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Irrational Quantities

In traditional philosophy, what is the fundamental philosophical operation? It is that which consists of constructing a miniature image of the world, as complete as possible, and whose aim is to capture the essential of the real world. The benefit of this approach is patently obvious: simplify to understand better. That is, taking everything together, we reduce so that we can retain more. We are, moreover, pushed to carry out such a task, which proves itself to be highly useful, for obvious reasons:

– there is, among other things, a *vital necessity* to know the relative importance of each thing, our own situation in the world as well as the place that is ours;

– let us also note that such a project is *democratic*;

– finally, everyone has the right to know who they are and where they are: the very procedure that allows this, in accordance with its objective, could only have emerged in a context that was conducive to its appearance (Ancient Greece).

In any event, this consists of factorizing all that is perceived into a certain number of classes and then, for each group being considered, to choose one or more distinct representatives[1]. The dimensional reduction, if it is correct, then becomes heuristic and leads to an undeformed model of the real world. But there are many ways of operating this concentration and, contrary to the old adage, in philosophy, alas, *less*, is *not* always more.

What then must be prioritized and preserved from the range of phenomena? From this colorful variety, these often garish shades of existence that were made up, in the time of the Ancient Greeks, by people, places and even the monuments around them – from all this, the earliest philosophers, the disciples of Pythagoras, only wished to retain the number.

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[1] Such a model has been described in detail in a publication [PAR 93a].

For them, the number was an able replacement for any thing as, with things and numbers in the same stratification, it was easy to substitute one for the other. Numbers themselves could be contained within the first four (the Mystic Tetrad) and, consequently, the entire universe can thus be contained within the beginning of the numerical series (1 + 2 + 3 + 4 = 10)[2]. How economical! Not only is the world condensed into symbols, but all we need is 4 of these to cover the full spectrum. To these thinkers, who were still naive, the alpha and omega in the real world were nothing but numbers and relations (*logoi*) between numbers. Reason itself, which is nothing but science being exercised, identified with these relations. It also bears the same name (*logos*). Reason, therefore, is essentially *proportion*. At the time, reason fell within the limits of the Pythagorean theory of medieties[3]. And, perhaps the thing that is most difficult for us 21st Century people to understand, *it is nothing else*.

1.1. The appearance of irrationals or the end of the Pythagorean dream

Naturally, this kind of a perspective, rather too radical, would not be tenable in the long run. A simple mathematical problem, the doubling of a square[4], brought up the first irrational, which we have named "square root of two" ($\sqrt{2}$) and which, geometrically, can be identified with the diagonal of the square on which is constructed the square that is the double of a square with side one. As we have seen, the Pythagoreans named such quantities *a-loga*, that is, strictly speaking, "without ratio" or "incommensurables". It is easy to demonstrate why $\sqrt{2}$ – or, as the Greeks really called it "the number whose product with itself gives 2" (as they did not know the expression ($\sqrt{a}$)) – is not rational. Any rational must be of the form p/q, an irreducible fraction. But the property of an irreducible fraction is that its numerator and denominator cannot both be even (if they were, we could of course further reduce this by dividing it by 2). Let us thus posit $\sqrt{2} = p/q$ is irreducible. We then have $p^2 = 2q^2$, which signifies that p^2 is even, and therefore, p is even. Let us then posit that $p = 2n$ and substitute this value in the equation. We obtain $4n^2 = 2q^2$, that is $q^2 = 2n^2$, thus q^2 is even, which signifies that q is even. We thus have a contradiction and $\sqrt{2}$ is not rational.

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[2] This, according to M. Ghyka [GHY 31], is what the Pythagoreans called "Tetractys".

[3] This theory is best known because of the texts of Archytas, Nicomachus of Gerasa and Theon of Smyrna (see [MIC 50]).

[4] The question of the appearance of incommensurables would provoke polemical debates among science historians. We have no precise trace for the discovery of irrationals in Ancient Greek – only the accounts of commentators (Pappus, Proclus, Iamblichus, etc.), who wrote their accounts close to 700 years after the facts they were reporting. Pappus certainly traces this discovery to the Pythagorean sect, relating it to the question of the diagonal of the square, and attributes it to Hyppasius. Proclus, however, attributes it to Pythagoras himself. As for Iamblichus, he considers that rather than the doubling of a square, this discovery of irrationals arose from the problem of dividing a segment into extreme and mean ratios, that is the golden number.

1.2. The first philosophical impact

We find many echoes of this discovery in Greek philosophy, especially in Platonic thought. In his weighty tome on Mathematical Philosophy [BRU 93], Léon Brunschvicg included the following observation:

"In Plato's Dialogues, there is more than one hint that the discovery of irrationals is not alien to the Platonic doctrine of science. In the introduction to *Thaetetus*, the dialogue that would mark the first degrees of analysis that went from perceptible appearance to truth, Plato recalled the writings of his tutor, Theodore[5], who established the irrationality of $\sqrt{5}$, $\sqrt{7}$, etc. and pursued the search for irrational square roots up to $\sqrt{17}$ [6]. In book VII of *The Laws*, he deplores, as a crime against the nation, that young Greeks were left ignorant (as he was left ignorant) of the distinction between commensurable quantities themselves and incommensurable quantities[7], a distinction that he used as the basis for the 'humanities'. Above all, the example of *Meno* must be highlighted: the problem, one of the simplest of those that could arise after the discovery of incommensurability, consists of determining the length of the side of a square that would be double that of another square with a surface of four feet. What is significant is the objective of this example: it was to prove the Reminiscence Theory of Knowledge. The Platonic Socrates introduces a slave who, it was claimed, without any direct learning, and using solely the effect of natural light which revealed itself, could find the veritable solution to the problem[8]. The first responses of the slave were borrowed from the framework of pure arithmetic: the square with double the area seems to have a side with double the length. But, double the length would be 4 and thus the doubled area would be 16. The side of the square would, thus, be greater than 2 and smaller than 4, that is, 3. But this response, which exhausts the truly numerical imagination, is still inexact: the square with a side of three feet would have an area of 9 feet. Socrates, thus, proposes an exclusively geometric reflection.

"Let the square be ABCD (Figure 1.1)[9]; we can juxtapose this with three equal squares so as to obtain the quadruple area AEGF. Taking the diagonals BC, CI, IH and HB, we divide into two each of these four areas, equal to a primitive square. The square BCIH is, therefore, double the primitive square; the side whose length would

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[5] Theodore of Cyrene, a mathematician who, according to Diogenes Laërtius (III, 6), taught Plato mathematics [note by D. Parrochia].

[6] *Thaeteus*, 147d. See the study by [ZEU 10, p. 395 onwards].

[7] Plato, *The Laws*, 820c.

[8] Plato, *Meno*, 82b.

[9] Here, L. Brunschvicg goes back to [CAN 07, p. 217] [note by D. Parrochia].

be equal to $\sqrt{8}$ is the line that the Sophists call the diameter: it is from the diameter, thus, that the doubled area is formed"[10].

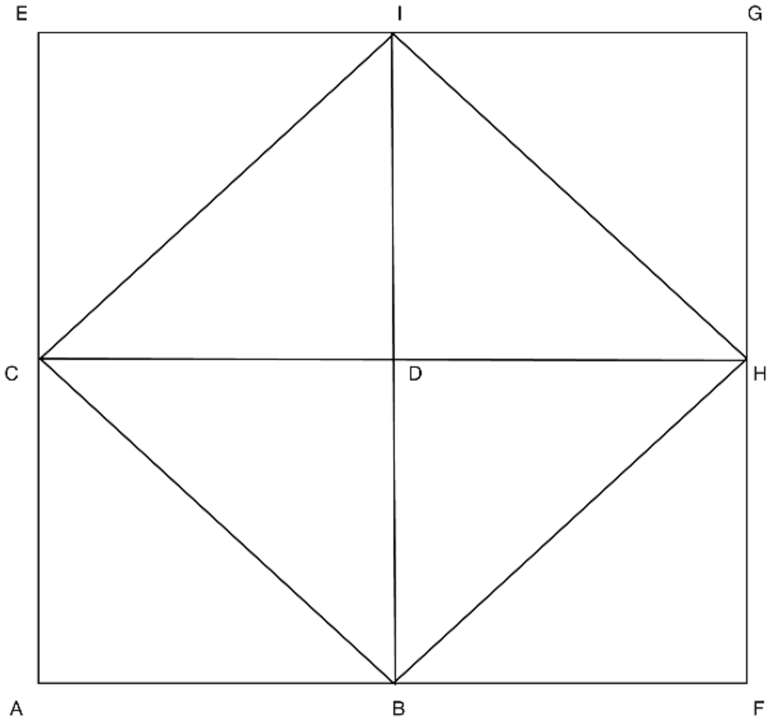

Figure 1.1. *Meno's square*

Plato's theory of science and, in particular, the idea of reminiscence are thus not anchored in mythology, as we believe only too often (appealing to the Myths is, as always, simply a pedagogical or psychagogic tactic Plato uses to express himself) but instead it is anchored in a rationality that is fundamental to the human mind; in the cognitive abilities of the mind which are expressed here, precisely, in the fundamental movement which, at the time of this "crisis" or "near-crisis" of the irrationals[11], quite suddenly saw the growth of an extension to the concept of the number.

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[10] *Meno, 85b.*

[11] The notion of "the irrational crisis" is contested today. Historians of mathematics tend to think that the authors of the 19th Century, who described this period in history, overstated the importance of the "trauma", being influenced themselves by the "set theory crisis" that they were living through. However, we can observe that the discovery of irrationals threw the Greek world into disarray, not so much within mathematics as in the external world, where the

1.3. Consequences of the discovery of irrationals

The Pythagorean discovery led to several important philosophical consequences.

1.3.1. *The end of the eternal return*

The end of “everything is numbers” was not simply the rejection of the Pythagorean hypothesis according to which only whole and rational numbers existed. This idea had quite concrete consequences. It brought about a general modification of the cosmological representation of the world, especially the representation of time. Indeed, as Charles Mugler once wrote, it brought about a veritable “cosmic drama”. It marked the collapse of the Pythagorean concept of circular time where the revolutions of different heavenly bodies, assumed to be expressed only in whole numbers, would give rise to the calculation of a lowest common multiple (LCM). They thus led to the Pythagorean concept of the Grand Year, at the end of which period it was assumed that the heavenly bodies had returned to their initial position and that life on earth, which depended on them, would recommence completely.[12] However, the end of this periodic cosmology and the presence of possible disorder in the celestial world were not the only consequences of the appearance of irrationals.[13]

1.3.2. *Abandoning the golden ratio*

Plato had already noted in classical Greek architecture that there were some distortions between the apparent and real proportions in certain monuments, even reproaching architects for having used falsehoods to get people the truth. The problem only worsened, as the Greek aesthetic shifted over time not only to an excess of refinement and mannerism but toward a renouncement of reserve, sobriety and equilibrium in favor of expressing a certain dramatic tension, a certain *pathos* or *hubris* of despair as in the famous group of Laocoon. This is a sculpture that presents people in a state of agony, muscles taut and bulging eyes with despair in their eyes, as they are defeated by serpents. With this sculpture, which dates from after the 2nd Century B.C., the Apollonian order was overthrown by much more troubling and tormented representations, which would, one day in the future, attract the German

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consequences of this discovery would force aesthetics, morality and medicine to change their view of the world, which was until then founded on the theory of proportions, that is, rational numbers alone.

[12] For more on this, see the preface by C. Mugler [MUG 69].

[13] Today, after the work of Kolmogorov (1954), we know, on the contrary, that the irrationality of the ratio $\alpha = T2/T1$ of two periods $T1$ and $T2$ of two different celestial bodies of the solar system increases its stability. If α is diophantian, the stability is still much better. This sounds the end of the Pythagorean harmony.

Romantics[14]. This is nothing but another consequence of the collapse of the existing order brought about by the undeniable existence of the irrational in mathematics.

1.3.3. *The problem of disorder in medicine, morals and politics*

As we know, Greek wisdom also recommended "nothing in excess". Stobaeus, before Plato, already judged that there must be proportion in the soul and in the city: "Once the rational count is found, revolts will die down and amity increase" (*Flor*. IV, I, 139). Alcmaeon, although he certainly lacked the means of experimentally taking measurements, tried to prove the equality of forces in the body [VOI 06, pp. 11–78]. According to a fragment gathered by Diels-Kranz, "Alcmaeon said that what maintains health is the balancing of forces (*tèn isônomian tôn dunaméôn*), humid, dry, cold, heat, bitterness, sweet and others, the domination of any one of them (*en autois monarchian*) causing disease; this was because the domination of a single is corrupting... health is the combination of qualities in the correct proportion (*tèn summetron tôn oiôn krasin*)"[15].

With the appearance of the irrationals, the end of Greek life was, in the long run, programmed. Nothing would be in proportion anymore: not in the cosmos, nor in the human soul, nor in the city. The potential presence of a destabilizing element, introducing the incommensurable (irrational movement, unrestrained passion, tyranny, etc.) would, every time, threaten to bring about a rapid downfall.

It was, therefore, useful to put up the defenses and fight back – starting with the field of mathematics itself.

1.4. Possible solutions

How does one rid oneself of these irrationals? Squaring them, that is, turning *aloga* into *dunamei monon rêta*, is an easy solution. However, this will, obviously, change the value of the numbers. To remain faithful to the data given in the problem, we will

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[14] The Laocoon group strongly inspired a movement in German art and authors as diverse as Winckelmann, Lessing, Herder, Goethe, Novalis and Schopenhauer covered this in their commentaries. In Winckelmann's classic expression, which again inspired Lessing, Herder and Goethe, the Laocoon group illustrated the concept of the strength of the soul and the aesthetic rule of tempered expressions. Contrary to this, however, we have Novalis. In line with Heinse and Moritz who saw this instead as "the most violent horror and the strongest emotion", Novalis saw here the very sign of an aesthetics of excess (see [OST 03]). Much later, Spengler would see in this sculpture the decline of Greek society and the end of the art and values of classical antiquity.

[15] Fragment DK 24 B 4.

try to find approximation formulas that would make it possible to turn the irrationals into rationals.

One of the procedures, while it may not have been entirely known to the Pythagoreans, was, nonetheless, anticipated by them. This was the procedure of continuous fractions, which seems to have been explicitly introduced by the Hindu mathematician Aryabhata (550–476 AD). This is written (using the modern forms) as:

$$\sqrt{2} = 1 + \frac{1}{2 + \frac{1}{2+\frac{1}{2+\ldots}}}$$

Today, we can easily obtain this formula in the following manner. Because $1 < \sqrt{2} < 2$, we first posit, as the first approximation:

$$\sqrt{2} = 1 + \frac{1}{a}, \quad \text{with } a \neq 0 \qquad [1.1]$$

From this, we then find the value for a. Hence:

$$a = \frac{1}{\sqrt{2} - 1} = \sqrt{2} + 1 \qquad [1.2]$$

However, since:

$$\sqrt{2} = 1 + \frac{1}{a}$$

upon replacing $\sqrt{2}$ in expression [1.2] by its value, we also immediately have:

$$a = 2 + \frac{1}{a} \qquad [1.3]$$

By then substituting this value for a in expression [1.1] and then in the successive expressions, wherever a appears, we obtain the desired formula.

It appears that we cannot find an explicit trace for continuous fractions in Greek mathematics earlier than the work of Aristarchus (3rd Century B.C.) and Heron (1st Century A.D.). However, Paul-Henri Michel was able to suggest that the procedures to dimidiate unity and other approximations that Thomas L. Heath was able to report[16] contributed to anticipating them.

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[16] “Not only did the Pythagoreans discover the irrationality of $\sqrt{2}$; they demonstrated, as we have seen, how to approach, as closely as we wish, their numerical value” (see [HEA 21, p. 167]).

Jules Vuillemin, following in their path, noted more recently that “the Pythagoreans made use of infinite sets in their polygonal number tables and in the definitions of progressions” [VUI 01, p. 11]. Reflecting on the use of these procedures to demonstrate the irrationality of $\sqrt{2}$, especially in the algorithms called “Theon’s algorithms” and “alternate division”, he observed that we could thus easily deduce the laws of continuous fractions from these [VUI 01, p. 71]. The Pythagorean use of triangular number tables also meant that it was not necessary to explicitly know these algorithms, but to know them, “only through certain properties of their approximation” [VUI 01, p. 71].

Despite their lack of resolution, according to Vuillemin himself, these procedures seem to have served as models for the Platonic method of division. In his own words, again, “while logical rigor is lacking in this initial recourse to finite sets, and while these difficulties inherent to continuous fractions also affect their rudiments, let us remember that the chief obstacle Greek mathematics came up against is the idea of the real number and we will see Theodore conceiving of roots of natural, non-squared whole numbers as the limits of the infinite series of rational approximationx [VUI 01, p. 106].

1.5. A famous example: the golden number

Among the ratios that the Pythagoreans loved, one could pass for a clever compromise: this was a ratio and, at the same time, corresponded to an irrational quantity. This ratio is defined in the following manner. We posit:

$$\frac{a+b}{a} = \frac{a}{b}$$

But this is equal to:

$$1 + \frac{b}{a} = \frac{a}{b} \iff \frac{a}{b} + 1 = (\frac{a}{b})^2 \iff (\frac{a}{b})^2 - \frac{a}{b} - 1 = 0 \qquad [1.4]$$

We then posit:

$$k = \frac{a}{b}$$

And equation [1.4] becomes:

$$k^2 - k - 1 = 0$$

One of these two solutions (the positive solution) to this new equation (which would, moreover, find many applications in the field of aesthetics, notably architecture) has been the subject of reams of writing over history[17]. The solutions are classically obtained as follows. The discriminant of the equation is:

$$\Delta = b^2 - 4ac = 5$$

which gives, as the roots,

$$k' = \frac{1+\sqrt{5}}{2}, \qquad k'' = \frac{1-\sqrt{5}}{2}$$

k' is conventionally called the "golden number" or the "golden section". We ordinarily designate this by the letter ϕ.

We then observe that $k'.k'' = -1$, and that $k' + k'' = 1$. From this, we then can write:

$$-k'' = \frac{1}{k'} = \frac{1}{1-k''} = \frac{1}{1+\frac{1}{k'}}$$

Hence, the rational approximation of the "golden number":

$$\phi = \frac{1+\sqrt{5}}{2} = 1 + \frac{1}{1+\frac{1}{1+\frac{1}{1+\dots}}}$$

By developing the successive approximations of ϕ, we then have the set of ratios formed by the numbers belonging to a famous mathematical series[18]:

$$\frac{1}{1}, \frac{2}{1}, \frac{3}{2}, \frac{5}{3}, \frac{8}{5}, \frac{13}{8}, \frac{21}{13}, \frac{34}{21}, \frac{55}{34}, \frac{89}{55}, \quad \text{etc.}$$

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[17] It must be noted that since the work of A. Zeising [ZEI 54], in the mid-1800s, and of M. Ghyka [GHY 31], in the first half of the 20th Century, the golden number has met with much success. While this study was able to stimulate some research (see [CLE 09, pp. 121–123]), it must be admitted that a certain number of errors were also propagated through it and that the omnipresence attributed to it in nature or art was often mythical (see Neveux and Huntley's critical study [NEV 95]). Generally speaking, the work on this subject – which most often repeats itself – flourishes. We can cite, among many others, Herz-Fischler [HER 98]. The mystical aspect of the number seems to have been emerged in the 19th Century with the translation of Pacioli [PAC 80]. The expression "golden section" does not seem to date back further than the 19th Century and Martin Ohm's work.

[18] The series of numbers 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, etc., discovered by the Italian mathematician Leonardo Fibonacci in the 13th Century, is characterized by the fact that each number, starting from two, is the sum of the two previous numbers in the series. This series has its origins in a question that Fibonacci asked in his *Liber Abaci* in 1202: "How many pairs of rabbits can be produced in a year from a single pair if each pair produces a new pair every month, each new pair reproducing from the second month?" The first numbers in the series respond to this question. The formal definition of the Fibonacci series, given by Kepler in the 15th Century, is:

$$F(1) = F(2) = 1 \qquad F(n) = F(n-1) + F(n-2)$$

Knowing that the above ratios may be denoted in a generic manner by $\frac{F(n+1)}{F(n)}$, we can demonstrate that:

$$\lim_{n\to\infty} \frac{F(n+1)}{F(n)} = \phi$$[19]

1.6. Plato and the dichotomic processes

The processed called the "dimidiation" of unity, a characteristic of the golden number, finds an obvious parallel in language with the *dichotomic processes*, which Plato would use increasingly often in his dialogues. In fact, Plato compared ideas to numbers and a wrong calculation to an identification error (*Thaetetus*, 199c). And like the Pythagoreans, who saw a simple correspondence between numbers and things, Plato believed for a long time that it was possible to establish a simple correspondence between the intelligible and the tangible, ideas and their referents in the world. This belief, however, threw up many difficulties, as can be seen in the conversation between the young Socrates and Parmenides in the eponymous dialogue.

The *Parmenides*, and then the *Sophist*, acknowledged this failure and the later dialogues raised other objections. Thus, the *Philebus* showed that certain ideas, such as those of pleasure, which had multiple variables and that could only be attained through excess or failure, could be difficult to identify with intangible, full or whole realities. Irrational numbers are like this: they cannot be used in clearly defined relations and are not commensurable.

To approach them, we must have a process analogous to the dimidiation of unity. The processes that play this kind of a role are dichotomic processes (or procedures for binary divisions in language). Thanks to such "algorithms", the definitions of concepts, which are no longer considered in a granular manner like whole numbers, but rather as a mixture of the limited and unlimited that is, like irrationals, may be approached through successive divisions that converge on a final, stable element. These procedures, which originated in the *Gorgias* (450a), and had become widely used in dialogues that followed the *Parmenides*, are the exact transposition of continuous fractions or, as we have said, algorithms that anticipated these[20].

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[19] Botanists consider that the ratios given express the most common phyllotaxis. Furthermore, a mathematical study has also made it possible to suggest that there may be some relation between the Fibonacci numbers and the logarithmic spiral – a curve that is found in nature [JEA 78, p. 75; ARC 94].

[20] The posterity of continuous fractions is remarkable. They were generalized and applied to the approximate calculation of π and e and were recently used in the theory of dynamic systems (see [YOC 06, pp. 403–437]). A continuous fraction's formula, discovered by Ramanujan and that he proposed to Hardy in 1913, even introduces a relation between e,π and ϕ.

1.7. The Platonic generalization of ancient Pythagoreanism

1.7.1. *The Divided Line analogy*

Let us now consider Plato's extremely famous text from the *Republic* (VI, 509e–511e), called "The Divided Line analogy", in which Plato describes the world by comparing it to a line segment that he constructs as follows:

"Now take a line which has been cut into two unequal parts, and divide each of them again in the same proportion, and suppose the two main divisions to answer, one to the visible and the other to the intelligible, and then compare the subdivisions in respect of their clearness and want of clearness, and you will find that the first section in the sphere of the visible consists of images. And by images I mean, in the first place, shadows, and in the second place, reflections in water and in solid, smooth and polished bodies and the like: Do you understand? – Yes, I understand! – Imagine, now, the other section, of which this is only the resemblance, to include the animals which we see, and everything that grows or is made. – I can imagine it, he said. – Would you not admit that both the sections of this division have different degrees of truth, and that the copy is to the original as the sphere of opinion is to the sphere of knowledge? – Most undoubtedly! – Next proceed to consider the manner in which the sphere of the intellectual [noetic] is to be divided. – In what manner? – Thus: There are two subdivisions, in the lower of which the soul uses the figures given by the former division as images; the enquiry can only be hypothetical, and instead of going upwards to a principle descends to the other end; in the higher of the two, the soul passes out of hypotheses, and goes up to a principle which is above hypotheses, making no use of images as in the former case, but proceeding only in and through the ideas themselves".

A line is, thus, divided into two unequal parts which are, in nature, visible and intelligible. Each section is then divided in the same ratio. This, which is visible, relative to a relation of clarity and obscurity, makes it possible to confront real bodies (animals, things) with their images (shadows and reflections of these bodies, sometimes called "simulacra"). With regard to the intelligible segment, this is also divided into two parts: on one side we have ideas (in the Platonic sense of the term: idea of beauty, idea of justice, etc.) and mathematical objects.

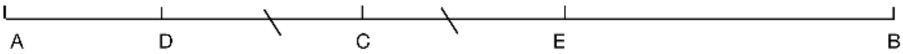

Figure 1.2. *Plato's line, from the Republic*

Before any philosophical interpretation, we must understand what exactly Plato has done and why he divided this line in this manner. On the one hand, we have

$$\frac{AC}{CB}=\frac{AD}{DC}=\frac{CE}{EB} \qquad \text{hence : } DC=\frac{AD.CB}{AC} \qquad [1.5]$$

On the other hand, we have

$$\frac{AC}{AB}=\frac{AD}{AC}=\frac{CE}{CB} \qquad \text{hence : } CE=\frac{AD.CB}{AC} \qquad [1.6]$$

As we can see, from [1.5] and [1.6], two central segments in Plato's lines are equal[21]. In other words:

$$DC = CE \qquad [1.7]$$

The most plausible interpretation of this situation is that if these two central segments represent objects from the tangible world and mathematical ideals, then the equality suggests that mathematics can be *rigorously* applied to the tangible world. In the tangible world, it is the world of living beings and real things. The problem that then arises is knowing whether or not images (or what Deleuze could call, using an inappropriate term, "simulacra") can be compared to rationality. This was the grand problem that Plato would work on till the end of his life. His final philosophy, the theory of "ideas and numbers" (which we know through Aristotle) replied in the affirmative: the world, from the purest ideas to the most mixed tangible complexities, is a hierarchy of mixed objects that inform tangible reality, even if less and less effectively. But nothing in the world is entirely lacking in reason.

1.7.2. *The algebraic interpretation*

Let us say that k is the ratio between the line and its subsegments.

It is then remarkable that the configuration of Plato's division makes it possible to have any kind of ratio (or almost) k.

Let us posit:

$$AC = a,\ AD = a_1,\ DC = a_2$$

$$CB = b,\ CE = b_1,\ EB = b_2$$

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[21] This result was known to authors of that period and, notably, to commentators on Plato's work.

Plato's hypotheses were:

$$a = a_1 + a_2$$
$$b = b_1 + b_2$$

$$b = ka,\ b_2 = kb_1,\ a_2 = ka_1$$

We can then write:

$$b = ka \Rightarrow b_1 + b_2 = k(a_1 + a_2) \Rightarrow b_1 + kb_1 = a_2 + ka_2 \Rightarrow b_1(k+1) = a_2(k+1)$$

And for any $k \neq -1$:

$a_2 = b_1$ or, as we saw earlier, $DC = CE$

1.7.2.1. *Impossibilities*

A case where $k = -1$, which assumes the existence of negative numbers and oriented segments, corresponds to a situation that Plato could not imagine but that can be considered in algebra. This, however, leads to an impossible line. Similarly, a case where $k = 0$ also makes the line and its segments impossible. In any other situation, including when $k = 1$ (divided equally[22]), Plato's hypotheses remain valid. This is, therefore, a very general situation.

1.7.2.2. *The case where* $k = \phi$

Let us now consider – one example among many others, but perfectly compatible with Plato's hypotheses – a case where the ratio k represents the golden number itself. For this, it is enough to posit that

$$k = \phi$$

Among all possible ratios that define the division of a line AB into two segments AD and DB, there is, in fact, one where the ratio of the smaller segment to the larger segment is the same as that of the larger segment to the undivided line. This precisely defines the ratio associated with the "golden number". We have

$$\phi = \frac{AD}{DB} = \frac{DB}{AB}$$

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[22] Thus, the text being ambiguous over whether we should read "*isos*" or "*anisos*" (equal or unequal) for the division of the line does not influence the result of the operation in any way.

This is the characteristic ratio of the golden section and Plato's line can be divided in this ratio (among others) and this ratio accorded a significant importance by the Pythagoreans, being the key aspect governing the construction of the famous pentacle of the Pythagorean society.

In Figure 1.3, we have the following ratios:

$$\frac{BE}{DE} = \frac{DE}{FE} = \frac{FE}{JF} = \frac{JF}{OF} = \frac{OF}{ON}$$

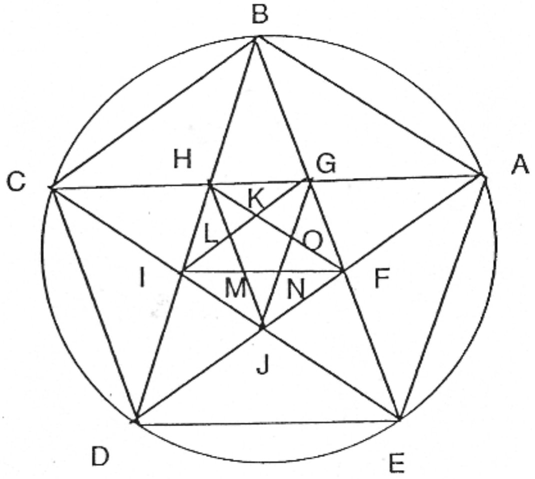

Figure 1.3. *The pentacle of the Pythagoreans*

By dividing his line as he did, Plato thus retained the general principle of harmony, but extended this harmony to any kind of ratio. We can, thus, say that he quite considerably generalized Pythagorean philosophy[23].

1.8. Epistemological consequences: the evolution of reason

Based on the facts we have seen, it may be said that the evolution of science periodically requires revising reason, which is not ingraved in stone, independent of

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[23] Here, we diverge from J. Vuillemin's interpretation [VUI 01, p. 89], which assumes that the ratio in which the line is divided in the *Republic* is the golden number. We cannot see anything in this study that would validate such an interpretation. On the contrary, admitting that this ratio is one of the possible ratios is absolutely exact, and admitting that the line may also be divided in this ratio poses no problem.

the activities through which it works and which oblige it to be transformed. Gaston Bachelard noted this in a study that is a perfect summary of our opinions:

"In sum, science instructs reason. Reason must obey science, the most evolved science, the evolving science. Reason has no right to overvalue an immediate experience; on the contrary it must be balanced with the most richly structured experience. In all circumstances, the immediate must give way to the constructed. Destouches often repeats: if arithmetic, in distant developments, is revealed to contradict itself, we shall reform reason to wipe out the contradiction and preserve arithmetic and keep it intact. Arithmetic has demonstrated so many instances of efficiency, exactitude [and] coherence that we cannot think of abandoning its organization. Faced with a sudden contradiction, or more exactly, faced with a sudden necessity for a contradictory use of arithmetic, there would arise the problem of a non-arithmetic, a panarithmetic, that is a dialectical extension of intuitions regarding numbers that can make it possible to span both the classic doctrine as well as the new doctrine" [BAC 40, p. 144].

We see here that the Pythagorean reason, which to begin with was essentially proportion, ratio, coincides, at this period, with its use within fundamental mathematics. In order to evolve, it was necessary that mathematics also evolve. With the *aloga*, the Greeks discovered that there are numbers and reason beyond what they had held, so far, as being reason (*logos* as proportion). They thus needed to evolve their conception of reason and admit the incommensurable. Mathematically, this situation would lead directly to analysis: it not only led the way out of Greek geometry but required the discovery of another type of rationality than the rationality of proportion, thus opening the door to something that challenged reason within itself, namely, not only folly but anything that would, in general, disturb human reason: the infinite, the sublime, asymmetry, etc.

What was also revealed by the "crisis" or "pseudo-crisis" of irrationals is that there is, therefore, no *single* reason from the start to the finish of history. Here again, Bachelard, has clearly appraised the phenomenon. Thus, in *Le nouvel esprit scientifique* (The New Scientific Spirit), he observes that the revolutionary growth of science "must have a profound reaction on the nature of the mind" and that "the mind has a variable structure from the time that knowledge has had a history". There is, therefore, irreversible progress in the development of scientific knowledge, which is enough to distinguish science from the other modes of understanding the real world:

"Human history may, in its passions and prejudices, in everything that is born of immediate impulses, be an eternal beginning; however, there are thoughts that do not start over; these are thoughts that have been rectified, broadened, completed. They do not go back to their restricted or faltering beginnings. Moreover, the scientific spirit is essentially a rectification of knowledge, a broadening of the framework of knowledge. It judges its past and condemns it. Its structure is the awareness of its historical faults.

Scientifically, we think of the truth as the historical rectification of one long error, we think of experience as the rectification of the common and primary illusion. All the intellectual life of science plays, dialectically, on this differential in knowledge, on the frontiers of the unknown. The very essence of reflection is to understand that we have not understood" [BAC 73, pp. 173–174].

Reason, therefore, evolves based on science. We do indeed say "based on science" and not on god knows what other discourse or activity (theological, philosophical, literary, poetic, etc.) whose transformational virtue is not proven. On the other hand, science and technology, which confront true realities, and especially mathematics, which encounters the most formal aspect of reality, have unequalled power to transform. With no offence to certain sociologists (who wish to consider scientific discourse as nothing more or less than any other social product, such as art or religion), it is science that forces the most radical transformations and that sets them into motion itself[24]. With science, borders that had seemed insurmountable melt away, like the fragile constructions of charlatans of the mind. After a mathematical revolution, we can no longer think like we did before.

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[24] It is possible that art (when not retrograde, "pompier" (The French term for Academic art), or purely negative, as it is often is today) can anticipate these transformations; religion, however, trails far behind and tries, at best, to "limit damages".