1

Fourier Series

1.1. Theoretical background

1.1.1. *Orthogonal functions*

1.1.1.1. *Orthogonal vectors*

Let us consider a vector space with n dimensions and vectors $\mathbf{x_1}, \mathbf{x_2}, \ldots \mathbf{x_n}$ as the orthogonal basis.

$$\mathbf{x_i^\dagger}.\mathbf{x_j} = \begin{cases} 0 & \text{if } i \neq j \\ k_i & \text{if } i = j \end{cases} \qquad [1.1]$$

k_i is the norm for vector $\mathbf{x_i}$.

$\mathbf{x_i^\dagger} = \mathbf{x}^{\mathbf{T*}}$ is the Hermitian vector (conjugate and transposed) of vector $\mathbf{x_i}$.

This collection of vectors is assumed to be complete once there is no way of finding any more values for $\mathbf{x_k}$ such as $\mathbf{x_k^\dagger}.\mathbf{x_i} = \mathbf{0}$

Consider vector $\mathbf{A}$ within this space and $A_1, A_2, \ldots A_n$ this vector's components in relation to n basis vectors. If the basis is complete, we can pose:

$$\mathbf{A} = \sum_{i=1}^{i=n} A_i \mathbf{x_i} \qquad [1.2]$$

We obtain coefficients A_i using to the following relation:

$$A_i = \frac{\mathbf{A}^{\dagger}.\mathbf{x_i}}{\mathbf{x_i}^{\dagger}.\mathbf{x_i}} = \frac{\mathbf{A}^{\dagger}.\mathbf{x_i}}{k_i} \quad [1.3]$$

1.1.1.2. *Vector-function analogy*

Consider two functions $f_i(t)$ and $f_j(t)$ that are orthogonal on interval $[t_1, t_2]$ if:

$$\int_{t_1}^{t_2} f_i(t) f_j^*(t) dt = \begin{cases} o & \text{if } i \neq j \\ k_i & \text{if } i = j \end{cases} \quad [1.4]$$

k_i is the squared norm of function $f_i(t)$.

The vector space is complete if we can no longer find any further functions $f(t)$ that are orthogonal to the previous ones.

When this space is complete and infinite, we can establish an exact representation of any function as a series on interval $[t_1, t_2]$:

$$g(t) = \sum_{i=1}^{i=\infty} C_i f_i(t) \quad t \in [t_1, t_2] \quad [1.5]$$

The coefficients of the decomposition are found using the following relation:

$$C_i = \frac{\int_{t_1}^{t_2} g(t) f_i^*(t)}{\int_{t_1}^{t_2} f_i(t) f_i^*(t)} = \frac{\int_{t_1}^{t_2} g(t) f_i^*(t)}{k_i} \quad [1.6]$$

1.1.2. *Fourier Series*

1.1.2.1. *Trigonometric series*

Functions $\{\cos n\omega_0 t\}$ and $\{\sin n\omega_0 t\}$ form, on interval $[t_0, t_0 + T]$, an infinite complete collection of orthogonal functions with $T = \dfrac{2\pi}{\omega_0}$.

It is thus possible to represent a function $f(t)$ on interval $[t_0, t_0 + T]$:

$$f(t) = a_0 + \sum_{n=1}^{n=\infty} a_n \cos n\omega_0 t + \sum_{n=1}^{n=\infty} b_n \sin n\omega_0 t \quad t \in [t_0, t_0 + T] \qquad [1.7]$$

a_0 is the mean value of $f(t)$ on interval $[t_0, t_0 + T]$

The coefficients are found using the following relations:

$$\begin{cases} a_0 = \dfrac{1}{T} \displaystyle\int_{t_0}^{t_0+T} f(t)\, dt \\ a_n = \dfrac{2}{T} \displaystyle\int_{t_0}^{t_0+T} f(t) \cos n\omega_0 t \, dt \\ b_n = \dfrac{2}{T} \displaystyle\int_{t_0}^{t_0+T} f(t) \sin n\omega_0 t \, dt \end{cases} \qquad [1.8]$$

1.1.2.2. *Exponential series*

Functions $\{e^{jn\omega_0 t}\}$ form an infinite complete collection of orthogonal functions on interval $[t_0, t_0 + T]$.

On this same interval, a function $f(t)$ would be noted:

$$f(t) = \sum_{n=-\infty}^{n=\infty} C_n e^{jn\omega_0 t} \quad t \in [t_0, t_0 + T] \qquad [1.9]$$

The coefficients for the decomposition are found using the following:

$$C_n = \frac{1}{T}\int_{t_0}^{t_0+T} f(t)e^{-jn\omega_0 t}\, dt \qquad [1.10]$$

In the case where $f(t)$ is real, we know that the following is true:

$$C_n = C_{-n}^{*} \qquad [1.11]$$

Thus $|C_n| = |C_{-n}|$ et $Arg(C_n) = -Arg(C_{-n})$

This gives us an example of Hermitian symmetry.

1.1.2.3. *Relations between two different forms of series*

The following relations allow us to change between one form of Fourier series and another:

$$\begin{cases} a_0 = C_0 & \\ a_n = c_n + C_{-n} & b_n = j\left(C_n - C_{-n}\right) \\ C_n = \frac{1}{2}\left(a_n - jb_n\right) & C_{-n} = \frac{1}{2}\left(a_n + jb_n\right) \end{cases} \qquad [1.12]$$

There is an alternative form for the trigonometric series:

$$\begin{cases} f(t) = a_0 + \sum_{n=1}^{n=\infty} A_n \cos\left(n\omega_0 t + \phi_n\right) \\ A_n = \sqrt{a_n^2 + b_n^2} \quad \text{et} \quad tg\phi_n = -\frac{b_n}{a_n} \end{cases} \qquad [1.13]$$

Table 1.1 presents a recap of these different formulas.

Series form	Calculation of coefficients
$f(t) = \sum_{n=-\infty}^{n=\infty} C_n e^{jn\omega_0 t}$	$C_n = \frac{1}{T} \int_{t_0}^{t_0+T} f(t) e^{-jn\omega_0 t}\, dt$
$f(t) = a_0 + \sum_{n=1}^{n=\infty} a_n \cos n\omega_0 t + \sum_{n=1}^{n=\infty} b_n \sin n\omega_0 t$ $= a_0 + \sum_{n=1}^{n=\infty} A_n \cos\left(n\omega_0 t + \phi_n\right)$	$a_0 = \frac{1}{T} \int_{t_0}^{t_0+T} f(t)\, dt$ $a_n = \frac{2}{T} \int_{t_0}^{t_0+T} f(t) \cos n\omega_0 t\, dt$ $b_n = \frac{2}{T} \int_{t_0}^{t_0+T} f(t) \sin n\omega_0 t\, dt$
Formulas for switching between forms	
$a_0 = C_0$	
$a_n = C_n + C_{-n}$	$b_n = j\left(C_n - C_{-n}\right)$
$C_n = \frac{1}{2}\left(a_n - jb_n\right)$	$C_{-n} = \frac{1}{2}\left(a_n + jb_n\right)$
$A_n = \sqrt{a_n^2 + b_n^2} = 2\lvert C_n \rvert$	$\phi_n = Arg\left(C_n\right) = -Arctg\left(\frac{b_n}{a_n}\right)$

Table 1.1. *Recap of Fourier series formulas*

1.1.3. *Periodic functions*

A periodic function with a period T is a function that repeats itself identically every T seconds.

$$f(t + nT) = f(t) \quad \forall t \qquad n \in Z \qquad [1.14]$$

Consider a periodic function of period T represented on interval $\left[t_0, t_0 + T\right]$ by a Fourier series, the representation remains valid regardless of whether $t \in \left]-\infty, \infty\right[$.

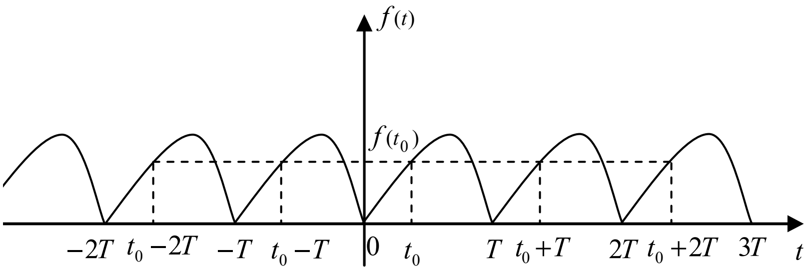


Figure 1.1. *Periodic function*

The value for t_0 is thus irrelevant. In practice, we often set $t_0 = 0$ or $t_0 = -\frac{T}{2}$, the integrals of formulas 8 and 10 thus become:

$$\int_0^T \text{ and } \int_{-T/2}^{T/2}$$

1.1.4. *Properties of Fourier series*

1.1.4.1. *Time domain translation*

If $f(t)$ is represented by the following Fourier series:

$$f(t) = \sum_{n=-\infty}^{n=\infty} C_n e^{jn\omega_0 t}$$

When translated in time, function $f(t-\tau)$ can be noted as follows:

$$\begin{cases} f(t-\tau) = \sum_{n=-\infty}^{n=\infty} C_n^{'} e^{jn\omega_0 t} \\ C_n^{'} = C_n e^{-jn\omega_0 \tau} \end{cases} \quad [1.15]$$

1.1.4.2. *Even functions*

If $f(t)$ is an even function, that is $f(t) = f(-t)$, its decomposition into a Fourier series will not contain any sine values.

$$f(t) = a_0 + \sum_{n=1}^{n=\infty} a_n \cos n\omega_0 t$$

It is also possible to cut the integration interval by half:

$$\begin{cases} a_0 = \dfrac{2}{T}\displaystyle\int_0^{T/2} f(t)\,dt \\ a_n = \dfrac{4}{T}\displaystyle\int_0^{T/2} f(t)\cos n\omega_0 t\ dt \end{cases} \qquad [1.16]$$

1.1.4.3. *Odd functions*

If $f(t)$ is an odd function, i.e. $f(t) = -f(-t)$, its decomposition into a Fourier series will only contain sine values.

$$\begin{cases} f(t) = \displaystyle\sum_{n=1}^{n=\infty} b_n \sin n\omega_0 t \\ b_n = \dfrac{4}{T}\displaystyle\int_0^{T/2} f(t)\sin n\omega_0 t\ dt \end{cases} \qquad [1.17]$$

An absence of function parity can be masked by the mean value of function $f(t)$. Therefore, we must analyze this property in the case of function $f(t) - a_0$.

1.1.4.4. *Rotational symmetry*

In this case, function $f(t)$ is composed of two identical half-periods of opposing signs: $f(t \pm \frac{T}{2}) = -f(t)$.

The decomposition into a Fourier series contains only odd harmonics: n odd.

This property can also be masked by the mean value of the function.

1.1.5. *Discrete spectra. Power distribution*

A periodic function $f(t)$ has a frequency spectrum which provides it with a representation on the frequency domain. This spectrum only exists for discrete values of ω. It is either a discrete or a line spectrum.

The amplitude of each spectral line is proportional to the value set by the function.

1.1.5.1. *Physical spectrum. Complex spectrum*

The physical spectrum represents values A_n and ϕ_n. It corresponds to positive values of frequency.

The complex spectrum represents C_n. There are therefore two spectra, one is the amplitude spectrum and the other is the phase spectrum. This highlights negative pulses; the amplitude of the spectral lines at $n\omega_0$ is actually the combination of all the spectral lines at $\pm n\omega_0$.

In a case where function $f(t)$ is real, $C_n = C_{-n}^*$ the amplitude spectrum will be symmetric, while the phase spectrum will be antisymmetric to the origin of the frequency.

The phase spectra are identical while the amplitude spectra are linked through the two following equations $\begin{cases} C_0 = a_0 \\ 2\left|C_n\right| = A_n \end{cases}$

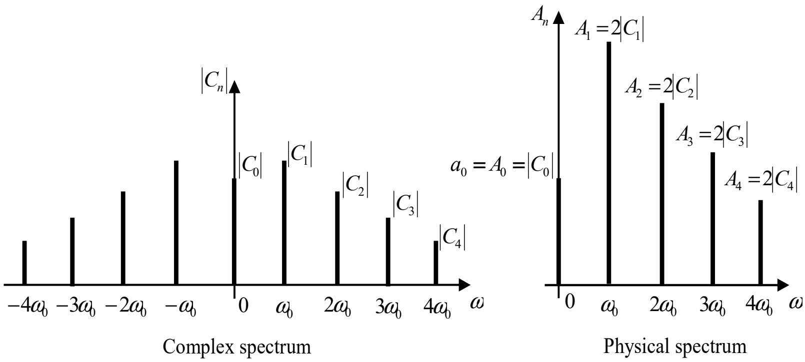


Figure 1.2. *Physical spectrum and complex spectrum*

1.1.5.2. *Power spectrum*

The mean quadratic value of a periodic function $f(t)$ or root mean square (RMS), is given by the following equation:

$$F_{eff}^2 = \frac{1}{T}\int_0^T f^2(t)\,dt$$

This value is directly linked to the power of the signal:

$$F_{eff}^2 = \frac{1}{T}\int_0^T f^2(t)dt = \sum_{n=-\infty}^{n=+\infty} |C_n|^2 \qquad [1.18]$$

Parseval's theorem demonstrates how the power contained within a signal is the sum of the powers contained in all the spectral lines of its Fourier decomposition.

It is possible to trace a power spectrum representing $|C_n|^2$.

1.2. Exercises

The exercises presented in this chapter come from B.P. Lathi [LAT 66].

1.2.1. *Exercise 1.1. Examples of decomposition calculations*

For each of the following periodic functions, calculate a Fourier decomposition and trace the outline of the complex spectrum.

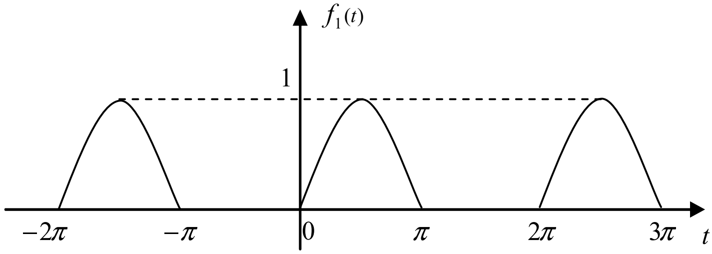


Figure 1.3. *Sinusoidal arc function*

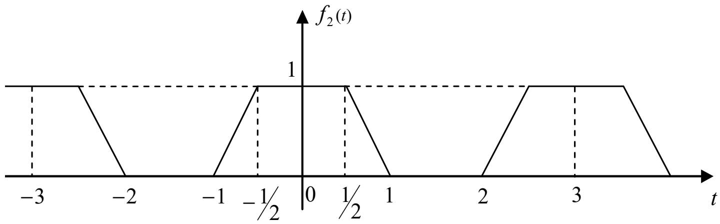


Figure 1.4. *Trapezoid function*

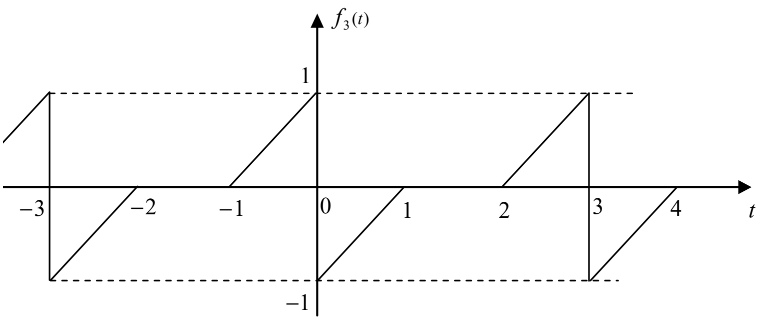


Figure 1.5. *Saw tooth function*

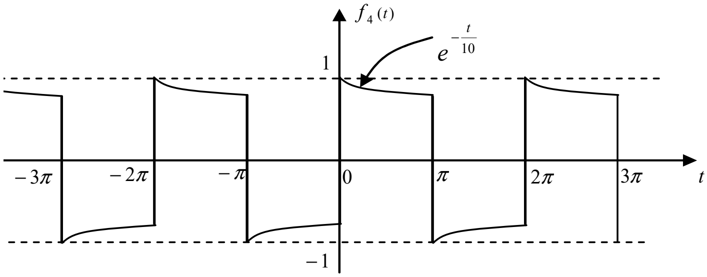


Figure 1.6. *Exponential function.*

In the case of Figure 1.6, what limit does this decomposition approach when the exponential time constant becomes infinite?

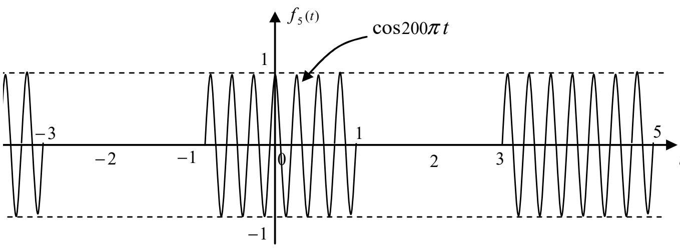
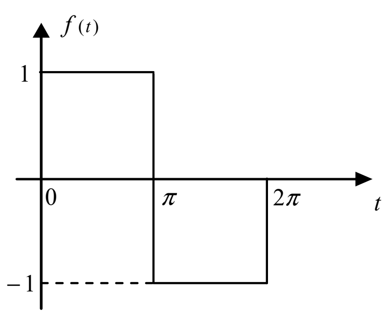


Figure 1.7. *Modulated Sinusoidal function*

1.2.2. *Exercise 1.2*

Consider the following rectangular function:



– Demonstrate that on interval $\left[0, 2\pi\right]$, $f(t)$ is orthogonal to $\cos nt$ with n integer number. What conclusion would you draw from this regarding the Fourier decomposition of $f(t)$?

– Demonstrate that on the same interval, error function $f_e(t) = f(t) - \frac{4}{\pi}\sin t$ is orthogonal to $\sin t$. Explain this result.

1.2.3. *Exercise 1.3*

Consider two functions $f_1(t)$ and $f_2(t)$ that are orthogonal on interval $\left[t_1, t_2\right]$.

We define the energy of a function on this interval with the following equation:

$$E = \int_{t_1}^{t_2} f^2(t)dt$$

Demonstrate that the energy contained in the sum, or difference, of these two functions $f_1(t)$ and $f_2(t)$ is the sum of the energies contained in each of these functions.

1.2.4. *Exercise 1.4*

Demonstrate that for an even periodic function, the coefficients of the Fourier decomposition are real and pure imaginary for an odd function.

1.2.5. *Exercise 1.5*

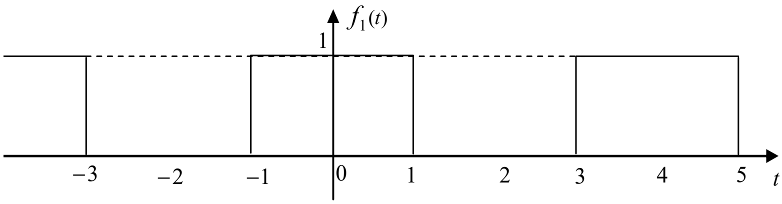
– Give an example of two periodic functions that have the same Fourier series with the exception of component a_0.

– What are the differences between Fourier developments of functions that are identical in everything other than their periods? Make a comparison between a function that has a period of 2π

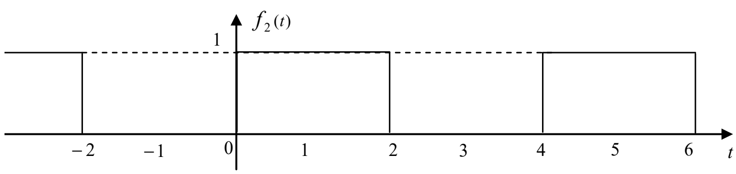
seconds with another function that is identical except that it has a period of π milliseconds.

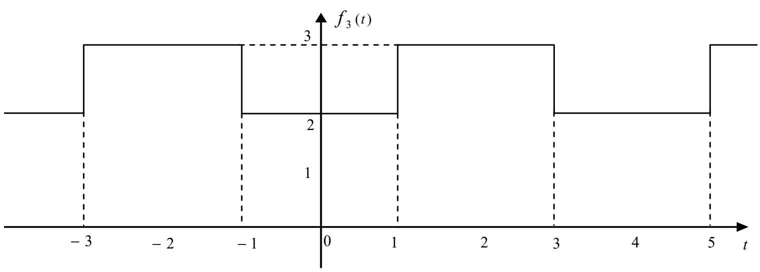
1.2.6. *Exercise 1.6. Decomposing rectangular functions*

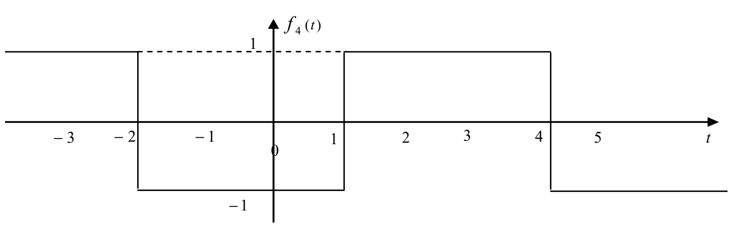
– Calculate the Fourier decomposition of the following function:



– Using this result, establish in the simplest way possible the decompositions of the following periodic functions:

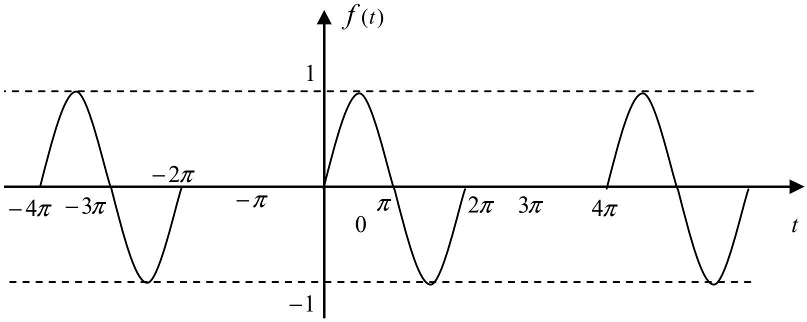






1.2.7. *Exercise 1.7. Translation and composition of functions*

Consider the following function $f(t)$:

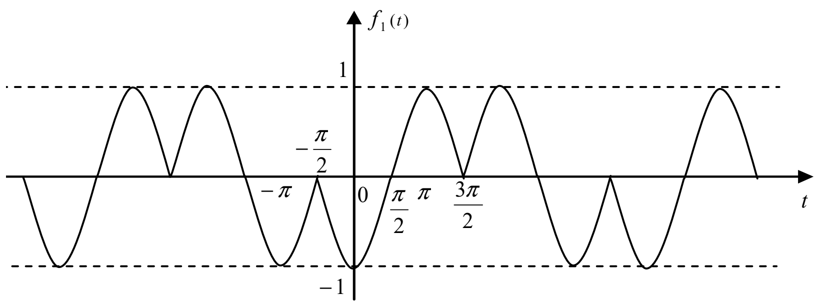


– directly calculate the Fourier decomposition of $f(t)$;

– perform a translation of π to the left, or $f_i(t) = f(t+\pi)$. Decompose $f_i(t)$ into a Fourier series and then find the result for $f(t)$ starting from $f_i(t)$;

– same exercise but this time performing a translation of π to the right, $f_i^{'}(t) = f(t-\pi)$;

– consider the following function $f_1(t)$:



Demonstrate that $f_1(t)$ can be expressed as a function of $f(t)$. Use the previous results to determine the decomposition of $f_1(t)$.

1.2.8. *Exercise 1.8. Time derivation of a function*

Demonstrate that if function $f(t)$ is periodic and derivable, function $f^{'}(t)=\dfrac{df(t)}{dt}$ is also periodic.

From this, show that if $f(t)=\sum_{n=-\infty}^{n=\infty} C_n e^{jn\omega_0 t}$, the Fourier decomposition of the derived function is:

$$\frac{df(t)}{dt}=\sum_{n=-\infty}^{n=\infty} jn\omega_0 C_n e^{jn\omega_0 t}$$

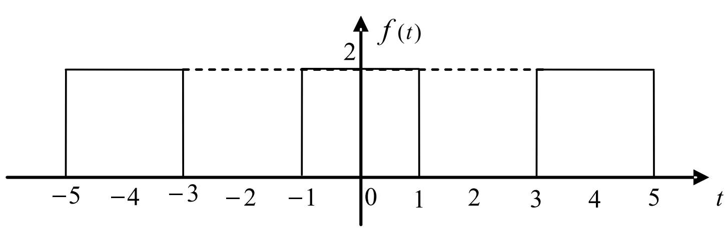
1.2.9. *Exercise 1.9. Time integration of functions*

Consider a periodic function $f(t)$. Demonstrate that integral function $F(t)=\int f(t)dt$ is periodic on condition that the mean value of $f(t)$ be 0.

In this case, demonstrate that $F(t)=\sum_{n=-\infty}^{n=\infty} \dfrac{C_n}{jn\omega_0} e^{jn\omega_0 t}$.

1.2.10. *Exercise 1.10*

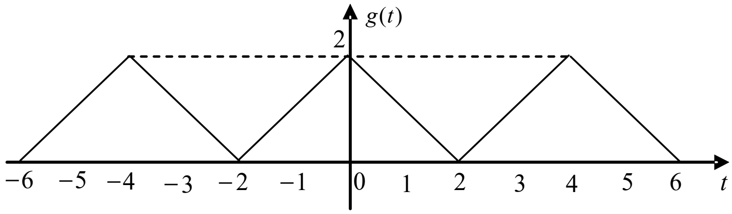
Consider the following periodic function $f(t)$:



With the following Fourier decomposition:

$$f(t) = \sum_{n=-\infty}^{n=+\infty} \frac{\sin n\frac{\pi}{2}}{n\frac{\pi}{2}} e^{jn\frac{\pi}{2}t}$$

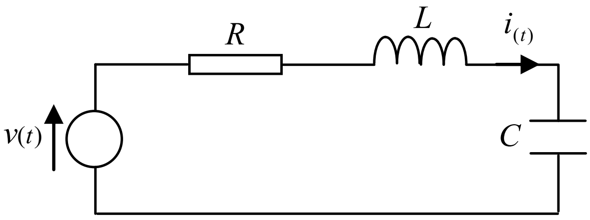
Using the previous function, establish the decomposition of the following function $g(t)$:



We can use the results from exercise 1.9.

1.2.11. *Exercise 1.11. Applications in electronic circuits*

Let us consider a periodic voltage $v(t)$ of period $T = 1s$, applied to the following circuit:

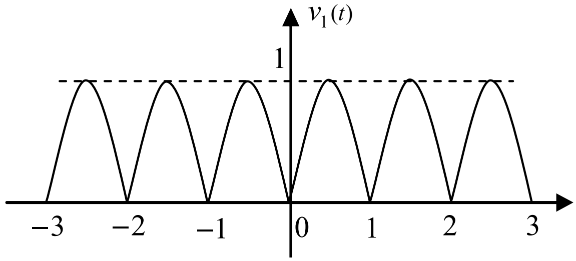


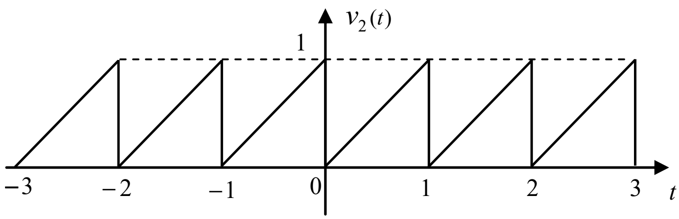
$R = 0.01\Omega$

$L = 1H$

$C = \frac{1}{16\pi^2}F$

Find the Fourier decomposition of current $i(t)$ when $v(t)$ takes the following forms. What can you say about these results?





1.3. Solutions to the exercises

1.3.1. *Exercise 1.1. Examples of decomposition calculations*

– Function $f_1(t)$:

The analytical form of function $f_1(t)$ of Figure 1.3 during a single period is:

$$f_1(t) = \begin{cases} \sin t & t \in [0, \pi] \\ 0 & t \in [\pi, 2\pi] \end{cases}$$

The period here is $T = 2\pi$, the fundamental pulse is $\omega_0 = 1$.

Let us find the decomposition in trigonometric form:

$$a_0 = \frac{1}{T}\int_0^T f_1(t)dt = \frac{1}{2\pi}\int_0^\pi \sin t dt$$

or: $\boxed{a_0 = \frac{1}{\pi}}$

$$a_n = \frac{2}{T}\int_0^T f_1(t)\cos n\omega_0 dt = \frac{2}{2\pi}\int_0^\pi \sin t \cos ntdt$$

$$= \frac{1}{2\pi}\int_0^\pi [\sin(n+1)t - \sin(n-1)t]dt$$

Hence $a_n = \frac{-1}{2\pi}\left[\frac{\cos(n+1)t}{n+1} - \frac{\cos(n-1)t}{n-1}\right]_0^\pi$ for $n \neq 1$

or: $\boxed{a_n = \frac{1}{\pi}\frac{1+\cos n\pi}{1-n^2}}$

If n is even, we can set $n = 2p$, thus: $a_{2p} = \frac{1}{\pi}\frac{2}{1-4p^2}$

If n is odd, then $n = 2p+1$, thus: $a_{2p+1} = 0$

We now perform this calculation again with $n = 1$:

$$a_1 = \frac{1}{\pi}\int_0^\pi \sin t \cos tdt = \frac{1}{2\pi}\int_0^\pi \sin 2tdt = \frac{-1}{\pi}\left[\frac{\cos 2t}{2}\right]_0^\pi = 0$$

We therefore deduce that: $a_{2p+1} = 0 \quad \forall p$

$$b_n = \frac{2}{T}\int_0^T f_1(t)\sin n\omega_0 dt = \frac{2}{2\pi}\int_0^\pi \sin t \sin ntdt$$

$$= \frac{1}{2\pi}\int_0^\pi [\cos(n+1)t - \cos(n-1)t]dt$$

$$= \frac{1}{2\pi}\left[\frac{\sin(n+1)t}{n+1} - \frac{\sin(n-1)t}{n-1}\right]_0^\pi \quad n \neq 1$$

Therefore: $\boxed{b_n = 0 \quad \forall n \neq 1}$

Using the calculation for b_1:

$$b_1 = \frac{1}{\pi}\int_0^{\pi} \sin^2 t dt \;\text{ or }\; b_1 = \frac{1}{2}$$

Hence the trigonometric decomposition of $f_1(t)$:

$$\boxed{f_1(t) = \frac{1}{\pi} + \frac{1}{2}\sin t - \frac{2}{\pi}\sum_{p=1}^{p=\infty}\frac{1}{4p^2-1}\cos 2pt}$$

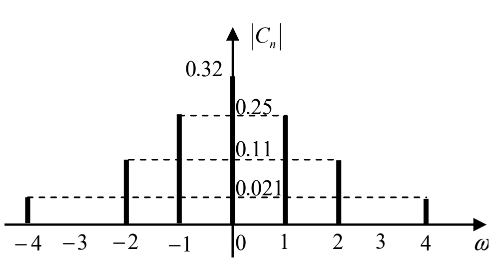
The coefficients for the complex spectrum appear as follows:

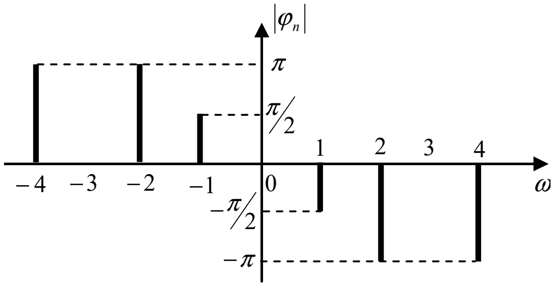
$$\begin{cases} C_0 = a_0 \\ C_n = \frac{1}{2}(a_n + jb_n) \\ C_{-n} = \frac{1}{2}(a_n - jb_n) \end{cases}$$

Thus:

$$\boxed{\begin{matrix} C_0 = \frac{1}{\pi} & C_1 = C_{-1}^* = -j\frac{1}{4} \\ C_{2p} = \frac{-1}{\pi(4p^2-1)} & \end{matrix}}$$

From this we can plot the amplitude and phase spectra:





– Function $f_2(t)$:

Function $f_2(t)$ is even, the series decomposition will therefore not include any sine values. The integration will happen over half a period.

As the function is defined in segments, we choose to divide the integration interval into 3: $\left[0,\frac{1}{2}\right]\cup\left[\frac{1}{2},1\right]\cup\left[1,\frac{3}{2}\right]$

The period is $T=3$ and the fundamental pulse is thus $\omega_0=\frac{2\pi}{3}$:

$$a_0=\frac{1}{2} \qquad b_n=0$$

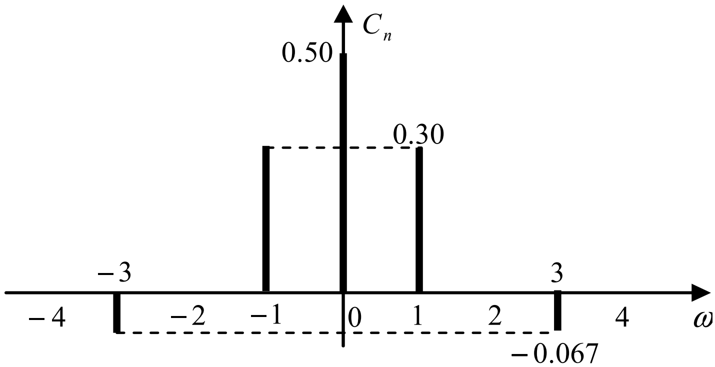
Before calculating a_n, it should be noted that this function presents a rotational symmetry masked by mean value a_0. This means that even harmonic pairs will have a null amplitude.

$$a_{2p}=0$$

$$a_{2p+1}=(-1)^p\frac{12}{\pi^2(2p+1)^2}\sin(2p+1)\frac{\pi}{6}$$

$$f_2(t)=\frac{1}{2}+\frac{12}{\pi^2}\sum_{p=1}^{p=\infty}\frac{(-1)^p}{(2p+1)^2}\sin(2p+1)\frac{\pi}{6}\cos(2p+1)\frac{2\pi}{3}t$$

The complex spectrum is: $C_{2p+1} = \frac{1}{2} a_{2p+1}$ and only includes real components. It will be represented by a single plot.



The magnitude of the spectral lines quickly decreases with the order of the harmonics. This is due to the similarity in shape between this function and a sinusoidal one.

– Function $f_3(t)$:

Function $f_3(t)$ is odd, coefficients a_n are null. Integration will take place over half a period.

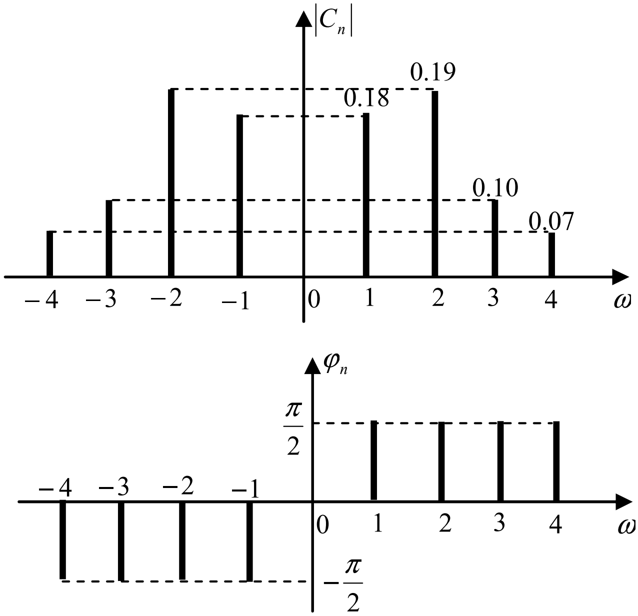
As the function is defined in segments, we choose to divide the integration interval into 2: $[0,1] \cup \left[1, \frac{3}{2}\right]$

The period is $T = 3$, fundamental pulse is thus $\omega_0 = \frac{2\pi}{3}$

$$\boxed{\begin{array}{l} a_0 = a_n = 0 \\ b_n = \dfrac{-2}{n\pi} + \dfrac{3}{n^2\pi^2} \sin \dfrac{2n\pi}{3} \end{array}}$$

$$f_3(t) = \frac{1}{\pi} \sum_{n=1}^{n=\infty} \left(\frac{-2}{n} + \frac{3}{n^2\pi} \sin \frac{2n\pi}{3} \right) \sin \frac{2n\pi}{3} t$$

The complex spectrum is: $C_n = \frac{-j}{2} b_n$ and only includes imaginary components.



– Function $f_4(t)$:

Function $f_4(t)$ presents a rotational symmetry $f_4(t) = f_4(t \pm \frac{T}{2})$, this means we will only encounter odd harmonics.

Expressing this function in its analytical form on half a period suggests it would be better to use the exponential form of the Fourier series. After integration, we find the following result:

$$C_{2p+1} = \frac{j\left(1+e^{-\frac{\pi}{10}}\right)e^{-j(2p+1)\frac{\pi}{2}}}{\pi\left(\frac{1}{10}+j(2p+1)\right)}\sin(2p+1)\frac{\pi}{2}$$

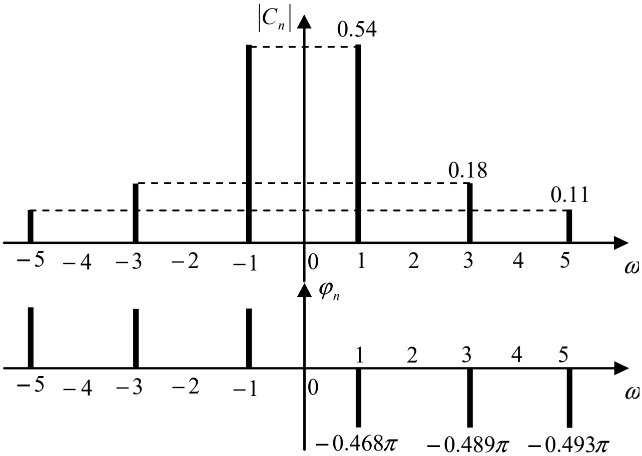
This function can then be noted as follows:

$$f_4(t) = j\frac{1+e^{-\frac{\pi}{10}}}{\pi}\sum_{p=-\infty}^{p=+\infty}\frac{e^{-j(2p+1)\frac{\pi}{2}}}{\frac{1}{10}+j(2p+1)}\sin(2p+1)\frac{\pi}{2}e^{j(2p+1)t}$$

When the exponential time constant becomes infinite, C_{2p+1} becomes:

$$C_{2p+1} = \frac{\sin(2p+1)\frac{\pi}{2}}{(2p+1)\frac{\pi}{2}}e^{-j(2p+1)\frac{\pi}{2}}$$

This statement corresponds to the decomposition of a rectangular signal with a period of 2π and with a duty cycle of 2.



– Function $f_5(t)$:

Function $f_5(t)$ is even, coefficients b_n are null. Integration will take place over half a period, that period being $T=4$ and the fundamental pulse stands at $\omega_0 = \dfrac{\pi}{2}$

$$a_n = \frac{4}{4}\int_0^1 \cos 200\pi t \cos n\frac{\pi}{2} t \, dt$$

This gives us:

$$a_n = \frac{2}{\pi}\frac{(-1)^{n+1} n}{(400+n)(400-n)} \text{ for } n \neq 400$$

If we repeat this operation for *n=400*, we obtain:

$$a_{400} = \frac{1}{2}$$

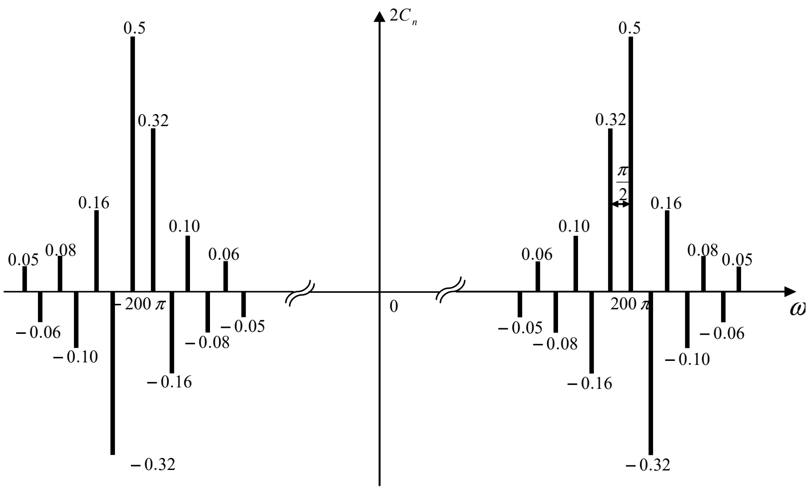
What's more: $a_0 = b_n = 0$

The Fourier decomposition will appear as follows:

$$f_5(t) = \frac{1}{2}\cos 200\pi t + \sum_{n=1,\ n\neq 400}^{n=\infty} \frac{2}{\pi}\frac{(-1)^{n+1} n}{(400+n)(400-n)} \cos n\frac{\pi}{2} t$$

The spectral lines will be centered around *n=±400* thus $\omega = \pm 200\pi$.

As the spectrum is real, we will trace $a_n = 2C_n$.



1.3.2. *Exercise 1.2*

Let us begin by proving that $f(t)$ is orthogonal to $\cos nt$ on interval $[0,2\pi]$:

$$I = \int_0^{2\pi} f(t)\cos nt\,dt = \int_0^{\pi} \cos nt\,dt + \int_{\pi}^{2\pi} -\cos nt\,dt$$

The integral is null, $f(t)$ is hence orthogonal to $\cos nt$.

This tells us that the Fourier decomposition will not include any cosine values.

Let us now show that the error function $f_e(t)$ is orthogonal to $\sin t$ on interval $[0,2\pi]$.

$$I' = \int_0^{2\pi} f_e(t)\sin t\,dt = \int_0^{\pi}\left(1-\frac{4}{\pi}\sin t\right)dt + \int_{\pi}^{2\pi}\left(-1-\frac{4}{\pi}\sin t\right)dt$$

The integral is null and $f_e(t)$ is hence orthogonal to $\sin t$.

The error function therefore cannot be expressed according to sin t. This means that $\frac{4}{\pi}\sin t$ is the fundamental value of the Fourier decomposition of $f(t)$.

1.3.3. *Exercise 1.3*

The energy of the function is noted as follows:

$$E = \int_{t_1}^{t_2} f^2(t)dt = \int_{t_1}^{t_2} \left(f_1(t) + f_2(t)\right)^2 dt = \int_{t_1}^{t_2} f_1^2(t)dt + \int_{t_1}^{t_2} f_2^2(t)dt + \int_{t_1}^{t_2} 2f_1(t)f_2(t)dt$$

The two first terms represent the energy of each function. The third is interaction energy between the two functions.

If both functions are orthogonal on interval $[t_1, t_2]$ then integral $\int_{t_1}^{t_2} f_1(t)f_2(t)dt = 0$

The level of interaction energy is null. The resultant is immediately apparent.

1.3.4. *Exercice 1.4*

If function $f(t)$ is even, terms b_n must be null. The terms of the exponential decomposition appear as follow:

$$C_n = \frac{1}{2}(a_n - jb_n) = \frac{1}{2}a_n$$

This indicates they are real.

Similarly, when $f(t)$ is odd, terms a_n are null.

Hence: $C_n = \frac{1}{2}(a_n - jb_n) = -j\frac{1}{2}b_n$

The terms are therefore imaginary.

1.3.5. *Exercise 1.5*

Both functions $f(t)$ and $g(t)$ have the same Fourier decomposition except in their mean value. The difference between these two functions must be orthogonal to $\cos n\omega_0 t$ and $\sin n\omega_0 t$ regardless of n with $n \neq 0$.

We obtain $f(t) - g(t) = K = cte$, hence the solution: $f(t) = g(t) + K$

Two identical functions with different periods will have the same Fourier decomposition coefficients and only the frequencies of the various harmonics will differ.

$$T_1 = 2\pi \ s \Rightarrow \omega_1 = 1 \ {rd}/{s} \Rightarrow \omega_n = n \ {rd}/{s}$$
$$T_2 = \pi \ ms \Rightarrow \omega_1 = 2.10^3 \ {rd}/{s} \Rightarrow \omega_n = 2n.10^3 \ {rd}/{s}$$

1.3.6. *Exercise 1.6*

– Function $f_1(t)$ is even and therefore $b_n = 0$. We have to integrate it into a half-period.

The period is $T = 4$ and the fundamental pulse is $\omega_0 = \dfrac{\pi}{2}$

$$a_0 = \frac{1}{2}\int_0^1 dt = \frac{1}{2} \qquad a_n = \int_0^1 \cos n\frac{\pi}{2} t dt = \frac{\sin n\frac{\pi}{2}}{n\frac{\pi}{2}}$$

Thus: $$f_1(t) = \frac{1}{2} + \sum_{n=1}^{n=\infty} \frac{\sin n\frac{\pi}{2}}{n\frac{\pi}{2}} \cos n\frac{\pi}{2} t$$

All even harmonics are null. This is due to the rotational symmetry of the function that is being masked by the mean value.

We can rewrite the Fourier decomposition as:

$$f_1(t) = \frac{1}{2} + \sum_{p=0}^{p=\infty} \frac{\sin(2p+1)\frac{\pi}{2}}{(2p+1)\frac{\pi}{2}} \cos(2p+1)\frac{\pi}{2}t$$

The terms of the exponential decomposition then appear as follows:

$$C_{2p+1} = \frac{1}{2} \frac{\sin(2p+1)\frac{\pi}{2}}{(2p+1)\frac{\pi}{2}}$$

– Function $f_2(t)$ corresponds to the translation of function $f_1(t)$:

$$f_2(t) = f_1(t-1)$$

The coefficients in the exponential decomposition appear as expected, the amplitudes and the phases differ to $n\omega_0\tau = n\frac{\pi}{2}$:

$$C'_n = C_n e^{-jn\frac{\pi}{2}} \Rightarrow C'_{2p+1} = \frac{1}{2} \frac{\sin(2p+1)\frac{\pi}{2}}{(2p+1)\frac{\pi}{2}} e^{-j(2p+1)\frac{\pi}{2}}$$

$$C'_0 = \frac{1}{2}$$

– Function $f_3(t)$ is the result of translating and changing the mean value of $f_1(t)$:

$$f_3(t) = 2 + f_1(t-2)$$

These are the coefficients of the exponential decomposition:

$$C_n^{'} = C_n e^{-jn\omega_0\tau} \Rightarrow C_{2p+1}^{'} = \frac{1}{2}\frac{\sin(2p+1)\frac{\pi}{2}}{(2p+1)\frac{\pi}{2}}e^{-j(2p+1)\pi}$$

$$C_0^{'} = \frac{5}{2}$$

– Function $f_4(t)$ is the result of changing the mean value, amplitude, period and performing a time-shift on $f_1(t)$:

New amplitude: 2

Mean value: 0

Period: $T = 6 \Rightarrow \omega_0 = \frac{\pi}{3}$

Time-shift: $\tau = 2.5 \Rightarrow \omega_0\tau = \frac{5\pi}{6}$

$$C_n^{'} = 2C_n e^{-jn\omega_0\tau} \Rightarrow C_{2p+1}^{'} = \frac{\sin(2p+1)\frac{\pi}{3}}{(2p+1)\frac{\pi}{3}}e^{-j(2p+1)\frac{5\pi}{6}}$$

$$C_0^{'} = 0$$

1.3.7. *Exercise 1.7. Translation and composition of functions*

– Period is $T = 4\pi$, fundamental pulse is $\omega_0 = \frac{1}{2}$.

The exponential decomposition can be obtained from the following integral:

$$C_n = \frac{1}{4\pi}\int_0^{2\pi} \sin t e^{-jn\frac{t}{2}} dt$$

After calculations:

$$\begin{cases} C_{2p} = 0 \;\text{ for }\; p \neq \pm 1 & C_{2p+1} = \dfrac{2}{\pi}\dfrac{1}{4-(2p+1)^2} \\ C_2 = \dfrac{1}{4j} & C_{-2} = \dfrac{-1}{4j} \end{cases}$$

The exponential decomposition will therefore appear as follows:

$$f(t) = \frac{1}{2}\sin t + \sum_{p=-\infty}^{p=+\infty} \frac{1}{2\pi\left(1-\left(p+\frac{1}{2}\right)^2\right)} e^{j\left(p+\frac{1}{2}\right)t}$$

– $f_i(t)$ is an odd function. The decomposition contains only terms with sine. The terms of the exponential decomposition are easily obtained by performing a phase change of $-n\omega_0\tau$ that corresponds to time-shift $\tau = -\pi$, which would be $-(2p+1)\frac{\pi}{2}$.

We then have: $f_i(t) = -\frac{1}{2}\sin t + j\frac{2}{\pi}\sum_{p=-\infty}^{p=+\infty} \frac{(-1)^p}{4-(2p+1)^2} e^{j(2p+1)\frac{t}{2}}$

– The same method can be applied to function $f_i'(t)$, the phase shift that we apply here has the opposite sign to function $f_i(t)$ which would be $(2p+1)\frac{\pi}{2}$

We then have: $f_i'(t) = -\frac{1}{2}\sin t - j\frac{2}{\pi}\sum_{p=-\infty}^{p=+\infty} \frac{(-1)^p}{4-(2p+1)^2} e^{j(2p+1)\frac{t}{2}}$

– Function $f_1(t)$ can also be presented as follows:

$$f_1(t) = f(t - \frac{3\pi}{2}) - f(t + \frac{\pi}{2})$$

Starting with the decomposition of $f(t)$ and adding to that a phase change, we obtain the decompositions for $f(t-3\frac{\pi}{2})$ and $f(t+\frac{\pi}{2})$.

If we then subtract the decomposition for $f(t)$:

$$f_1(t) = -\frac{1+j}{\sqrt{2}}\sum_{p=-\infty}^{p=+\infty}\frac{e^{jp\frac{\pi}{2}}}{1-\left(p+\frac{1}{2}\right)^2}e^{j\left(p+\frac{1}{2}\right)t}$$

1.3.8. *Exercise 1.8. Time derivation of functions*

The derived function can be expressed as follows:

$$f'(t+nT) = \lim_{\varepsilon\to 0}\frac{f(t+nT+\varepsilon)-f(t+nT)}{\varepsilon}$$

If the period for the function is T, the following can be stated:

$$f'(t+nT) = \lim_{\varepsilon\to 0}\frac{f(t+\varepsilon)-f(t)}{\varepsilon} = f'(t)$$

This means $f'(t)$ is periodic and has a period of T.

$f'(t)$ can then be decomposed into a Fourier series:

$$f'(t) = \sum_{n=-\infty}^{n=+\infty} C'_n e^{jn\omega_0 t}$$

With $C'_n = \frac{1}{T}\int_0^T \frac{df(t)}{dt}e^{-jn\omega_0 t}dt$

After part by part integration we obtain: $C'_n = jn\omega_0 C_n$

Hence: $f(t) = \sum_{n=-\infty}^{n=+\infty} C_n e^{jn\omega_0 t} \Rightarrow \frac{df_{(t)}}{dt} = \sum_{n=-\infty}^{n=+\infty} jn\omega_0 C_n e^{jn\omega_0 t}$

1.3.9. *Exercise 1.9. Time integration of functions*

Consider the following periodic function $f(t)=\sum_{n=-\infty}^{n=+\infty} C_n e^{jn\omega_0 t}$ and its integral function $F(t)$, so that:

$$F(t)=\int f(t)dt$$

We can calculate $F(t+nT)-F(t)=\left[F(x)\right]_t^{t+nT} = \int_t^{t+nT} f(x)dx = nTC_0$

C_0 being the mean value of $f(t)$.

The integral function of $f(t)$ is only periodic with a period of T if the mean value for $f(t)$ is null.

$F(t)$ can then be decomposed into a Fourier series:

$$F(t)=\sum_{n=-\infty}^{n=+\infty} C_n^{'} e^{jn\omega_0 t}$$

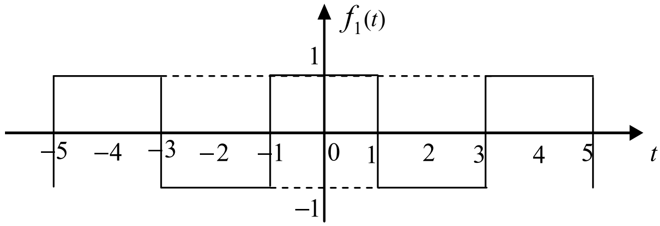
with: $C_n^{'}=\frac{1}{T}\int_0^T F(t)e^{-jn\omega_0 t}dt$

After part-by-part integration: $C_n^{'}=\frac{C_n}{jn\omega_0}$

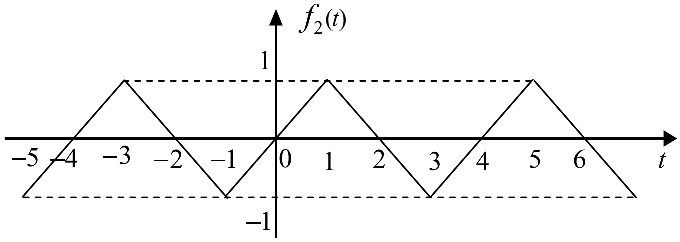
1.3.10. *Exercise 1.10*

It only takes a few simple operations to get from $f(t)$ to $g(t)$:

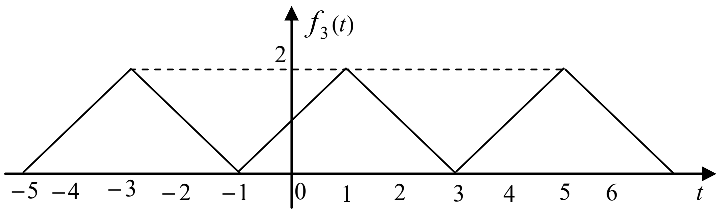
– Remove mean value: $f_1(t)=f(t)-1$



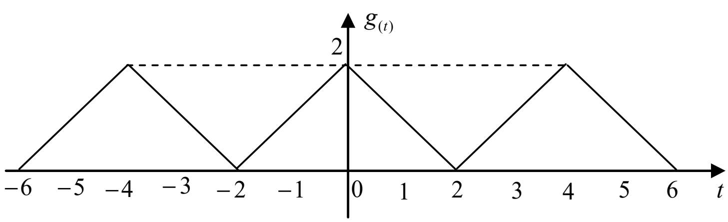
– Integrate: $f_2(t) = \int f_1(t)dt$



– Introduce mean value: $f_3(t) = f_2(t) + 1$



– Translate to the left: $g(t) = f_3(t+1)$



These various operations in the time-domain then translate very simply into the frequency domain.

$$f(t) = 1 + \sum_{n=-\infty}^{n=\infty} \frac{\sin n\frac{\pi}{2}}{n\frac{\pi}{2}} e^{jn\frac{\pi}{2}t}\Bigg|_{n\neq 0}$$

$$f_1(t) = \sum_{n=-\infty}^{n=\infty} \frac{\sin n\frac{\pi}{2}}{n\frac{\pi}{2}} e^{jn\frac{\pi}{2}t}\Bigg|_{n\neq 0}$$

$$f_2(t) = \sum_{n=-\infty}^{n=\infty} \frac{\sin n\frac{\pi}{2}}{j\left(n\frac{\pi}{2}\right)^2} e^{jn\frac{\pi}{2}t}\Bigg|_{n\neq 0}$$

$$f_3(t) = 1 + \sum_{n=-\infty}^{n=\infty} \frac{\sin n\frac{\pi}{2}}{j\left(n\frac{\pi}{2}\right)^2} e^{jn\frac{\pi}{2}t}\Bigg|_{n\neq 0}$$

$$g(t) = 1 + \sum_{n=-\infty}^{n=\infty} \frac{\sin n\frac{\pi}{2}}{j\left(n\frac{\pi}{2}\right)^2} e^{jn\frac{\pi}{2}} e^{jn\frac{\pi}{2}t}\Bigg|_{n\neq 0}$$

Note that the even terms are null. We obtain:

$$g(t) = 1 + \sum_{p=-\infty}^{p=\infty} \frac{4}{\pi^2(2p+1)^2} e^{j(2p+1)\frac{\pi}{2}t}$$

1.3.11. *Exercise 1.11*

The impedance of a circuit as a sinusoidal system will appear:

$$Z(j\omega) = R + jL\omega + \frac{1}{jC\omega} = 0.01 + j\omega + \frac{16\pi^2}{j\omega}$$

We obtain the components of the Fourier decomposition of $i(t)$ by dividing those for $v(t)$ by $Z_{(j\omega)}$ set at the pulsation.

$$\begin{cases} v(t) = \sum_{n=-\infty}^{n=+\infty} C_n e^{jn\omega_0 t} \\ Z(j\omega) = 0.01 + jn\omega_0 + \dfrac{16\pi^2}{jn\omega_0} \\ i(t) = \sum_{n=-\infty}^{n=+\infty} \dfrac{C_n}{Z(jn\omega_0)} e^{jn\omega_0 t} = \sum_{n=-\infty n}^{n=+\infty} C'_n e^{jn\omega_0 t} \end{cases}$$

Both voltages have the same period $T = 1$ meaning $\omega_0 = 2\pi$.

The impedance at the different pulsations of the decomposition can be expressed as follows:

$$Z_{(jn\omega_0)} = 0.01 + j2\pi \frac{n^2 - 4}{n}$$

For $n=0$ which is the same as saying $\omega = 0$, the impedance becomes infinite. The continuous component, or mean value of the current will be null which is explained by the presence of a capacitor in the circuit.

The impedance is real and equal to 0.001 for $n = \pm 2$ or $\omega = \pm 4\pi = \pm 2\omega_0$ which is the resonant frequency of the RLC series circuit.

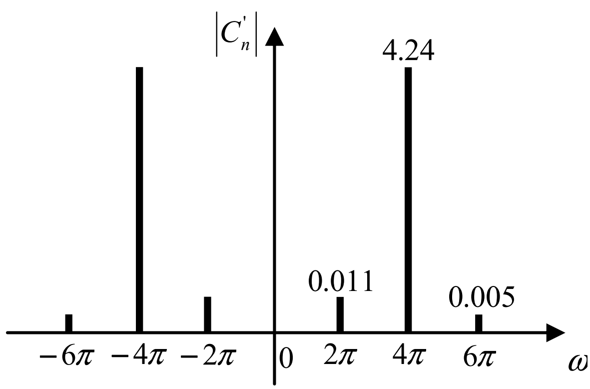
The Fourier decompositions for the different voltages are:

$$v_1(t) \Rightarrow C_n = -\frac{2}{\pi}\frac{1}{4n^2-1}$$

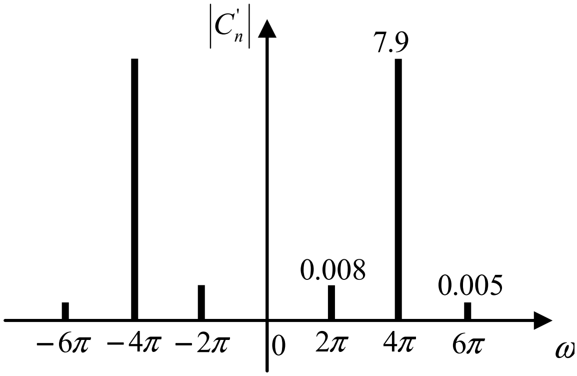
$$v_2(t) \Rightarrow \begin{cases} C_n = \dfrac{-1}{j2\pi n} \\ C_0 = \dfrac{1}{2} \end{cases}$$

This gives us the expression of the Fourier decompositions of the different currents and the spectral representation of the first three harmonics:

$$i_1(t) \Rightarrow \begin{cases} C_0^{'} = 0 \\ C_n^{'} = \dfrac{2}{\pi}\dfrac{1}{1-4n^2}\dfrac{1}{0.01 + j2\pi\dfrac{n^2-4}{n}} \end{cases}$$



$$i_2(t) \Rightarrow \begin{cases} C_0^{'} = 0 \\ C_n^{'} = \dfrac{-1}{j2\pi n\left(0.01 + j2\pi\dfrac{n^2-4}{n}\right)} \end{cases}$$



Spectral components other than $\omega = \pm 4\pi$ have a very low amplitude. This means the current is practically sinusoidal with a frequency double that of the input signal. This comes down to the selectivity of the resonant RLC series circuit.