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The Flow of Viscous Fluids. Flow in the Vicinity of a Wall: Boundary Layers and Films

1.1. Introduction

An important class of problems concerns the interaction between a viscous flow and a boundary: an open solid boundary (plate) or a closed one (tube), a fluid at rest, another flow or another fluid.

Given their common characteristics, these flows can be grouped in the category of boundary layers:

– *Outer boundary layers*: interaction between a flow of indefinite extension and a flat, curved, cylindrical or other shaped wall

– *Inner boundary layers*: interaction between a flow of finite extension and a partially or fully closed wall: pipe flow, open channel flow

– *"Mixing" boundary layers*: jets, etc.

These flows can be laminar or turbulent. In this respect, viscosity plays a key role.

1.2. Characteristics and classification of boundary layers

In essence, boundary layers are "thin" flows. Therefore, in any boundary layer, a distinction between "lateral" and "longitudinal" scales is established. In particular, the gradients of parameters (velocity, temperature, concentration) are much stronger along the lateral dimensions than along the longitudinal ones.

In the case of outer boundary layers, a two-step calculation method is applicable, which has already been mentioned in Chapter 3 of [LED 17]:

a) When a solid body is placed in a uniform flow, *the so-called "potential" flow is calculated* (see the kinematics review) based on a perfect inviscid fluid theory.

b) Viscosity in the "thin" boundary layer is then taken into account.

The "inner" boundary layers are present in areas where the regime sets in. Therefore, the scales to be compared are the radius of a pipe relative to the length L of this inflow area.

The jet boundary layers relate the lateral dimension of a jet, expressed by a lateral profile of velocity, and the longitudinal dimension of the jet.

The analytical approaches in this manual are focused on the outer or inner boundary layer. The numerical approach is better suited for jet problems.

A prototype of outer boundary layers consists of the flow developed by a flat plate in the uniform flow of velocity U_e [or locally uniform flow if $U_e = U_e\left(x\right)$ has an axial variation]. In this case, there is a non-disturbed flow (or "potential" flow; this problem has been mentioned in Chapter 3 of [LED 17]) and a border layer flow, where the flow connects to the wall area. This connection area with thickness δ is "thin" compared with the longitudinal dimension L of the body being considered.

There are longitudinal scales (along Ox in Cartesian coordinates), which are always larger than the lateral scales (along Oy in Cartesian coordinates).

Or expressed in orders of magnitude:

$$\delta << L \qquad [1.1]$$

There are two consequences for the *orders of magnitude* of the terms in the equations of fluid mechanics:

The boundary layer is "thin"; therefore, the flow is "nearly parallel to the flow". In Cartesian coordinates and for a plane flow, a first relation can be established between the longitudinal component u and lateral component v of velocity: $u << v$.

NOTE.– As will be seen, to ensure continuity, this cannot be rigorously valid. This will particularly be the case when employing integral methods. It will be used in certain cases as boundary "property" to obtain approximate results: see, for example, Stokes' first problem presented further on.

There is a similar relation, in terms of order of magnitude, between longitudinal and lateral derivatives of the same function f:

$$\frac{\partial f}{\partial x} << \frac{\partial f}{\partial y} \tag{1.2}$$

A simplification of the equations of fluid mechanics is thus possible, though to a limited extent.

Indeed, in orders of magnitude:

$$\frac{\partial u}{\partial x} \approx \frac{u}{L}; \frac{\partial v}{\partial y} \approx \frac{v}{\delta}; v << u; \delta << L \tag{1.3}$$

$$\text{But: } \frac{u}{L} \approx \frac{v}{\delta}; \frac{\partial u}{\partial x} \approx \frac{\partial v}{\partial y} \tag{1.4}$$

$$u\frac{\partial u}{\partial x} \approx u\frac{u}{L}; v\frac{\partial u}{\partial y} \approx u\frac{v}{\delta}; \frac{\partial^2 u}{\partial y^2} \approx \frac{u}{\delta^2} \tag{1.5}$$

These three terms have the same order of magnitude.

For a plane incompressible steady flow above a flat plate, we have:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{1.6}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} \tag{1.7}$$

It is worth noting that not all the terms containing the derivative with respect to x disappear, as these equations contain products of small terms $\left(\frac{\partial u}{\partial x}, v\right)$ and large terms $\left(u, \frac{\partial u}{\partial y}, \frac{\partial^2 u}{\partial y^2}\right)$.

A physicist's approach to these order of magnitude-related notions is recommended. In the outer boundary layers, when the distance from the wall increases, velocity reaches quite rapidly a value close to that of the potential flow, U_e. Rigorously speaking, it is an asymptotic variation, meaning that u tends to U_e, when x tends to infinity. In wind tunnel, there would be infinities of several centimeters!

The thickness of the boundary layer, although denoted by δ, is not infinitely small. In wind tunnel applications, for a plate with length of several tens of centimeters, δ is also measured in centimeter. Given a ship with length $L = 250\,m$, δ is measured in meter. In a geographical area, the atmospheric boundary layer can measure 1–2 km!

Moreover, an infinitely small δ compared to L often comes down to ratios $\frac{\delta}{L}$ of the order of 10^{-2}.

1.2.1. *Boundary layers – various approaches*

For laminar flows, the boundary layer equations can be solved. Several examples, such as the Blasius theory and the Stokes theory, will be given below.

There are various approaches to turbulent flows: semi-empirical theories, the mixing length theory, the Boussinesq viscosity models and numerical approaches (Reynolds decomposition, $k-\varepsilon$ methods). References to the very rich literature on this subject will be provided.

Reasonable results can also be achieved with "lower costs", by using the integral methods. Although quite old, this approach is still worth being known. It was our intention to initiate the interested reader into these techniques, and a specific paragraph is dedicated to this purpose.

1.3. The outer boundary layers: an analytical approach

1.3.1. *The laminar boundary layer developed by a flat plate in a uniform flow*

Although laminar flows may be perceived as rather academic from a practical point of view, the reasoning developed in regards to this model is highly inciting and formative for the thinking process of fluid mechanics scientists.

Let us consider a fluid with constant physical properties (density μ and kinematic viscosity ν). The potential flow is then obviously a uniform flow of velocity U_e. We shall consider the case of a plane steady flow. An orthonormal frame of reference is attached to the system. The origin is on the leading edge of the plate; x and y axes are parallel and perpendicular to the plate, respectively.

The velocity vector $\vec{V}(u,v)$ verifies the following equations:

Continuity equation:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \qquad [1.8]$$

Impulse equation:

$$u\frac{\partial u}{\partial x}+v\frac{\partial v}{\partial y}=-\frac{1}{\rho}\frac{dp_G}{dx}+\nu\frac{\partial^2 u}{\partial y^2} \qquad [1.9]$$

The term containing $\frac{dp_G}{dx}$ calls for two remarks:

The flow is not rigorously parallel in the boundary layer. In effect, $v(x,y)$ is smaller than $u(x,y)$, but not rigorously null. This makes physical sense: the plate “slows down” the fluid, which is “chased away from the plate” and into the free flow. Nevertheless, it is close enough to a parallel flow to allow us to consider that perpendicularly to these flow lines, and therefore to the plate, the lateral gradient of p_G, $\frac{\partial p_G}{\partial y}$, is null.

A $"full"d$ (full derivative) with respect to x is justified here. Similarly, the value of $p_G(x)$ will be to the same as the one observed for the same coordinate x in the free flow.

From a physical perspective, U_e can vary if a wind tunnel of variable cross-section is considered, for example. In the potential flow, where the fluid is deemed perfect, Bernoulli’s theorem is verified. Therefore:

$$\frac{d}{dx}\left(\rho\frac{U_e^2}{2}+p_G\right)=0 \qquad [1.10]$$

which offers the possibility to replace $-\frac{1}{\rho}\frac{dp_G}{dx}$ with

$$-\frac{1}{\rho}\frac{dp_G}{dx} = U_e \frac{dU_e}{dx} \quad [1.11]$$

When the potential flow is rigorously uniform, this term is null, a case which will be considered in what follows.

The Blasius problem

Blasius solved the problem of the laminar boundary layer of a fluid with constant physical properties over a flat plate of indefinite width placed in the path of wind of a uniform flow.

The system of equations that relate the two unknowns $u(x,y)$ and $u(x,y)$ becomes

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad [1.12]$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial y} = \nu \frac{\partial^2 u}{\partial y^2} \quad [1.13]$$

The boundary conditions being written

$$y = 0;\ u = v = 0 \quad [1.14.a]$$

$$y \to \infty;\ u \to U_e \quad [1.14.b]$$

It is worth noting that the problem states an asymptotic variation of velocity toward the outer flow “to infinity”. In fact, rapid convergence will be noted. A boundary layer thickness is defined by the point in which $u\,(x,\delta) = 0,99U_e$. This entails $\delta = \delta(x)$. We shall verify in the following that $\frac{\delta(x)}{x}$ is small.

Blasius showed that the problem becomes self-similar through the definition of a dimensionless composite variable $\eta(x,y)$:

$$\eta = y\sqrt{\frac{U_e}{\nu x}} \tag{1.15}$$

A dimensionless function $f(\eta)$ emerges, which verifies the following differential equation, in which the symbol " ' " signifies one differentiation:

$$f''' + 2ff'' = 0 \tag{1.16}$$

Accompanied by the boundary conditions

$$f(0) = 0; f'(0) = 0; f'(\infty) \to 1 \tag{1.17}$$

this equation is solved numerically.

The results to be retained are the profile of $f(\eta)$, which gives the profiles of the velocity components u, v and the wall friction expressed by a friction coefficient C_f defined by $C_f = \frac{2\tau_W}{\rho U_e^2}$, where τ_W is the wall shear stress.

We show that the reduced velocity $\frac{u}{U_e}$ is equal to:

$$\frac{u}{U_e} = f'(\eta) \tag{1.18}$$

This leads to the emergence of a self-similarity of velocity profiles.

In particular, for any value of x, the thickness of the boundary layer will be given for

$$f'(\eta) = 0{,}99 \tag{1.19}$$

which is verified for:

$$\eta = 4{,}96 \tag{1.20}$$

$$\eta = \delta\sqrt{\frac{U_e}{\nu x}} = 4{,}96 \tag{1.21}$$

$$\delta(x) = 4{,}96\sqrt{\frac{\nu x}{U_e}} \tag{1.22}$$

which is often written as:

$$\frac{\delta(x)}{x} = \frac{4{,}96}{\sqrt{R_x}} \tag{1.23}$$

where R_x is the Reynolds number for the x axis:

$$R_x = \frac{U_e\, x}{\nu} \tag{1.24}$$

Using the value of $f''(0)$, we obtain:

$$\tau_W = \mu \left.\frac{\partial u}{\partial y}\right|_{y=0} = \mu U_e \left(\left.\frac{df'}{d\eta}\right|_{\eta\,=\,0} \right) \frac{\partial \eta}{\partial y} = \mu U_e \sqrt{\frac{U_e}{\nu x}}\, f''(0) \tag{1.25}$$

$$f''(0) = 0{,}332 \tag{1.26}$$

which finally gives:

$$\frac{C_f}{2} = \frac{\tau_W}{\rho U_e^2} = \frac{0{,}332}{\sqrt{R_x}} \tag{1.27}$$

1.3.2. *The turbulent boundary layer*

The turbulent boundary layer, which is more interesting from a more practical perspective, cannot be treated in a similar manner. The analytical approach can only yield mean values of the parameters.

Therefore, an important operation is to calculate these mean values:

We define the mean velocity as:

$$\bar{u}\,(x,y)=\frac{1}{T}\int_0^T u\,(t,\,x,\,y) \qquad [1.28]$$

This time-averaged value theoretically calculated over infinite time is in fact experimentally established over physically long durations (of the order of several minutes).

A Reynolds decomposition ensues:

$$u\,(x,y)=\bar{u}\,(x,y)+u' \qquad [1.29]$$

Time-averaged u' is null, but not its root mean square; therefore, fluctuations can be calculated by:

$$\frac{1}{T}\int_0^T u'^2\,(t,\,x,\,y) \qquad [1.30]$$

Similarly, various correlations will play an important role, such that:

$$\frac{1}{T}\int_0^T u'(t,\,x,\,y)\,v'(t,\,x,\,y) \qquad [1.31]$$

We shall however see that within the framework of integral methods, similar approaches are applicable to laminar and turbulent layers.

Historically, several approaches have been conducted, generally because of the advancement of metrology. For the boundary layer above a flat plate:

The first studies using the Pitot tube have led to researching mean velocity profiles expressed as power laws (that can be used particularly with integral methods):

$$\frac{\bar{u}\,(x,y)}{U_e}=\left(\frac{y}{\delta(x)}\right)^n \qquad [1.32]$$

where n is a coefficient that varies in principle with Reynolds number. For the usual Reynolds numbers, n takes predominantly the value $\frac{1}{7}$ (1/7th power law).

The development of anemometry (hot-wire anemometry, then Laser Doppler anemometry) facilitated a more detailed rendering of profiles and particularly of the flow structure.

Although the matter is beyond the scope of this initiation manual, it is worth noting that spectral approaches to fluctuations have gained attention. This was made possible by the progressive decrease in sensor response time, from microscopic hot wire to the optical measurement of velocity (Laser Doppler anemometry). We shall mention here the most prominent result, although it will not be used further on.

Variables x^+, y^+ and the dimensionless velocities u^+, v^+ have then been defined by a reference velocity, called friction velocity:

$$U_f = \sqrt{\frac{2\tau_W}{\rho}} = U_e \sqrt{\frac{C_f}{2}} \quad [1.33]$$

$$x^+ = \frac{U_f x}{\nu}; \; y^+ = \frac{U_f y}{\nu} \quad [1.34]$$

$$u^+ = \frac{\overline{u}}{U_f}; \; v^+ = \frac{\overline{v}}{U_f} \quad [1.35]$$

The velocity profile has then been divided into three regions.

Film region corresponding to a “viscous” laminar sub-layer with linear velocity profile:

$$u^+ = y^+$$

Wall region comprises most of the boundary layer. This wall law leads to a logarithmic profile with reduced variables:

$$u^+ = \frac{1}{\kappa} \ln y^+ + C \quad [1.36]$$

where κ is von Kármán’s constant $\kappa \approx 0.4$.

Region of connection to the potential flow

Using Reynolds decomposition method, the equations for the components of mean velocity can be written as follows:

Using the same calculation technique for the instantaneous equation, the equations can be written in classical form as:

$$\frac{\partial \overline{u}}{\partial x} + \frac{\partial \overline{v}}{\partial y} = 0 \qquad [1.37]$$

$$\overline{u}\frac{\partial \overline{u}}{\partial x} + \overline{v}\frac{\partial \overline{u}}{\partial y} = -\frac{1}{\rho}\frac{\partial \overline{p}_G}{\partial y} + \nu\frac{\partial^2 \overline{u}}{\partial y^2} - \frac{\partial}{\partial y}\left(\overline{u'v'}\right) \qquad [1.38]$$

The system cannot be reduced to these equations of mean velocities.

It is possible to establish an equation for this correlation $\overline{u'v'}$ (this will not be done here). Then, it is noted that the process is divergent: for each differential equation written for a correlation of order n, correlations of the order $n+1$ emerge. This is natural. The information lost in the statistical approach (creation of terms in u' and v') requires more than chance to reemerge through simple mathematical manipulations.

Therefore, these equations must be closed, which means that the information must be fed back in through a supplementary relation. This closure can be performed at various n levels.

The best known closure, and one of the simplest, is of the gradient type, such as that proposed by Boussinesq:

$$\overline{u'v'} = -\varepsilon_\nu \frac{\partial \overline{u}}{\partial y} \qquad [1.39]$$

where ε_ν is turbulent viscosity. Let us note that ε_ν has the dimension of a kinematic viscosity.

The significance of the term $\overline{u'v'}$ in the equations is that of shear stress, and hence it is called “Reynolds stress”. It contains the mechanism of “turbulent friction” illustrated by the Boussinesq closure.

The viscosity of a fluid is due to the transfer of momentum between the layers of fluid that are in relative translation by molecular agitation. Similarly, eddies of the turbulent flow carry momentum through the velocity gradient. In this case, the scale of the phenomenon is the scale of the eddy. This was used by Prandtl in his “mixing length” theory.

Moreover, the values obtained for turbulent viscosity are much higher; “molecular” viscosity will subsequently be often neglected (with the exception of the laminar sub-layer).

In the field of inner boundary layers, the problem will most often be solved empirically. Since the established regime is not predominant, Moody charts and a Darcy type of approach will be used to determine frictions.

The values that are often retained for the thickness of the boundary layer and the friction over a flat plate are deduced from the value of the friction coefficient:

$$\frac{\delta}{x} = \frac{0,382}{R_x^{0.2}} \qquad [1.40]$$

$$C_f = \frac{0,0594}{R_x^{0,2}} \qquad [1.41]$$

where $R_x = \dfrac{U_e\ x}{\nu}$ is the Reynolds number for the x axis.

Many semi-empirical methods have been tried in the past for the treatment of turbulent flows. The predominant method currently applied is the numerical approach, which is commonly used in many industrial applications. On this subject, the reader is invited to see the chapters dedicated to numerical processing.

Several problems pertaining to this class of approaches will be provided below. In particular, an introduction to the treatment of non-Newtonian fluids will be offered.

1.4. Examples of analytical approach: outer flows

Comprehensive works have been dedicated to the subject of boundary layer theory. For an introduction on the subject, it is worth presenting the reader with some rather atypical and original problems. Therefore, this section focuses on unsteady boundary layers (two classical problems are proposed) and an introduction to non-Newtonian fluids. As we shall see, when dealing with boundary layers or tubes, it is worth considering the rheological specificity of certain fluids in a simple manner, without reducing it to a Newtonian fluid hypothesis, which may sometimes prove too reductive.

For an approach of unsteady flows, a higher dimension should be considered. This may prove a complex issue, as the dimension of the problem becomes $n+1$, where n is the number of space dimensions considered.

In-depth study of unsteady flows is not the object of this book. Several classical examples that are physically useful will be presented here for the dimension n reduced to 1. These solutions appear as welcome simplifications of more complex problems of unsteady boundary layers. We shall establish in passing an interesting connection between an unsteady setting and a steady boundary layer.

As far as non-Newtonian fluids are concerned, we shall consider power law fluids or Eyring fluids.

EXAMPLE 1.1 (Unsteady boundary layer. Stokes' first problem).–

Let us consider the uniform flow of velocity U of an incompressible fluid of density ρ and dynamic viscosity μ. Kinematic viscosity of this fluid will be expressed as $\nu = \dfrac{\mu}{\rho}$. The flow lines are considered horizontal.

At time $t = 0$, a flat plate of infinite extension is placed in this fluid, parallel to the flow lines. Given the symmetry of the problem relative to the plate, only the flow over the upper face will be studied. The flow lines are also considered to remain parallel to the wall at any time subsequent to $t = 0$.

The axis perpendicular to the wall will be denoted by Oy, the origin of the coordinates of which being located on this wall.

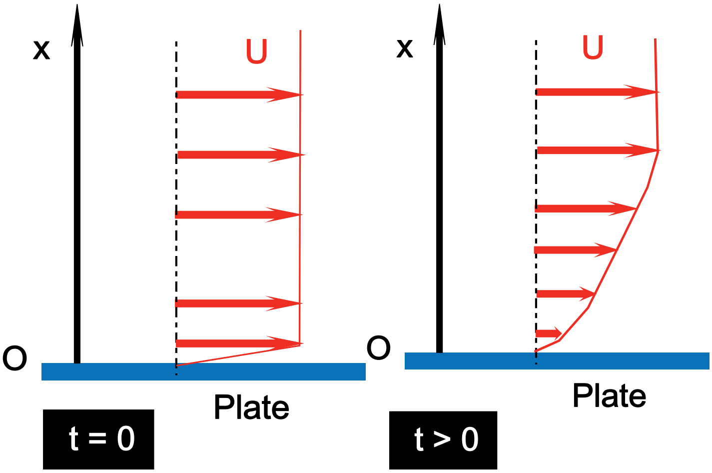

Figure 1.1. *Stokes' first problem. Representation of the velocity profile at two moments of time*

1) Show that velocity assumes the form $\vec{V}(y,t)$ in Eulerian description.

Write the differential equation which is satisfied by $u(y,t)$ and the boundary conditions of the problem for $t > 0$.

2) What does the form of this equation remind you of?

3) Find the expression of $u(y,t)$. It is useful to introduce the variable η:

$$\eta = \frac{y}{\sqrt{4\nu t}} \qquad [1.42]$$

4) Let us now consider the boundary layer developed by a uniform flow of outer velocity U_e over a semi-infinite plate placed under the path of wind. By modeling a section of thickness dx perpendicular to the plate, whose displacement starting from the leading edge is observed, and which plays the role of the previous flow, finds an approximate shape of the velocity $u(y)$ profile for a given abscissa x, which will be an approximation of $u(x,y)$ in the boundary layer.

Deduce from this an approximate formula for the thickness of the dynamic boundary layer $\delta(x)$. Compare this form with the exact result of the Blasius analysis.

Solution:

1) The flow is plane.

The velocity $\vec{V}$ has only one component u. For an incompressible fluid, the continuity equation is:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \qquad [1.43]$$

As v is null, $\frac{\partial u}{\partial x}=0$ and u can only be a function of y and t, since the flow is unsteady. Then, the projection on Ox of the equation of (dynamic) impulse for a plane flow of Newtonian fluid can be written as:

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\mu\frac{\partial^2 u}{\partial y^2} \qquad [1.44]$$

where v is null, and so is the term containing pressure. Introducing kinematic viscosity, the above relation becomes:

$$\frac{\partial u}{\partial t}= \nu\frac{\partial^2 u}{\partial y^2} \qquad [1.45]$$

Two boundary conditions are associated to this equation:

$$t>0\ y=0;u=0 \qquad [1.46.a]$$

$$t>0\ y\rightarrow\infty;u\rightarrow U \qquad [1.46.b]$$

2) This equation is formally identical to the equation of heat under one-dimensional and unsteady regime, in which ν plays the role of thermal diffusivity a.

Although it is beyond the scope of this work, it is worth noting that the symmetry of the roles of ν and a is reflected by the impulse and energy equations written for fluids with constant physical properties. ν and a have the same dimension, L^2T^{-1}, and their ratio defines one of the most important numbers in the theory of thermal convection, the Prandtl number $P_r = \frac{\nu}{a}$

3) To solve this problem, the approach taken in the theory of thermal conduction will be referred to as a model. The following variable is introduced:

$$\eta = \frac{y}{\sqrt{4\nu t}} \tag{1.47}$$

This variable is the analogue of $\eta = \frac{y}{\sqrt{4ax}}$ in the theory of unsteady state heat conduction (in this case, a is thermal diffusivity).

Differentiations are performed as follows:

$$\frac{\partial \eta}{\partial t} = -\frac{y}{\sqrt{4\nu}} \frac{1}{2t^{3/2}} = -\frac{\eta}{2t} \tag{1.48}$$

$$\frac{\partial \eta}{\partial y} = \frac{1}{\sqrt{4\nu t}}; \frac{\partial^2 \eta}{\partial y^2} = 0 \tag{1.49}$$

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial t}; \frac{\partial u}{\partial y} = \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial y}; \frac{\partial^2 u}{\partial y^2} = \frac{\partial u}{\partial \eta}\frac{\partial^2 \eta}{\partial y^2} + \frac{\partial^2 u}{\partial \eta^2}\left(\frac{\partial \eta}{\partial y}\right)^2 \tag{1.50}$$

$$\frac{\partial u}{\partial \eta}\frac{\eta}{2t} = -\frac{\partial^2 u}{\partial \eta^2}\frac{1}{4\nu t} \tag{1.51}$$

Time is eliminated from this equation and u appears as a unique function of η, which satisfies the equation accompanied by its boundary conditions:

$$\frac{d^2 u}{d\eta^2} = -2\eta \frac{\partial u}{\partial \eta} \tag{1.52}$$

$$\eta = 0; u = 0; \eta \to \infty; u \to U \tag{1.53}$$

The solution yields immediately:

$$\frac{\frac{d^2u}{d\eta^2}}{\frac{\partial u}{\partial \eta}} = -2\eta \qquad [1.54]$$

$$Ln\frac{\partial u}{\partial \eta} = -\eta^2 + Ln\,C_1 \qquad [1.55]$$

$$\frac{\partial u}{\partial \eta} = C_1 \exp - \eta^2 \qquad [1.56]$$

$$u = \int C_1 \exp - \xi^2 + C_2 \qquad [1.57]$$

The boundary conditions yield, knowing that:

$$\int_0^\infty \exp - \xi^2 = \frac{\sqrt{\pi}}{2} \qquad [1.58]$$

$$u = C_1 \int_0^\eta \exp - \xi^2\, d\xi \qquad [1.59]$$

$$C_1 = \frac{2U}{\sqrt{\pi}} \qquad [1.60]$$

The error function can be identified in [1.59], up to a coefficient:

$$erf\left(\eta\right) = \frac{2}{\sqrt{\pi}} \int_0^\infty \exp - \xi^2\, d\xi \qquad [1.61]$$

$u\left(y,t\right)$ is finally expressed as:

$$u = U\, erf \frac{y}{\sqrt{4\nu t}} \qquad [1.62]$$

4) The approximation mechanism proposed in this section should be properly understood. It amounts to applying to a thin section a result established for an indefinite medium. In the proposed pattern, which follows the section in its motion, x is identified as Ut.

Then, $u(t,y)$ is identical to $u(x,y)$ and the field of velocity within the framework of this approximation can be written as:

$$\frac{u(x,y)}{U} = erf\left(y\sqrt{\frac{U}{4\nu x}}\right) \qquad [1.63]$$

The conventional limit of this boundary layer is reached for $u = 0.99U$, which is an argument of the erf function equal to 1.82. This yields:

$$y\sqrt{\frac{U}{4\nu\delta}} = 1,82 \qquad [1.64]$$

$$\frac{\delta}{x} = 3,64\sqrt{\frac{\nu x}{U}} = \frac{3,64}{R_x} \qquad [1.65]$$

Reynolds number for the x axis $R_x = \sqrt{\frac{Ux}{\nu}}$ has been introduced. This result is comparable to that of the Blasius analysis:

$$\frac{\delta}{x} = \frac{4,95}{R_x} \qquad [1.66]$$

The error is 26%, for quite an elementary and rapid solving.

EXAMPLE 1.2 (Stokes' second problem).–

A horizontal flat plate of length L and indefinite width oscillates parallel to itself. A frame of reference Oxy is defined (this is obviously a two-dimensional phenomenon). Ox is parallel to the plate along its axis of movement and Oy is perpendicular to the plate.

The velocity of the plate is given by $U(t) = U_0 \cos \omega t$

This plate is immersed in an incompressible fluid of density ρ and dynamic viscosity μ. The fluid region set in motion is considered thin relative to the longitudinal dimension L of the plate (which validates the hypothesis of the indefinite plate). It can be readily admitted that fluid velocity above the wall has only a horizontal component.

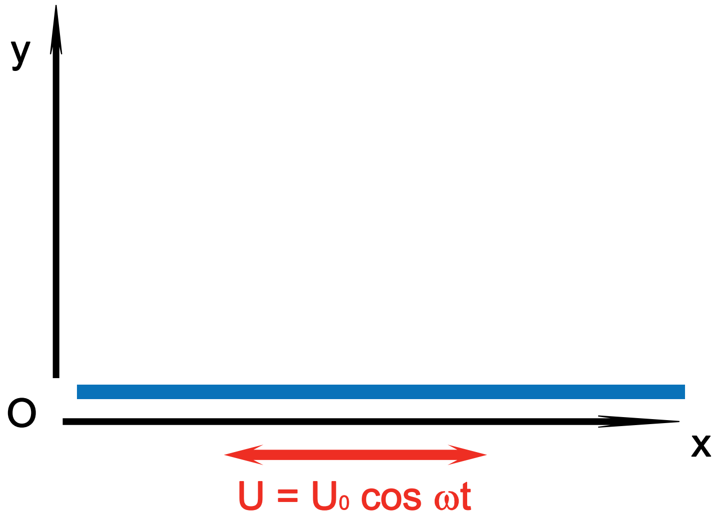

Figure 1.2. *Stokes' second problem*

1) The frequency of vibration f is high enough for the Strouhal number to be considered very large, which will be defined as:

$$S = \frac{Lf}{U_0} \qquad [1.67]$$

where L becomes a length-scale characteristic to the problem.

Considering the orders of magnitude for the problem of various terms of the impulse equation, find its very simplified form.

Quickly show that velocity above the plate has the form $u(y,t)$.

Write the equation that is satisfied by u.

Write the boundary conditions for y.

NOTE.– The problem is considered in the context of the plate vibrating for a sufficiently long time. The flow is therefore in "steady-state regime".

All unsteady problem of this type involves a transient solution (from the moment the plate is set in motion) and a "steady-state" solution. This latter type of solution is of interest here, given the previous hypothesis.

2) Find the form of $u\left(y,t\right)$

Finding this solution will be significantly facilitated by using complex numbers. $u\left(y,t\right)$ will then be the real part of the complex solution, which is written as:

$$u_{Complex}=\Phi(y)e^{i\omega t} \quad [1.68]$$

Let us recall the identity:

$$\sqrt{i}=\frac{1+i}{\sqrt{2}} \quad [1.69]$$

3) Show that there is a phase difference $\phi(y)$ between the velocities of the two planes of fluid parallel to the wall. What is the distance δ between two in-phase planes?

What is the distance l from the wall above which the velocity amplitude Φ will be one hundredth of U_0?

4) Numerical application.

The plate oscillates in water with $f=500\,Hz.$ We know that $\rho=1000\,kg.m^{-3}$ and $\mu=10^{-3}\,Pl.$ Find the values of δ and $S=l.$

Show that this device can be used to measure viscosity. What type of viscosity is directly measured? If δ is measurable up to 10%, what is the accuracy for ν?

Solution:

1) The flow is plane

The velocity $\vec{V}$ has only one component u.

For an incompressible fluid, the continuity equation is:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \quad [1.70]$$

as v is null:

$$\frac{\partial u}{\partial x}=0 \quad [1.71]$$

and u can only be a function of y and t, since the flow is unsteady.

Then, the projection on Ox of the equation of (dynamic) impulse for a plane flow of Newtonian fluid can be written as:

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\mu\frac{\partial^2 u}{\partial y^2} \qquad [1.72]$$

v is null, and so is the term containing pressure. Introducing kinematic viscosity, the above relation becomes:

$$\frac{\partial u}{\partial t}=\nu\frac{\partial^2 u}{\partial y^2} \qquad [1.73]$$

Two boundary conditions are associated to this equation:

$$t>0\ y=0;u=U_0\cos\omega t \qquad [1.74]$$

$$t>0\ y\to\infty;u\to 0 \qquad [1.75]$$

2) The context is that of “steady-state regime” (out of transient regime), and we search for u under an associated complex form $u_{Complex}$:

$$u_{Complex}=\Phi(y)e^{i\omega t} \qquad [1.76]$$

In order to satisfy the differential equation, $\Phi(y)$ has to verify:

$$i\omega\,\Phi(y)e^{i\omega t}=\nu\frac{d^2\Phi}{d\,y^2}e^{i\omega t} \qquad [1.77]$$

$$\frac{d^2\Phi}{dy^2}-\frac{i\omega}{\nu}\Phi=0 \qquad [1.78]$$

A classical equation with the general solution:

$$\Phi(y)=A\exp\sqrt{\frac{i\omega}{\nu}}y+B\exp-\sqrt{\frac{i\omega}{\nu}}y \qquad [1.79]$$

$$\Phi(y)=A\exp\frac{1+i}{\sqrt{2}}\sqrt{\frac{\omega}{\nu}}y+B\exp-\frac{1+i}{\sqrt{2}}\sqrt{\frac{\omega}{\nu}}y \qquad [1.80]$$

$$\Phi(y) = A\exp\left(\sqrt{\frac{\omega}{2\nu}}y\right)\exp i\left(\omega t + \sqrt{\frac{\omega}{2\nu}}y\right) + B\exp\left(-\sqrt{\frac{\omega}{2\nu}}\right)\exp i\left(\omega t - \sqrt{\frac{\omega}{2\nu}}y\right) \qquad [1.81]$$

$\Phi(y)$ has to tend to 0 at infinity; therefore, A is null.

$u_{Complex} = \Phi(y)e^{i\omega t}$ can then be written as:

$$u_{Complex} = B\exp\left(-\sqrt{\frac{\omega}{2\nu}}y\right)\exp i\left(\omega t - \sqrt{\frac{\omega}{2\nu}}y\right) \qquad [1.82]$$

$$u_{Complex} = B\exp\left(-\sqrt{\frac{\omega}{2\nu}}y\right)\left[\cos\left(\omega t - \sqrt{\frac{\omega}{2\nu}}y\right) + i\sin\left(\omega t - \sqrt{\frac{\omega}{2\nu}}y\right)\right] \qquad [1.83]$$

Taking the real part:

$$u_{Complex} = B\exp\left(-\sqrt{\frac{\omega}{2\nu}}y\right)\left[\cos\left(\omega t - \sqrt{\frac{\omega}{2\nu}}y\right)\right] \qquad [1.84]$$

The wall condition requires that $B = U_0$

$$u_{Complex} = U_0\exp\left(-\sqrt{\frac{\omega}{2\nu}}y\right)\left[\cos\left(\omega t - \sqrt{\frac{\omega}{2\nu}}y\right)\right] \qquad [1.85]$$

It can be noted that the plate imposes a disturbance that decreases exponentially with the distance to the plate.

3) Moreover, all the planes that are parallel to the plate vibrate at the same frequency, but are out of phase. The phase difference between two planes separated by y is:

$$\phi = \sqrt{\frac{\omega}{2\nu}}y \qquad [1.86]$$

Two planes in phase are separated by δ:

$$\phi = 2\pi = \sqrt{\frac{\omega}{2\nu}}\delta \qquad [1.87]$$

$$\delta = \sqrt{\frac{8\pi^2\nu}{\omega}} \qquad [1.88]$$

The velocity equals one hundredth of U_0 for:

$$\exp\left(-\sqrt{\frac{\omega}{2\nu}}l\right) = 10^{-2}; \ \sqrt{\frac{\omega}{2\nu}}l = Ln10^{-2} = 4,6 \qquad [1.89]$$

$$l = 4,6\sqrt{\frac{2\nu}{\omega}} \qquad [1.90]$$

4) Numerical application

$f = 500\,Hz$; $\omega = 1000\pi; \nu = 10^{-6}\,m.s^{-2}$

Let:

$$\delta = 1,59.10^{-4}\,m = 159\,\mu m \qquad [1.91]$$

$$l = 4,6\sqrt{\frac{2\nu}{\omega}} = 1,1.10^{-4}\,m = 110\,\mu m \qquad [1.92]$$

For a relative error of 10% on δ, the accuracy of the measurement of kinematic viscosity is:

$$\frac{\Delta\nu}{\nu} = 2\frac{\Delta\delta}{\delta} \qquad [1.93]$$

The error is 20%.

1.5. Examples of analytical approach: inner flows

EXAMPLE 1.3 (Couette flow. Newtonian and non-Newtonian perfect fluids).–

The Couette flow is generated between two horizontal, parallel, flat plates of indefinite dimensions. One of the plates moves in parallel to itself with a velocity U_0, while the other plate is at rest. The two plates are separated by a fluid of thickness e.

This physical situation models what happens in the gap between the two cylinders of a Couette viscometer, if we consider that the radius of the cylinders is very large compared with their radius difference.

For this reason, the pressure will be considered unchanged in the direction parallel to the plates.

A frame of reference of axes Oxy is attached to the system (the flow is obviously plane), with Ox parallel to the plate. The origin O is located on the lower plate, which moves to the right with velocity U_0.

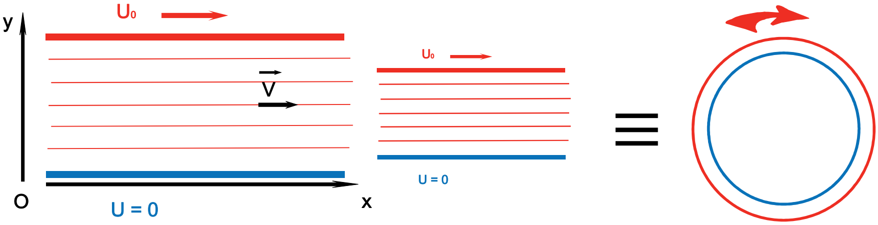

Figure 1.3. *Couette flow. On the left: Couette flow between two plates. On the right: application of this model as an approximation to Couette viscometer*

1) A perfect fluid is placed between the two plates: What happens?

2) Then an incompressible fluid of density ρ and dynamic viscosity μ is placed between the plates. Find the expression of the components u and v of the velocity field $\vec{V}$. Find the expression of the shear stress τ_W on each plate.

3) An Ostwald–de Waele fluid is placed between the plates. Consider question 2 for this type of fluid.

4) Resume question 2 for an Eyring fluid.

REMINDER.–

Ostwald and de Waele showed that for certain non-Newtonian fluids, the relationship between shear stress and a one-dimensional velocity gradient $\frac{du}{dy}$ can be expressed as:

$$\tau = k\left(\frac{du}{dy}\right)^n \quad [1.94]$$

where τ is the shear stress, k is the "coefficient of viscosity" (or compacity, which does not have the dimension of a dynamic viscosity) and n is a coefficient.

If $n < 1$, the fluid is called pseudo-plastic; if $n > 1$, it is called dilatant.

Based on the works of Eyring, for certain non-Newtonian fluids, the following relation between shear stress $n < 1$ and shear strain rate $\frac{du}{dy}$ can be written as:

$$\tau = A\,sh^{-1}\left(\tau_0 \frac{du}{dy}\right) \qquad [1.95]$$

where τ_0 is a constant and $sh^{-1}x$ (or $\arg shx$) is the inverse function of hyperbolic sine shx.

The Powell–Eyring general law is more complex:

$$\tau = \mu \frac{du}{dy} \qquad [1.96]$$

where:

$$\mu - \mu_\infty = \frac{(\mu_0 - \mu_\infty)}{\tau_0 \frac{du}{dy}} sh^{-1}\left(\tau_0 \frac{du}{dy}\right) \qquad [1.97]$$

The form provided here corresponds to a pseudo-plastic fluid, for which μ_∞ is very low. μ_∞ and τ_0 are constants. In this case, μ_∞ will be neglected.

Solution:

1) The perfect fluid hypothesis

By definition, in a perfect fluid, the tangential component of surface forces is null. Therefore, the fluid does not exert any tangential force on the walls, so there is no braking or drive. Similarly, following Newton's "third" law, referring to action and reaction, the plate exerts only normal forces (pressure) on the fluid, and therefore will exert no drive or braking action on the fluid. The plates will "slip perfectly" over the fluid, which remains perfectly still. Referring to this particular point, it is worth noting that the proposed hypothesis is contradicted by experience. This contradiction makes the object of the so-called "d'Alembert's paradox".

NOTE.– Let us recall that too many students similarly forget to consider Newton's "second" law, also known as the fundamental principle of dynamics. The quotation marks are a reminder that Newton's "first" law is in fact an inertia principle that had already been formulated by Galileo, a "principle" that can be proven starting from Newton's "second" law.

2) Newtonian fluid

Although not explicitly provided in the text, each time we introduce a fluid of density ρ and dynamic viscosity μ, the assumption being made is that the fluid is Newtonian (in general, a non-Newtonian fluid is defined based on coefficients, and not on viscosity; to be extremely rigorous, a formula is provided for the viscosity, which relates it to stresses). Moreover, the other questions of the problem leave no room for ambiguity.

The fluid being incompressible, we readily proceed to defining its kinematic viscosity ν, which appears in the equation of impulse: $\nu = \frac{\mu}{\rho}$

Kinematic remark: For plates of indefinite dimensions, it is not difficult to see that the flow is parallel (obviously non-uniform).

Therefore, the component v of the flow will necessarily be null.

In order to get information, let us use the continuity principle and the second law of dynamics (impulse equation).

The continuity equation gives:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad [1.98]$$

$$v = 0; \frac{\partial u}{\partial x} = 0; u = u(y) \qquad [1.99]$$

There is only one component of velocity left, $u(y)$, exclusively dependent on y, which will satisfy the impulse equation written for an incompressible Newtonian fluid (two-dimensional Navier–Stokes equation):

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + 2\nu\frac{\partial^2 u}{\partial x^2} + \nu\left[\frac{\partial^2 u}{\partial y^2} + \frac{\partial v}{\partial x \partial y}\right] + F_{Vx} \qquad [1.100]$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial y}+\nu\left(\frac{\partial^2 u}{\partial x\,\partial y}+\frac{\partial^2 v}{\partial x^2}\right)+\frac{\partial}{\partial y}2\nu\left(\frac{\partial^2 u}{\partial y^2}\right)+F_{V_y} \qquad [1.101]$$

When projecting the equation on Oy, all the terms containing velocities will be null. The volume forces are reduced to the gravity term, which, given that Oy is a vertical coordinate, can be written as: $F_{V_y}=-g$

This equation comes down to:

$$-\frac{1}{\rho}\frac{\partial p}{\partial y}-g=0 \qquad [1.102]$$

Introducing $p_G = p+\rho y$, we get:

$$-\frac{1}{\rho}\frac{\partial p+\rho y}{\partial y} = -\frac{\partial p_G}{\partial y}=0 \qquad [1.103]$$

By projecting the equation on Ox, we obtain:

$$\frac{d^2u}{dy^2}=0 \qquad [1.104]$$

Here we are using full derivatives, since u depends only on y.

This relation also expresses that:

$$\frac{d}{dy}\tau_w=0 \qquad [1.105]$$

Since in particular (see the text):

$$\frac{\partial p}{\partial x}=0 \qquad [1.106]$$

u is readily deduced from the simple differential equation. Two boundary conditions are needed and they readily obtain:

$$\frac{d^2u}{dy^2}=0 \qquad [1.107]$$

$$y=0 \Rightarrow u=0;\; y=e \Rightarrow u=U_0 \qquad [1.108]$$

The solution is trivial. The fact that the second derivative is null indicates that u is a linear function. The application of boundary conditions yields:

$$u(y)=U_0\frac{y}{e} \tag{1.109}$$

For a Newtonian fluid, shear stress is expressed as:

$$\tau_W=\mu\frac{du}{dy}=\mu\frac{U_0}{e} \tag{1.110}$$

It has the same "absolute value" on each of the plates.

3) The Ostwald–de Waele fluid

Given the flow structure and the results of question 2, we have:

$$u=u(y);\, v=0 \tag{1.111}$$

$$\frac{d}{dy}\tau=0 \tag{1.112}$$

The stress can be written as:

$$\tau=k\left(\frac{du}{dy}\right)^n \tag{1.113}$$

The equation to be solved and its boundary conditions become:

$$\frac{d}{dy}k\left(\frac{du}{dy}\right)^n=0 \tag{1.114}$$

$$y=0\Rightarrow u=0;\, y=e\Rightarrow u=U_0 \tag{1.115}$$

The resolution is simple:

$$k\left(\frac{du}{dy}\right)^n=C_1 \tag{1.116}$$

$$\frac{du}{dy}=\left(\frac{C_1}{k}\right)^{1/n}=C_1' \tag{1.117}$$

After integration:

$$u = C_1' y + C_2' \qquad [1.118]$$

where C_1, C_1' and C_2' are constants.

In order to satisfy the boundary conditions:

$$u(y) = U_0 \frac{y}{e} \qquad [1.119]$$

which is the same result as for the Newtonian fluid.

Similarly, the wall stresses will be:

$$\tau_W = \tau = k\left(\frac{du}{dy}\right)^n = k\left(\frac{U_e}{e}\right)^n \qquad [1.120]$$

4) Eyring fluid

Given the flow structure and the results of question 2, we have:

$$u = u(y); v = 0 \qquad [1.121]$$

$$\frac{d}{dy}\tau = 0 \qquad [1.122]$$

The stress has the form:

$$\tau = A\, sh^{-1}\left(\tau_0 \frac{du}{dy}\right) \qquad [1.123]$$

where

$$A = \frac{(\mu_0 - \mu_\infty)}{\tau_0} \qquad [1.124]$$

In this case, $\mu_\infty << \mu$

The equation to be solved and its boundary conditions become:

$$\frac{d}{dy} A\, sh^{-1}\left(\tau_0 \frac{du}{dy}\right) = 0 \qquad [1.125]$$

$$y = 0 \Rightarrow u = 0;\; y = e \Rightarrow u = U_0 \qquad [1.126]$$

A first integration yields:

$$sh^{-1}\left(\tau_0 \frac{du}{dy}\right) = C_1 \qquad [1.127]$$

which is equivalent to

$$\frac{du}{dy} = \frac{sh\, C_1}{\tau_0} = C'_1 \qquad [1.128]$$

$$y = 0\; u = 0;\; y = e\; u = U_0 \qquad [1.129]$$

The boundary conditions determine C'_1, and this refers us to the two previous problems:

$$u(y) = U_0 \frac{y}{e} \qquad [1.130]$$

$$\tau_W = \tau = A\, sh^{-1}\left(\tau_0 \frac{du}{dy}\right) = \tau = A\, sh^{-1}\left(\tau_0 \frac{U_e}{e}\right) \qquad [1.131]$$

In a Couette viscometer, U_e is determined by the velocity of the cylinder and the shear strain rate is proportional to the $\frac{U_e}{e}$ ratio. Even though the velocity profile is, in theory, linear (which allows for a direct determination of the shear strain rate), in the case of a Newtonian fluid, the Couette viscometer will demonstrate the expected linearity between stress and shear strain rate, while for de Waele and Eyring fluids this relation is nonlinear.

EXAMPLE 1.4 (Non-Newtonian flow in a pipe).–

We have already recalled that Ostwald and de Waele have shown that for certain non-Newtonian fluids, the relation between the shear stress and a one-dimensional velocity gradient $\frac{du}{dy}$ can be expressed as:

$$\tau = k\left(\frac{du}{dy}\right)^n \qquad [1.132]$$

where τ is the shear stress, k is a “coefficient of viscosity” (which does not have the dimension of a dynamic viscosity) and n is a coefficient.

This problem is restricted to fluids for which:

$$(-1)^n = -1 \qquad [1.133]$$

The objective here is to study the flow of such fluid as laminar flow through a horizontal tube of radius R.

The following hypotheses are accepted:

The flow is steady

The regime is steady-state:

The paths of the particles of fluid are lines parallel to the axis of the cylinder.

The flow is of axial revolution.

The fluid does not slip on the solid wall.

The shear stress retains a finite value.

A cylindrical coordinate system (x, r, θ) is used. The stagnation pressure on a plane normal to the axis of the tube is denoted by p_G.

Let us consider an elementary annular space between the two cylinders of radii r and r + dr, limited by two planes normal to the axis of the tube, separated by dx.

1) Find and write the expressions of the forces exerted on this annular space.

2) Write Euler’s theorem for this space.

3) Deduce the differential equation relating $u(r)$ and $p_G(x)$ from it. Solve this equation. As a particular case, show that $\frac{dp_G}{dx}$ is a constant, which will be denoted by $-K$.

4) Find the expression of $u(r)$. In order to correctly make explicit the term $\left|\frac{du}{dy}\right|$, we shall consider that the profile of the velocity $u(r)$ is a monotone decreasing function of r. This hypothesis will subsequently be validated.

4.1) Find the expression of $u(r)$ as a function of k, n, R and K. Verify the validity of the hypothesis for $u(r)$.

4.2) Find the expression of the flow velocity V_q.

4.3) Then, express K as a function of the volume flow rate Q_V, as well as of k, n and R.

5) The head loss is to be expressed according to Darcy

Find the relation between p_G and Q_V. Write it as a Darcy-type expression, where J is the head loss per unit length and ψ is a coefficient of head loss per unit length, which can be expressed as:

$$J = \psi \frac{V_q^2}{2gD} \qquad [1.134]$$

6) For non-Newtonian fluids, an equivalent Reynolds number is often introduced, with the following definition:

$$R_{en} = 8^{1-n} \left(\frac{4n}{3n+1} \right)^n \frac{\rho V_q^{2-n} D^n}{k} \qquad [1.135]$$

Show that R_{en} is dimensionless ($D = 2R$ is the diameter of the tube).

What is the purpose of this definition?

How are these expressions written when $n = 1$?

What is the physical case they relate to?

Solution:

1) This is a case of established flow. The velocity $\vec{V}$ is reduced to its axial component u.

Let us project these forces on Ox. The only forces with non-null projection are the surface forces: viscosity forces and pressure forces.

The resultant of pressure and gravity forces is:

$$-2\pi r\,dr\,\frac{dp_G}{dx}\,dx \qquad [1.136]$$

We are using here p_G, which takes into account the effect of gravity forces on the local pressure. The balance of pressure forces is then applied to opposite surface elements of the same elevation. The result thus obtained can be extended to a tube that is inclined toward the horizontal direction. Let us calculate the viscosity forces.

On the inner cylinder of radius r, they equal:

$$-2\pi r\,dx\,\tau = -2\pi r\,dx\,k\left(\frac{du}{dy}\right)^n \qquad [1.137]$$

On the outer cylinder (radius $r + d\,r$), they equal:

$$-2\pi r\,dx\,k\left(\frac{du}{dy}\right)^n + \frac{d}{dr}\left(-2\pi r\,dx\,k\left(\frac{du}{dy}\right)^n\right)dr \qquad [1.138]$$

Only fluids in which $(-1)^n = -1$ are considered. The sign to be attached to each of these expressions is the same, irrespective of the local direction of variation of u with y.

The projection on Ox of the final balance of forces is:

$$-2\pi r\,dr\,\frac{dp_G}{dx}\,dx - 2\pi r\,dx\,k\left(\frac{du}{dy}\right)^n + \frac{d}{dr}\left(-2\pi r\,dx\,k\left(\frac{du}{dy}\right)^n\right)dr$$

$$= -2\pi r\,dr\,\frac{dp_G}{dx}\,dx - \frac{d}{dr}\left(2\pi r\,dx\,k\left(\frac{du}{dy}\right)^n\right)dr \qquad [1.139]$$

2) Let us write Euler's theorem for a volume of fluid limited by:

two cylinders of radii r and $r + d\,r$ and two planes perpendicular to Ozr (therefore vertical) separated by a distance dx.

Each particle of fluid of this element moves with constant velocity (vector quantity defined by direction, orientation and hourly speed). Therefore, its acceleration is null. The momentum entering the reference volume is equal to the momentum leaving it, thereby making the resultant of forces applied to it null.

$$-2\pi r\,dr\frac{dp_G}{dx}dx - \frac{d}{dr}\left(2\pi r\,dx\,k\left(\frac{du}{dy}\right)^n\right)dr = 0 \qquad [1.140]$$

3) After simplification by constant terms and several rearrangements, the sought-for differential equation is finally found:

$$\frac{1}{r}\frac{d}{dr}\left(r\;k\left(\frac{du}{dy}\right)^n\right) = \frac{dp_G}{dx} \qquad [1.141]$$

The solving of this system draws on Poiseuille's theory:

The equation equalizes a term exclusively dependent on r and the one exclusively dependent on z. These terms cannot be equal unless they are constant. Let us denote this constant K. It has been chosen to be positive and it can be predicted that $\frac{dp_G}{dx}$ will be negative:

$$\frac{1}{r}\frac{d}{dr}\left(rk\left(\frac{du}{dy}\right)^n\right) = -K \qquad [1.142]$$

$$\frac{dp_G}{dx} = -K \qquad [1.143]$$

4) Resolving the equation for $u(r)$

4.1) The equation for u is a second-order equation and it calls for the following two boundary conditions:

A no-slip condition in $r = R$: $r = R; u = 0$

The expression of volume flow rate: $Q_v = \int_0^R u(r)\, 2\pi r dr$

Let us find the general form of $u(r)$:

$$\frac{1}{r}\frac{d}{dr}\left(rk\left(\frac{du}{dy}\right)^n \right) = -K \qquad [1.144]$$

$$\frac{d}{dr}\left(rk\left(\frac{du}{dy}\right)^n \right) = -K\,r \qquad [1.145]$$

$$rk\left(\frac{du}{dy}\right)^n = -K\frac{r^2}{2} + C_1 \qquad [1.146]$$

$$\frac{du}{dy} = \left(-\frac{K}{k}\frac{r}{2} + \frac{C_1}{r} \right)^{\frac{1}{n}} \qquad [1.147]$$

Since the derivative should obviously be finite when $r = 0$ (continuity of profile), C_1 is nullified.

$$u = \left(-\frac{K}{2k} \right)^{\frac{1}{n}} \frac{n}{n+1} r^{\frac{n+1}{n}} + C_2 \qquad [1.148]$$

Given that $(-1)^n = -1$, the no-slip condition leads to:

$$C_2 = \left(\frac{K}{2k} \right)^{\frac{1}{n}} \frac{n}{n+1} R^{\frac{n+1}{n}} \qquad [1.149]$$

$u(r)$ is written as:

$$u = \left(-\frac{K}{2k} \right)^{\frac{1}{n}} \frac{n}{n+1} \left(R^{\frac{n+1}{n}} - r^{\frac{n+1}{n}} \right) \qquad [1.150]$$

or in a more canonical form:

$$u=\left(-\frac{K}{2k}\right)^{\frac{1}{n}}\frac{n}{n+1}R^{\frac{n+1}{n}}\left[1-\left(\frac{r}{R}\right)^{\frac{n+1}{n}}\right]=U_0\left[1-\left(\frac{r}{R}\right)^{\frac{n+1}{n}}\right] \tag{1.151}$$

with

$$U_0=\left(-\frac{K}{2k}\right)^{\frac{1}{n}}\frac{n}{n+1}R^{\frac{n+1}{n}} \tag{1.152}$$

where U_0 is the velocity at $r=0,$ that is, velocity on the axis of the tube. It is an extremum (maximum).

We verify that $u(r)$ is indeed a monotone decreasing function of r.

4.2) Let us calculate the flow rate Q_v

$$Q_v=\int_0^R U_0\left[1-\left(\frac{r}{R}\right)^{\frac{n+1}{n}}\right]2\pi r\,dr=2\pi U_0\int_0^R\left[1-\left(\frac{r}{R}\right)^{\frac{n+1}{n}}\right]rdr \tag{1.153}$$

In order to calculate the definite integral

$$\int_0^R\left[1-\left(\frac{r}{R}\right)^{\frac{n+1}{n}}\right]rdr \tag{1.154}$$

we consider $\xi=\frac{r}{R}.$ Then, $r\,dr=\ R^2\,d\,\xi,$ the limits become 0 and 1, and we have:

$$\int_0^R\left[1-\left(\frac{r}{R}\right)^{\frac{n+1}{n}}\right]rdr=R^2\int_0^1\left(1-\xi^{\frac{n+1}{n}}\right)\xi\,d\xi$$

$$=R^2\left(\frac{\xi^2}{2}-\frac{n}{3n+1}\xi^{\frac{3n+1}{n}}\right)\Bigg|_0^1=\frac{n+1}{6n+2}R^2 \tag{1.155}$$

$$Q_v = 2\pi R^2 U_0 \frac{n+1}{6n+2} \qquad [1.156]$$

The velocity V_q is by definition:

$$Q_v = \pi R^2 V_q \qquad [1.157]$$

Therefore:

$$V_q = \frac{n}{6n+2} U_0 \qquad [1.158]$$

4.3) After several calculations, from the expression of Q_v, we find K:

$$K = 2k\left(\frac{3n+1}{\pi n}\right)^n R^{-(3n+1)} Q_V^n \qquad [1.159]$$

which can also be written as:

$$K = 2k\left(\frac{3n+1}{\pi n}\right)^n \frac{V_q^n}{R^{n+1}} \qquad [1.160]$$

5) By definition, the coefficient of head loss per unit length can be written as:

$$J = \frac{d}{dx}\left(\frac{u^2}{2g} + \frac{p_G}{\rho g}\right) = \frac{1}{\rho g}\frac{dp_G}{dx} = \frac{K}{\rho g} \qquad [1.161]$$

since u does not vary with x. By replacing K and U_0 with their values, we get:

$$J = \frac{2k}{\rho g}\left(\frac{3n+1}{n}\right)^n R^{-(n+1)} V_q^n \qquad [1.162]$$

$$J = \left[\frac{8k}{\rho}\left(\frac{3n+1}{n}\right)^n 2^n D^{-(n)} V_q^n\right] \frac{V_q^2}{2g(2R)} \qquad [1.163]$$

$$\psi = \frac{2^{n+3}\,k}{\rho}\left(\frac{3n+1}{n}\right)^{n} D^{-(n)}\,V_q^{n-2} = 2^{n+3}\,k\left(\frac{3n+1}{4n}\right)^{n}\frac{k}{\rho\,D^{n}\,V_q^{2-n}} \qquad [1.164]$$

By introducing the equivalent Reynolds number R_{en}:

$$R_{en} = 8^{1-n}\left(\frac{4n}{3n+1}\right)^{n}\frac{\rho V_q^{2-n}\,D^{n}}{k} \qquad [1.165]$$

we can identify:

$$\psi = \frac{64}{R_{en}} \qquad [1.166]$$

Given the adopted definitions, a relation that is formally identical to the one resulting from Poiseuille’s law can be found.

If $n=1$, the fluid is Newtonian, k is identical to dynamic viscosity μ and we are within the framework of Poiseuille’s theory. We find the classical results:

$$R_{en} = \frac{\rho V_q\,D}{\mu} \qquad [1.167]$$

$$J = \left[\frac{8\mu}{\rho}(4)2DV_q\right]\frac{V_q^2}{2g(2R)} = \psi\frac{V_q^2}{2gD} \qquad [1.168]$$

$$\psi = 2^4\,\mu(1)\frac{\mu}{\rho D V_q} = \frac{64}{R_{en}} \qquad [1.169]$$

EXAMPLE 1.5 (Flow in an annular space).–

We study the laminar flow of an incompressible Newtonian fluid of density ρ and dynamic viscosity μ in the annular space limited by two cylindrical walls of radii R_1 and R_2 $(R_1 < R_2)$. The analysis technique employed here is derived from Poiseuille’s theory.

The flow rate Q_V in the annular space is given.

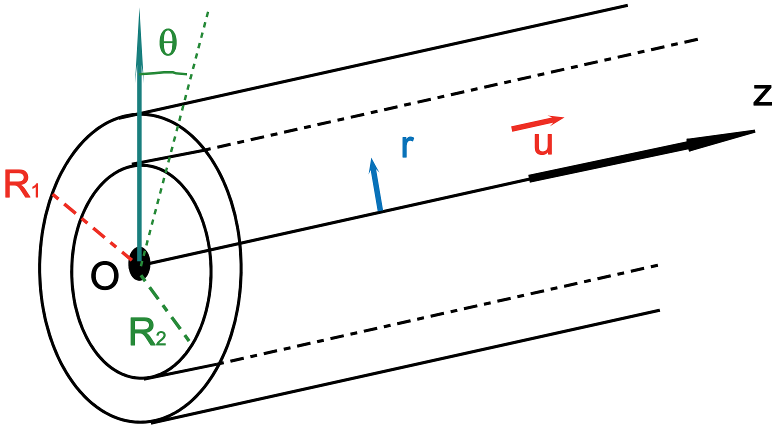

Figure 1.4. *Flow in annular space*

The flow is established: velocity $\vec{V}$ reduces to its axial component u, which depends only on r.

A cylindrical coordinate system (x, r, θ) is used.

To simplify the expressions written, the axis Ox is placed horizontally.

1) Find the differential equation that relates $u(r)$ and the stagnation pressure p_G.

To find it, the law of dynamics is written for a fluid volume limited by two cylinders of radii r and $r+dr$ and two planes perpendicular to Oz (therefore vertical).

2) Writing $u(r)$ as $U_0\ f(r)$, find the function $f(r)$. Find a dimensionless representation of the velocity profile. A constant denoted by K emerges. What does U_0 stand for?

3) Find the expression of $\dfrac{dp_G}{dx}$ as a function of K. How can K be determined? We shall simply indicate the method, without making the calculations.

It is important to note that this method is in fact an implicit application of Euler's theorem (momentum input equals momentum output), and hence an opportunity to understand its significance again.

Solution:

1) This is an established flow. Velocity $\vec{V}$ is reduced to its axial component u .

Let us write the law of dynamics for a fluid volume limited by:

- two cylinders of radii r and $r + dr$,
- two planes perpendicular to Oz (therefore vertical) separated by a distance L

Each particle of fluid of this element moves with constant velocity (vector quantity defined by direction, orientation, hourly speed), therefore its acceleration is null. The acceleration of the fluid volume is null, therefore the resultant of the forces exerted on it is null. Let us project these forces on Ox . Only the surface forces have a non-null projection. These forces consist of viscosity forces and pressure forces. The pressure and gravity forces have the following resultant:

$$-2\pi r\,dr\frac{dp_G}{dx}dx$$

The relation employs p_G, which takes into account the effect of gravity forces on the local pressure. The balance of pressure forces involves opposite surface elements of the same elevation. The result obtained can readily be extended to a tube that is inclined relative to the horizontal. Let us calculate the viscosity forces.

On the inner cylinder of radius r , they are equal to:

$$-2\pi r\,L\frac{du}{dr}$$

On the outer cylinder (radius $r + dr$), they are equal to:

$$2\pi r\,L\frac{du}{dr}+\frac{d}{dr}\left(2\pi r\,L\frac{du}{dr}\right)dr$$

The same sign should be attached to both of these expressions, irrespective of the local orientation of the variation of u with r.

The projection on Ox of the final balance of forces is:

$$-2\pi r\,dr\frac{dp_G}{dx}dx-2\pi r\,L\frac{du}{dr}+\left[2\pi r\,L\frac{du}{dr}+\frac{d}{dr}\left(2\pi r\,L\frac{du}{dr}\right)dr\right]=0 \qquad [1.170]$$

After simplification by constant terms and several rearrangements, the sought-for differential equation is:

$$\frac{\mu}{r}\frac{d}{dr}\left(r\,\frac{du}{dr}\right)=\frac{dp_G}{dx} \qquad [1.171]$$

This system is solved in the classical manner:

The equation equalizes a term exclusively dependent on r with a term exclusively dependent on x. These terms cannot be equal unless they are constant. Let us denote this constant by K. It has been chosen to be positive and it can be predicted that $\frac{dp_G}{dx}$ will be negative.

$$\frac{\mu}{r}\frac{d}{dr}\left(r\,\frac{du}{dr}\right)=-K \qquad [1.172]$$

$$\frac{dp_G}{dx}=-K \qquad [1.173]$$

2) The equation for u is of second order and it calls for two boundary conditions, which will be the no-slip conditions in R_1 and R_2.

$$r=R_1\Rightarrow u=0;\, r=R_2\Rightarrow u=0 \qquad [1.174]$$

Solving this equation draws on Poiseuille's theory:

$$\frac{d}{dr}\left(r\,\frac{du}{dr}\right)=-\frac{K}{\mu}r \qquad [1.175]$$

$$r\,\frac{du}{dr}=-\frac{K}{\mu}\frac{r^2}{2}+C_1 \qquad [1.176]$$

$$\frac{du}{dr}=-\frac{K}{\mu}\frac{r}{2}+\frac{C_1}{r} \qquad [1.177]$$

hence the form of u:

$$u=-\frac{K}{\mu}\frac{r^2}{4}+C_1\,Ln\,r+C_2 \qquad [1.178]$$

The application of boundary conditions yields:

$$-\frac{K}{\mu}\frac{R_1^{\,2}}{4}+C_1\,Ln\,R_1+C_2=0;-\frac{K}{\mu}\frac{R_2^{\,2}}{4}+C_1\,Ln\,R_2+C_2=0 \qquad [1.179]$$

By subtracting the terms of the two equalities, C_1 and C_2 are obtained:

$$C_1\,Ln\frac{R_2}{R_1}-\frac{K}{\mu}\frac{R_2^{\,2}-R_1^{\,2}}{4}=0 \qquad [1.180]$$

$$C_1=\frac{K}{4\mu}\frac{R_2^{\,2}-R_1^{\,2}}{Ln\dfrac{R_2}{R_1}} \qquad [1.181]$$

$$C_2=\frac{K}{4\mu}R_2^2-\frac{K}{4\mu}\frac{R_2^{\,2}-R_1^{\,2}}{Ln\dfrac{R_2}{R_1}}Ln\,R_2=\frac{K}{4\mu}R_1^2-\frac{K}{4\mu}\frac{R_2^{\,2}-R_1^{\,2}}{Ln\dfrac{R_2}{R_1}}Ln\,R_1 \qquad [1.182]$$

$$C_2=\frac{K}{4\mu}\frac{R_2^2+R_1^2}{2}-\frac{K}{4\mu}\frac{R_2^{\,2}-R_1^{\,2}}{Ln\dfrac{R_2}{R_1}}\frac{Ln\,R_2R_1}{2} \qquad [1.183]$$

Or the alternative form:

$$C_2=\frac{K}{4\mu}\left[R_2^2-\frac{R_2^{\,2}-R_1^{\,2}}{Ln\dfrac{R_2}{R_1}}Ln\,R_2\right]=\frac{K}{4\mu}\left[R_1^2-\frac{R_2^{\,2}-R_1^{\,2}}{Ln\dfrac{R_2}{R_1}}Ln\,R_1\right] \qquad [1.184]$$

$$C_2=\frac{K}{4\mu}\left[\frac{R_2^2+R_1^2}{2}-\frac{R_2^{\,2}-R_1^{\,2}}{Ln\dfrac{R_2}{R_1}}\frac{Ln\,R_2R_1}{2}\right] \qquad [1.185]$$

There are three possible expressions for C_2. The first two use the boundary condition in R_1 or in R_2, while the third is the half-sum of the previous two. It is worth noting that since the flow has no point on the axis of the tube, there is no reason for cancelling out the logarithmic term. This will render further calculations more complicated.

3) In Poiseuille's calculation, which treats the flow in a tube, three constants emerge: C_1, C_2 and K. Two pieces of information are used: a boundary condition (at the wall) which yields C_1, a physical remark on the necessity to have a finite flow rate $(C_2 = 0)$ and an "integral" condition, the flow rate value, which yields K. This solving method is quite unusual and may confuse some readers. It is unexpected to see three constants emerging when solving a second-order differential equation. This is just a paradox, which, as always, is a wrongly posed question. In fact, there are two coupled differential equations, a second order one for u and a first order one for p_G. Therefore, three constants are indeed necessary.

In the present problem, there are also three constants: two integration constants, C_1 and C_2, and a third constant, K. Two boundary conditions have been used. Similarly to the "classical" Poiseuille's theory, K will result from the calculation of the volume flow rate Q_V:

$$Q_V = \int_{R_1}^{R_2} u(r) 2\pi r \, dr = 2\pi \int_{R_1}^{R_2} \left(-\frac{K}{\mu}\frac{r^2}{4} + C_1 \, Ln\, r + C_2 \right) r \, dr \qquad [1.186]$$

C_1 and C_2 are known, therefore K can be determined.

1.6. Outer boundary layers: integral methods

1.6.1. *Principle of the integral method*

Integral methods, which probably are not very well known, are an elegant approach to solve certain transfer problems (including heat and mass transfer) rapidly in laminar and turbulent boundary layers.

They save efforts in terms of direct solving of the equations when the searched-for results are limited to the scales of the boundary layer and to wall transfer. As will be seen, it is sufficient to suppose a self-similarity of profiles; the accuracy will depend on how close the considered profile is to the real one. This is how profiles that are close to the one resulted from the Blasius analysis can be reintroduced in a heat transfer problem with high success rate.

Similarly, in turbulent regime, the so-called "power law" profiles will prove particularly useful. These methods will not be presented here, as we focus on dynamic boundary layers. It should be kept in mind that these methods are equally effective in heat and mass transfer problems, including in the presence of chemical reactions in the walls. Moreover, the reader might be surprised by the attention given here to laminar layers, the value of which is rather academic than practical, from a technological perspective. If time is deservedly devoted, solving the proposed exercises will challenge the reader to consider lines of reasoning that he/she is not always accustomed to. The study of boundary layer extensively calls for physicist skills in understanding mechanics and practicing orders of magnitude. The latter aspect is decisive when handling the equations, which may prove impracticable in the absence of reasonable simplification.

Integral methods for an outer boundary layer in Cartesian coordinates:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \qquad [1.187]$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{dp_G}{dx}+\nu\frac{\partial^2 u}{\partial y^2}=U_e\frac{dU_e}{dx}+\nu\frac{\partial^2 u}{\partial y^2} \qquad [1.188]$$

Here, the possibility of a variation of the outer flow velocity[1] $U_e(x)$ is taken into account.

The identity $-\frac{1}{\rho}\frac{dp_G}{dx}=U_e\frac{dU_e}{dx}$ [1.189]

has been used here. Let us recall (see Chapter 1, section 1.3.1 in [LED 17]) that here we have Bernoulli's equation for outer flow, which is considered perfect. Moreover, we are within the framework of boundary layer approximations; the flow being "very close" to a parallel flow, p_G y is constant in any plane perpendicular to the flow lines. Let us integrate the continuity and impulse equations on y between 0 and infinity:

The first one yields:

$$\int_0^\infty \frac{\partial u}{\partial x}dy=-\int_0^\infty \frac{\partial v}{\partial y}dy=-v_\infty \qquad [1.190]$$

---

[1] Integral methods have to a large extent been developed by von Kármán, notably for treating problems of interaction between a real fluid and obstacles. In this case, the potential (outer) flow is eminently variable (see the reminders on this subject in Chapter 3 of [LED 17]).

v_∞ is not null (the flow is "deflected" to the outside, because of the "deficit" of mass flow rate in the boundary layer):

$$\int_0^\infty \left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} \right) dy = \int_0^\infty \nu \frac{\partial^2 u}{\partial y^2} dy = \nu \frac{\partial u}{\partial y}\Big|_0^\infty = -\frac{1}{\rho}\mu\frac{\partial u}{\partial y} + U_e\frac{dU_e}{dx} = \frac{\tau_w}{\rho} + U_e\frac{dU_e}{dx} \qquad [1.191]$$

where τ_W is the shear stress. The friction coefficient C_f is defined as:

$$C_f = \frac{2\tau_w}{\rho U_e^2} \qquad [1.192]$$

$$\int_0^\infty \left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} \right) dy = \nu \frac{\partial u}{\partial y}\Big|_0^\infty = -\frac{1}{\rho}\mu\frac{\partial u}{\partial y} = \frac{\tau_w}{\rho} \qquad [1.193]$$

$$\int_0^\infty \left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} \right) dy = \int_0^\infty \left(\frac{\partial u^2}{\partial x} - u\frac{\partial u}{\partial x} + \frac{\partial uv}{\partial y} - u\frac{\partial v}{\partial y} \right) dy = \int_0^\infty \left(\frac{\partial u^2}{\partial x} + \frac{\partial uv}{\partial y} \right) dy \qquad [1.194]$$

$$\text{As } -u\frac{\partial u}{\partial x} - u\frac{\partial v}{\partial y} = -u\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) = 0 \qquad [1.195]$$

given the continuity equation. Moreover:

$$\int_0^\infty \frac{\partial u v}{\partial y} dy = U_e v_\infty = -U_e \int_0^\infty \frac{\partial u}{\partial x} dy \qquad [1.196]$$

$$\int_0^\infty \left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} \right) dy = = \int_0^\infty \left(\frac{\partial u^2}{\partial x} + \frac{\partial uv}{\partial y} \right) dy = \int_0^\infty \left(\frac{\partial u^2}{\partial x} - U_e\frac{\partial u}{\partial x} \right) dy = \frac{\tau_w}{\rho} \qquad [1.197]$$

$$\int_0^\infty \left(\frac{\partial u^2}{\partial x} - U_e\frac{\partial u}{\partial x} \right) dy = \frac{d}{dx}\int_0^\infty \left(u^2 - U_e u \right) dy = \frac{\tau_w}{\rho} \qquad [1.198]$$

These expressions are dimensionless when divided by U_e^2. Momentum thickness is defined as:

$$\delta_2 = \int_0^\infty \left(1 - \frac{u}{U_e} \right) \frac{u}{U_e} dy \qquad [1.199]$$

and we introduce the friction coefficient $C_f = \dfrac{2\tau_W}{\rho U_e^2}$ [1.200]

This finally leads to an expression given by von Kármán:

$$\frac{d\delta_2}{dx} = \frac{C_f}{2} \qquad [1.201]$$

The integral method is deduced from this expression: given a hypothesis of form for the profile $\dfrac{u}{U_e} = f\left(\dfrac{y}{\delta(x)}\right)$ [1.202]

a more or less approximate value is calculated, depending on the validity of the expression chosen for δ_2 as a function of $\delta(x)$, which, through derivation, leads to calculate the friction coefficient. Let us note that this expression imposes a flow self-similarity. The effectiveness of the method results from the integration over the boundary layer, which minimizes the effects of an error on the profile, compared to a direct calculation of the wall stress at the wall by local derivation at $y = 0$. In turbulent regime, the same method will be applied to the mean velocity $\overline{u}(x, y)$.

1.6.2. *Applications of integral methods*

EXAMPLE 1.6 (Laminar boundary layer of a Newtonian fluid).–

Let us consider the boundary layer developed by a flat plate in the path of wind in a uniform flow of non-Newtonian fluid. The velocity U_e of the outer flow is constant.

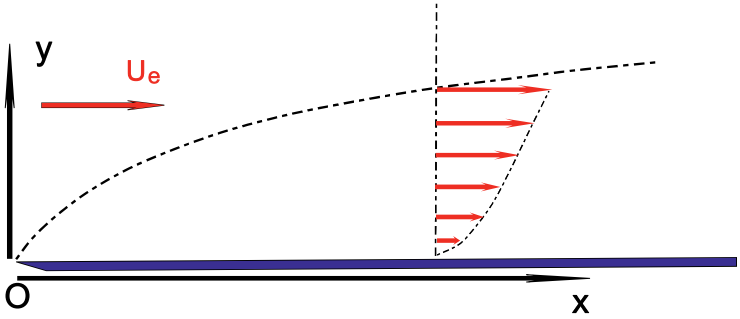

Figure 1.5. *Flat plate in the path of wind*

We shall evaluate the evolution of the thickness of the boundary layer $\delta(x)$ and the friction (through the value of the friction coefficient C_f) using the integral method. Three approximations of the form of velocity profile are proposed. By applying the integral method for the three forms of profile hypothesized below, give in each instance an approximate expression for the evolution of the thickness of the boundary layer, using the ratio $\frac{\delta(x)}{x}$ and the friction coefficient $C_f(x)$. Compare these results with those of the Blasius analysis. Which one of these calculations do you find more satisfactory?

1) The first approximation is the most elementary we could imagine

$$\text{for } y \le \delta; \frac{u}{U_e} = \frac{y}{\delta} \qquad [1.203.a]$$

$$\text{for } y > \delta; \frac{u}{U_e} = 1 \qquad [1.203.b]$$

2) The second approximation relies on a more realistic profile of the boundary layer

$$\text{for } y \le \delta; \frac{u}{U_e} = \frac{3y}{2\delta} - \frac{1}{2}\left(\frac{y}{\delta}\right)^3 \qquad [1.204.a]$$

$$\text{for } y > \delta; \frac{u}{U_e} = 1 \qquad [1.204.b]$$

3) The third approximation, which will prove equally realistic, takes the following form:

$$\text{for } y \le \delta; \frac{u}{U_e} = \sin\left(\frac{\pi}{2}\frac{y}{\delta}\right) \qquad [1.205.a]$$

$$\text{for } y > \delta; \frac{u}{U_e} = 1 \qquad [1.205.b]$$

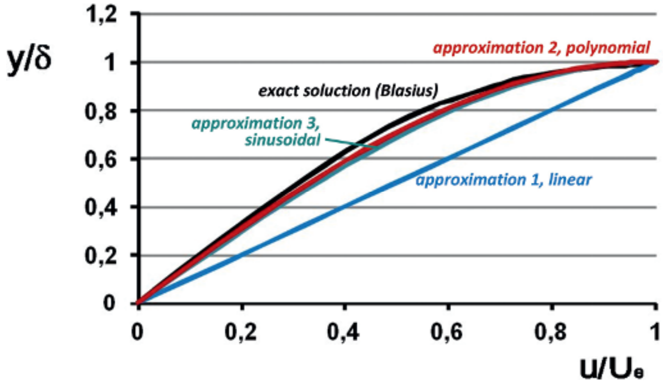

Figure 1.6. *Various approximations of the Blasius profile for boundary layer. For a color version of this figure, see www.iste.co.uk/ledoux/fluid.zip*

Solution:

Let us recall von Kármán equation, which expresses the fundamental law of dynamics in an integral form. In the case of a flow without outer velocity gradient, this equation can be written as:

$$\frac{d\delta_2}{dx}=\frac{C_f}{2} \qquad [1.206]$$

where

$$\delta_2 = \int_0^\infty \left(1-\frac{u}{U_e}\right)\frac{u}{U_e}dy \qquad [1.207]$$

$$C_f = \frac{2\tau_w}{\rho U_e^2} \qquad [1.208]$$

1) Linear approximation

for $y \le \delta;\frac{u}{U_e}=\frac{y}{\delta}$ [1.209.a]

for $y > \delta;\frac{u}{U_e}=1$ [1.209.b]

The definition of momentum thickness δ_2 leads to finding its relation with the boundary layer thickness δ:

$$\delta_2 = \int_0^\infty \left(1 - \frac{u}{U_e}\right)\frac{u}{U_e}\,dy = \int_0^\delta \left(1 - \frac{y}{\delta}\right)\frac{y}{\delta}\,\delta\, d\frac{y}{\delta} \qquad [1.210]$$

Let us make the change of variables $\xi = \frac{y}{\delta}$. We integrate from 0 to 1, because the integrand is null beyond $y = \delta$:

$$\delta_2 = \delta \int_0^1 (1 - \xi)\,\xi\, d\xi \qquad [1.211]$$

$$\delta_2 = \delta \left(\frac{\xi^2}{2} - \frac{\xi^3}{3}\right)\Bigg|_0^1 = \frac{\delta}{6} \qquad [1.212]$$

Moreover, the slope of the velocity profile allows us to write the wall shear stress, and then C_f.

NOTE.– It is at this level that the main source of error in this type of calculation is found.

$$\tau_W = \mu \left.\frac{\partial u}{\partial y}\right|_{y=0} = \mu \left.\frac{\partial}{\partial y} U_e \frac{u}{U_e}\right|_{y=0} = \frac{\mu U_e}{\delta} \qquad [1.213]$$

It is at this level that the main source of error in this type of calculation is found. The derivation depends in effect on the chosen form of profile. It is particularly worth noting that the approximations 2 and 3 are far more adapted from this point of view. In this case, we note that the shear stress is in fact constant in the boundary layer:

$$C_f = \frac{2\tau_w}{\rho U_e^2} \qquad [1.214]$$

$$C_f = \frac{2\mu U_e}{\rho U_e^2\, \delta} = \frac{2\mu}{\rho U_e \delta} \qquad [1.215]$$

Von Kármán equation can then be written as:

$$\frac{1}{6}\frac{d\delta}{dx} = \frac{\mu}{\rho U_e \delta} \qquad [1.216]$$

$$\delta \frac{d\delta}{dx} = \frac{6\mu}{\rho U_e} \qquad [1.217]$$

By introducing the axial Reynolds number $R_x = \frac{U_e x}{\nu} = \frac{\mu U_e x}{\rho}$, and taking into account a boundary layer whose thickness is null at the origin, the differential equation readily yields:

$$\frac{\delta^2}{2} = \frac{6\mu x}{\rho U_e} \qquad [1.218]$$

$$\delta(x) = \sqrt{\frac{12\nu x}{U_e}} \qquad [1.219]$$

$$\frac{\delta(x)}{x} = \frac{3,46}{R_x^{1/2}} \qquad [1.220]$$

C_f can then be written as:

$$C_f = \frac{2\mu}{\rho U_e \delta} = \frac{2\nu}{U_e}\sqrt{\frac{U_e}{12\nu x}} \qquad [1.221]$$

$$\frac{C_f}{2} = \frac{\nu}{U_e}\sqrt{\frac{U_e}{12\nu x}} = \frac{0,289}{R_x^{1/2}} \qquad [1.222]$$

These results are comparable to those of Blasius:

$$\frac{\delta(x)}{x} = \frac{4,95}{R_x^{1/2}};\ \frac{C_f}{2} = \frac{0,332}{R_x^{1/2}} \qquad [1.223]$$

We shall note that the form of $\frac{\delta(x)}{x}$ and $\frac{C_f}{2}$, up to the constants, is found.

This comparison yields an error of 30% on δ and 13% on the friction coefficient C_f. We shall note that this relatively rough approximation yields a result that is already acceptable in the case of friction, which is the most important from a practical standpoint.

2) Polynomial approximation

$$\text{for } y \le \delta;\ \frac{u}{U_e} = \frac{3y}{2\delta} - \frac{1}{2}\left(\frac{y}{\delta}\right)^3 \qquad [1.224.a]$$

$$\text{for } y > \delta;\ \frac{u}{U_e} = 1 \qquad [1.224.b]$$

The same reasoning steps as above are applicable

Let us establish the relation between momentum thickness δ_2 and boundary layer thickness δ.

$$\delta_2 = \int_0^\infty \left(1 - \frac{u}{U_e}\right)\frac{u}{U_e}dy = \int_0^\delta \left(1 - \frac{3y}{2\delta} + \frac{1}{2}\left(\frac{y}{\delta}\right)^3\right)\left[\frac{3y}{2\delta} - \frac{1}{2}\left(\frac{y}{\delta}\right)^3\right]\delta d\frac{y}{\delta} \qquad [1.225]$$

Let us make the change of variables $\xi = \frac{y}{\delta}$. We integrate from 0 to 1, because the integrand is null beyond $y = \delta$:

$$\delta_2 = \delta \int_0^1 \left(1 - \frac{3\xi}{2} + \frac{1}{2}\xi^3\right)\left[\frac{3\xi}{2} - \frac{1}{2}\xi^3\right] d\xi \qquad [1.226]$$

By expanding the product of brackets in the integrand and making a term-by-term integration of the monomials obtained, we have:

$$\delta_2 = \delta \left[-\frac{\xi^7}{28} + \frac{3\xi^5}{10} - \frac{\xi^4}{8} - \frac{9\xi^3}{12} + \frac{3\xi^2}{4} \right]\Bigg|_0^1 \qquad [1.227]$$

$$\delta_2 = 0.139\,\delta \qquad [1.228]$$

Moreover, the slope of the velocity profile allows us to write the wall shear stress, then C_f:

$$\tau_W = \mu \left.\frac{\partial u}{\partial y}\right|_{y=0} = \mu \frac{\partial}{\partial y} U_e \left.\frac{u}{U_e}\right|_{y=0}$$

$$= \mu \frac{\partial}{\partial y} U_e \left.\left(\frac{3y}{2\delta} - \frac{1}{2}\left(\frac{y}{\delta}\right)^3 \right)\right|_{y=0} = \mu U_e \left.\left(\frac{3y}{2\delta} - \frac{3}{2}\frac{y^2}{\delta^3} \right)\right|_0^1 \qquad [1.229]$$

$$\tau_W = \frac{3\mu U_e}{2\delta} \qquad [1.230]$$

$$\frac{C_f}{2} = \frac{\tau_w}{\rho U_e^2} = \frac{3\nu}{2\delta U_e} \qquad [1.231]$$

Then, von Kármán equation can be written as:

$$0.139 \frac{d\delta}{dx} = \frac{3\nu}{2\delta U_e} \qquad [1.232]$$

$$\frac{d\delta^2}{dx} = \frac{21.58\nu}{U_e}; x = 0; \delta = 0 \qquad [1.233]$$

By introducing the axial Reynolds number $R_x = \frac{U_e x}{\nu} = \frac{\mu U_e x}{\rho}$, the differential equation readily yields:

$$\delta^2 = 21{,}58 \frac{\nu x}{U_e} \qquad [1.234]$$

$$\delta(x) = \sqrt{\frac{21{,}58\, \nu x}{U_e}} = 4{,}64\sqrt{\frac{\nu x}{U_e}} \qquad [1.235.a]$$

$$\frac{\delta(x)}{x} = \frac{4{,}64}{R_x^{1/2}} \qquad [1.235.b]$$

Then, the friction coefficient comes down to

$$\frac{C_f}{2} = = \frac{3\nu}{2\delta U_e} = \frac{3\nu}{2*4{,}64\, U_e}\sqrt{\frac{U_e}{\nu x}} \qquad [1.236.a]$$

$$\frac{C_f}{2} = \; = \frac{0{,}322}{R_x^{1/2}} \qquad [1.236.b]$$

This comparison yields an error of 6,3% on δ and 2,9% on the friction coefficient C_f. It can be noted that the form of profile adopted, which is close to the Blasius profile, gives an excellent approximation.

3) Sinusoidal approximation

For $y \le \delta; \frac{u}{U_e} = \sin\left(\frac{\pi y}{2\delta}\right)$ [1.237.a]

For $y > \delta; \frac{u}{U_e} = 1$ [1.237.b]

The same approach as previously discussed is applicable.

Let us explore the relation between momentum thickness δ_2 and border layer thickness δ:

$$\delta_2 = \int_0^\infty \left(1 - \frac{u}{U_e}\right)\frac{u}{U_e} dy = \int_0^\delta \left(1 - \sin\left(\frac{\pi}{2}\frac{y}{\delta}\right)\right)\sin\left(\frac{\pi}{2}\frac{y}{\delta}\right) dy \qquad [1.238]$$

We consider $\xi = \frac{y}{\delta}$ [1.239]

$$\delta_2 = \delta \int_0^1 \left(1 - \sin\left(\frac{\pi}{2}\xi\right)\right)\sin\left(\frac{\pi}{2}\xi\right) d\xi \qquad [1.240]$$

The integration is made from 0 to 1, because the integrand is null beyond $y = \delta$.

Moreover, let us recall that $\cos 2\alpha = \cos^2\alpha - \sin^2\alpha = 2\cos^2\alpha - 1$

$$\delta_2 = \delta \int_0^1 \left(\sin\left(\frac{\pi}{2}\xi\right) - \sin^2\left(\frac{\pi}{2}\xi\right)\right) \, d\xi$$

$$= \delta \int_0^1 \left(\sin\left(\frac{\pi}{2}\xi\right) - \frac{1 - \cos 2\frac{\pi}{2}\xi}{2} \right) d\xi \qquad [1.241]$$

$$\delta_2 = \delta \left[\left. -\frac{2}{\pi}\cos\left(\frac{\pi}{2}\xi\right)\right|_0^1 \left. -\frac{\xi}{2}\right|_0^1 + \left.\frac{\sin \pi\xi}{2\pi}\right|_0^1 \right]$$

$$= \delta\left(\frac{2}{\pi} - \frac{1}{2}\right) + 0 = \frac{4-\pi}{2\pi}\delta = 0.137\,\delta \qquad [1.242]$$

$$\delta_2 = 0{,}137\,\delta \qquad [1.243]$$

The friction is related to the thickness of the boundary layer by:

$$\tau_W = \mu \left.\frac{\partial u}{\partial y}\right|_{y=0} = \mu \frac{\partial}{\partial y} U_e \frac{\pi}{2} \cos\left.\left(\frac{\pi}{2}\xi\right)\right|_{y=0} = \frac{\pi}{2}\frac{\mu U_e}{\delta} \qquad [1.244]$$

$$C_f = \frac{2\tau_w}{\rho U_e^2} = \pi \frac{\nu}{U_e \delta} \qquad [1.245]$$

$$\frac{C_f}{2} == \frac{\pi}{2}\frac{\nu}{U_e \delta} \qquad [1.246]$$

von Kármán equation can then be written as:

$$0.137 \frac{d\delta}{dx} = \frac{\pi}{2}\frac{\nu}{U_e \delta} \qquad [1.247]$$

$$\delta \frac{d\delta}{dx} = 11.47 \frac{\nu}{U_e \delta} \qquad [1.248]$$

$$\frac{d\delta^2}{dx} = 22.93 \frac{\nu}{U_e \delta}; \; x = 0; \; \delta = 0 \qquad [1.249]$$

This simple differential equation associated to its boundary condition yields:

$$\delta^2 = 22{,}93\frac{\nu x}{U_e} \qquad [1.250]$$

$$\delta = 4{,}79\sqrt{\frac{\nu x}{U_e}} \qquad [1.251]$$

or, still further, by introducing the axial Reynolds number $R_x = \frac{U_e\, x}{\nu}$,

$$\frac{\delta}{x} = 4{,}79\sqrt{\frac{\nu x}{U_e}} = \frac{4{,}79}{\sqrt{R_x}} \qquad [1.252]$$

The friction coefficient C_f then results as:

$$\frac{C_f}{2} = \frac{\pi}{2}\frac{\nu}{U_e\,\delta} = 0{,}328\sqrt{\frac{\nu}{U_e\, x}} = \frac{0{,}328}{\sqrt{R_x}} \qquad [1.253]$$

These results are comparable to those of Blasius. The expressions of $\frac{\delta}{x}$ and C_f are quite identical in the form to those of Blasius. By adopting this third profile, the thickness of the boundary layer is underestimated by $3{,}2\,\%$ and the friction coefficient by $1{,}2\,\%$. This validates another very good approximation, which introduces circular functions in the calculation.

EXAMPLE 1.7 (Plate in the path of wind. Friction with injection of wall fluid).–

The objective is to study the boundary layer produced by a far-off uniform flow of velocity U_e and a flat porous plate placed in the path of wind, parallel to the streamlines of potential flow. The fluid is incompressible with density ρ and dynamic viscosity μ. All these physical properties are constant. Fluid is taken out all along the plate with a constant velocity of module W. The flow being plane, a system of axes Oxy (Ox along the plate) will be used as frame of reference. The origin O is chosen on the leading edge. The velocity has two components u and v, directed along Ox and Oy, respectively.

Starting from a certain abscissa x_0, the flow is supposed to be self-similar, meaning that the longitudinal component of the local velocity depends only on the distance to the wall, y: $u = u(y)$. The region near the leading edge $(x < x_0)$ will be called non-similar region. The problem will be solved first by an analytical method, and then by an integral method.

1) Analytical method

Only the similar region of the flow is of interest here.

1.1) Deduce from the expression of continuity the profile of the lateral component v of the velocity in the boundary layer.

1.2) Deduce from it the simple form taken by the impulse equation.

1.3) Find the profiles of velocity and friction.

a) Find the profile of velocity $u(y)$

b) Deduce from it the thickness of the dynamic boundary layer.

c) Write the expressions of the wall shear stress τ_W and the friction coefficient C_f.

2) Integral method

2.1) Unless it has not already been solved, treat question 1.1

2.2) Establish for this problem the relation between δ_2, C_f and W. This relation will be established for the whole boundary layer, including the non-similar region. Let us recall that in this region $u = u(x, y)$; $v = v(x, y)$. The integral method will be used in order to find the friction coefficient C_f in the self-similar region. For this purpose, the previously described analysis used for establishing von Kármán relation will be resumed, taking into account the fact that v is not null at the wall. Moreover, the following considerations are made:

The thickness of the dynamic boundary layer δ is constant in this region.

The profile of $u(y)$ is self-similar:

$$\frac{u}{U_e} = f\left(\frac{y}{\delta}\right) \qquad [1.254]$$

where $f\left(\frac{y}{\delta}\right)$ is a function that needs not be *a priori* given.

Compare the expression found for $C_f\left(x\right)$ to the one obtained at question 1.

Solution:

1) Analytical method

1.1) Profile of $v\left(y\right)$

For a two-dimensional plane and incompressible flow, the continuity equation is written as:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \qquad [1.255]$$

Here, it comes down to:

$$u=u\left(y\right);\frac{\partial u}{\partial x}=0 \qquad [1.256]$$

$$\frac{\partial v}{\partial y}=0 \qquad [1.257]$$

v is constant in the boundary layer, therefore y will have a fixed value at the wall:

$$v=-W \qquad [1.258]$$

1.2) The impulse equation for the two-dimensional plane and incompressible boundary layer is written in a simplified form as:

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\frac{\partial^2 u}{\partial y^2} \qquad [1.259]$$

In the absence of a longitudinal gradient of the outer velocity:

$$-\frac{1}{\rho}\frac{\partial p}{\partial x}=U_e\frac{dU_e}{dx}=0 \qquad [1.260]$$

The final form thus results:

$$0 - W\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} \qquad [1.261]$$

$$y = 0; u = 0 \qquad [1.262]$$

1.3) Finding velocity and friction

a) Velocity profile

The equation in $u(y)$ is readily integrated:

$$0 - W\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} \qquad [1.263]$$

$$\frac{\frac{\partial^2 u}{\partial y^2}}{\frac{\partial u}{\partial y}} = -\frac{W}{\nu} \qquad [1.264]$$

$$Ln\left(\frac{\partial u}{\partial y}\right) = -\frac{W}{\nu}y + Ln\,C_1 \qquad [1.265]$$

Taking the exponential of the two terms:

$$\frac{\partial u}{\partial y} = C_1 \exp -\frac{W}{\nu}y \qquad [1.266]$$

$$u = \int_0^y C_1 \exp -\frac{W}{\nu}dy + C_2 \qquad [1.267]$$

$$u = -\frac{\nu}{W}C_1 \;\exp -\frac{W}{\nu}y\Bigg|_0^y = -\frac{\nu}{W}C_1\left(\exp -\frac{W}{\nu}y - 1\bigg|\right) + C_2 \qquad [1.268]$$

Let us note that we could have also written $u = \int_{C_2}^y C_1 \exp -\frac{W}{\nu}dy$, which obviously leads to the same result. Applying the boundary conditions:

$$y = 0;\; u = 0; C_2 = 0 \qquad [1.269]$$

$$y \to \infty ; u \to U_e ; \; U_e = -\frac{\nu}{W} C_1 \left(0 - 1 \right) + C_2 = \frac{\nu}{W} C_1 ; C_1 = \frac{W U_e}{\nu} \tag{1.270}$$

Finally:

$$u = U_e \left(1 - \exp - \frac{W}{\nu} y \right) \tag{1.271}$$

Let us note that $\frac{W}{\nu} y$ is a Reynolds number.

b) Thickness of the boundary layer

According to the definition of the limit of the boundary layer:

$$y = \delta ; \frac{u}{U_e} = 0{,}99 \tag{1.272}$$

$$1 - \exp - \frac{W}{\nu} \delta = 0{,}99 \tag{1.273}$$

$$\delta = 4{,}605 \frac{\nu}{W} \tag{1.274}$$

Or if the result is expressed in dimensionless form, by means of a Reynolds number:

$$\frac{W \delta}{\nu} = 4{,}605 \tag{1.275}$$

We note that in the self-similar region of the flow, the thickness of the boundary layer is constant, in the same way as the Reynolds number $\frac{W \delta}{\nu}$

c) Friction

The wall shear stress can be calculated directly from the velocity profile:

$$\tau = \mu \frac{du}{dy} = \mu \frac{d}{dy} U_e \left(1 - \exp - \frac{W}{\nu} y \right)_{y=0} = \rho W U_e \tag{1.276}$$

$$C_f = \frac{2\tau_W}{\rho U_e^2} = 2\frac{W}{U_e} \qquad [1.277]$$

2) Integral method

The integral method was presented in 1.6.1 and in the previous examples for a fluid-proof plate. The analysis will be slightly more complicated for the present case. The Von Kármán expression is not directly applicable. This analysis must be resumed.

2.1) Let us recall that in 1.1, it was established that the lateral component of velocity is constant in the boundary layer:

$$v = -W \qquad [1.278]$$

2.2) Integral method for an outer boundary layer in Cartesian coordinates. Let us apply the reasoning of 1.6.1 in the case of the plate with transpiration:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad [1.279]$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} \qquad [1.280]$$

Let us integrate the continuity and impulse equations over y between 0 and infinity:

The first one yields:

$$\int_0^\infty \frac{\partial u}{\partial x}dy = -\int_0^\infty \frac{\partial v}{\partial y}dy = -\left[v_\infty - v(y=0)\right] = W - v_\infty \qquad [1.281]$$

v_∞ is not null (outer flow deflection, due to the deficit of mass rate flow in the limit boundary):

$$\int_0^\infty \left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right)dy = \nu\frac{\partial u}{\partial y}\Big|_0^\infty = -\frac{1}{\rho}\mu\frac{\partial u}{\partial y} = \frac{-\tau_w}{\rho} \qquad [1.282]$$

where τ_w is the wall stress.

The friction coefficient C_f is defined as:

$$C_f = \frac{2\tau_w}{\rho U_e^2} \qquad [1.283]$$

$$\int_0^\infty \left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} \right) dy = \nu \frac{\partial u}{\partial y}\Big|_0^\infty = -\frac{1}{\rho}\mu\frac{\partial u}{\partial y} = \frac{-\tau_w}{\rho} \qquad [1.284]$$

$$\int_0^\infty \left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} \right) dy = \int_0^\infty \left(\frac{\partial u^2}{\partial x} - u\frac{\partial u}{\partial x} + \frac{\partial uv}{\partial y} - u\frac{\partial v}{\partial y} \right) dy = \int_0^\infty \left(\frac{\partial u^2}{\partial x} + \frac{\partial uv}{\partial y} \right) dy \qquad [1.285]$$

since $-u\frac{\partial u}{\partial x} - u\frac{\partial v}{\partial y} = -u\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right)$ is null because of the continuity equation. Moreover:

$$\int_0^\infty \frac{\partial uv}{\partial y} dy = U_e\, v_\infty = U_e \left(W - \int_0^\infty \frac{\partial u}{\partial x} dy \right) \qquad [1.286]$$

$$\int_0^\infty \left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} \right) dy = \int_0^\infty \left(\frac{\partial u^2}{\partial x} + \frac{\partial uv}{\partial y} \right) dy = \int_0^\infty \left(\frac{\partial u^2}{\partial x} - U_e\frac{\partial u}{\partial x} \right) dy + W\,U_e = \frac{\tau_w}{\rho} \qquad [1.287]$$

$$\int_0^\infty \left(\frac{\partial u^2}{\partial x} - U_e\frac{\partial u}{\partial x} \right) dy = \frac{d}{dx}\int_0^\infty \left(u^2 - U_e\, u \right) dy = \frac{\tau_w}{\rho} - W\,U_e \qquad [1.288]$$

These expressions are rendered dimensionless by dividing the two terms of the equation by U_e^2. The momentum thickness is then defined as:

$$\delta_2 = \int_0^\infty \left(1 - \frac{u}{U_e} \right) \frac{u}{U_e} dy \qquad [1.289]$$

and the friction coefficient is introduced:

$$C_f = \frac{2\tau_W}{\rho U_e^2} \qquad [1.290]$$

The form of von Kármán equation specific to our problem is finally obtained:

$$\frac{d\delta_2}{dx} = \frac{C_f}{2} - \frac{W}{U_e} \qquad [1.291]$$

Let us establish the relation between δ and δ_2 by using, as previously, its definition:

$$\delta_2 = \int_0^\infty \left(1 - \frac{u}{U_e}\right)\frac{u}{U_e} dy = \delta \int_0^\infty \left(1 - f\left(\frac{y}{\delta}\right)\right) f\left(\frac{y}{\delta}\right) d\frac{y}{\delta} = F\delta \qquad [1.292]$$

$$F = \int_0^\infty \left(1 - f\left(\frac{y}{\delta}\right)\right) f\left(\frac{y}{\delta}\right) d\frac{y}{\delta} \qquad [1.293]$$

F is a definite integral, the value of which, as we shall see, does not need to be known. In effect, *taking into account the invariability of δ in the self-similar region*, von Kármán law can be written as:

$$\frac{d\delta_2}{dx} = F\frac{d\delta}{dx} = 0 = \frac{C_f}{2} - \frac{W}{U_e} \qquad [1.294]$$

This obviously yields:

$$C_f = \frac{2W}{U_e} \qquad [1.295]$$

We obtain the result that has already been established by the analytical approach.

1.7. Channels and films

In a monophasic fluid, the outer boundary layers lead to distinguish a boundary layer region, in which the viscosity phenomena are predominant, and an outer flow region, which is often treated as a perfect fluid. In a film or channel, a liquid flow is in contact, through one or more of its boundaries, with a solid wall, and through another boundary it is in contact with a gaseous fluid. At the level of the solid, viscosity forces need to be taken into account. At the level of gas, the gas-to-liquid dynamic viscosity ratios lead to consider that friction is negligible at the limit. Moreover, the thickness of the liquid flow can often be considered constant. Examples of this situation will be found in the following.

EXAMPLE 1.8 (The Pont du Gard in Roman times).–

"Indeed, in my opinion the three most magnificent works of Rome, in which the greatness of her empire is best seen, are the aqueducts, the paved roads and the construction of sewers. I say this with respect not only to the usefulness of the work (concerning which I shall speak in the proper place), but also to the magnitude of the cost…." (Dionysius of Halicarnassus)

The renowned Pont du Gard, near Nîmes, in France, is the vestige of an aqueduct that was built during the 1st Century AD, under the Emperor Claudius, with the purpose of bringing water from the "source d'Eure" spring, near Uzès, to the city of Nîmes, at 50 km distance. The altitude Z of the source is 12.5 m higher than that of Nîmes. Let us conceive a model of this structure.

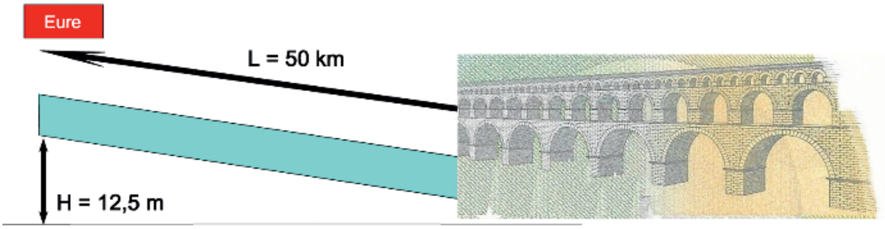

Figure 1.7. *Pont du Gard. Left: diagram of the aqueduct. Right: Pont du Gard, such as it is depicted on the €5 note*

First part. Pont du Gard – an ideal use

According to Figure 1.7, let us suppose that the Nîmes aqueduct can be assimilated to a rectilinear channel of width $l = 1\,m$, between two points whose difference in altitude is $H = 12,5\,m$. The length of the channel is $L = 50\,km$.

The flow in this channel is considered rigorously parallel and of constant depth h. Let viscosity and density of water be $\mu = 1,8.10^{-3}\,Pl$ and $\rho = 1000\,kg.m^{-3}$, respectively. Air will be considered a perfect fluid.

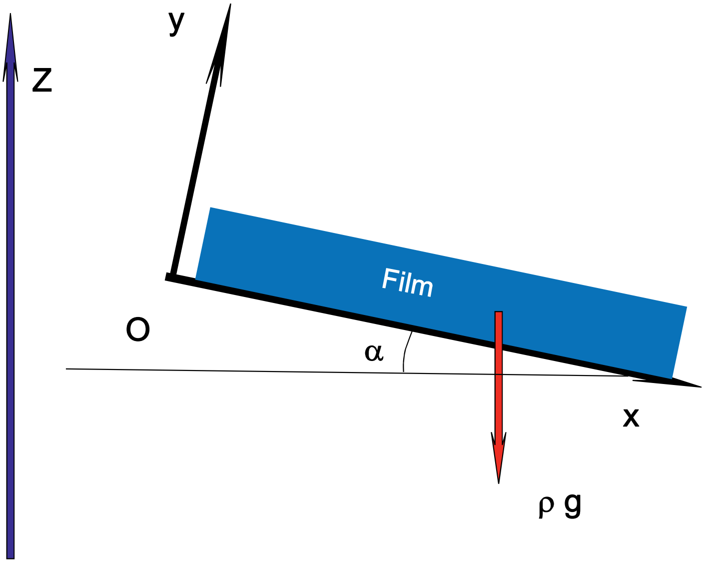

Figure 1.8. *Pont du Gard: the system of coordinates*

1) A frame of reference Ox, Oy is attached to the system, where Ox is located along the bottom of the channel.

1.1) Show that the velocity $\vec{V}(u,v,w)$ comes down to a single component u, which varies only with y.

1.2) Find the form of variation of the stagnation pressure $p_G(x)$.

1.3) Write the differential equation satisfied by $u(y)$.

1.4) Find the expression of $u(y)$.

1.5) Find the expression of the volume rate flow q_v in the channel, in literal form.

1.6) According to ancient texts, the flow rate from the source supplying Nîmes was $36\,000\,m^3.per\ day$. What was the depth h of the water flow?

Solution:

1) The flow satisfies the continuity equation

1.1) For an incompressible fluid, it should be written in two dimensions:

$$div\vec{V} = 0; \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad [1.296]$$

We have considered a parallel film flow. Therefore, velocity reduces to its component parallel to the wall, u, which entails $v = 0$. It follows that

$$\frac{\partial u}{\partial x} + 0 = 0 \quad [1.297]$$

and $u = u(y)$ is a function of only y

1.2) Let us define a vertical axis OZ. This axis should be clearly distinguished from an axis 0z, which is not represented here and which would complete the orthonormal system of reference (Ox, Oy, Oz).

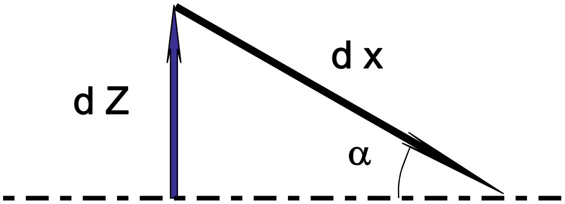

Figure 1.9. *Relation between dz and dx*

Obviously, an angle exists between OZ and Oy. Let us denote by α the angle of the channel relative to the horizontal. It can be readily calculated, knowing that the channel goes down 12.5 m over a length of 50 km:

$$\sin\alpha = \frac{12.5}{50\,000} = 2.5.10^{-4}; \alpha = 1.43.10^{-2}\,° \quad [1.298]$$

Therefore: $dZ = -dx\sin\alpha$ [1.299]

The derivative with respect to x of the stagnation pressure can then be written as:

$$\frac{dp_G}{dx} = \frac{d(p + \rho g Z)}{dx} = \frac{dp}{dx} + \rho g\frac{dZ}{dx} = \frac{dp}{dx} - \rho g\sin\alpha \quad [1.300]$$

1.3) The equation of dynamics, or the impulse equation, for a two-dimensional flow of incompressible Newtonian fluid can be written as:

$$\nu\left[\frac{\partial^2 u}{\partial y^2}\right] + \rho g \sin\alpha = 0 \qquad [1.301]$$

u depends only on y, the partial derivatives can be transformed into full derivatives.

$$\nu\left[\frac{d^2 u}{dy^2}\right] + \rho g \sin\alpha = 0 \qquad [1.302]$$

The volume forces reduced to gravity and relative to a system of reference whose axis $0x$ is at an angle α with the horizontal can be written as:

$$F_{Vx} = \rho g \sin\alpha \qquad [1.303]$$

Taking into account that both $\frac{\partial u}{\partial x}$ and v are null, the equation of dynamics is finally reduced to two equivalent expressions:

$$-\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left[\frac{d^2 u}{dy^2}\right] + \rho g \sin\alpha = 0 \qquad [1.304]$$

or:

$$-\frac{1}{\rho}\frac{\partial p_G}{\partial x} + \nu\left[\frac{d^2 u}{dy^2}\right] = 0 \qquad [1.305]$$

In any plane perpendicular to the flow lines of the film, the stagnation pressure p_G is constant. Therefore, for any abscissa x, it is equal to its value on the free surface. At this point, the static pressure is equal to the atmospheric pressure p_a. We therefore have:

$$\frac{dp_G}{dx} = \frac{dp}{dx} - \rho g \sin\alpha = \frac{dp_a}{dx} - \rho g \sin\alpha = 0 - \rho g \sin\alpha \qquad [1.306]$$

The impulse equation can be written as:

$$\mu\left[\frac{d^2u}{dy^2}\right]+\rho g\sin\alpha=0 \qquad [1.307]$$

The boundary conditions can be written simply as:

a) non-slip condition:

on the bottom of the channel, velocity is null: $y=0; u=0$ [1.308]

b) air is a perfect fluid: the shear stress it applies on the flow surface.

Given the law of action and reaction, the shear stress in water at the free surface is null:

$$y=h;\ \mu\frac{du}{dy}=0 \qquad [1.309]$$

1.4) The second-order linear equation can be readily solved. We proceed by a double integration in y, which will yield the two constants C_1 and C_2 resulted from the boundary conditions:

$$\mu\left[\frac{d^2u}{dy^2}\right]+\rho g\sin\alpha=0 \qquad [1.310]$$

$$\frac{d^2u}{dy^2}=-\frac{g}{\nu}\sin\alpha=0 \qquad [1.311]$$

Let us integrate once:

$$\frac{du}{dy}=-\left(\frac{g}{\nu}\sin\alpha\right)y+C_1 \qquad [1.312]$$

Let us integrate a second time:

$$u=-\left(\frac{g}{\nu}\sin\alpha\right)\frac{y^2}{2}+C_1\,y+C_2 \qquad [1.313]$$

Let us now apply the boundary conditions:

$$y = 0; u = 0 = -\left(\frac{g}{\nu}\sin\alpha\right) * \frac{0^2}{2} + C_1 * 0 + C_2; C_2 = 0 \qquad [1.314]$$

$$y = h;\ \mu\frac{du}{dy} = 0; \frac{du}{dy} = 0 = -\left(\frac{g}{\nu}\sin\alpha\right)h + C_1; C_1 = \left(\frac{g}{\nu}\sin\alpha\right)h \qquad [1.315]$$

Given the expression of u :

$$u(y) = -\left(\frac{g}{\nu}\sin\alpha\right)\frac{y^2}{h} + \left(\frac{gh}{\nu}\sin\alpha\right)y \qquad [1.316]$$

This expression can be reformulated under a dimensionless form, by evidencing a reduced velocity $\tilde{u} = \frac{u}{U_0}$ and a reduced ordinate $\xi = \frac{y}{h}$

$$u(y) = -\left(\frac{g}{\nu}\sin\alpha\right)\frac{y^2}{2}\frac{h^2}{h^2} + \left(\frac{gh}{\nu}\sin\alpha\right)y\frac{h}{h} \qquad [1.317]$$

$$u(y) = U_0\left(-\frac{y^2}{2h^2} + \frac{y}{h}\right) = U_0\left(\xi - \frac{\xi^2}{2}\right) \qquad [1.318]$$

$$U_0 = \frac{g\,h^2}{\nu}\sin\alpha \qquad [1.319]$$

$$\tilde{u}(\xi) = \xi - \frac{\xi^2}{2} \qquad [1.320]$$

1.5) To calculate the volume flow rate q_v in the aqueduct, we shall divide a section into small horizontal strips of width l of the aqueduct and height dy, therefore of surface area $d\,S = l\,dy$, in order to take into account the variation of u with y. It can therefore be written in a classical manner:

$$q_v = \int_0^h u(y)\,dS = \int_0^h u(y)\,l\,dy \qquad [1.321]$$

After a change of variable:

$$\xi = \frac{y}{h};\ d\xi = \frac{dy}{h} \tag{1.322}$$

$$q_v = \int_0^h u(y)\,dS = \int_0^1 u(y)\,lhd\xi \tag{1.323}$$

$$q_v = \int_0^1 u(y)\,lhd\xi = \int_0^1 U_0\left(\xi - \frac{\xi^2}{2}\right)lhd\xi \tag{1.324}$$

$$U_0 = \frac{gh^2}{\nu}\sin\alpha \tag{1.325}$$

$$q_v = lhU_0\int_0^1 \left(\xi - \frac{\xi^2}{2}\right)d\xi = lhU_0 \left.\frac{\xi^2}{2} - \frac{\xi 3}{6}\right|_0^1 \tag{1.326}$$

The final expression of the flow rate is:

$$q_v = \frac{lhU_0}{3} = \frac{glh^3}{3\nu}\sin\alpha \tag{1.327}$$

1.6) Numerical application

It is known that the flow rate of the source supplying Nîmes was 36000 m^3 per day, or $q_v = \frac{36000}{24*3600} = 0,417\,m^3s^{-1}$ [1.328]

Using the numerical values in SI:

$$q_v = \frac{lhU_0}{3} = \frac{glh^3}{3\nu}\sin\alpha \tag{1.329}$$

$$\sin\alpha = 2,5.10^{-4};\ \nu = \frac{1,8.10^{-3}}{1000} = 1.8.10^{-6} \tag{1.330}$$

$$0,417 = \frac{9,81*1*2,5.10^{-4}}{3*1,8.10^{-6}}h^3 \tag{1.331}$$

Therefore, the height of water in the aqueduct results:

$$h = 9,72.10^{-2}\, m = 9,72\, cm \qquad [1.332]$$

Second part. Hackers of the aqueduct

From a purely "academic" perspective, this part of the problem should definitely find its place in Chapter 4. In effect, what follows is a classical flow problem pertaining to the mechanics of perfect fluids (problem related to the one of "the invert"). For the sake of some aestheticism, this example has been related to the previous one, of the film. It is known, particularly from the work "*De Aquaeductu Urbis Romae*" of Frontinus, that despite the formal ban enforced by the Roman emperor and the very harsh repressive measures (among other punishments, those who were caught risked having one hand cutoff), unscrupulous citizens engaged in water diversion.

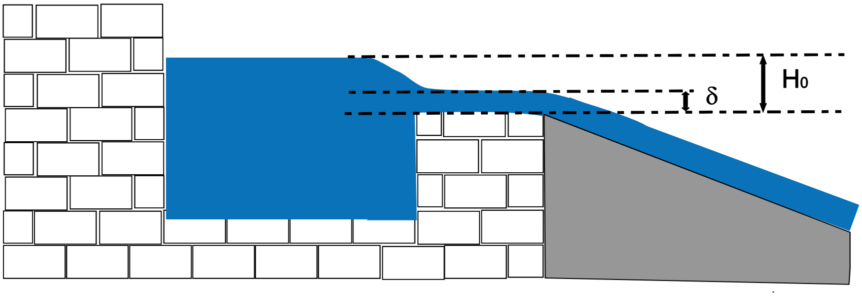

Figure 1.10. *Hacking the aqueduct*

Figure 1.10 represents an instance of water diversion. To estimate the flow rate of the water diversion, we shall study the flow in a particular region where there is a definite flow rate and the fluid can be considered perfect. While legitimate up to an approximation in this case, this hypothesis would obviously not apply to calculating the kinematics of the film. This will not be considered here. It would actually come down to resuming the line of reasoning in part 1 of the example. In this model, the aqueduct is considered a tank whose water height is estimated at $H_0 = 8\, cm$, constant in time (in fact, it is of the order of the height of the film determined in the first part of this example).

NOTE.– We shall verify that water diversion is not done over the whole height of the aqueduct channel.

The water in a tank is considered at rest. Therefore, in this model, the flow in the channel is considered to have no effect on the hydrodynamics of water diversion.

The discharge is considered to lead to the following kinematic configuration: between the “tank” and the “film” flowing into the diversion channel, a rigorously parallel flow is established in a region AB, in which the water depth is δ. At this level, the diversion has a width l_0 (perpendicular to the figure) of 50 cm. The elevations H_0 and δ are illustrated in Figure 1.10.

If δ is not known, it will be subsequently determined. It is simply known that, out of all the values δ arbitrarily considered to take, the only one to be observed is the one leading to the maximum flow rate. In other words, nature automatically “optimizes” the output flow rate.

2.1) Show that the flow is uniform in the AB region. Find the velocity u_f and the volume flow rate q_{vf} of this flow for a hypothetical value of δ.

Then, deduce the observed value of δ.

2.2) Given $H_0 = 8\,cm$, what is the flow rate of the water diversion expressed in m^3 *per day*?

2.3) It is estimated that during the 3rd Century water diversions had reduced the flow rate of water supply to the city of Nîmes to $13\,000 m^3$ *per day*! Making a rough estimation that all water diversions have the same order of magnitude, which we have just calculated, and counting one diversion per family, how many uncivic households were there in the countryside around Nîmes?

Solution:

2.1) Let us attach to the flow in the region AB, which is obviously two-dimensional, a system of axes Ox, Oy. The axis Ox is along the bottom of the channel, and hence it is horizontal in this region. The axis Oy is therefore vertical, and y is an elevation. In this system, H_0 is the elevation of the free surface. In this tank, the fundamental law of hydrostatics is verified:

$$p + \rho g y = Cst = p_a + \rho g H_0 \qquad [1.333]$$

NOTE.– Let us recall that in this "modeling", we forget that the "tank" is in fact a "flow". We shall accept the hypothesis of the tank, which serves the purpose of simplification. On each flow line in the AB flow coming out of the tank, between a point C of the tank and a point D of the flow, the following can be written as:

$$p_C + \rho g y_C + 0 = p_D + \rho g y_D + \rho \frac{u^2(y_D)}{2} \qquad [1.334]$$

The stagnation pressure $p_G = p + \rho g y$ is evidenced, constant in any plane perpendicular to Ox. For any plane perpendicular to Ox, we have:

$$p_G = p + \rho g y = p_a + \rho g \delta \qquad [1.335]$$

Given that the film on AB has a constant thickness δ, the stagnant pressure is constant throughout the flow. A synthesis of what has just been stated gives:

$$p_C + \rho g y_C + 0 = p_a + \rho g H_0 = p_{GD} + \rho \frac{u^2(y_D)}{2} = p_a + \rho g \delta + \rho \frac{u^2(y_D)}{2} \qquad [1.336]$$

$$\rho \frac{u^2(y_D)}{2} = \rho g (H_0 - \delta) \qquad [1.337]$$

$$u(y_D) = \sqrt{2g(H_0 - \delta)} \qquad [1.338]$$

The velocity is therefore constant throughout the AB flow region. This result could have been intuitively predicted by noting that the flow has constant energy. The flow rate $q_{vf}(\delta)$ calculated for the as yet unknown value of δ can be readily deduced as:

$$q_{vf}(\delta) = Su = l_0 \delta \sqrt{2g(H_0 - \delta)} \qquad [1.339]$$

The observed value will be the maximum of this function of δ. For a more convenient calculation, we shall note that this will also be the extremum of the function $\delta^2 (H_0 - \delta)$:

$$\frac{d}{d\delta} \delta^2 (H_0 - \delta) = 2\delta (H_0 - \delta) - \delta^2 = \delta(2H_0 - 3\delta) \qquad [1.340]$$

The derivative is annulled for $\delta = 0$, which corresponds to a null flow rate, hence the extremum is a minimum obtained for $\delta = \frac{2H_0}{3}$ [1.341]

which is the sought-for solution.

2.2) The flow rate of the diversion will consequently be:

$$q_{vf}\left(\delta\right) = Su = l_0 \frac{2H_0}{3}\sqrt{2g\left(H_0 - \frac{2H_0}{3}\right)} \qquad [1.342]$$

$$q_{vf}\left(\delta\right) = l_0 \frac{2H_0}{3}\sqrt{2g\left(H_0 - \frac{2H_0}{3}\right)} = \sqrt{g}\, l_0 \left(\frac{2H_0}{3}\right)^{\frac{3}{2}} \qquad [1.343]$$

The "unit" flow rate of the diversion will hence be:

$$q_{vf}\left(\delta\right) = = \sqrt{g}\, l_0 \left(\frac{2H_0}{3}\right)^{\frac{3}{2}} = 1{,}93.10^{-2}\, m^3 s^{-1} = 1666\, m^3.day^{-1} \qquad [1.344]$$

2.3) The illegal diversions lead to a daily loss of water of $36\,000 - 13\,000 = 23\,000\, m^3.day^{-1}$. Their number can therefore be estimated at:

$$n = \frac{23000}{1666} = 138 \qquad [1.345]$$

This is practically the equivalent of three diversions per kilometer!

The solution brought by the two previous examples can be generalized to many practical problems. It is particularly worth noting the extension of the channel problem to the film problem, a thin flow on the wall, which is distinct from both the boundary layer of a flat plate because of the fluid discontinuity, and from the jet because of the constant thickness of the liquid layer in the flow.

In order to show how the above can be used, an additional example will be provided. The approach of this example extracted from daily life accidents is perfectly applicable to the industrial field, notably to chemical engineering.

EXAMPLE 1.9 (A household incident).–

It is recommended to first solve the previous example. While painting a staircase, a do-it-yourselfer spills the paint pot. The result is that a vertical film of liquid flows down the riser. The paint will be assimilated to a Newtonian fluid of viscosity $\mu = 1{,}8.10^{-3}\,Pl$ and density $\rho = 1000\,kg.m^{-3}$. The flow rate of the film is $q_v = 8{,}10^{-6}\,m^3.s^{-1}$. Its width is $l = 5\,cm$. What is the thickness δ, considered constant, of the falling film?

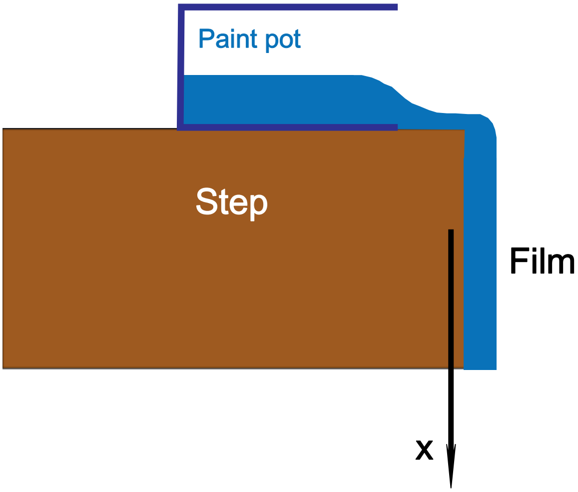

Figure 1.11. *A household incident*

Solution:

The results obtained in part 1 of the example entitled “Pont du Gard” can be readily applied here, noting that the angle $\alpha = \dfrac{\pi}{2}$. The equation will be written in a rigorously similar manner, and will yield the same result, with $\sin\alpha = 1$. The role of h is assumed by δ.

Hence:

$$q_v = \frac{gl\delta^3}{3\nu} \qquad [1.346]$$

$$\delta^3 = \frac{3 * 1{,}8.10^{-6} * 8.10^{-4}}{9{,}81 * 5.10^{-2}} \qquad [1.347]$$

$$\delta = 2,065.10^{-3}\, m = 2,06\, mm \qquad [1.348]$$

This analysis is applicable in industrial settings, notably in the field of chemical engineering, in which the theory of the falling film is developed. In this context, the film facilitates the heat and mass transfers. In practice, this film is often located on an inner cylindrical surface of vertical axis. The previous result is still applicable, provided the thickness of the film remains lower than the radius of the tube.