1

Introduction, Generalities, Definitions of Systems

This chapter explores the representation, modeling and identification of signals, transmission systems and filtering concepts.

A number of mathematical concepts introduced in this chapter (distributions) are an extension of the conventional notion of functions. The comprehensive study of distributions is not absolutely essential if some unproven results are accepted. The objective of this chapter is to introduce basic notions, temporal relations and transformations that enable this correspondence to be established.

1.1. Introduction

Modeling is a very important step in linear systems control. To properly control a system, the knowledge of a good model is necessary. For example, to drive a car, the more accurate the knowledge of its dynamic behavior or its model is (by learning or training), the better it will be controlled at high speed and therefore the better it will be driven and will show the best performance. The dynamic model is acquired by learning or by identifying the system after knowing the structure of this model.

During the development of an application for automation purposes, we follow the following steps:

1) modeling;

2) identification;

3) behavior analysis;

4) controller synthesis;

5) control implementation;

6) analysis and study of the closed-loop system;

7) verification of the performance and eventually repeat steps 2, 3 or 4.

The modeling stage becomes crucial when the requirements are strict regarding performances and when the control implemented proves to be complex.

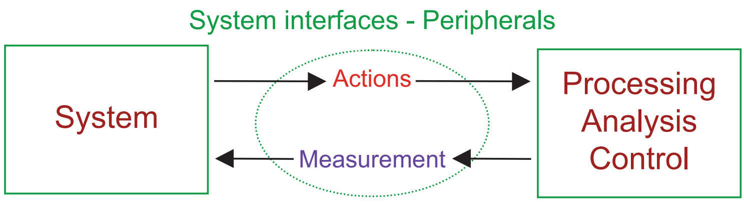

Figure 1.1. *Peripherals of a system*

1.2. Signals and communication systems

In electronics, as for most other areas, a signal designates any electromagnetic or physics phenomenon used as medium for information to be transmitted. This signal is used to characterize a physical quantity captured by reflecting its evolution over time and or in space, amplitude, energy or power. For example, this is the case in a thermometer with the indicated level or the electrical signal supplied by a thermocouple.

Electrical signals are typically provided by a system called sensor, consisting of an element sensitive to a physical effect that it converts into an electrically measurable quantity and an adapter amplifier that thereof provides the equivalence in the form of a signal. This is the case for pressure, temperature, radiation, speed, position and acceleration sensors. This signal is then manipulated by an analog or a digital system.

Sensor: Physical effect$\longrightarrow$Converter$\longrightarrow$Amplifier$\longrightarrow$Adaptor$\longrightarrow$Signal

In general, in the case of an observation of a physical phenomenon, a signal can be defined as being the variation of a physical quantity (measured by a sensor) in time (t) or in space (x) - (observation). In this case, the processing may involve the separation of signal and noise, extraction of information, extraction of frequency and temporal characteristics, etc.

Observation: Sensor $\longrightarrow$ Electronic system $\longrightarrow$ Signal $\longrightarrow$ Information

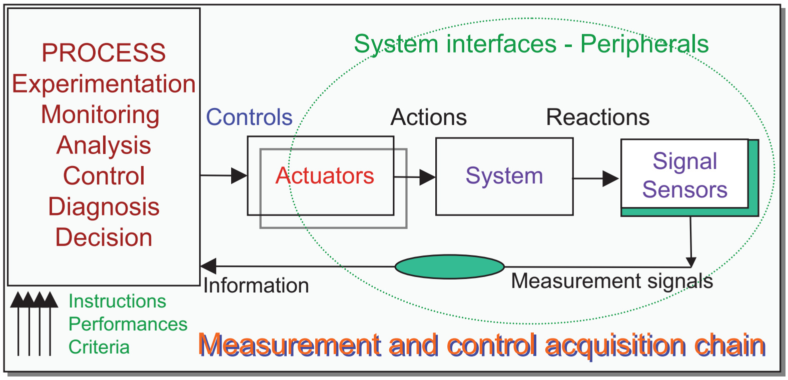

Figure 1.2. *Acquisition, sensors and triggers*

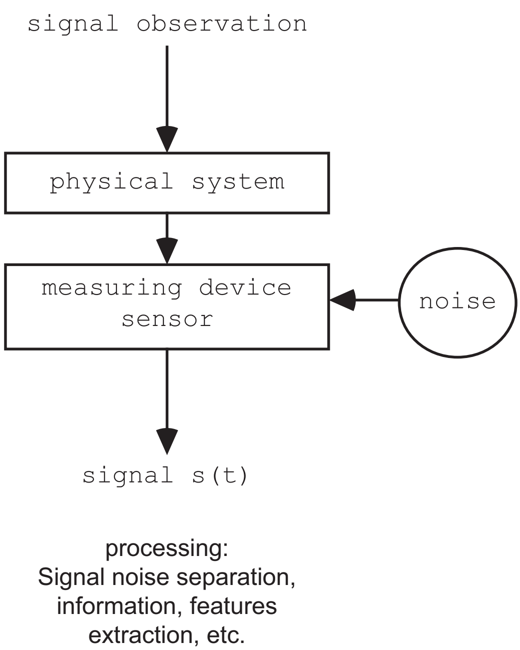

Figure 1.3. *Observation sequence*

A communication system is usually a means of communication between an information source and a recipient (electronic, optical or mechanical system)

Communication sequence:

Emission: Message $\longrightarrow$ Coder $\longrightarrow$ Electronic system $\longrightarrow$ Signal $\longrightarrow$ Transmitter $\longrightarrow$ Channel

Reception:

Channel $\longrightarrow$ Receiver $\longrightarrow$ Electronic system $\longrightarrow$ Decoder $\longrightarrow$ Signal $\longrightarrow$ Processing

Processing:

Signal processing $\longrightarrow$ Information processing $\longrightarrow$ Decision

– The variable is usually time t (it can be space ξ or any other physical parameters).

– The signal is denoted as $s(t)$, $y(t)$ (or $z(\xi)$).

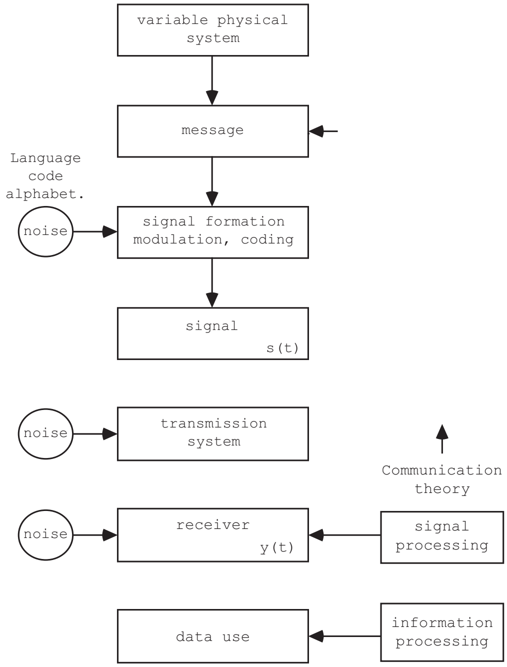

Figure 1.4. *Communication sequence*

1.3. Signals and systems representation

1.3.1. *Signal*

A signal is a physical representation of a phenomenon that evolves in time or in space. It can be represented in time and frequency domains. The frequency representation of a signal is interesting because it provides more information about the signal. The tool used to shift from the time domain to the frequency domain is the Fourier transform (FT).

Analog macroscopic measurements provide curves (in time) of the relevant signal. This signal is represented by a mathematical function $x(t)$, most often, with real values of one real variable (time t).

These functions, since they reflect physical quantities with finite energy or finite average power, are:

– real;

– bounded functions and with bounded support (integration without difficulties);

– continuous and differentiable in any point (derivation with no difficulties);

– they can be periodical $x(t) = x(t + T)$ if T is the period;

– in addition, for causal physical signals, we get: $x(t) = 0$ for $t < 0$.

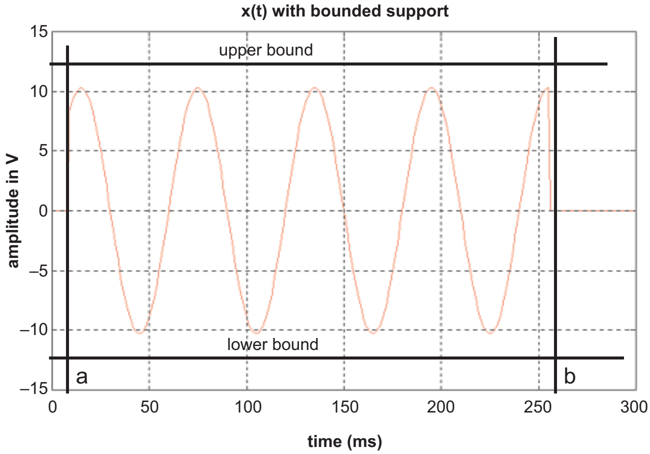

Figure 1.5. *Time signal with bounded support*

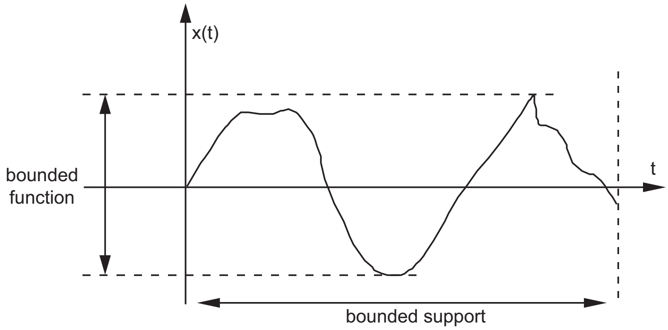

Figure 1.6. *Signal defined over a bounded medium*

Subsequently, in order to simplify calculations and mathematical manipulations, we will consider that signals can be represented (or modeled) by functions having the following properties:

– bounded functions;

– functions defined in $t \in R =]-\infty, +\infty[$;

– functions with discontinuities or piecewise continuous (rectangle or gate, sawtooth signals, etc.);

– real- or complex-valued functions.

Often, the set of all functions having the above properties is restricted either to that of absolutely summable functions, $x(t)$ such that $\int_{-\infty}^{+\infty} |x(t)|\, dt$ does exist, or that of square-integrable functions such that quantity $\int_{-\infty}^{+\infty} |x(t)|\, dt$ exists and is finite. The advantage of square-integrable functions is that they make it possible to represent finite-energy signals (case of most frequently encountered real signals). It can be shown that this set constitutes a vector space.

1.3.2. *Functional space L*$_2$

Consider the vector space of square-integrable functions:

$$L_2 = \{\,\text{function}\, f(t) : \mathbb{R} \longrightarrow \mathbb{C}\,\text{such that}: \int_{-\infty}^{+\infty} |f(t)|^2\, dt < \infty\} \qquad [1.1]$$

If $f^*(t)$ is the conjugate of $f(t)$, we get

$$< f(t), g(t) >= \int_{-\infty}^{+\infty} f(t)g^*(t)dt \qquad [1.2]$$

the scalar product of the two functions. If $f(t)$ and $g(t)$ are real, it is a scalar product, otherwise a Hermitian product.

Schwartz inequality:

$$|< f, g >|^2 \leq < f, f >< g, g > \qquad [1.3]$$

The equality is obtained if and only if $f = \lambda.g$ with λ scalar.

PROOF.– $f(t)$ and $g(t)$ are two functions of the space of finite-energy functions L_2, and λ is a constant parameter arbitrarily chosen. Let $\lambda \in \mathbb{C}$ such that $q =< f + \lambda g, f + \lambda g >^2 \geq 0$.

Let: $\alpha = \int_{-\infty}^{+\infty} |f(t)|^2 dt$, $\gamma = \int_{-\infty}^{+\infty} |g(t)|^2 dt$ and $\beta = \int_{-\infty}^{+\infty} f(t)g^*(t)dt$.

$\alpha =< f, f > \ \gamma =< g, g >$

$\beta =< f, g > \ \beta^* =< g, f >$

$$\forall \lambda \in \mathbb{C}, \ q = \alpha + \lambda^*\beta + \lambda\beta^* + \lambda\lambda^*\gamma \geq 0. \qquad [1.4]$$

1) if $\gamma = 0 \Longrightarrow \forall\lambda \in \mathbb{C}, q = \alpha + 2Re(\lambda^*\beta) \geq 0 \Longrightarrow \beta = 0$

the inequality is then verified: $q =< f, f >= \alpha$

2) if $\gamma \neq 0$, multiply by g the two members of the equation [1.4]: $\gamma\alpha + \lambda^*\beta\gamma + \lambda\beta^*\gamma + \lambda\lambda^*\gamma\gamma > 0$

$$(\lambda\gamma + \beta)\lambda^*\gamma + \beta^*\lambda\gamma + \alpha\gamma + \beta\beta^* - \beta^*\beta \geq 0 \qquad [1.5]$$

$$(\lambda\gamma + \beta)(\lambda^*\gamma + \beta^*) + \alpha\gamma - \beta\beta^* \geq 0 \qquad [1.6]$$

$$|\lambda\gamma + \beta|^2 + \alpha\gamma - |\beta|^2 \geq 0 \qquad [1.7]$$

or even

$$\lambda\gamma + \beta 2 + \alpha\gamma \geq |\beta|^2 \qquad [1.8]$$

taking $\lambda = -\frac{\beta}{\gamma}$, we get the Schwartz inequality: $\alpha\gamma \geq |\beta|^2$

The equality corresponds to $q = 0$ where

$$\int_{-\infty}^{+\infty} |f(t) + \lambda g(t)|^2 \, dt = 0 \Longrightarrow f(t) + \lambda g(t) = 0 \,\text{where}\, f(t) = -\lambda g(t) \quad [1.9]$$

EXAMPLE 1.1.– *The output signal of a harmonic oscillator can be represented by the sinusoidal function represented by the curve below. This signal (assumed as deterministic) can be modeled by the following equation:* $y(t) = \sin(2\pi f_o t + \varphi)$.

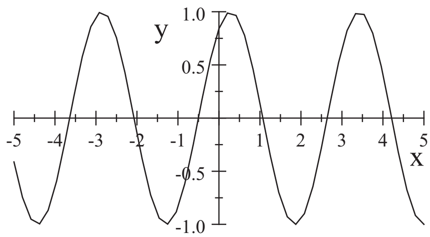

Figure 1.7. *Harmonic oscillator output signal*

This function is continuous, differentiable, bounded, and periodic of period $T = 1/f_o$ but with unbounded support and square non-integrable (infinite energy and finite mean power). In this case, for the study, we will rather consider *the average power* in a period and for the spectral representation either the FT in the sense of distributions or Fourier series will be used.

Therefore, for the study of signals, abstract mathematical modeling will be used in order to take advantage of the power of the theoretical tools available. Often, this abstract representation expands the properties of the signal being considered and its definition (finite average power signals, distributions). The interpretation of the results obtained with such mathematical models must be made with care, taking into account physical considerations of the problems in order to face realistic situations after a theoretical study.

1.3.3. *Dirac distribution*

It is possible to summarily define $\delta(t)$ by its properties because they are most often sufficient for the processing technique under consideration.

DEFINITION 1.1.– *Consider the function defined by* $f_l(t) = 0$ *if* $t < -l/2$ *or* $t > l/2$ *and* $f_l(t) = 1/l$ *if* $t \geq -l/2$ *and* $t \leq l/2$*. The appearance of this function is* *represented here for different values of l. The limit of this function as l tends to zero gives us the Dirac distribution. When the width of the curve l is made to tend to zero, an infinitely high and narrow rectangle is obtained whose area is always equal to* 1. *When the limit is reached, we get a mathematical object that is not a function of R, because it is undefined for $t = 0$.* This is the Dirac delta function or impulse symbol *denoted as $\delta(t)$.*

Conventionally, it is represented by an arrow of height 1 *at $t = 0$.*

$\delta(t - t_1)$ represents the impulse translated of t_1 on the time axis. $\delta(t).x(t)$ is also a distribution of the same kind that represents a mass point $x(0)$ concentrated in $t = 0$. It shows the following features.

For its application in physics, the interesting particularity of Dirac's delta function is the finite area (equal to 1), representing, for example, a point mass concentrated in $t = 0$. This distribution is very useful for the mathematical modeling of physical phenomena: point sources in optics, point mass in mechanics, percussion in acoustics, point charge in electricity, acceleration during shock. Before stating its properties, we define the convolution product $*$ by the composition product.

The delta function is graphically represented by convention as an arrow with a unit height. It should be noted that the height is here connected with the mass of the distribution (surface under the curve) and not with amplitude, as is the case for functions.

DEFINITION 1.2.– *The convolution product of two functions $f(t)$ and $g(t)$ denoted as $f(t) * g(t)$ is defined by the integral (when it makes sense):*

$$h(t) = x(t) * y(t) = \int_{-\infty}^{+\infty} x(t - \tau)y(\tau)d\tau \qquad [1.10]$$

PROPOSITION 1.1.– The delta function shows the following particularities (Table 1.1).

The area under the curve is equal to 1: $\int_{-\infty}^{+\infty} f_l(t)dt = 1$ for any width l.	
$x(t) * \delta(t) = x(t)$	$\delta(t)$neutral element of the convolution
$x(t) * \delta(t - t_o) = x(t - t_o)$	(offset)
$x(t - t_1) * \delta(t - t_2) = x(t - t_1 - t_2)$	
$\delta(t - t_2) * \delta(t - t_1) = \delta(t - t_2 - t_1)$	
$\delta(at) = \lvert a\rvert^{-1} \delta(t)$	

Table 1.1. *The delta function*

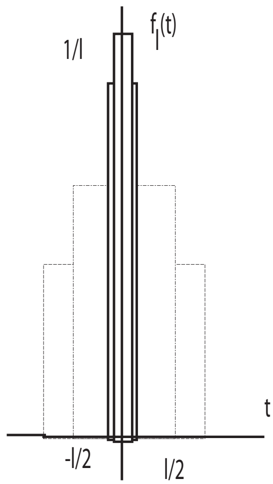

Figure 1.8. *Unit impulse*

1.4. Convolution and composition products – notions of filtering

In Table 1.1, let us recall that $*$ denotes the convolution product and δ(t) is the delta function (impulse symbol), which is equal to zero everywhere except at zero where it is infinite.

1.4.1. *Convolution or composition product*

PROPOSITION 1.2.– Convolution or product composition has the following proprieties:

– commutativity: $x(t) * y(t) = y(t) * x(t)$ (change in variable $u = t - \tau$);

– distributivity: $x(t) * (y(t) + z(t)) = x(t) * y(t) + x(t) * z(t)$;

– associativity: $(x(t) * y(t)) * z(t)) = x(t) * (y(t) * z(t))$.

Dirac function properties:

The graphic representation of a pulse is by convention as shown in Figure 1.9 for $\delta(t - t_o)$.

a) $\delta(t) = 0 \; \forall t \neq 0$ and $\delta(t) = \infty$ for $t = 0$;

b) $\int_{-\infty}^{+\infty} \delta(t)dt = 1 \; \int_{-\infty}^{+\infty} \delta(t)x(t)dt = x(0)$; $\int \delta(t - t_o)x(t)dt = x(t_o)$

δ(t-t$_o$) represents the impulse offset by to on the axis of time t. The quantity $\delta(t$ t_o)x(t) is also a distribution of the same kind representing a point mass $x(to)$ concentrated in $t = t_o$.

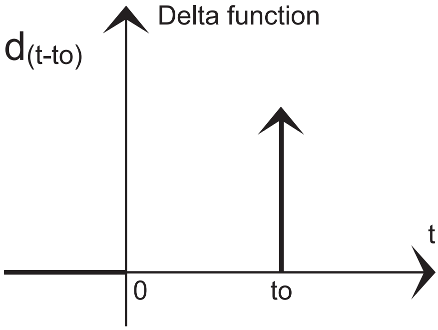

Figure 1.9. *Dirac's delta function*

Figure 1.10. *Input/output of a system*

(c) $x(t) * \delta(t) = x(t)$	Neutral element of the convolution
(d) $x(t) * \delta(t - t_o) = x(t - t_o)$	Offset of a time function
(e) $x(t - t_1) * \delta(t - t_o) = x(t - t_1 - t_o)$	Offsets cumulation
$\delta(t - t_1) * \delta(t - t_o) = \delta(t - t_1 - t_o)$	
(f) $\delta(at) = \lvert a \rvert^{-1} \delta(t)$	Change of scale
(g) $\delta(t - t_o) = \delta(t_o - t)$	Symmetry of the pulse
(h) $\delta(t - t_o) = \frac{d}{dt}(u(t - t_o))$	Where $u(t - t_o)$ is the unit level offset in t_o

Table 1.2. *Properties of the delta function*

The delta function can also be physically approximated by a triangular or exponential function whose area under the curve is equal to the unit.

1.4.2. *System*

A system can be represented by a filter whose response is the convolution product of its transfer function and the input signal. A system can be described by time-differential equations and algebraic equations or by a transfer function in the complex plane (frequency domain). The transfer function represents the Laplace transform of the impulse response of the system.

1.5. Transmission systems and filters

Consider a system S, with an input a signal $x(t)$ assumed as real (for the moment) and an output $y(t)$ (the response of the system to input $x(t)$). The system S can be represented by an operator that we will also define as $S : X \longrightarrow Y$. We will then write $y(t) = S(x(t))$ to mean that S transforms $x(t)$ of the input signals vector space X into $y(t)$ of the output vector space Y (under certain assumptions of reliability, stability, signals boundedness, etc.):

– the system is linear if the operator associated with it is linear:

$x_1(t) \longrightarrow y_1(t)$ and $x_2(t) \longrightarrow y_2(t)$, then at $x_1(t)+b.x_2(t) \longrightarrow a.y_1(t)+b.y_2(t)$ $\forall a, b \in R$

– the system is time invariant (stationary) if its behavior is independent of the time origin:

$x_1(t) \longrightarrow y_1(t)$, then $x_1(t-\tau) \longrightarrow y_1(t-\tau)$ $\forall \tau \in R$

– a system is known as causal if its impulse response (response to an impulse) is zero for negative times ($h(t) = 0$ if $t < 0$).

REMARK 1.1.– *As a first approximation, almost all systems are (very often) considered as linear for weak signals. A linear system realizes an application from a vector space X in an another Y. Generalizing to the spaces of complex signals (spaces defined in C). In general, X and Y define the same vector space (or two subspaces), the application S is then a linear operator and Y is the image of X by S.*

Consequently, any signal, $x(t) \in X$, can be written according to the base elements of the vector space $x_i(t)$:

$x(t) = \sum_i a_i x_i(t) \longrightarrow y(t) = \sum_i a_i y_i(t)$; with $S : x_j(t) \longrightarrow y_j(t)$ for $j = 1, 2, 3...$

The response of the system therefore will be written based on the $y_i(t)$ images of the elements of the basis of the space X by the operator S (linear combination with the same coefficients as the decomposition of $x(t)$). To know the answer to any random action, is suffices that the image by operator S is known for an enumerable collection of function $x_i(t)$ (basis of X).

1.5.1. *Convolution and filtering*

The physical justification of the convolution or composition product for filtering uses the concept of impulse response of a system or filter:

– $h(t)$ is the *impulse response* of the filter or system (H) that is the response of this system when an impulse is applied on its input, such as Dirac's delta function $\delta(t)$, $S : \delta(t) \longrightarrow h(t)$.

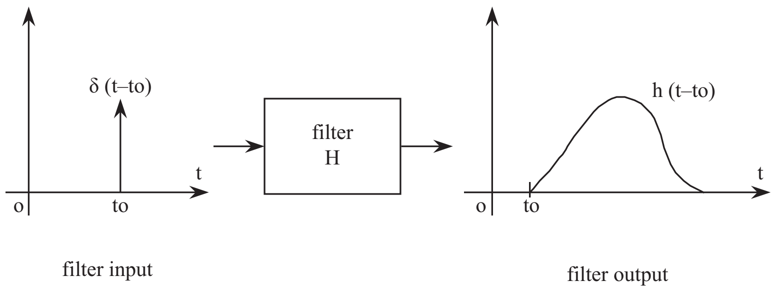

Figure 1.11. *Impulse response of the filter or system*

– When an impulse is applied to the system offset by t_o, its response will be shifted as much: $S : \delta(t - t_o) \longrightarrow h(t - t_o)$.

– When applying a signal $x(t)$ on input of this system, the input signal is subdivided with a step $\Delta\tau$, into a set of elementary impulse, as shown in Figure 1.12.

– Considering the kth shifted impulse of $k\Delta\tau = \tau$, it has mass (weight or area under the curve) $x(k\Delta\tau).\Delta\tau$; the response to this impulse will also be shifted in time and amplitude $(x(k\Delta_\tau).\Delta\tau).h(t - k\Delta\tau)$.

– Then, the response to any random signal $x(t)$ will be the sum of the terms consisting of the responses to all the impulses that constitute $x(t)$; that is the sum of $x(k\Delta\tau).\Delta\tau.h(t - k\Delta\tau)$ for all values of k obtained during the subdivision, that is $\sum_k (x(k\Delta\tau).\Delta\tau.h(t - k\Delta\tau))$:

- whence by passing to the limit when $\Delta\tau$ tends to zero, and by replacing $k\Delta\tau$ by τ and $\Delta\tau$ by $d\tau$, we get the response of the filter on input x(t): $y(t) = \int_0^{+\infty} (x(\tau).h(t - \tau))d\tau$, that is, because signals are causal:

$$y(t) = \int_{-\infty}^{+\infty} (x(\tau).h(t - \tau))d\tau = x(t) * h(t) = h(t) * x(t) \quad [1.11]$$

In conclusion, for a system the response $y(t)$ to an input $x(t)$ is expressed as the convolution product of the input and the impulse response of the system. Therefore, it is possible to represent a system by its impulse response inasmuch as it is sufficient to determine its response to any input $x(t)$. It is the *representation of a system by the impulse response*.

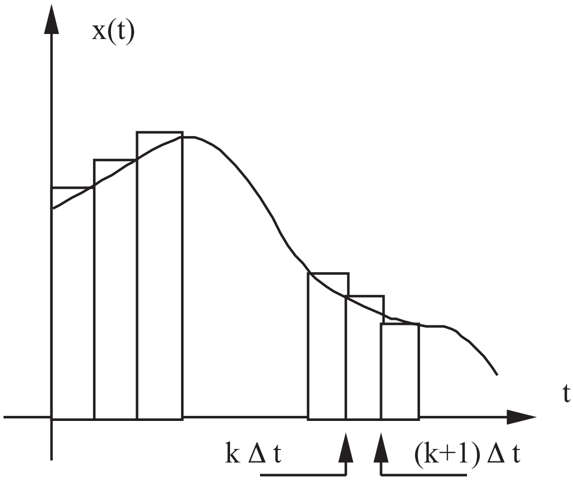

Figure 1.12. *Input signal with a step* $\Delta\tau$

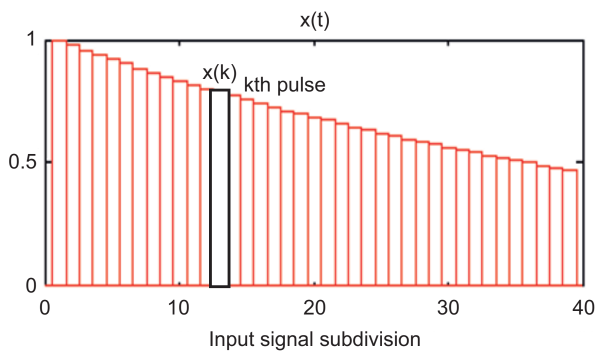

Figure 1.13. *Signal subdivision*

In this section, we have shown the interest of functions, distributions and operators and mathematical tools for modeling signals and systems. In the following section, we will present some types of signals and systems as well as their modeling and representations.

1.6. Deterministic signals – random signals – analog signals

1.6.1. *Definitions*

1.6.1.1. *Deterministic signals*

Signals originate from phenomena for which the knowledge of initial conditions and physical laws allows the anticipation of the result of the measurement and represents a set of results in the form of a function $x(t)$ (for example falling body, filter response with a known x). Such a signal is considered as finite when it is possible to determine its value at any time t. The description of a signal may be non-parametric (recording, graphics, etc.) or parametric.

We refer to parametric representation or signal model when one is able to define a set of parameters that make it possible to trace the evolution of the signal in time and determine its values at any moment.

Sinusoid	$x(t) = A.sin(\omega t) + B$	Parameters: A, ω, B
Damped oscillating signal	$s(t) = A(1 - e^{-b.t}).cos(\omega t + \varphi)$	Parameters: A, b, ω,φ
Square signal	$y(t) = A.signe(sin(\omega t)) + b$	Parameters: A, b, ω

Table 1.3. *Deterministic signals*

EXAMPLE 1.2.–

A deterministic signal may, in principle, be rigorously reproduced identical to itself.

1.6.1.2. *Random signals (or probabilistic)*

Signals for which the result of a measurement (test) is not predictable and that can only be characterized by using statistical quantities (random distribution), cannot be determined by instantaneous values. For example, the value of the temperature in a geographical point is impossible to determine in advance (before measurement), it constitutes a random signal. A prediction can only be made in the statistical sense with a probability not equal to the unity. A range of values is associated with a probability of occurrence (for example probability $(10 < \theta \leq 20) = 0.3$). *A random signal is not strictly reproducible.*

1.6.1.3. *Signal and noise*

Signal = a quantity carrying information.

Noise = a quantity carrying no information or unnecessary to the user.

It is the recipient who considers if there is or isn't information. For instance, thermal noise from the sky is a signal for the radio astronomer; on the other hand, it is a disruptive noise for telecommunication engineers.

1.6.2. *Some deterministic analog signals*

Heaviside signal (unit step), denoted as $H(t)$, $G(t)$ or $u(t)$, will subsequently be written as $u(t)$. It is defined by:

$$u(t) = \begin{cases} 1, \text{ if } t \geq 0 \\ 0, \text{ if } t < 0 \end{cases} \quad [1.12]$$

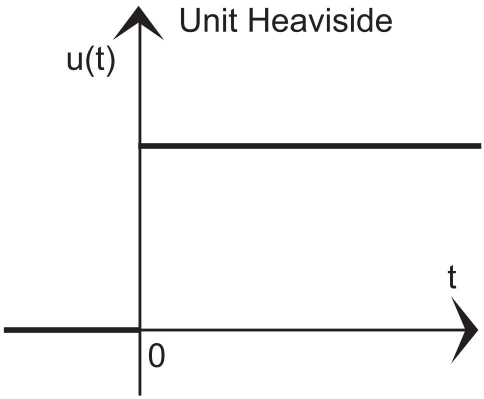

Figure 1.14. *Heaviside*

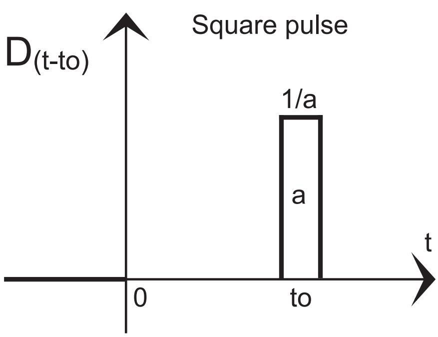

Figure 1.15. *Rectangular pulse*

Its derivative is the Dirac delta function $\delta(t)$

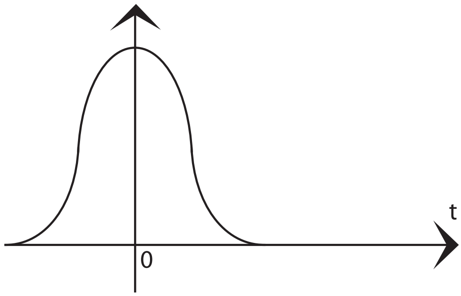

Figure 1.16. *Gaussian*

1.6.2.1. *Exponential signals*

This class includes polynomial and sinusoid signals. There are used in the solution of differential equations with constant coefficients; if $\rho=\frac{d}{dt}$ denotes the derivation operator:

$$\rho^n x(t) + a_{n-1}.\rho^{n-1} x(t) + ... + a_1.\rho x(t) + a_0.x(t) = 0 \quad \rho x(t) = \frac{d}{dt} x(t) \qquad [1.13]$$

The general solution of this type of equation is of the form (a combination of particular solutions $e^{\alpha_n . t}$):

$$x(t) = \sum_{i=1}^{i=q} e^{\alpha_i . t} \sum_{j=1}^{n_i - 1} c_{ij}.t^j \qquad [1.14]$$

where c_{ij} are constants depending on initial conditions and are thereof the complex or real roots of the characteristic equation:

$$\rho^n + a_{n-1}.\rho^{n-1} + ... + a_1.\rho + a_0 = 0 = \prod_{i=1}^{q} (\alpha - \alpha_i)^{n_i} \quad , \sum_{i=1}^{q} n_i = n \qquad [1.15]$$

1.6.2.2. *Impulse signals*

Different forms of impulse functions can be considered. Figures 1.17 and 1.23 give some examples where pulse height and width are chosen such that the area under the curve is unitary. All these functions have in common the properties of Dirac's delta function previously defined.

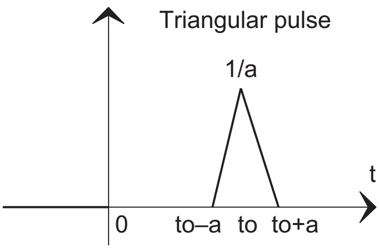

Figure 1.17. *Triangular pulse*

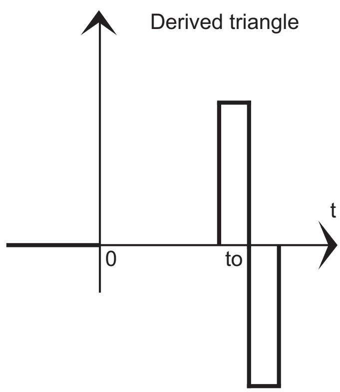

Figure 1.18. *Derivative of the triangular signal*

1.6.2.2.1. Examples

1) Rectangular function centered in t_o $f_l(t - t_o)$

2) Triangular function of height $1/a$ and width at the base $2a$, also centered in t_o.

3) Note that the derivative of this impulse yields two square impulses of opposite signs.

4) Exponential function, for example, if $u(t)$ is the step function: $\Delta(t) = (1/a)e^{-t/a}.u(t)$. All these functions can be used to decompose a signal into a series of pulses because at the limit they give $\delta(t)$, that is: $\int_{-\infty}^{+\infty} \delta(t - t_o)x(t)dt = x(t_o)$.

5) Polynomial function: $\Delta(t) = (1/2\pi)a/(a^2 + t^2)$.

6) Gaussian-shaped function: $\Delta(t) = (\frac{1}{a\sqrt{2\pi}})e^{-t^2/a^2}$ Gaussian form

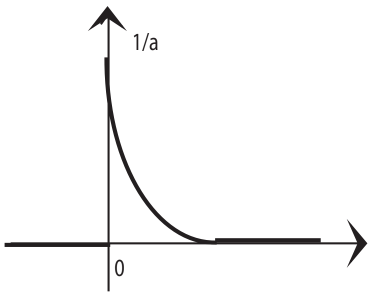

Figure 1.19. *Exponential*

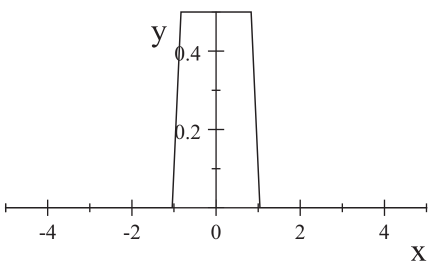

Figure 1.20. *Rectangular pulse*

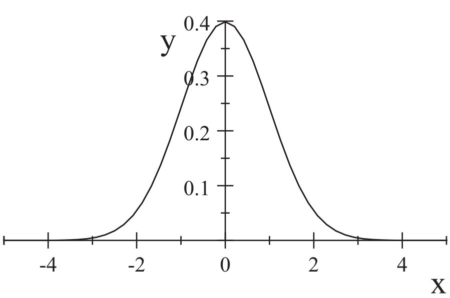

Figure 1.21. *Gaussian form*

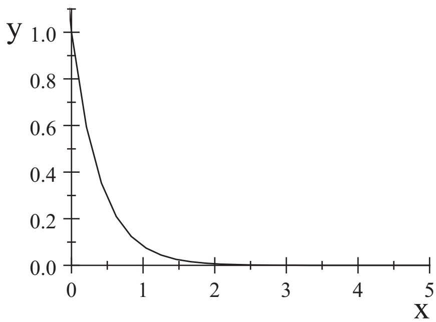

Figure 1.22. *Exponential form*

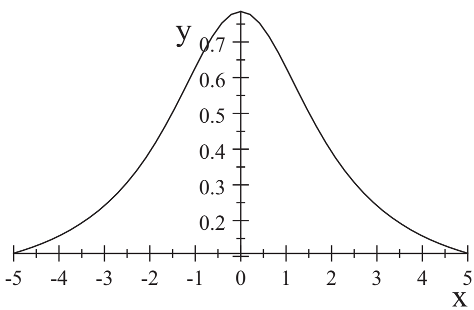

Figure 1.23. *Polynomial form*

1.6.3. *Representation and modeling of signals and systems*

In order to introduce the different types of modeling, we are going to study a few examples.

1.6.3.1. *Representation by polynomial equations*

The signal is defined by a polynomial in t or a function of t such as exponential functions. These signals are generally of the same type as sinusoid or exponential signals.

EXAMPLE 1.3.–

a) $y(t) = \frac{K}{(a^2+t^2)^n}$

b) $y(t) = \cos(\omega t) = (e^{j\omega t} + e^{-j\omega t})/2$

1.6.3.2. *Representation by differential equations*

They are defined by differential equations and they are their solutions; for example, for a signal:

$$\rho^n x(t) + a_{n-1}.\rho^{n-1} x(t) + ... + a_1.\rho x(t) + a_0 x(t) = 0,\ \rho x(t) = \frac{dx(t)}{dt} \qquad [1.16]$$

and for an input $u(t)$ and output system $x(t)$, we get:

$$\begin{aligned}\rho^n x(t)\ + a_{n-1}.\rho^{n-1} x(t) + ... + a_0 x(t) &= b_m.\rho^m u(t)\\ &+ b_{m-1}.\rho^{m-1} u(t) + ... + b_1.\rho u(t) + b_0 u(t)\end{aligned} \qquad [1.17]$$

1.6.3.3. *Representation by state equations*

Let a vector of dimension n be defined (minimal representation for a system of order n), the knowledge of which together with the initial state makes it possible to determine the state of the system at any time using the equation of the system. The state equation of a system for the linear case is defined by a vector differential equation of order one of the original differential equation linking the input to the output. It is presented in the following form for a system whose input is $u(t)$ and output is $y(t)$.

DEFINITION 1.3.– *The minimal dimension of the state of a system corresponds to the number of initial conditions necessary to integrate its differential equation (evolution equation). For a system of order n (degree of its differential equation), this dimension is equal to n. The first-order vector differential equation describing the dynamics of a state vector is a state representation of the system.*

State equation of the system with input u and output y is written as

$$\dot{X} = A.X + Bu(t) \qquad [1.18]$$

Observation equation

$$y(t) = C.X \qquad [1.19]$$

In the case of a signal, it suffices to cancel input u in the above-mentioned equation.

1.6.3.4. *Graphic representations*

Among graphic representations, the most important and the most commonly used are as follows:

– the time representation of the evolution of the signal (plot of the impulse or step response, or the evolution of the signal in time);

– representation in the phase plane (the plot in a coordinate system defined by the components of the system state vector, for example, the derivative or velocity as a function of the position for a second-order system), which is rather interesting in automatic control as in signal processing;

– frequency representations (Bode, Black, Nyquist) that are studied in automation after the time-frequency transformations;

– the time-frequency representation (three-dimensional) that is rather useful for non-stationary signals.

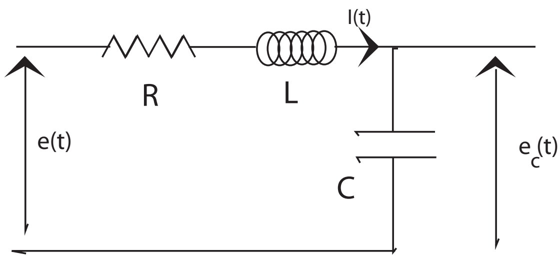

Figure 1.24. *A R L C circuit*

1.6.3.4.1. Examples

EXAMPLE 1.4.– Electric RLC circuit

The differential equation giving the behavior model of this circuit is written as:

$$C\frac{de_c(t)}{dt} = i(t) \quad \text{and} \tag{1.20}$$

$$e(t) = Ri(t) + L\frac{di(t)}{dt} + e_c(t) \tag{1.21}$$

The state vector of the system is given by $X(t) = \binom{e_c(t)}{i(t)}$. This vector defines the internal state of the system at the moment t. This allows us to obtain the system state representation, which is a temporal representation. This representation is not unique. In this case, matrix A and the state representation are defined by:

$$\dot{X} = A\begin{pmatrix} e_c(t) \\ i(t) \end{pmatrix} + Bu \qquad [1.22]$$

$$e_c = (1;0)X = (1;0)\begin{pmatrix} e_c(t) \\ i(t) \end{pmatrix}; \qquad [1.23]$$

with $\quad A = \begin{pmatrix} 0 & \frac{1}{C} \\ -\frac{1}{L} & -\frac{R}{L} \end{pmatrix} \quad B = \begin{pmatrix} 0 \\ \frac{1}{L} \end{pmatrix}$

EXAMPLE 1.5.– Butterworth filter case

A normalized low-pass, fourth-order Butterworth filter, with a cut-off pulse $1\,\mathrm{rad/s}$, whose input is $u(t)$ and output $y(t)$, has a differential equation as:

$$(\rho^4 + 2.6131\rho^3 + 3.4142\rho^2 + 2.6131\rho + 1)y(t) = u(t)\rho \text{ and } \quad x(t) = \frac{dx(t)}{dt} \qquad [1.24]$$

It can be written in the state form:

$$\dot{X} = AX + Bu; \quad y = CX \qquad [1.25]$$

with $A = \begin{pmatrix} -2.6131 & -3.4142 & -2.6131 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} B = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} C = \begin{pmatrix} 0 \; 0 \; 0 \; 1 \end{pmatrix}$

1.6.4. *Phase–plane representation*

Consider the case of the second-order signal or system in free regime, defined by its differential equation:

$$\ddot{y}(t) = f(\dot{y}(t), y(t)) \qquad [1.26]$$

By considering $x_1 = y(t)$ and $x_2 = \dot{y}(t)$ as components of the state vector, it is possible to associate thereto the following state representation:

$$\dot{X}(t) = \begin{pmatrix} x_2 \\ -f(x_1, x_2) \end{pmatrix} = AX; \quad y(t) = [1;0].X \qquad [1.27]$$

State variables x_1 and x_2 are also called phase variables; they determine at every moment the state of the system that can be represented by a point M in the plane (x_1, x_2). This plane is called the phase plane. Point M is characterized by state vector $X(t)$ and its evolution in time from an initial point M_o describes a phase curve. Phase curves can be graded in time and depend on the initial point M_o (the initial state). By eliminating the time variable, an equation is obtained that is parameterized according to the coordinates (x_1, x_2), defining a network of curves. The plot of these curves provides a visualization of the evolution of the system.

A large number of linear or nonlinear systems can be approximated by a second order. This method of representation allows us to easily conclude about the evolution and stability of a system. These disadvantages lie in the fact that graphical representation is impossible for systems with an order higher than 2 and is only applicable for autonomous systems (without input control and whose differential equation does not explicitly depend on time) having a unique solution (for the state equation). It should be noted that the principle remains valid for systems of order greater than 2, although the graphical representation is impossible. In the following section, we illustrate this method for the case of a second-order linear system.

1.6.4.1. *Case of a second-order linear system*

For the representation of a second-order system in the phase plane, we consider the equation:

$$\ddot{y}(t) + 2z\omega_o\dot{y}(t) + \omega_o^2 y(t) = 0 \qquad [1.28]$$

z is the damping and ω_o is the angular frequency of the system. The poles p_1, p_2 are the roots of the characteristic equation:

$$r + 2z\omega_o r + \omega_o = 0 \qquad [1.29]$$

According to damping values z, several possible cases can be distinguished.

Case 1: If $p_1$and p_2 are two real negative roots of the characteristic equation:

$$y(t) = x_1 = C_1 e^{p_1 t} + C_2 e^{p_2 t} \qquad [1.30]$$

$$\dot{y}(t) = x_2 = p_1 C_1 e^{p_1 t} + p_2 C_2 e^{p_2 t} \qquad [1.31]$$

hence

$$e^{p_1 t} = \frac{[p_2 x_1 - x_2]}{[p_2 - p_1]C_1} \quad \text{and} \quad e^{p_2 t} = \frac{[p_1 x_1 - x_2]}{[p_1 - p_2].C_2} \qquad [1.32]$$

wherefrom by raising these two equations to the power p_1 and p_2, we get:

$$(p_2x_1 - x_2)^{\frac{p_2}{p_1}} = C(p_1x_1 - x_2) \qquad [1.33]$$

Constant C depends on constants C_1 and C_2. This equation represents a family of parabolas tending to zero in t, as shown in Figure 1.25.

Case 2: If p_1 and p_2 are two real positive roots, then it yields, by proceeding as previously shown, the result in Figure 1.26.

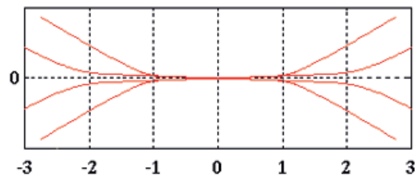

Figure 1.25. *p_1 and p_2 are two real negative roots of the characteristic equation*

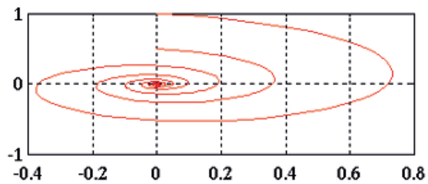

Figure 1.26. *p_1 and p_2 are two real positive roots*

Case 3: In the case where p_1 and p_2 are two real roots of opposite signs, equation [1.33] this time belongs to a family of hyperbolas, because $p_2/p_1 < 0$, whose asymptotes have equations:

$$x_2 = p_1.x_1 \quad \text{and} \qquad x_2 = p_2.x_1 \qquad [1.34]$$

One of the asymptotes (the one corresponding to p_i <0) has its branches directed toward the origin (x_1=0, x_2=0). The representation is shown in Figure 1.25.

Case 4: If $p_1$and p_2 are two complex roots with real negative parts, the system is stable and damped oscillating. The solutions are of the form:

$$x_1 = Ae^{p_1}.\sin(\omega t + \varphi) \text{ and } x_2 = Be^{p_2}.\sin(\omega t + \psi) \quad [1.35]$$

A, B, φ and ψ depend on initial conditions. Trajectories have the shape of spirals going toward zero as shown in Figure 1.26.

Case 5: If p_1 and p_2 are two complex roots with real positive parts, this yields spirals ranging from zero to infinity (see Figure 1.27).

Case 6: For two pure imaginary roots, we then obtain ellipses as presented in Figure 1.28:

$$x_1 = A.\sin(\omega t + \varphi) \text{ and } x_2 = B.\sin(\omega t + \psi) \quad [1.36]$$

We have previously presented the phase–plane representation. This representation is useful for the stability analysis of a system but it is limited to the second-order case.

1.6.5. *Dynamic system*

1.6.5.1. *Definitions relating to the equilibrium and stability of a dynamic system*

Consider a free dynamic system described by the differential vector equation:

$$\dot{x} = f(x,t) \, \forall t \in R \quad [1.37]$$

where $x(t)$ is the state vector of dimension n of the system, defining a point in the phase plane, and $f(x,t)$ is a vector function in R that may be nonlinear. We consider here the case of free signals or systems. Note that this system is free but it is not generally autonomous since its equation depends explicitly on time. The equation of an autonomous system can be written in the form $\dot{x} = f(x)$ without explicit dependence of time. Thus, a free time variant system is not autonomous.

Let $\Phi(t_o, x_o, t)$, the vector function, non-perturbated solution of equation [1.37], differentiable with respect to time and such that for any $x \in R^n$, $t \in R$ ($x(t_o) = x_o$ initial state and t_o initial moment), we have:

$$\Phi(t_o, x_o, t) = x = x(t) \quad [1.38]$$

$$\frac{d(\Phi(t_o, x_o, t))}{dt} = f[\Phi(t_o, x_o, t), t] \quad [1.39]$$

$\Phi(t_o, x_o, t)$ defines a curve or trajectory of the system in the phase space for an initial state $x_o \in \mathbb{R}^n$ and $t \in \mathbb{R}$.

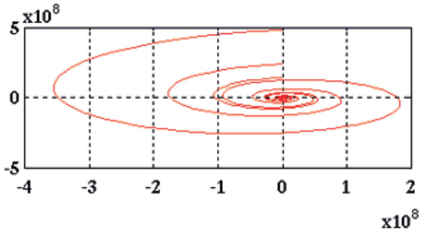

Figure 1.27. *p_1 and p_2 are two complex roots with real positive parts*

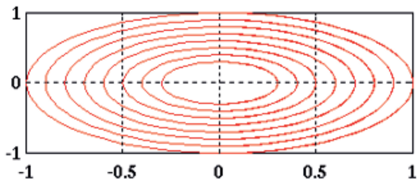

Figure 1.28. *Two pure imaginary roots*

1.6.5.2. *Equilibrium state of a system*

DEFINITION 1.4.– *A state x_e of the free system is called equilibrium point if:*

$$f(x_e, t) = 0 \, \forall t \in \mathbb{R}, t > to \quad [1.40]$$

This expresses that if x_e is an equilibrium point, then, in a case without disturbance, the system tends to remain in a nearby neighborhood because at this point velocity $\dot{x}_e = f(x_e, t)$ is zero. When the equilibrium point is $x_e \neq 0$, it can be brought by transformation to the origin of the phase space. For a linear time-invariant system, $\dot{x} = f(x, t) = A.x$, if A is not singular, then $x_e = 0$ is the point of equilibrium. If A is singular, there is an infinite number of equilibrium points. The search for equilibrium points does not require solving differential equations ($\dot{x} = f(x, t) = 0$).

1.6.5.2.1. Stable equilibrium state

DEFINITION 1.5.– *An equilibrium state is stable if, after deviating from this equilibrium, the system remains in the neighborhood or tends to return to this* *equilibrium point. An equilibrium point will be known as unstable equilibrium if after deviating from this point, the system tends to move away from it.*

For example, a pendulum has a stable equilibrium (low vertical position) and an unstable equilibrium (high vertical position, modulo $2k\pi$).

1.6.5.3. *Stable system: bounded input bounded output*

DEFINITION 1.6.– *A system is said to be stable if for any bounded input $e(t)$ it establishes a correspondence to a bounded output $s(t)$. This notion of stability is called "Bounded Input Bounded Output stability".*

If $h(t)$ is the impulse response of the system, the following can be stated:

$$s(t) = \int_{-\infty}^{+\infty} e(\tau).h(t-\tau)d\tau = e(t) * h(t) \qquad [1.41]$$

Knowing that $e(t)$ is bounded, $M \in R$ we then obtain:

$$|s(t)| \leq \int_{-\infty}^{+\infty} |e(\tau)| \,.\, |h(t-\tau)| \, d\tau \leq M. \int_{-\infty}^{+\infty} |h(\tau)| \, d\tau \qquad [1.42]$$

hence the system is stable if $h(t)$, its impulse response, is a summable function. A sufficient condition for stability is that $h(t)$ is summable ($\int_{-\infty}^{+\infty} |h(\tau)| \mathrm{d}\tau$ is finite).

1.7. Comprehension and application exercises

EXERCISE 1.–

1) Write the differential equation of a system composed of mass M suspended by a spring of stiffness k. We consider that the mass moves along a vertical axis without friction.

2) Represent this system in the form of state taking a state vector composed of the position and the velocity of mass M.

EXERCISE 2.–

An oscillator delivers a signal composed of two frequencies f_1 and f_2:

1) give all possible representations to define this signal and the parameters corresponding to them;

2) the amplitude of the oscillations of both frequencies decreases by 10% after 5 h; what happens to the representations of question 1?;

EXERCISE 3.–

1) a low-pass filter has impulse response $h(t) = Ae^{\frac{-t}{t_1}} + Be^{\frac{-t}{t_2}}$ to determine other representations capable of describing this system;

2) give the conditions about the parameters of the system providing the stability of the filter;

3) give different graphical representation of this system.