PART 1

Theory of Stochastic Processes

1

Stochastic Processes. General Properties. Trajectories, Finite-dimensional Distributions

1.1. Definition of a stochastic process

Let $(\Omega, \mathcal{F}, \mathrm{P})$ be a probability space. Here, Ω is a sample space, i.e. a collection of all possible outcomes or results of the experiment, and $\mathcal{F}$ is a σ-field; in other words, $(\Omega, \mathcal{F})$ is a measurable space, and P is a probability measure on $\mathcal{F}$. Let $(\mathcal{S}, \Sigma)$ be another measurable space with σ-field Σ, and let us consider the functions defined on the space $(\Omega, \mathcal{F})$ and taking their values in $(\mathcal{S}, \Sigma)$. Recall the notion of random variable.

DEFINITION 1.1.– *A* random variable *on the probability space* $(\Omega, \mathcal{F})$ *with the values in the measurable space* $(\mathcal{S}, \Sigma)$ *is a measurable map* $\Omega \xrightarrow{\xi} \mathcal{S}$*, i.e. a map for which the following condition holds: the pre-image* $\xi^{-1}(B)$ *of any set* $B \in \Sigma$ *belongs to* $\mathcal{F}$*.* Equivalent forms of this definition are: for any $B \in \Sigma$, we have that

$$\xi^{-1}(B) \in \mathcal{F},$$

or, for any $B \in \Sigma$, we have that

$$\{\omega : \xi(\omega) \in B\} \in \mathcal{F}.$$

Consider examples of random variables.

1) The number shown by rolling a fair die. Here,

$$\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5, \omega_6\}, \mathcal{F} = 2^{\Omega}, \mathcal{S} = 1, 2, 3, 4, 5, 6, \Sigma = 2^{\mathcal{S}}.$$

2) The price of certain assets on a financial market. Here, $(\Omega, \mathcal{F})$ can depend on the model of the market, and the space $\mathcal{S}$, as a rule, coincides with $\mathbb{R}_+ = [0, +\infty)$.

3) Coordinates of a moving airplane at some moment of time. People use different coordinate systems to determine the coordinates of the airplane that has three coordinates at any time. The coordinates are time dependent and random, to some extent, because they are under the influence of many factors, some of which are random. Here, $\mathcal{S} = \mathbb{R}^3$ for the Cartesian system, or $\mathcal{S} = \mathbb{R}^2 \times [0, 2\pi]$ for the cylindrical system, or $\mathcal{S} = \mathbb{R} \times [0, \pi] \times [0, 2\pi]$ for the spherical system.

Now, we formalize the notion of a stochastic (random) process, defined on $(\Omega, \mathcal{F}, \mathrm{P})$. We will treat a random process as a set of random variables. That said, introduce the parameter set $\mathbb{T}$ with elements $t : t \in \mathbb{T}$.

DEFINITION 1.2.– *Stochastic process on the probability space* $(\Omega, \mathcal{F}, \mathrm{P})$*, parameterized by the set* $\mathbb{T}$ *and taking values in the measurable space* $(\mathcal{S}, \Sigma)$*, is a set of random variables of the form*

$$X_t = \{X_t(\omega), t \in \mathbb{T}, \omega \in \Omega\},$$

where $X_t(\omega) : \mathbb{T} \times \Omega \to S$.

Thus, each parameter value $t \in \mathbb{T}$ is associated with the random variable X_t taking its value in $\mathcal{S}$. Sometimes, we call $\mathcal{S}$ a phase space. The origin of the term comes from the physical applications of stochastic processes, rather than from the physical problems which stimulated the development of the theory of stochastic processes to a large extent.

Here are other common designations of stochastic processes:

$$X(t),\ \xi(t),\ \xi_t,\ X = \{X_t, t \in \mathbb{T}\}.$$

The last designation is the best in the sense that it describes the entire process as a set of the random variables. The definition of a random process can be rewritten as follows: for any $t \in \mathbb{T}$ and any set $B \in \Sigma$

$$X_t^{-1}(B) \in \mathcal{F}.$$

Another form: for any $t \in \mathbb{T}$ and any set $B \in \Sigma$

$$\{\omega : X_t(\omega) \in B\} \in \mathcal{F}.$$

In general, the space $\mathcal{S}$ can depend on the value of t, $\mathcal{S} = \mathcal{S}_t$, but, in this book, space $\mathcal{S}$ will be fixed for any fixed stochastic process $X = \{X_t, t \in \mathbb{T}\}$. If $\mathcal{S} = \mathbb{R}$, then the process is called real or real-valued. Additionally, we assume in this case that $\Sigma = \mathcal{B}(\mathbb{R})$, i.e. $(\mathcal{S}, \Sigma) = (\mathbb{R}, \mathcal{B}(\mathbb{R}))$, where $\mathcal{B}(\mathcal{S})$ is a Borel σ-field on $\mathcal{S}$. If

$\mathcal{S} = \mathbb{C}$, the process is called complex or complex-valued, and if $\mathcal{S} = \mathbb{R}^d, d > 1$, the process is called vector or vector-valued. In this case, $(\mathcal{S}, \Sigma) = (\mathbb{C}, \mathcal{B}(\mathbb{C}))$ and $(\mathcal{S}, \Sigma) = (\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$, respectively.

Concerning the parameter set $\mathbb{T}$, as a rule, it is interpreted as a time set. If the time parameter is continuous, then usually either $\mathbb{T} = [a, b]$, or $[a, +\infty)$ or $\mathbb{R}$. If the time parameter is discrete, then usually either $\mathbb{T} = \mathbb{N} = 1, 2, 3, \ldots$, or $\mathbb{T} = \mathbb{Z}^+ = \mathbb{N} \cup 0$ or $\mathbb{T} = \mathbb{Z}$.

The parameter set can be multidimensional, e.g. $\mathbb{T} = \mathbb{R}^m, m > 1$. In this case, we call the process a random field. The parameter set can also be mixed, the so-called time–space set, because we can consider the processes of the form $X(t, x) = X(t, x, \omega)$, where $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^d$. In this case, we interpret t as time and $x \in \mathbb{R}^d$ as the coordinate in the space $\mathbb{R}^d$.

There can be more involved cases, e.g. it is possible to consider random measures $\mu(t, A, \omega)$, where $t \in \mathbb{R}_+, A \in \mathcal{B}(\mathbb{R}^d)$, or random processes defined on the groups, whose origin comes from physics. We will not consider in detail the theory of such processes.

In what follows, we consider the real-valued parameter, i.e. $\mathbb{T} \subset \mathbb{R}$, so that we can regard the parameter as time, as described above.

1.2. Trajectories of a stochastic process. Some examples of stochastic processes

1.2.1. *Definition of trajectory and some examples*

A stochastic process $X = \{X_t(\omega), t \in \mathbb{T}, \omega \in \Omega\}$ is a function of two variables, one of them being a time variable $t \in \mathbb{T}$ and the other one a sample point (elementary event) $\omega \in \Omega$. As mentioned earlier, fixing $t \in \mathbb{T}$, we get a random variable $X_t(\cdot)$. In contrast, fixing $\omega \in \Omega$ and following the values that $X_\cdot(\omega)$ takes as the function of parameter $t \in \mathbb{T}$, we get a trajectory (path, sample path) of the stochastic process. The trajectory is a function of $t \in \mathbb{T}$ and, for any t, it takes its value in $\mathcal{S}$. Changing the value of ω, we get a set of paths. They are schematically depicted in Figure 1.1.

Let us consider some examples of random processes and draw their trajectories. First, we recall the concept of independence of random variables.

DEFINITION 1.3.– *Random variables $\{\xi_\alpha, \alpha \in \mathcal{A}\}$, where $\mathcal{A}$ is some parameter set, are called* mutually independent *if for any finite subset of indices $\{\alpha_1, \ldots, \alpha_k\} \subset \mathcal{A}$ and, for any measurable sets $A_1, \ldots, A_k$, we have that*

$$\mathrm{P}\{\xi_{\alpha_1} \in A_1, \ldots, \xi_{\alpha_k} \in A_k\} = \Pi_{i=1}^k \mathrm{P}\{\xi_{\alpha_i} \in A_i\}.$$

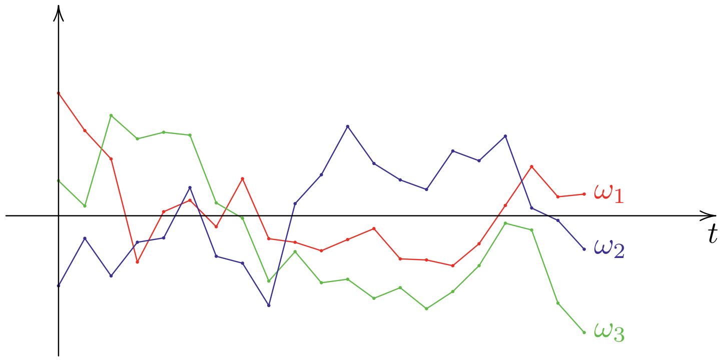

Figure 1.1. *Trajectories of a stochastic process. For a color version of the figure, see www.iste.co.uk/mishura/stochasticprocesses.zip*

1.2.1.1. *Random walks*

A *random walk* is a process with discrete time, e.g. we can put $\mathbb{T} = \mathbb{Z}^+$. Let $\{\xi_n, n \in \mathbb{Z}^+\}$ be a family of random variables taking values in $\mathbb{R}^d, d \geq 1$. Put $X_n = \sum_{i=0}^n \xi_i$. Stochastic process $X = \{X_n, n \in \mathbb{Z}_+\}$ is called a random walk in $\mathbb{R}^d$. In the case where $d = 1$, we have a random walk in the real line. In general, the random variables ξ_i can have arbitrary dependence between them, but the most developed theory is in the case of random walks with mutually independent and identically distributed variables $\{\xi_n, n \in \mathbb{Z}^+\}$. If, additionally, any random variable ξ_n takes only two values a and b with respective probabilities $\mathrm{P}\{\xi_n = a\} = p$ and $\mathrm{P}\{\xi_n = b\} = q = 1 - p \in (0, 1)$, then we have a Bernoulli random walk. If $a = -b$ and $p = q = \frac{1}{2}$, then we have a symmetric Bernoulli random walk. The trajectory of the random walk consists of individual points, and is shown in Figure 1.2.

1.2.1.2. *Renewal process*

Let $\{\xi_n, n \in \mathbb{Z}^+\}$ be a family of random variables taking positive values with probability 1. Stochastic process $N = \{N_t, t \geq 0\}$ can be defined by the following formula:

$$N_t = \begin{cases} 0, t < \xi_1; \\ \sup\{n \geq 1 : \sum_{i=1}^n \xi_i \leq t\}, t \geq \xi_1. \end{cases}$$

Stochastic process $N = \{N_t, t \geq 0\}$ is called a *renewal process*. Trajectories of a renewal process are step-wise with step 1. The example of the trajectory is represented in Figure 1.3.

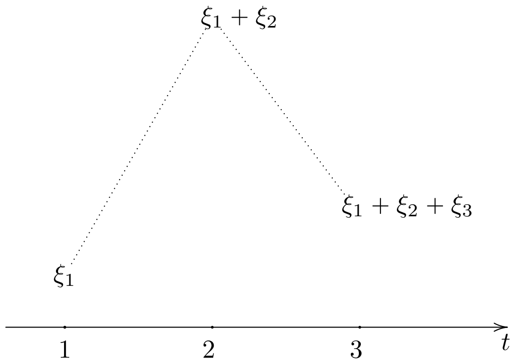

Figure 1.2. *Trajectories of a random walk*

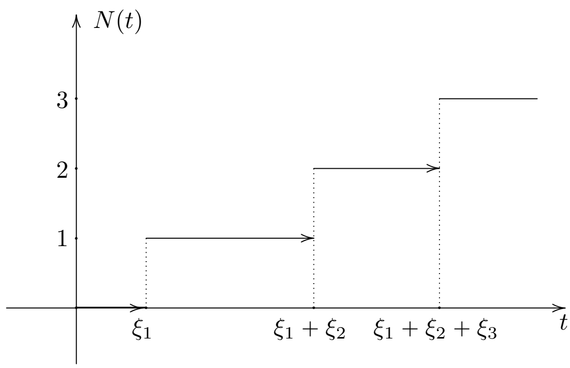

Figure 1.3. *Trajectories of a renewal process*

Random variables $T_1 = \xi_1, T_2 = \xi_1 + \xi_2, \ldots$ are called jump times, arrival times or renewal times of the renewal process. The latter name comes from the fact that the renewal processes were considered in applied problems related to moments of failure and replacement of equipment. Intervals $[0, T_1]$ and $[T_n, T_{n+1}], n \geq 1$ are called renewal intervals.

1.2.1.3. *Stochastic processes with independent values and those with independent increments*

DEFINITION 1.4.– *A stochastic process $X = \{X_t, t \geq 0\}$ is called a* process with independent values *if the random variables $\{X_t, t \geq 0\}$ are mutually independent.*

It will be shown later, in Example 6.1, that the trajectories of processes with independent values are quite irregular and, for this reason, the processes with independent values are relatively rarely used to model phenomena in nature, economics, technics, society, etc.

DEFINITION 1.5.– *A stochastic process* $X = \{X_t, t \geq 0\}$ *is called a* process with independent increments*, if, for any set of points* $0 \leq t_1 < t_2 < \ldots < t_n$*, the random variables* $X_{t_1}, X_{t_2} - X_{t_1}, \ldots, X_{t_n} - X_{t_{n-1}}$ *are mutually independent.*

Here is an example of a random process with discrete time and independent increments.

Let $X = \{X_n, n \in \mathbb{Z}^+\}$ be a random walk, $X_n = \sum_{i=0}^{n} \xi_i$, and the random variables $\{\xi_i, i \geq 0\}$ be mutually independent. Evidently, for any $0 \leq n_1 < n_2 < \ldots < n_k$, the random variables

$$X_{n_1} = \sum_{i=0}^{n_1} \xi_i, \quad X_{n_2} - X_{n_1} = \sum_{i=n_1+1}^{n_2} \xi_i, \ldots, \quad X_{n_k} - X_{n_{k-1}} = \sum_{i=n_{k-1}+1}^{n_k} \xi_i$$

are mutually independent; therefore, X is a process with discrete time and independent increments. Random processes with continuous time and independent increments are considered in detail in Chapter 2.

1.2.2. *Trajectory of a stochastic process as a random element*

Let $\{X_t, t \in \mathbb{T}\}$ be a stochastic process with the values in some set $\mathcal{S}$. Introduce the notation $\mathcal{S}^{\mathbb{T}} = \{y = y(t), \ t \in \mathbb{T}\}$ for the family of all functions defined on $\mathbb{T}$ and taking values in $\mathcal{S}$. Another notation can be $\mathcal{S}^{\mathbb{T}} = \times_{t \in \mathbb{T}} \mathcal{S}_t$, with all $\mathcal{S}_t = \mathcal{S}$ or simply $\mathcal{S}^{\mathbb{T}} = \times_{t \in \mathbb{T}} S$, which emphasizes that any element from $\mathcal{S}^{\mathbb{T}}$ is created in such a way that we take all points from $\mathbb{T}$, assigning a point from $\mathcal{S}$ to each of them. For example, we can consider $\mathcal{S}^{[0,\infty)}$ or $\mathcal{S}^{[0,T]}$ for any $T > 0$. Now, the trajectories of a random process X belong to the set $\mathcal{S}^{\mathbb{T}}$. Thus, considering the trajectories as elements of the set $\mathcal{S}^{\mathbb{T}}$, we get the mapping $X : \Omega \to \mathcal{S}^{\mathbb{T}}$, that transforms any element of Ω into some element of $\mathcal{S}^{\mathbb{T}}$. We would like to address the question of the measurability of this mapping. To this end, we need to find a σ-field $\Sigma^{\mathbb{T}}$ of subsets of $\mathcal{S}^{\mathbb{T}}$ such that the mapping X is $\mathcal{F}$-$\Sigma^{\mathbb{T}}$-measurable, and this σ-field should be the smallest possible. First, let us prove an auxiliary lemma.

LEMMA 1.1.– *Let* $\mathcal{Q}$ *and* $\mathcal{R}$ *be two spaces. Assume that* $\mathcal{Q}$ *is equipped with* σ*-field* $\mathcal{F}$*, and* $\mathcal{R}$ *is equipped with* σ*-field* $\mathcal{G}$*, where* $\mathcal{G}$ *is generated by some class* K*, i.e.* $\mathcal{G} = \sigma(K)$*. Then, the mapping* $f : \mathcal{Q} \to \mathcal{R}$ *is* $\mathcal{F}$*-*$\mathcal{G}$*-measurable if and only if it is* $\mathcal{F}$*-*K*-measurable, i.e. for any* $A \in K$*, the pre-image is* $f^{-1}(A) \in \mathcal{F}$*.*

PROOF.– Necessity is evident. To prove sufficiency, we should check that, in the case where the pre-images of all sets from K under mapping f belong to $\mathcal{F}$, the pre-images of all sets from $\mathcal{G}$ under mapping f belong to $\mathcal{F}$ as well. Introduce the family of sets

$$K_1 = \{B \in \mathcal{G} : f^{-1}(B) \in \mathcal{F}\}.$$

The properties of pre-images imply that K_1 is a σ-field. Indeed,

$$f^{-1}\left(\bigcup_{n=1}^{\infty} B_n\right) = \bigcup_{n=1}^{\infty} f^{-1}(B_n) \in \mathcal{F},$$

if $f^{-1}(B_n) \in \mathcal{F}$,

$$f^{-1}(C_2 \setminus C_1) = f^{-1}(C_2) \setminus f^{-1}(C_1) \in \mathcal{F},$$

if $f^{-1}(C_1) \in \mathcal{F}, i = 1, 2$, and $f^{-1}(\mathcal{R}) = \mathcal{Q} \in \mathcal{F}$. It means that $K_1 \supset \sigma(K) = \mathcal{G}$, whence the proof follows. □

Therefore, to characterize the measurability of the trajectories, we must find a "reasonable" subclass of sets of $\mathcal{S}^{\mathbb{T}}$, the inverse images of which belong to $\mathcal{F}$.

DEFINITION 1.6.– *Let the point $t_0 \in \mathbb{T}$ and the set $A \subset \mathcal{S}, A \in \Sigma$ be fixed.* Elementary cylinder *with base A over point t_0 is the following set from $\mathcal{S}^{\mathbb{T}}$:*

$$C(t_0, A) = \{y = y(t) \in \mathcal{S}^{\mathbb{T}} : y(t_0) \in A\}.$$

If $\mathcal{S} = \mathbb{R}$ and A is some interval, then $C(t_0, A)$ is represented schematically in Figure 1.4. Elementary cylinder consists of the functions whose values at point t_0 belong to the set A.

Let K_{el} be the class of elementary cylinders, and $\mathbb{K}_{el} = \sigma(K_{el})$, with the σ-field being generated by the elementary cylinders.

THEOREM 1.1.– *For any stochastic process $X = \{X_t, t \in \mathbb{T}\}$, the mapping $X : \Omega \to S^{\mathbb{T}}$, which assigns to any element $\omega \in \Omega$ the corresponding trajectory $X(\cdot, \omega)$, is $\mathcal{F}$–$\mathbb{K}_{el}$-measurable.*

PROOF.– According to lemma 1.1, it is sufficient to check that the mapping X is $\mathcal{F}$–K_{el}-measurable. Let the set $C(t_0, A) \in K_{el}$. Then, the pre-image $X^{-1}(C(t_0, A)) = \{\omega \in \Omega : X(t_0, \omega) \in A\} \in \mathcal{F}$, and the theorem is proved. □

COROLLARY 1.1.– *The σ-field $\mathbb{K}_{el}$, generated by the elementary cylinders, is the smallest σ-field $\Sigma^{\mathbb{T}}$ such that for any stochastic process X, the mapping $\omega \mapsto \{X_t(\Omega), t \in \mathbb{T}\}$ is $\mathcal{F}$-$\Sigma^{\mathbb{T}}$-measurable.*

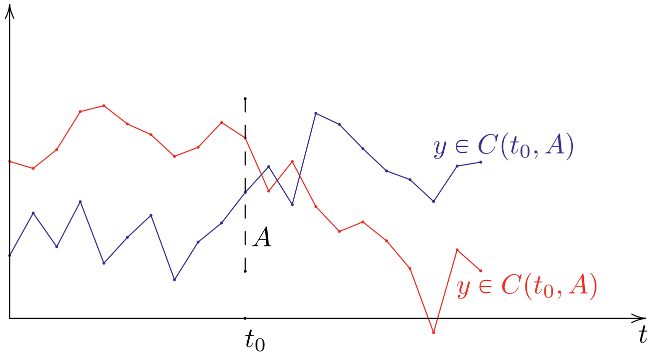

Figure 1.4. *Trajectories that belong to elementary cylinder. For a color version of the figure, see www.iste.co.uk/mishura/stochasticprocesses.zip*

1.3. Finite-dimensional distributions of stochastic processes: consistency conditions

There are two main approaches to characterizing a stochastic process: by the properties of its trajectories and by some number-valued characteristics, e.g. by finite-dimensional distributions of the values of the process. Of course, these approaches are closely related; however, any of them has its own specifics. Now we shall consider finite-dimensional distributions.

1.3.1. *Definition and properties of finite-dimensional distributions*

Let $X = \{X_t, t \in \mathbb{T}\}$ be a stochastic process taking its values in the measurable space $(\mathcal{S}, \Sigma)$. For any $k \geq 0$, consider the space $\mathcal{S}^{(k)}$, that is, a Cartesian product of $\mathcal{S}$:

$$\mathcal{S}^{(k)} = \underbrace{\mathcal{S} \times \mathcal{S} \times \ldots \times \mathcal{S}}_{k} = \times_{i=1}^{k} \mathcal{S}.$$

Let the σ-field $\Sigma^{(k)}$ of measurable sets on $\mathcal{S}^{(k)}$ be generated by all products of measurable sets from Σ.

DEFINITION 1.7.– *Finite-dimensional distributions of the process X is a family of probabilities of the form*

$$\mathbf{P} = \{\mathrm{P}\{(X_{t_1}, X_{t_2}, \ldots, X_{t_k}) \in A^{(k)}\}, k \geq 1, t_i \in \mathbb{T}, 1 \leq i \leq k, A^{(k)} \in \Sigma^{(k)}\}.$$

REMARK 1.1.– Often, especially in applied problems, *finite-dimensional distributions* are defined as the following probabilities:

$$\mathbf{P}_1 = \{\mathrm{P}\{X_{t_1} \in A_1, \ldots, X_{t_k} \in A_k\}, k \geq 1, t_i \in \mathbb{T}, A_i \in \Sigma, 1 \leq i \leq k\}.$$

Since we can write

$$\mathrm{P}\{X_{t_1} \in A_1, \ldots, X_{t_k} \in A_k\} = \mathrm{P}\left\{(X_{t_1}, \ldots, X_{t_k}) \in \times_{i=1}^k A_i\right\},$$

and $\times_{i=1}^k A_i \in \Sigma^{(k)}$, the following inclusion is evident: $\mathbf{P}_1 \subset \mathbf{P}$. The inclusion is strict, because the sets of the form $\times_{i=1}^k A^{(i)}$ do not exhaust $\Sigma^{(k)}$ unless $k = 1$. However, below we give a result where checking some properties for $\mathbf{P}$ is equivalent to checking them for $\mathbf{P}_1$.

1.3.2. *Consistency conditions*

Let $\pi = \{l_1, \ldots, l_k\}$ be a permutation of the coordinates $\{1, \ldots, k\}$, i.e. l_i are distinct indices from 1 to k. Denote for $A^{(k)} \in \Sigma^{(k)}$ by $\pi(A^{(k)})$ the set obtained from $A^{(k)}$ by the corresponding permutation of coordinates, e.g.

$$\pi\left(\times_{i=1}^k A_i\right) = \times_{i=1}^k A_{l_i}.$$

Denote also $\pi(X_{t_1}, \ldots, X_{t_k}) = (X_{t_{i_1}}, \ldots, X_{t_{i_k}})$ the respective permutation of vector coordinates $(X_{t_1}, \ldots, X_{t_k})$. Consider several consistency conditions which finite-dimensional distributions of random processes and the corresponding characteristic functions satisfy.

Consistency conditions (A):

1) For any $1 \leq k \leq l$, any points $t_i \in \mathbb{T}$, $1 \leq i \leq l$, and any set $A^{(k)} \in \Sigma^{(k)}$

$$\mathrm{P}\{(X_{t_1}, \ldots, X_{t_k}, X_{t_{k+1}}, \ldots, X_{t_l}) \in A^{(k)} \times \mathcal{S}^{(l-k)}\}$$
$$= \mathrm{P}\{(X_{t_1}, \ldots, X_{t_k}) \in A^{(k)}\}.$$

2) For any permutation π

$$\mathrm{P}\{\pi(X_{t_1}, \ldots, X_{t_k}) \in \pi(A^{(k)})\} = \mathrm{P}\{(X_{t_1}, \ldots, X_{t_k}) \in A^{(k)}\}. \quad [1.1]$$

REMARK 1.2.– Assume now that $\mathcal{S} = \mathbb{R}$ and consider the characteristic functions that correspond to the finite-dimensional distributions of stochastic process X. Denote

$$\psi(\lambda_1, \ldots, \lambda_k; t_1, \ldots, t_k) = \mathrm{E}\exp\left\{i\sum_{j=1}^k \lambda_j X_{t_j}\right\},$$

$\lambda_j \in \mathbb{R}$, $t_j \in \mathbb{T}$. Evidently, for $\psi(\lambda_1, \ldots, \lambda_k; t_1, \ldots, t_k)$, consistency conditions can be formulated as follows.

Consistency conditions (B):

1) For any $1 \leq k \leq l$ and any points $t_i \in \mathbb{T}$, $1 \leq i \leq l$, $\lambda_i \in \mathbb{R}$, $1 \leq i \leq k$

$$\psi(\lambda_1, \ldots, \lambda_k, \underbrace{0, \ldots, 0}_{l-k}; t_1, \ldots, t_k, t_{k+1}, \ldots, t_l) = \psi(\lambda_1, \ldots, \lambda_k; t_1, \ldots, t_k).$$

2) For any $k \geq 1, \lambda_i \in \mathbb{R}, t_i \in \mathbb{T}, 1 \leq i \leq k$

$$\psi(\pi(\overline{\lambda}); \pi(\overline{t})) = \psi(\lambda_1, \ldots, \lambda_k; t_1, \ldots, t_k),$$

where $\pi(\overline{\lambda}) = (\lambda_{i_1}, \ldots, \lambda_{i_k}), \pi(\overline{t}) = (t_{i_1}, \ldots, t_{i_k})$.

From now on, we assume that $\mathcal{S}$ is a metric space with the metric ρ, and Σ is a σ-field of Borel sets of S, generated by the metric ρ. We shall use the notation $(\mathcal{S}, \rho, \Sigma)$. Sometimes, we shall omit notations Σ and ρ yet assuming that they are fixed. Note that, for any $k > 1$, the space $\mathcal{S}^{(k)}$ is a metric space, where the metric ρ_k on the space $\mathcal{S}^{(k)}$ is defined by the formula

$$\rho_k(x, y) = \sum_{i=1}^{k} \rho(x_i, y_i), \tag{[1.2]}$$

and $x = (x_1, \ldots, x_k) \in \mathcal{S}^{(k)}$, $y = (y_1, \ldots, y_k) \in \mathcal{S}^{(k)}$. Moreover, we can define the σ-field $\Sigma^{(k)}$ of the Borel sets on $\mathcal{S}^{(k)}$, generated by the metric ρ_k. (Note that it coincides with the σ-field generated by products of Borel sets from $\mathcal{S}$.)

LEMMA 1.2.– Let the metric space $(\mathcal{S}, \rho, \Sigma)$ be separable and let the finite-dimensional distributions of the process X satisfy the following version of consistency conditions.

Consistency conditions $(A1)$

1) For any $1 \leq k \leq l$, any points $t_i \in \mathbb{T}$, $1 \leq i \leq l$ and any set $A^{(k)} = \times_{i=1}^{k} A_i, A_i \in \Sigma$, the following equality holds

$$\begin{aligned} &\mathrm{P}\{X_{t_1} \in A_1, \ldots, X_{t_k} \in A_k, X_{t_{k+1}} \in \mathcal{S}, \ldots, X_{t_l} \in \mathcal{S}\} \\ &\quad = \mathrm{P}\{X_{t_1} \in A_1, \ldots, X_{t_k} \in A_k\}. \end{aligned}$$

2) For any permutation $\pi = (i_1, \dots, i_k)$,

$$\mathrm{P}\{X_{t_{i_1}} \in A_{i_1}, \dots, X_{t_{i_k}} \in A_{i_k}\} = \mathrm{P}\{X_{t_1} \in A_1, \dots, X_{t_k} \in A_k\}.$$

Then the finite-dimensional distributions of the process X satisfy consistency conditions (A), where $\Sigma^{(k)}$ is a σ-field of Borel sets of $\mathcal{S}^{(k)}$. Therefore, for the stochastic process with the values in a metric separable space $(\mathcal{S}, \Sigma)$, consistency conditions for the families of sets $\mathbf{P}$ and $\mathbf{P}_1$ are fulfilled simultaneously.

PROOF.– The statement follows immediately from theorem A2.2 by noting that both sides of [1.1] are probability measures, and the sets of the form $\times_{i=1}^{k} A_i$, $A_i \in \Sigma$ form a π-system generating the σ-field $\Sigma^{(k)}$. $\square$

1.3.3. *Cylinder sets and generated σ-algebra*

DEFINITION 1.8.– *Let* $\{t_1, \dots, t_k\} \subset \mathbb{T}$*, the set* $A^{(k)} \in \Sigma^{(k)}$. Cylinder set *with base* $A^{(k)}$ *over the points* $\{t_1, \dots, t_k\}$ *is the set of the form*

$$C(t_1, \dots, t_k, A^{(k)}) = \{y = y(t) \in S^{\mathbb{T}} : (y(t_1), \dots, y(t_k)) \in A^{(k)}\}.$$

REMARK 1.3.– If $A^{(k)}$ is a rectangle in $\mathcal{S}^{(k)}$ of the form $A^{(k)} = \times_{i=1}^{k} A_i$, then $C(t_1, \dots, t_k, A^{(k)})$ is the intersection of the corresponding elementary cylinders:

$$C(t_1, \dots, t_k, A^{(k)}) = \{y = y(t) \in \mathcal{S}^{\mathbb{T}} : y(t_i) \in A_i, 1 \le i \le k\} = \bigcap_{i=1}^{k} C(t_i, A_i).$$

Denote by K_{cyl} the family of all cylinder sets.

LEMMA 1.3.–

1) The family of all cylinder sets K_{cyl} is an algebra on the space $\mathcal{S}^{\mathbb{T}}$.

2) If the set $\mathcal{S}$ contains at least two points, and the set $\mathbb{T}$ is infinite, then the family of all cylinder sets is not a σ-algebra.

PROOF.– 1) Let $C(t_1^1, \dots, t_k^1, A^{(k)})$ and $C(t_1^2, \dots, t_m^2, B^{(m)})$ be two cylinder sets, possibly with different bases and over different sets of points. We write them as cylinder sets with different bases but over the same set of points, namely over the set $\{t_1, \dots, t_l\} = \{t_1^1, \dots, t_k^1\} \cup \{t_1^2, \dots, t_m^2\}$. Specifically, define projections

$$p_1(x_1, \dots, x_l) = (x_i, t_i \in \{t_1^1, \dots, t_k^1\}), p_2(x_1, \dots, x_l) = (x_i, t_i \in \{t_1^2, \dots, t_m^2\}).$$

Then

$$C\left(t_1^1, \ldots, t_k^1, A^{(k)}\right) = C\left(t_1, \ldots, t_l, p_1^{-1}(A^{(k)})\right),$$
$$C\left(t_1^2, \ldots, t_m^2, B^{(m)}\right) = C\left(t_1, \ldots, t_l, p_2^{-1}(B^{(m)})\right),$$

so the set

$$C\left(t_1^1, \ldots, t_k^1, A^{(k)}\right) \cup C\left(t_1^2, \ldots, t_m^2, B^{(m)}\right)$$
$$= C\left(t_1, \ldots, t_l, p_1^{-1}(A^{(k)}) \cup p_2^{-1}(B^{(m)})\right)$$

belongs to K_{cyl}, because

$$p_1^{-1}(A^{(k)}) \cup p_2^{-1}(B^{(m)}) \in \Sigma^{(l)}.$$

Similarly, the set

$$C\left(t_1^1, \ldots, t_k^1, A^{(k)}\right) \setminus C\left(t_1^2, \ldots, t_m^2, B^{(m)}\right)$$
$$= C\left(t_1, \ldots, t_l, p_1^{-1}(A^{(k)}) \setminus p_2^{-1}(B^{(m)})\right)$$

belongs to K_{cyl}. Finally, for any $t_0 \in \mathbb{T}$

$$\mathcal{S}^{\mathbb{T}} = \{y = y(t) : y(t_0) \in \mathcal{S}\} \in K_{cyl},$$

whence it follows that the family of cylinder sets K_{cyl} is an algebra on the space $\mathcal{S}^{\mathbb{T}}$.

2) Let $\mathcal{S}$ contain at least two different points, say, s_1 and s_2, and let $\mathbb{T}$ be infinite. Then $\mathbb{T}$ contains a countable set of points $\{t_n, n \geq 1\}$. The set

$$\left(\bigcup_{i=1}^{\infty} C(t_{2i}, \{s_1\})\right) \cup \left(\bigcup_{i=1}^{\infty} C(t_{2i+1}, \{s_2\})\right)$$

is not a cylinder set because it cannot be described in terms of any finite set of points from $\mathbb{T}$. It means that, in this case, the family of cylinder sets K_{cyl} is not a σ-field on the space $\mathcal{S}^{\mathbb{T}}$. □

Denote by $\mathbb{K}_{cyl}$ the σ-algebra generated by the family K_{cyl} of cylinder sets: $\mathbb{K}_{cyl} = \sigma(K_{cyl})$.

LEMMA 1.4.– *For any* $k \geq 1$, $\mathbb{K}_{cyl} = \mathbb{K}_{el}$.

PROOF.– Evidently, σ-algebra $\mathbb{K}_{el} = \sigma(K_{el}) \subset \mathbb{K}_{cyl} = \sigma(K_{cyl})$, because any elementary cylinder is a cylinder set. Vice versa, for a fixed subset $\{t_1, \ldots, t_k\} \subset \mathbb{T}$, define the family

$$\mathcal{K} = \left\{ B \in \Sigma^{(k)} : C(t_1, \ldots, t_k, B) \in \mathbb{K}_{el} \right\}.$$

This is clearly a σ-algebra, which contains sets of the form $A_1 \times \cdots \times A_k$, $A_i \in \Sigma$, and therefore, $\mathcal{K} = \Sigma^{(k)}$. Consequently, we have $\mathbb{K}_{el} \supset K_{cyl}$, whence $\mathbb{K}_{el} \supset \mathbb{K}_{cyl}$, as required. □

1.3.4. *Kolmogorov theorem on the construction of a stochastic process by the family of probability distributions*

If some stochastic process is defined, then we know in particular its finite-dimensional distributions. We can say that a family of finite-dimensional distributions corresponds to a stochastic process. Consider now the opposite question. Namely, let us have a family of probability distributions. Is it possible to construct a probability space and a stochastic process on this space so that the family of probability distributions is a family of finite-dimensional distributions of the constructed process? Let us formulate this problem more precisely.

Let $(\mathcal{S}, \rho, \Sigma)$ be a metric space with Borel σ-field and $\mathbb{T}$ be a parameter set, and consider the family of functions

$$\left(\mathrm{P}\{t_1, \ldots, t_n, B^{(n)}\}, n \geq 1, t_i \in \mathbb{T}, 1 \leq i \leq n, B^{(n)} \in \Sigma^{(n)} \right), \qquad [1.3]$$

where $\Sigma^{(n)}$ is a Borel σ-field on $S^{(n)}$. Assume that, for any $t_1, \ldots, t_n \in \mathbb{T}$, the function $\mathrm{P}\{t_1, \ldots, t_n, \cdot\}$ is a probability measure on $\Sigma^{(n)}$.

THEOREM 1.2.– *[A.N. Kolmogorov] If* $(\mathcal{S}, \Sigma)$ *is a complete separable metric space with Borel* σ*-field* Σ*, and family* [1.3] *satisfies consistency conditions* $(A1)$*, then there exists a probability space* $(\Omega, \mathcal{F}, \mathrm{P})$ *and stochastic process* $X = \{X_t, t \in \mathbb{T}\}$ *on this space and with the phase space* $(\mathcal{S}, \Sigma)$*, for which* [1.3] *is the family of its finite-dimensional distributions.*

PROOF.– We divide the proof into several steps. *Step 1.* At first, recall that according to lemma 1.2, for the separable metric space $(\mathcal{S}, \rho, \Sigma)$ conditions (A) and $(A1)$ are equivalent and continue to deal with the condition (A). Put $\Omega = \mathcal{S}^{\mathbb{T}}$, $\mathcal{F} = \mathbb{K}_{cyl}$.

Recall also that $\mathbb{K}_{cyl} = \sigma(K_{cyl})$, where K_{cyl} is the algebra of cylinder sets. Let C be the arbitrary cylinder set, and let it be represented as

$$C = C(t_1, \dots, t_n, A^{(n)}).$$

Construct the following function defined on the sets of K_{cyl}:

$$\mathrm{P}'\{C\} = \mathrm{P}\{t_1, \dots, t_n, A^{(n)}\}.$$

Note that, generally speaking, the cylinder set C admits non-unique representation. In particular, it is possible to rearrange points $t_1, \dots, t_n$ and to "turn" the base $A^{(n)}$ accordingly. Moreover, it is possible to append any finite number of points $s_1, \dots, s_m$ and replace $A^{(n)}$ with $A^{(n)} \times \mathcal{S}^{(m)}$. However, consistency conditions guarantee that $\mathrm{P}'\{C\}$ will not change under these transformations; therefore, function $\mathrm{P}'\{\cdot\}$ on K_{cyl} is well defined.

Step 2. Now we want to prove that $\mathrm{P}'\{\cdot\}$ is an additive function on K_{cyl}. To this end, consider two disjoint sets

$$C_1 = C\left(t_1^1, \dots, t_n^1, A^{(n)}\right) \text{ and } C_2 = C\left(t_1^2, \dots, t_m^2, B^{(m)}\right),$$

and let $\{t_1, \dots, t_l\} = \{t_1^1, \dots, t_n^1\} \cup \{t_1^2, \dots, t_m^2\}$. Defining projection operators $p_1(x_1, \dots, x_l) = (x_i, t_i \in \{t_1^1, \dots, t_n^1\})$, $p_2(x_1, \dots, x_l) = (x_i, t_i \in \{t_1^2, \dots, t_m^2\})$, we have

$$C\left(t_1^1, \dots, t_n^1, A^{(n)}\right) \cup C\left(t_1^2, \dots, t_m^2, B^{(m)}\right)$$
$$= C\left(t_1, \dots, t_l, p_1^{-1}(A^{(n)}) \cup p_2^{-1}(B^{(m)})\right).$$

The bases $p_1^{-1}(A^{(n)})$ and $p_2^{-1}(B^{(m)})$ are disjoint, since the sets C_1 and C_2 are, so it follows from the fact that $\mathrm{P}\{t_1^1, \dots, t_n^1, \cdot\}$ is a measure with respect to the sets $A^{(n)}$ and also from consistency conditions that the following equalities hold:

$$\mathrm{P}\{C_1 \cup C_2\} = \mathrm{P}\{t_1, \dots, t_l, p_1^{-1}(A^{(n)}) \cup p_2^{-1}(B^{(m)})\}$$
$$= \mathrm{P}\{t_1, \dots, t_l, p_1^{-1}(A^{(n)})\} + \mathrm{P}\{t_1, \dots, t_l, p_2^{-1}(B^{(m)})\}$$
$$= \mathrm{P}\{t_1^1, \dots, t_n^1, A^{(n)}\} + \mathrm{P}\{t_1^2, \dots, t_m^2, B^{(m)}\} = \mathrm{P}'\{C_1\} + \mathrm{P}'\{C_2\}.$$

Step 3. Now we shall prove that P' is a countably additive function on K_{cyl}. Let the sets $\{C, C_n, n \geq 1\} \subset K_{cyl}, C_n \cap C_k = \emptyset$ for any $n \neq k$, and moreover, let $C = \bigcup_{n=1}^{\infty} C_n$. It is sufficient to prove that

$$\mathrm{P}'\{C\} = \sum_{n=1}^{\infty} \mathrm{P}'\{C_n\}. \qquad [1.4]$$

Let us establish [1.4] in the following equivalent form. Denote $D_n = \bigcup_{k=n}^{\infty} C_k$. Then $D_1 \supset D_2 \supset \ldots$, and $\bigcap_{n=1}^{\infty} D_n = \emptyset$. Besides this, it follows from the additivity of P' on K_{cyl} that

$$\mathrm{P}'\{C\} = \sum_{k=1}^{n-1} \mathrm{P}'\{C_k\} + \mathrm{P}'\{D_n\}.$$

Therefore, in order to prove [1.4], it is sufficient to establish that

$$\lim_{n\to\infty} \mathrm{P}'\{D_n\} = 0.$$

Since the sets D_n do not increase, this limit exists. By contradiction, let

$$\lim_{n\to\infty} \mathrm{P}'\{D_n\} = \alpha > 0.$$

Without any loss of generality, we can assume that the set of points, over which D_n is defined, is growing with n. Let the points be $\{t_1, \ldots, t_{k_n}\}$, and $B_n \in \Sigma^{(k_n)}$ be the base of D_n. In other words, let $D_n = C(t_1, \ldots, t_{k_n}, B_n)$. Taking into account the fact that $\mathcal{S}$ is a completely separable metric space, we get from theorem A1.1 that the space $\mathcal{S}^{(k_n)}$ is also a completely separable metric space. Therefore, according to theorem A1.2, there exists a compact set $K_n \in \Sigma^{(k_n)}$, such that $K_n \subseteq B_n$ and

$$\mathrm{P}\{t_1, \ldots, t_{k_n}, B_n \backslash K_n) < \frac{\alpha}{2^{n+1}}.$$

Now construct a cylinder set Q_n with the base K_n over the points $t_1, \ldots, t_{k_n}$ and consider the intersection $G_n = \bigcap_{i=1}^{n} Q_i$. Let M_n be the base of the set G_n. The sets G_n are non-increasing, and their bases M_n are compact. Indeed, the set M_n is an intersection of the bases of the sets $Q_i,\ 1 \le i \le n$, but in the case where all of them are presented as the cylinder sets over the points $\{t_1, \ldots, t_{k_n}\}$. With such a record, the initial bases K_i of the sets Q_i take the form $K_i \times \mathcal{S}^{(k_n - k_i)}$ and thus remain closed although perhaps no longer compact, while the set K_n is compact. An intersection of closed sets, one of which is compact, is a compact set as well; therefore, M_n is a compact set. The fact that G_n are non-increasing means that any element of $G_{n+p}, p > 0$ belongs to G_n. Their bases M_n are non-increasing in the sense that, for any point $(y(t_1), \ldots, y(t_{k_{n+p}})) \in M_{n+p}$, its "beginning" $(y(t_1), \ldots, y(t_{k_n})) \in M_n$. Now let us prove that the sets G_n and consequently M_n are non-empty. Indeed, it follows from the additivity of P' that

$$\mathrm{P}'\{D_n \backslash G_n\} = \mathrm{P}'\{D_n \setminus \bigcap_{i=1}^{n} Q_i\} = \mathrm{P}'\left\{\bigcup_{i=1}^{n} (D_n \setminus Q_i)\right\} \le \sum_{i=1}^{n} \mathrm{P}'\left\{D_n \setminus Q_i\right\}$$

$$\le \sum_{i=1}^{n} \mathrm{P}'\{D_i \setminus Q_i\} = \sum_{i=1}^{n} \mathrm{P}\{t_1, \ldots, t_{k_i}, B_i \setminus K_i\} \le \sum_{i=1}^{\infty} \frac{\alpha}{2^{i+1}} = \frac{\alpha}{2}.$$

It means that $P'\{G_n\} \geq \frac{\alpha}{2}$, whence the sets G_n are non-empty. In turn, it means that $M_n \neq \emptyset$, and we can choose the points $(y_1^{(n)}, \ldots, y_{l_n}^{(n)}) \in M_n$, and moreover, l_n is non-decreasing in n. Take the sequence $(y_1^{(n)}, \ldots, y_{l_n}^{(n)})$ and consider its "beginning" $(y_1^{(n)}, \ldots, y_{l_1}^{(n)})$. As has just been said, the sequence $(y_1^{(n)}, \ldots, y_{l_1}^{(n)}) \in M_1$. Therefore, it contains a convergent subsequence, and then any sequence $\{y_k^{(n)}, n \geq 1\}$ for $1 \leq k \leq l_1$ contains a convergent subsequence. Take $(y_1^{(n)}, \ldots, y_{l_2}^{(n)}) \in M_2$; at once, any "column" $\{y_k^{(n)}, n \geq 1\}$ for $1 \leq k \leq l_2$ contains a convergent subsequence. Finally, any "column" $\{y_k^{(n)}, n \geq 1\}$ for $k \geq 1$ contains a convergent subsequence. Denote by $y_k^{(0)}$ the limit of convergent subsequence $\{y_k^{(n_j)}, j \geq 1\}$. Applying the diagonal method, we can choose, for any n, a convergent subsequence of vectors

$$(y_1^{(n_j)}, y_2^{(n_j)}, \ldots, y_{k_n}^{(n_j)}) \to (y_1^{(0)}, y_2^{(0)}, \ldots, y_{k_n}^{(0)}).$$

Since all the points $(y_1^{(n_j)}, y_2^{(n_j)}, \ldots, y_{k_n}^{(n_j)}) \in M_n$ and the sets M_n are closed, we get that $(y_1^{(0)}, \ldots, y_{k_n}^{(0)}) \in M_n$. Now, define a function $y = y(t) \in \mathcal{S}^{\mathbb{T}}$ by the formula $y(t_k) = y_k^{(0)}, k \geq 1$ and define $y(t)$ in an arbitrary manner in the remaining points from $\mathbb{T}$. Then arbitrary vector $(y(t_1), \ldots, y(t_{k_n})) \in M_n$; therefore, for any $n \geq 1$, the function $y \in G_n \subset D_n$. This means that $\bigcap_{n=1}^{\infty} D_n \neq \emptyset$, which contradicts to the construction of sets D_n. It means that the assumption $\lim_{n\to\infty} P'(D_n) > 0$ leads to the contradiction and so is false. Therefore, P' is a countably additive function on K_{cyl}, and consequently, P' is a probability measure on the algebra K_{cyl}.

Step 4. According to the theorem on the extension of the measure from algebra to generated σ-algebra, there exists the unique probability measure P, that is, the extension of the measure P' from K_{cyl} to $\mathbb{K}_{cyl}$. Construct a stochastic process $X = \{X_t, t \in \mathbb{T}\}$ on $(\Omega, \mathcal{F}, P)$ in the following way (recall that the elements $\omega \in \Omega$ are presented by functions $y \in \mathcal{S}^{\mathbb{T}}$):

$$X_t(\omega) = \omega(t) := y(t).$$

We first check that $X = \{X_t, t \in \mathbb{T}\}$ is indeed a stochastic process. For any set $A \in \Sigma$ and for any $t_0 \in \mathbb{T}$, we have that

$$\begin{aligned} X_{t_0}^{-1}(A) &= \{\omega : X_{t_0}(\omega) \in A\} \\ &= \{y = y(t) : y(t_0) \in A\} = C(t_0, A) \in K_{cyl} \subset \mathbb{K}_{cyl} = \mathcal{F}. \end{aligned}$$

Further,

$$\begin{aligned} &P\{(X_{t_1}, \ldots, X_{t_k}) \in A^{(k)}\} = P\{(y(t_1), \ldots, y(t_k)) \in A^{(k)}\} \\ &= P\{C(t_1, \ldots, t_k, A^{(k)})\} = P'\{C(t_1, \ldots, t_k, A^{(k)})\} = P\{t_1, \ldots, t_k, A^{(k)}\}. \end{aligned}$$

Therefore, $X = \{X_t, t \in \mathbb{T}\}$ has the prescribed finite-dimensional distributions. The theorem is proved. □

In the case where $\mathcal{S} = \mathbb{R}$, the Kolmogorov theorem can be formulated as follows.

THEOREM 1.3.– *Let a family* $\psi(\lambda_1, \dots, \lambda_k; t_1, \dots, t_k), k \geq 1, \lambda_j \in \mathbb{R}, t_j \geq 0$ *of characteristic functions satisfy consistency conditions* (B). *Then there exists a probability space* $(\Omega, \mathcal{F}, \mathrm{P})$ *and a real-valued stochastic process* $X = \{X_t, t \geq 0\}$ *for which* $\mathrm{E} \exp\{i \sum_{j=1}^k \lambda_j X_{t_j}\} = \psi(\lambda_1, \dots, \lambda_k; t_1, \dots, t_k)$.

1.4. Properties of σ-algebra generated by cylinder sets. The notion of σ-algebra generated by a stochastic process

Let $\mathbb{T} = \mathbb{R}_+ = [0, +\infty)$, $(\mathcal{S}, \Sigma)$ be a measurable space and $X = \{X_t, t \in \mathbb{T}\}$ be an $\mathcal{S}$-valued stochastic process. Consider the standard σ-algebra $\mathbb{K}_{cyl}$ generated by cylinder sets, and for any finite or countable set of points, $\mathbb{T}' = \{t_n, n \geq 1\} \subset \mathbb{T}$ forms the algebra $K_{cyl}(\{t_n, n \geq 1\})$ of cylinder sets in the following way: $A \in K_{cyl}(\{t_n, n \geq 1\})$ if and only if there exists a subset $\{t_{n_1}, \dots, t_{n_k}\} \subset \{t_n, n \geq 1\}$ and $B^{(k)} \in \Sigma^{(k)}$ such that

$$A = C'(t_{n_1}, \dots, t_{n_k}, B^{(k)}) := \{y : \mathbb{T}' \to \mathcal{S} : (y(t_{n_1}), \dots, y(t_{n_k})) \in B^{(k)}\}.$$

Consider the generated σ-algebra $\mathbb{K}_{cyl}(\{t_n, n \geq 1\})$. We shall prove the statement that describes $\mathbb{K}_{cyl}$ in terms of the countable collections of points from $\mathbb{T}$.

LEMMA 1.5.– *The set* $A \subset \mathcal{S}^{\mathbb{T}}$ *belongs to* $\mathbb{K}_{cyl}$ *if and only if there exists a sequence of points* $\{t_n, n \geq 1\} \subset \mathbb{T}$ *and a set* $B \in \mathbb{K}_{cyl}(\{t_n, n \geq 1\})$, *such that the following equality holds:*

$$A = C(\{t_n, n \geq 1\}, B) := \{y \in \mathcal{S}^{\mathbb{T}} : (y(t_n), n \geq 1) \in B\}. \qquad [1.5]$$

PROOF.– Let C be any cylinder set from algebra K_{cyl},

$$C = \{y \in \mathcal{S}^{\mathbb{T}} : (y(z_1), \dots, y(z_m)) \in B^{(m)} \subset \mathcal{S}^{(m)}\}.$$

Then C admits the representation [1.5] if we consider the arbitrary sequence of points $\{t_n, n \geq 1\}$ such that $t_n = z_n, 1 \leq n \leq m$ and $B = A(B^{(m)})$. Therefore, if we denote by $\mathbb{K}$ the sets from $\mathbb{K}_{cyl}$ that admit the representation [1.5], then $K_{cyl} \subset \mathbb{K}$. Let us establish now that $\mathbb{K}$ is a σ-algebra. Indeed, $S^{\mathbb{T}} \in \mathbb{K}$, because we can take an arbitrary sequence

$$\mathbb{T}' = \{t_n, n \geq 1\} \text{ and } B = \mathcal{S}^{\mathbb{T}'} := \times_{n=1}^{\infty} \mathcal{S} \in \mathbb{K}_{cyl}(\mathbb{T}'),$$

and get that the set $\mathcal{S}^{\mathbb{T}}$ has a form $\mathcal{S}^{\mathbb{T}} = \{y \in S^{\mathbb{T}} : (y(t_n), n \geq 1) \in B = S^{\mathbb{T}'}\}$, admitting with evidence the representation [1.5]. Further, if $A_1, A_2 \in \mathbb{K}$, they are defined over the sequences of points $\mathbb{T}^1 = \{t_n^1, n \geq 1\}$ and $\mathbb{T}^2 = \{t_n^2, n \geq 1\}$ and have the bases B_1, B_2, correspondingly. Then, we can consider these sets as defined over the same sequence of points, setting $\mathbb{S} = \{t_n^1, n \geq 1\} \cup \{t_n^2, n \geq 1\}$ and introducing the maps $p_i : y \in \mathcal{S}^{\mathbb{S}} \mapsto y|_{\mathbb{T}^i} \in S^{\mathbb{T}^i}$, $i = 1, 2$. Then $A_i = C(\mathbb{S}, p_i^{-1}(B_i))$, $i = 1, 2$. The bases $p_i^{-1}(B_i)$, $i = 1, 2$, are measurable, since the maps p_i, $i = 1, 2$ are measurable (even continuous in the topology of pointwise convergence). Therefore,

$$A_1 \setminus A_2 = C(\mathbb{S}, p_1^{-1}(B_1) \setminus p_2^{-1}(B_2)) \in \mathbb{K}.$$

Similarly, if $\{A_r, r \geq 1\} \subset \mathbb{K}$, and they are defined over the sequences of points $\mathbb{T}^r = \{t_n^r, n \geq 1\}$ and bases $B_r, r \geq 1$, we can define $\mathbb{T}^0 = \bigcup_{r=0}^{\infty} \mathbb{T}^r$ and $p_r : y \in \mathcal{S}^{\mathbb{T}^0} \mapsto y|_{\mathbb{T}^r} \in S^{\mathbb{T}^r}$, $r \geq 1$, so that

$$\bigcup_{r=1}^{\infty} A_r = C\left(\mathbb{T}^0, \bigcup_{r=1}^{\infty} p_r^{-1}(B_r)\right).$$

Thus, we have that $\mathbb{K}$ is a σ-algebra that contains K_{cyl}, i.e. $\mathbb{K} \supset \mathbb{K}_{cyl}$, but $\mathbb{K} \subset \mathbb{K}_{cyl}$ by the definition. It means that $\mathbb{K} = \mathbb{K}_{cyl}$, whence the proof follows. □

DEFINITION 1.9.– *The σ-algebra, generated by the process X is the family of sets* $\mathcal{F}^X = \{X^{-1}(A), A \in \mathbb{K}_{cyl}\}$.

REMARK 1.4.– It follows from the properties of pre-images that for any σ-algebra $\mathcal{A}$, the family of sets $\{X^{-1}, A \in \mathcal{A}\}$ is a σ-algebra; therefore, definition 1.9 is properly formulated.

LEMMA 1.6.– $\mathcal{F}^X = \sigma\{X^{-1}(A), A \in K_{cyl}\}$

PROOF.– Denote $\mathcal{F}_1^X = \{X^{-1}(A), A \in K_{cyl}\}$. On the one hand, since $\mathcal{F}^X$ is a σ-algebra and $\mathcal{F}^X$ contains all pre-images $X^{-1}(A)$ under mapping $A \in K_{cyl}$, then $\mathcal{F}^X \supset \mathcal{F}_1^X$. On the other hand, consider $\mathcal{F}_1^X$ and note that the mapping X is $\mathcal{F}_1^X$-K_{cyl}-measurable; therefore, according to lemma 1.1, the mapping X is $\mathcal{F}_1^X$-$\mathbb{K}_{cyl}$-measurable, i.e. $X^{-1}(A) \in \mathcal{F}_1^X$ for any $A \in \mathbb{K}_{cyl}$. It means that $\mathcal{F}_1^X \supset \mathcal{F}^X$, and the lemma is proved. □

The following fact is a consequence of lemma 1.4.

COROLLARY 1.2.– *The σ-algebra generated by a stochastic process X is the smallest σ-algebra containing all the sets of the form*

$$\{\omega \in \Omega : X(t_1, \omega) \in A_1, ..., X(t_k, \omega) \in A_k\}, A_i \in \Sigma, t_i \in \mathbb{T}, 1 \leq i \leq k, k \geq 1.$$