.


**Figure 1.6.** *A cost lower than $2\delta h^2$. For a color version of this figure, see www.iste.co.uk/harmand/bioprocesses.zip*

One way of solving this problem is to *penalize* stiff slopes. Let us, for example, consider:

*Problem: Road*-2

$$\min_{x(\cdot)\in D} \int_0^L \big((x(t) - h)^2 + \gamma^2 \dot{x}^2(t)\big)\, dt$$

$$D = \{x(\cdot) \in \mathcal{C}^1 : x(0) = 0 \,;\, x(L) = 0\}$$

If we posit:

$$l(x_1, x_2) = (x_1 - h)^2 + \gamma^2 x_2^2$$

(this "$l$" must not be confused with the length of the road, "$L$"). The Euler–Lagrange equations are:

$$2(x(t) - h) - \frac{d}{dt} 2\gamma^2 \dot{x}(t) = 0$$

that is:

$$\gamma^2 \ddot{x}(t) - x(t) + h = 0$$

That is, a second-order differential equation with constant coefficients for which it can be checked that the solution satisfying the boundary conditions $x(0) = 0$ and $x(l) = 0$ is:

$$x(t) = h \left( 1 - \frac{1}{1 + \mathrm{e}^{L/\gamma^2}} \mathrm{e}^{t/\gamma^2} - \frac{\mathrm{e}^{L/\gamma^2}}{1 + \mathrm{e}^{L/\gamma^2}} \mathrm{e}^{-t/\gamma^2} \right) \qquad [1.20]$$

In Figure 1.7, for the case where $h = 1$ and $h = 10$, we have represented the three solutions corresponding to $\gamma = 2$, $\gamma = 1$ and $\gamma = 0.5$. We can see that we have profiles whose slope becomes steeper as the "penalty", $\gamma^2$, becomes weaker.

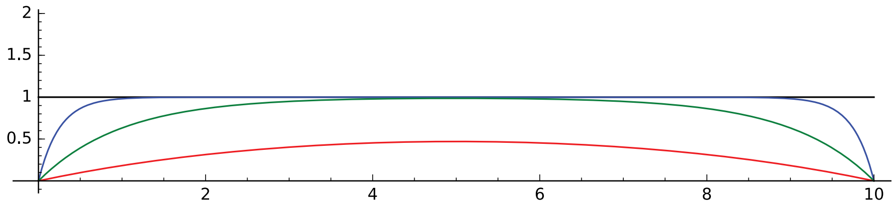


**Figure 1.7.** *Red $\gamma = 2$; green $\gamma = 1$; blue $\gamma = 0.5$. For a color version of this figure, see www.iste.co.uk/harmand/bioprocesses.zip*

The solution that we just saw, to our question of the optimal profile for the road, is not a satisfactory one as there is no direct relationship between the penalty and the slope. In other words, we do not know which $\gamma$ to choose. The ideal solution would be to consider:

*Problem: Road*-3

$$\min_{x(\cdot) \in D} \int_0^L (x(t) - h)^2 \, dt$$

$$D = \{x(\cdot) \in \mathcal{C}^0 \cap \mathcal{C}^1_{\text{pm}} : x(0) = 0 \;;\; x(L) = 0 \;;\; \left|\frac{dx}{dt}(t)\right| \leq \alpha\}$$

which expresses the *natural condition*:

– the profile of the road must be continuous (and not discontinuous, as steps): $x(\cdot) \in \mathcal{C}^0$;

– the slope of the road is not necessarily continuous (there may be breaks in the slope): $x(\cdot) \in \mathcal{C}^1_{\text{pm}}$;

– the slope must be lower than a given value: $|\frac{dx}{dt}(t)| \leq \alpha$

with some thought, we can see that the problem has the solution given below (see Figure 1.8):

$$\begin{cases} t \in [0, h/\alpha] & \Longrightarrow x(t) = \alpha\, t \\ t \in [h/\alpha, L - h/\alpha] & \Longrightarrow x(t) = h \\ t \in [l - h/\alpha, L] & \Longrightarrow x(t) = -\alpha(t - L) \end{cases} \qquad [1.21]$$


**Figure 1.8.** *Optimal profile for a road with a maximum slope* $\alpha$. *For a color version of this figure, see www.iste.co.uk/harmand/bioprocesses.zip*

Unfortunately, we cannot arrive at this solution within the framework of the computation of classical calculus of variations using Euler–Lagrange equations. This is because in order to establish these equations, there must first be no prior restriction on the value of $\frac{dx}{dt}(t)$. The constraint $|\frac{dx}{dt}| \leq \alpha$ is not acceptable in this framework.

In Chapter 2, we will see that the maximum principle is a framework within which we can resolve this kind of problem.

## 1.4. Hamilton's equations

### 1.4.1. *Hamilton's classical equations*

"Hamilton's equations" must not be confused with the "Hamilton–Jacobi–Bellman equation", which is another manner of approaching the question of optimal control.

We take a $\mathcal{C}^1$ class $\mathbb{R}^n \times \mathbb{R}^n$ function in $\mathbb{R}$:

$$(x_1, x_2) \mapsto l(x_1, x_2)$$

Let us recall how the problem of classical calculus of variations is formulated.

DEFINITION 1.2.– *Problem P0. Let* $T > 0$ *and* $D$ *be the set of* $\mathcal{C}^1$ $t \mapsto \gamma(t)$ *functions defined over* $[0\,,\,T]$ *such that* $\gamma(0) = \gamma_0$ *and* $\gamma(T) = \gamma_1$*. We look at the following optimization problem*

$$\min_{\gamma(.)\in D} J(\gamma(\cdot)) = \int_0^T l(\gamma(t), \dot{\gamma}(t))\, dt$$

In this problem, the unknown is presented as a differentiable arc, $\gamma(\cdot)$, but we could also have stated that our unknown is not the arc $\gamma(\cdot)$ but, instead, its derivative $\dot{\gamma}(\cdot)$, as we have the relationship:

$$\gamma(t) = \gamma_0 + \int_0^t \dot{\gamma}(s) ds$$

To clearly mark this change in point of view, we change the notations used. $\dot{\gamma}(\cdot)$ is replaced by $u(\cdot)$ and $\gamma(\cdot)$ by $x(\cdot)$; similarly, $\gamma_0$ is replaced by $x_0$ and $\gamma_1$ par $x_1$. Finally, the function $l(x_1, x_2)$ becomes $l(x, u)$. With these new notations, problem *P0* becomes:

DEFINITION 1.3.– *Problem P1. Let* $T > 0$ *and* $D$ *be the set of* $\mathcal{C}^1$ $t \mapsto u(t)$ *class functions defined over* $[0\,,\,T]$ *such that the solution to the Cauchy problem:*

$$\frac{dx}{dt}(t) = u(t), \quad x(0) = x_0$$

*satisfies the terminal condition* $x(T) = x_1$*. We look at the following optimization problem*

$$\min_{u(.)\in D} J(u(\cdot)) = \int_0^T l(x(t), u(t))\, dt$$

Using these new notations, the Euler–Lagrange equations that the solutions to problem P1 must satisfy can be rewritten as:

$$\frac{d}{dt}\left[\frac{\partial l}{\partial u}(x(t), u(t))\right] = \frac{\partial l}{\partial x}(x(t), u(t)) \qquad [1.22]$$

They make up a system of second-order differential equations in $x$, which we will transform into a *two-dimensional* system of *first-order* equations. In order to do this, we consider the quantity that we call the *Hamiltonian*:

$$H(x, p, u) = < p, u > -l(x, u) \qquad [1.23]$$

where $p$ is a vector of the same dimension as $x$ and $< p, u >$ denotes the scalar product:

$$< p, u >= p^\top u = \sum_{i=1}^{n} p_i u_i$$

We postulate the following hypotheses:

HYPOTHESIS 1.1.–

1) $l(x, u)$ is a function of class $\mathcal{C}^2$;

2) the function $u \mapsto l(x, u)$ is quadratic in $u$.

These hypotheses are satisfied in mechanics, where we have $l(x, u) = \frac{1}{2}u^2 - V(x)$, which is indeed quadratic in the variable $u$. From hypothesis 1.1, it immediately follows that for any $(x, p)$ the function $u \mapsto H(x, p, u)$ has a unique maximum (in effect, $H$ is concave in $u$ and, as $l$ is quadratic in $u$, when $\|u\|$ tends to infinity, $H$ tends to minus infinity).

$u(x, p)$ denotes the value for which this maximum is attained:

$$\max_{u \in \mathbb{R}^n} H(x, p, u) = H(x, p, u(x, p)) \qquad [1.24]$$

and we write:

$$\overline{H}(x, p) = H(x, p, u(x, p)) \qquad [1.25]$$

As the maximization of $H$ is carried out over the whole of $\mathbb{R}^n$ (this would not be true if we looked for the maximum over a bounded subset of $\mathbb{R}^n$), the gradient must be 0:

$$\frac{\partial H}{\partial u}(x, p, u(x, p)) = 0 \qquad [1.26]$$

From these hypotheses, it follows that $u(\cdot,\cdot)$ is of the class $\mathcal{C}^1$. We have:

$$\frac{\partial \overline{H}}{\partial x}(x,p) = \frac{\partial H}{\partial x}(x,p,u(x,p)) + \frac{\partial H}{\partial u}(x,p,u(x,p))\frac{\partial u}{\partial x}(x,p,u(x,p))$$

Thus:

$$\frac{\partial \overline{H}}{\partial x}(x,p) = \frac{\partial H}{\partial x}(x,p,u(x,p)) \qquad [1.27]$$

and similarly:

$$\frac{\partial \overline{H}}{\partial p}(x,p) = \frac{\partial H}{\partial p}(x,p,u(x,p)) \qquad [1.28]$$

From the definition of $H$, it follows that:

$$\frac{\partial H}{\partial x}(x,p,u) = -\frac{\partial l}{\partial x}(x,u) \quad \text{and} \quad \frac{\partial H}{\partial p}(x,p,u) = u$$

thus:

$$\begin{aligned} \frac{\partial \overline{H}}{\partial x}(x,p) &= -\frac{\partial l}{\partial u}(x,u) \\ \frac{\partial \overline{H}}{\partial p}(x,p) &= u \end{aligned} \qquad [1.29]$$

Let $(x(t),u(t))$ be the solution to problem P1, and hence the solution to [1.22]. Let us write:

$$p(t) = \left[\frac{\partial l}{\partial u}(x(t),u(t))\right]^\top \qquad [1.30]$$

According to [1.22], we have:

$$\frac{dp}{dt} = \left[\frac{\partial l}{\partial x}(x,u)\right]^\top = -\frac{\partial \overline{H}}{\partial x}^\top(x,p)$$

and on the other hand, we have:

$$\frac{dx}{dt} = u = \frac{\partial \overline{H}}{\partial p}^\top(x,p)$$

We have thus demonstrated:

THEOREM 1.2.– We assume that $t \mapsto x(t)$ is a solution to problem P1. Thus, with the notations and hypotheses given above, there exists a $t \mapsto p(t)$ such that the pair $(x(t), p(t))$ satisfies the differential system:

$$\begin{aligned} \frac{dx}{dt} &= \frac{\partial \overline{H}}{\partial p}^{\top}(x, p) \\ \frac{dp}{dt} &= -\frac{\partial \overline{H}}{\partial x}^{\top}(x, p) \end{aligned} \qquad [1.31]$$

We clearly have (all we need to do is differentiate) $\overline{H}(x(t), p(t)) = C$, where $C$ is a constant. The vector $p(t)$ is called the *adjoint vector*.

In this statement, the variable $u$, which comes into play in problem P1, has disappeared, but it can be reintroduced by stating the following theorem, equivalent to the previous theorem.

THEOREM 1.3.– We assume that $t \mapsto u^{\star}(t)$ is a solution to problem P1. Thus, with the notations and hypotheses given above, there exists a function $t \mapsto p(t)$ such that:

$$\begin{aligned} \frac{dx}{dt} &= \frac{\partial H}{\partial p}^{\top}(x, p, u^{\star}(t)) \\ \frac{dp}{dt} &= -\frac{\partial H}{\partial x}^{\top}(x, p, u^{\star}(t)) \end{aligned} \qquad [1.32]$$

where $u^{\star}(t)$ verifies,

$$H(x(t), p(t), u^{\star}(t)) = \max_{u \in \mathbb{R}^n} H(x(t), p(t), u)$$

Problem P1 may be approached through the maximum principle. We will see that this principle gives the above formulation.

### 1.4.2. *The limitations of classical calculus of variations and small steps toward the control theory*

The Euler–Lagrange equations are conditions of optimality for a functional $J$, which attains its minimum at a point $\gamma^{\star}$ that is found in the interior of the domain of definition of $J$. In other words, there must not be any *constraints* on the set of functions in the neighborhood of $\gamma^{\star}$. This point is not realized in the example of the profile of the road as, at optimum, the arc $\gamma^{\star}$ is at the boundary of the acceptable domain ($\frac{dx}{dt}(t) = \alpha$ over $[0, \frac{h}{\alpha}]$). Moreover, the demonstration of theorem 1.1 clearly assumes that it is possible to create *positive or negative* $\delta$ disturbances (while, to establish [1.14], we state that the derivative of the function $J_\delta(\varepsilon)$ must be 0), which is not possible when the constraints are saturated.

We must, thus, state a necessary condition for optimality that can be applied when the domain of admissible arcs is not open. One way of moving ahead here is to reformulate problem P1. When reformulating problem P1, the relationship between the "control" $u$ and the "trajectory" $x$ is a trivial differential equation, indeed, a simple quadrature:

$$\frac{dx}{dt} = u$$

Control theory aims to replace this differential equation with a "true differential equation":

$$\frac{dx}{dt} = f(x, u)$$

and our problem then becomes:

DEFINITION 1.4.– *Problem P2. Let* $T > 0$ *and* $\mathcal{U}$ *be the set of* $(t \mapsto u(t))$ *defined over* $[0\,,\,T]$ *such that the solution to the Cauchy problem:*

$$\begin{aligned} \frac{dx}{dt}(t) &= f(x(t), u(t))\ ; \quad u \in U \\ x(0) &= x_0 \end{aligned} \qquad [1.33]$$

*satisfies the terminal condition* $x(T) = x_1$. *We examine the following optimization problem:*

$$\min_{u(.)\in\mathcal{U}} J(u(.)) = \int_0^T l(x(t), u(t))\, dt$$

We can see that the road problem is a problem of this kind.

Pontryagin's maximum principle gives the necessary conditions for optimality for this problem. These conditions take a different form from the Euler–Lagrange equations. In effect, the Euler–Lagrange equations can only be formulated in the case where the variables $u$ and $x$ are of the same dimension. On the other hand, in the formulation in Hamiltonian terms, we can always formally write:

$$H(x, p, u) =< p, f(x, u) > -L(x, u)$$

The maximum principle demonstrates that this transformation is not only formal and that theorem 1.3 is valid for problem P2 with this new Hamiltonian.

## 1.5. Historical and bibliographic observations

Pierre de Fermat (1607–1665) is credited with stating the fact that the derivative of a function must be 0 at a local minimum. However, attributing this discovery to one individual does not really make sense and all the famous inventors in the field of infinitesimal calculus – including Galileo (1564–1642), Descartes (1596–1650), Newton (1643–1727) and Leibnitz (1646–1716) – should be mentioned here. More interestingly, given the subject of this textbook, we have Jean Bernoulli (1667–1748) who, in 1696, posed the problem of the *brachistochrone* curve to his mathematical colleagues. This problem required determining a curve in a plane such that a mass, subject only to its own weight, would take the minimum time to go from a point A to a point B situated lower down. Using modern notations, in the plane along the axes $(x, y)$ we need to find, among the curves $[0, T] \ni t \mapsto (x(t), y(t))$:

$$
\begin{aligned}
&(x(0), y(0)) = (0, 0); \qquad (x(T), y(T)) = (a, b) \\
&\tfrac{1}{2}(\dot{x}^2(t) + \dot{y}^2(t)) = gy(t)
\end{aligned}
$$

the curve with the smallest possible $T$. This challenge was taken up by some of the greatest mathematicians of the age: Leibniz, Jacob Bernoulli (Jean's brother), Ehrenfried von Tschirnhaus, the Marquis de l'Hospital and Newton himself; the solution, a cycloid arc, was found through various means. In a recent article [SUS 97], two major contemporary contributors to control theory, Hector J. Sussmann and Jan C. Willems, show us how this problem may be considered to be the foundation of the optimal control theory, even though it seems to belong more to the calculus of variations, where several prominent thinkers left their mark. To name just a few who appeared in this chapter: Euler (1707–1783), Lagrange (1736–1813) and Hamilton (1805–1865). The path we trace (which we hope will be educational) that leads to the maximum principle (toward the 1950s) resembles the historical route. We would highly recommend that the reader familiarizes themselves with this history by turning to the above-cited article [SUS 97].