PART 1

Introduction

1

Review of Linear Algebra

We will denote the fields of real and complex numbers by $\mathbb{R}$ and $\mathbb{C}$ respectively. If there is no need to distinguish between them, we will instead simply refer to the field $\mathbb{K}$ of scalars. The set E, equipped with the two operations of addition and scalar multiplication, denotes a vector space over $\mathbb{K}$ (or a $\mathbb{K}$-vector space).

1.1. Vector spaces

1.1.1. *General definitions*

DEFINITION.– *A vector space over the field $\mathbb{K}$ is a set E equipped with the two following operations:*

– addition, which equips E with the structure of a commutative group;

– an "outer" product of an element of E by an element of $\mathbb{K}$, satisfying the following properties:

- $\forall \lambda, \mu \in \mathbb{K}$, $\forall x \in E : (\lambda\mu)x = \lambda(\mu x)$;
- $\forall \lambda, \mu \in \mathbb{K}$, $\forall x \in E : (\lambda + \mu)x = \lambda x + \mu x$;
- $\forall \lambda \in \mathbb{K}$, $\lambda(x + y) = \lambda x + \lambda y$;
- $\forall x \in E : 1x = x$ *(where* 1 *is the identity element of* $\mathbb{K}$*).*

The elements of E are called vectors.

DEFINITION.– *Let F be a subset of the vector space E. F is a vector subspace of E if it is closed under the operations of E. In other words:*

– $\forall x, y \in F, x + y \in F$;

– $\forall \lambda \in \mathbb{K}, \forall x \in F, \lambda x \in F$.

DEFINITION.– *Let E be a $\mathbb{K}$-vector space, and suppose that F and G are two vector subspaces of E.*

1) The sum of F and G, written as $F + G$, is defined as the set:

$$F + G = \{x = y + z,\ y \in F,\ z \in G\}.$$

This is a vector subspace of E.

2) $F + G$ is said to be a direct sum if $F \cap G = \{0\}$. If so, we write this sum as $F \oplus G$.

3) If we also have that $E = F \oplus G$, we say that F and G are supplementary subspaces.

THEOREM.– *Suppose that F and G are supplementary. Then, for every element x of E, there exists a unique pair (y, z) in $F \times G$ such that $x = y + z$.*

1.1.2. *Free families, generating families and bases*

DEFINITION.– *Let $B = \{x_1, \ldots, x_p\}$ be a family of vectors in E.*

– We say that B is related if one of its vectors is a linear combination of the others, i.e.:

$$\exists(t_1, \ldots, t_p) \in \mathbb{K}^p,\ (t_1, \ldots, t_p) \neq (0, \ldots, 0) \text{ such that } \sum_{i=1}^{p} t_i x_i = 0.$$

– We say that B is free if it is not related, in which case its vectors are said to be linearly independent.

– We say that B is a generating family of E (or generates E) if every element of E is a linear combination of the elements of B.

DEFINITION.– *A family $B = \{e_1, \ldots, e_p\}$ of elements in a vector space E is said to be a basis of E if it is free and generates E.*

The canonical basis is one particular example of a basis, which is defined as follows:

DEFINITION.– *The canonical basis is the basis of vectors $\{e_i\}_{i=1\ldots n}$ such that the j-th element of e_i is 0 except when $i = j$, in which case it is equal to 1.*

Thus, every vector x in $\mathbb{R}^n$ may be decomposed with respect to the canonical basis as follows:

$$\forall x \in \mathbb{R}^n, \; x = \sum_{i=1}^{n} x_i e_i.$$

THEOREM.– *In a vector space generated by a finite family of elements, every basis has the same number of elements.*

DEFINITION.– *The dimension of a vector space E generated by a finite family is defined as the number of elements in any given basis of E. This value is denoted as "dim E".*

In any vector space with finite dimension n, we always use the same basis, $B = \{e_1, \ldots, e_n\}$. Thus, each vector x of E may be uniquely decomposed with respect to B as follows:

$$x = x_1 e_1 + \cdots + x_n e_n.$$

The element $X = (x_1, \ldots, x_n)$ in $\mathbb{K}^n$ may therefore be unambiguously chosen as a representation of x.

THEOREM.– *Let E be an n-dimensional vector space, and suppose that F and G are two vector subspaces of E. Then:*

1) every free family of n vectors is a basis;

2) every generating family of n vectors is a basis;

3) dim $F \leq$ dim E;

4) if $F \cap G = \{0\}$, then dim $F+$ dim $G \leq$ dim E;

5) in particular, if $E = F \oplus G$, then dim $F+$ dim $G =$ dim E.

1.2. Linear mappings

DEFINITION.– *Let E and F be two vector spaces over the field K. A mapping $u\colon E \to F$ is said to be a linear mapping if it satisfies the following properties:*

– $u(x+y) = u(x) + u(y)$ $\forall x, y \in E$;

– $u(\lambda x) = \lambda u(x)$ $\forall x \in E$, $\forall \lambda \in K$.

The set of linear mappings from E to F is denoted as $\mathcal{L}(E, F)$.

The linear mappings are the mappings that preserve the vector space structure.

DEFINITION.–

1) The kernel of u, written as Ker(u), is the vector subspace of E defined by:

$$\mathrm{Ker}(u) = \{x \in E \mid u(x) = 0\}.$$

2) The image of u, written as Im(u), is the vector subspace of F defined by:

$$\mathrm{Im}(u) = \{y \in F \mid \exists x \in E \text{ such that } y = u(x)\}.$$

THEOREM.–

– *u is injective if and only if Ker$(u) = \{0\}$.*

– *u is surjective if and only if Im$(u) = F$.*

DEFINITION.– *Let I_E (respectively I_F) be the identity mapping of E (respectively F). The linear mapping u from E to F is said to be invertible if there exists a linear mapping u^{-1} from F to E such that:*

$$u^{-1} \circ u = i_E \quad \text{and} \quad u \circ u^{-1} = i_F. \qquad [1.1]$$

It follows that every invertible linear mapping is bijective, i.e. injective and surjective.

THEOREM.– *Let $u \in \mathcal{L}(E, E)$. The following are equivalent:*

– *u is injective;*

– *u is surjective;*

– *u is bijective.*

THEOREM.– *Let $u \in \mathcal{L}(E, F)$ and suppose that $B = \{e_1, \ldots, e_n\}$ is a basis of E. Then:*

– *if u is injective, $\{u(e_1), \ldots, u(e_n)\}$ is a basis of Im(u);*

– *if u is surjective, $\{u(e_1), \ldots, u(e_n)\}$ is a generating family of F;*

– *the following relation holds:*

$$\mathit{dim}\, E = \mathit{dim}\, \mathit{Ker}(u) + \mathit{dim}\, \mathit{Im}(u).$$

DEFINITION.– *The rank of a linear mapping, denoted as "rank u", is the dimension of Im(u).*

1.3. Matrices

In this section, E, F and G are three vector spaces over the field $\mathbb{K}$, with finite dimensions n, p and q respectively. The families $B_E = \{e_1, \ldots, e_n\}$, $B_F = \{f_1, \ldots, f_p\}$ and $B_G = \{g_1, \ldots, g_q\}$ are the bases of E, F and G.

DEFINITION.– *Let $u \in \mathcal{L}(E, F)$. The matrix of u with respect to the bases B_E and B_F is defined as an array A of scalars (i.e. elements of $\mathbb{K}$) with p rows and n columns such that the j-th column of A is given by the components of the vector $u(e_j)$ with respect to the basis B_F.*

If a_{ij} is the element of A at the intersection of the i-th row and the j-th column, then:

$$u(e_j) = \sum_{i=1}^{p} a_{ij} f_i. \qquad [1.2]$$

The matrix A, which has p rows and n columns, is said to be of format or type (p, n), or is called a $p \times n$ (p-by-n) matrix.

It does not make much sense to prove results on matrices without referring to the linear mappings that they represent. We will use this link between mappings and matrices to define operations on matrices.

1.3.1. *Operations on matrices*

DEFINITION.– *Let A and B be two $p \times n$ matrices. The sum $A + B$ of A and B is the $p \times n$ matrix C with coefficients c_{ij} defined by:*

$$c_{ij} = a_{ij} + b_{ij}.$$

The matrix C thus obtained is the matrix of the linear mapping obtained by summing the two linear mappings represented by A and B.

Similarly, we define the product of a scalar λ and a matrix A as the matrix λA obtained by multiplying each coefficient of A by λ. The set of matrices with p rows and n columns is a vector space, written as $\mathcal{M}_{p,n}(\mathbb{K})$ (or simply $\mathcal{M}_{p,n}$ when the underlying field is explicit).

DEFINITION.– *Let* $u \in \mathcal{L}(F, G)$ *and* $v \in \mathcal{L}(E, F)$. *Define* $w = u \circ v$ *(and therefore* $v \in \mathcal{L}(E, G)$*). Let* A *be the matrix of* u *with respect to the bases* B_F *and* B_G*, and* B *the matrix of* v *with respect to the bases* B_E *and* B_F. *By definition, the matrix of* w *with respect to the bases* B_E *and* B_G *is equal to the product of* A *and* B*, written as* AB.

THEOREM.– *Let* $A \in \mathcal{M}_{n,p}$ *and* $B \in \mathcal{M}_{p,q}$*, and define* $C = AB$. *Then,* $C \in \mathcal{M}_{n,q}$ *and its elements are given by the formula:*

$$c_{ij} = \sum_{k=1}^{p} a_{ik} b_{kj} \quad i = 1, \ldots, n, \; j = 1, \ldots, q. \tag{1.3}$$

DEFINITION.– *The matrix* A *is said to be invertible (or regular or non-singular) if the linear mapping associated with* A *is invertible. If* $A \in \mathcal{M}_{n,n}$ *is invertible, then there exists an inverse matrix, denoted as* A^{-1}*, such that:*

$$AA^{-1} = A^{-1}A = I,$$

where I *is the identity matrix in* $\mathcal{M}_{n,n}$.

Properties

If the matrices A, B and C have dimensions that allow them to be multiplied, then, given any $\alpha \in \mathbb{R}$, the following relations hold:

– $C(A + B) = CA + CB$;

– $(A + B)C = AC + BC$;

– $A(BC) = (AB)C = ABC$;

– $\alpha(AB) = (\alpha A)B = A(\alpha B)$.

1.3.2. *Change-of-basis matrices*

DEFINITION.– *Let* E *be an* n*-dimensional vector space equipped with a basis* $B_E = \{e_1, \ldots, e_n\}$. *Consider another basis of* E*, denoted as* $B'_E = \{e'_1, \ldots, e'_n\}$. *The change-of-basis matrix is defined as the matrix* P *whose columns are the components of the elements of* B'_E *with respect to the original basis* B_E. *Therefore:*

$$e'_j = \sum_{i=1}^{n} p_{ij} e_i.$$

We cannot directly calculate the new components of a vector $\vec{x}$ as a function of its original components using the change-of-basis matrix. Suppose that:

$$x = \sum_{i=1}^{n} x_i e_i = \sum_{i=1}^{n} x_i' e'_i.$$

Let X be the vector in $\mathbb{K}^n$ formed by the components of x with respect to the original basis B_E, and write X' for the vector in $\mathbb{K}^n$ formed by the components of x with respect to the new basis B_E'. It can be shown that:

$$X = PX',$$

or, in other words, noting that change-of-basis matrices are always invertible:

$$X' = P^{-1}X.$$

THEOREM.– *Suppose that $u \in \mathcal{L}(E, E)$ has the matrix A with respect to the basis B_E. Then, the matrix A' of u with respect to the basis B_E' satisfies the following formula:*

$$A' = P^{-1}AP.$$

Change-of-basis operations are extremely important in practice, since they allow square matrices to be expressed in more “favorable” forms (diagonal, upper triangular or lower triangular structure).

1.3.3. *Matrix notations*

For any $A \in \mathcal{M}_{p,n}$, we will adopt the following notation, unless otherwise stated:

– A_j is the j-th column of the matrix A, which can therefore be represented by:

$$A = (A_1, A_2, \ldots, A_n).$$

– A^T is the transpose of the matrix A (obtained by switching the rows and columns of A). Thus, $A^T \in \mathcal{M}_{n,p}$ and:

$$(A^T)_{ij} = a_{ji} \quad \forall i, j,\ 1 \leq i \leq p, 1 \leq j \leq n.$$

– If $A \in \mathcal{M}_{np}(\mathbb{C})$, we write $\overline{A}$ for the complex conjugate matrix of A (whose coefficients are the complex conjugates of the coefficients of A).

– If A is a square matrix ($p = n$), we define the trace of A to be the sum of its diagonal elements:

$$\mathrm{tr}(A) = \sum_{i=1}^{n} a_{ii}.$$

– We write $D = \mathrm{diag}(d_1, \dots, d_n)$ for the diagonal matrix D.

– Every vector in $\mathbb{K}^n$ may be identified with an $n \times 1$ matrix (with n rows and one single column). If X is a column vector, then X^T is a row vector (a $1 \times n$ matrix, with one single row and n columns).

DEFINITION.– *Any matrix $A \in \mathcal{M}_{n,n}$ satisfying:*

$$a_{ij} = a_{ji} \quad \forall i, j \in \{1, \dots, n\}$$

is called a symmetric matrix, in which case $A^T = A$.

1.4. Determinants

To define the notion of determinant, we require the notions of permutation and signature, so we will present them first.

DEFINITION.– *Let $I_n = \{1, 2, \dots, n\}$. A permutation is a bijective mapping from I_n to itself. A transposition is a permutation that only exchanges two consecutive elements (we write τ_j for the permutation that exchanges j and $j + 1$).*

We write $\mathcal{P}_n$ for the set of permutations of I_n. Every permutation may be written as a product of transpositions. This product is not unique, but the number of transpositions in any given decomposition always has the same parity.

DEFINITION.– *The signature of a permutation σ is defined as the value sign(σ), which is $+1$ if the permutation may be decomposed into an even number of transpositions, and -1 otherwise.*

DEFINITION.– *The determinant of a matrix $A \in \mathcal{M}_{nn}$, written as det(A), is the number:*

$$\det(A) = \sum_{\sigma \in \mathcal{P}} \mathrm{sign}\ (\sigma) a_{\sigma(1),1} a_{\sigma(2),2} \dots a_{\sigma(n),n}. \qquad [1.4]$$

There are only finitely many permutations of a finite set, and the cardinal of $\mathcal{P}_n$ is finite. Therefore, the sum in formula [1.4] is finite.

Properties

The most important properties of determinants are as follows:

1) $\det(\overline{A}) = \overline{\det(A)}$;

2) $\det(A) = \det(A^T)$;

3) $\forall \sigma \in \mathcal{P}_n$, $\det(A_{\sigma(1)}, A_{\sigma(2)}, \ldots, A_{\sigma(n)}) = \text{sign}(\sigma)\det(A)$.

4) The mapping that sends any given column of A to the determinant of A is linear. In particular, if one of the columns of A is a linear combination of the others, then $\det(A) = 0$.

The mapping that sends the columns of the matrix A to the determinant of A is an alternating multilinear mapping. The multilinearity of this mapping follows from the aforementioned property 4. Hence:

$$\det(\alpha A) = \alpha^n \det(A).$$

This mapping is said to be an alternating mapping because, by the aforementioned property 3, permuting any two consecutive columns of the matrix changes the sign of the determinant.

Properties

1) In general, $\det(A + B) \neq \det(A) + \det(B)$.

2) Let A and B be two matrices in $\mathcal{M}_{nn}$. Then:

$$\det(AB) = \det(A)\det(B).$$

Let $A \in \mathcal{M}_{nn}(K)$. We write $A_{|i,j|} \in \mathcal{M}_{n-1,n-1}(K)$ for the minor of A obtained by deleting the i-th row and the j-th column. In "practice", determinants are usually calculated using the following result:

THEOREM.– *The determinant can be calculated by expanding along a column:*

$$\forall j \in \{1, \ldots, n\} \quad \mathit{det}(A) = \sum_{i=1}^{n} a_{ij}(-1)^{i+j} \mathit{det}(A_{|i,j|}). \qquad [1.5]$$

Alternatively, we can also expand along rows using an analogous formula, by the aforementioned property 2.

DEFINITION.– *The scalar* $(-1)^{i+j} \det\,(A_{|i,j|})$ *is said to be the cofactor of the element* a_{ij}*. The matrix of cofactors is called the comatrix, which is written as* $co(A)$.

THEOREM.– *A square matrix A is invertible (or regular or non-singular) if and only if its determinant is non-zero. If so, the inversion formula may be written as:*

$$A^{-1} = \frac{1}{det(A)}[co(A)]^T. \quad [1.6]$$

1.5. Scalar product

DEFINITION.– *A scalar product on E is a mapping from $E \times E$ to $\mathbb{R}$:*

$$(\vec{x}, \vec{y}) \to < \vec{x}, \vec{y} >= (\vec{x}, \vec{y})_E$$

satisfying the following properties:

– $< \vec{x}, \vec{x} > \geq 0$;

– $< \vec{x}, \vec{x} >= 0 \Rightarrow \vec{x} = 0$;

– $< \vec{x}, \vec{y} >=< \vec{y}, \vec{x} >$;

– $< \vec{x}, \vec{y} + \vec{z} >=< \vec{x}, \vec{y} > + < \vec{x}, \vec{z} >$;

– $< \vec{x}, \alpha\vec{y} >= \alpha < \vec{x}, \vec{y} > \forall \alpha \in \mathbb{R}$.

LEMMA.– *The scalar product is a symmetric bilinear form on E.*

1.6. Vector norm

The norm of a vector may be defined in several different ways. The most common is, of course, the Euclidean norm of a vector $\vec{x} = (x_1, x_2, \dots, x_n)^T$:

$$\|\vec{x}\| = \sqrt{< \vec{x}, \vec{x} >} = \sqrt{\sum_{i=1}^{n} x_i^2}, \quad x_i \in \mathbb{R}.$$

Geometrically, this norm represents the length of the vector. However, norms are defined more generally.

DEFINITION.– *The mapping $\|.\|: E \to \mathbb{R}$ (here, E is a real vector space) is said to be a vector norm if the following properties hold:*

1) $\| \vec{x} \| \geq 0$, $\forall \vec{x} \in E$ *and* $\| \vec{x} \| = 0 \Leftrightarrow \vec{x} = 0$;

2) $\| \lambda\vec{x} \| = | \lambda | \| \vec{x} \|$, $\forall a \in \mathbb{R}$, $\vec{x} \in E$;

3) $\| \vec{x} + \vec{y} \| \leq \| \vec{x} \| + \| \vec{y} \|$, $\forall \vec{x}, \vec{y} \in E$.

It can easily be verified that the Euclidean norm defined above satisfies these properties. As mentioned above, we can define several different norms on the same vector space. One well-known example of norms on $\mathbb{R}^n$ is the family of p-norms. Given any vector $\vec{x}$, the p-norms are defined as follows:

$$\| \vec{x} \|_p = \left(\sum_{i=1}^{n} |x_i^p| \right)^{1/p} .$$

When $p = 2$, we recover the definition of the Euclidean norm. Another example is the infinity norm, which is defined as follows:

$$\|\vec{x}\|_\infty = \max_{1 \leq i \leq n} |x_i|.$$

The infinity norm is one of the most widely used norms, together with the 2-norm and the 1-norm. Other more exotic norms can also be defined (e.g. elliptic norms); however, we will simply work with the norms defined above.

EXAMPLE.– The 1-norm, 2-norm and ∞-norm of the vector $\vec{x} = (1, 0, 1, 4)^T$ may be calculated as follows:

$$\|\vec{x}\|_1 = \sum_{i=1}^{4} |x_i| = 6;$$

$$\|\vec{x}\|_2 = \sqrt{\sum_{i=1}^{4} x_i^2} = \sqrt{18} = 3\sqrt{2};$$

$$\|\vec{x}\|_\infty = \max_{1 \leq i \leq 4} | x_i | = 4.$$

This example shows that different norms can take different values.

1.7. Matrix eigenvectors and eigenvalues

1.7.1. *Definitions and properties*

Throughout the rest of this section, E denotes a vector space over $\mathbb{C}$.

DEFINITION.– *Let $u \in \mathcal{L}(E, E)$. We say that $\lambda \in \mathbb{C}$ is an eigenvalue of u if there exists a non-zero vector $\vec{y}$ such that:*

$$u(\vec{y}) = \lambda \vec{y}. \qquad [1.7]$$

The vector $\vec{y}$ is said to be an eigenvector of the eigenvalue λ.

With respect to any given basis of E, the linear mapping u may be represented by a matrix $A \in \mathcal{M}_{nn}$, and the eigenvector y may be represented by an element Y of $\mathbb{C}^n$. Therefore:

$$AY = \lambda Y.$$

This allows us to introduce the following definition:

DEFINITION.– *Let $A \in \mathcal{M}_{nn}(\mathbb{C})$. We say that $\lambda \in \mathbb{C}$ is an eigenvalue of A if there exists a non-zero $Y \in \mathbb{C}^n$ such that:*

$$AY = \lambda Y. \qquad [1.8]$$

The vector Y is said to be an eigenvector of A for the eigenvalue λ.

THEOREM.– *The complex number λ is an eigenvalue of A if and only if:*

$$det(\lambda I - A) = 0. \qquad [1.9]$$

If A is not invertible, then 0 is an eigenvalue of A.

DEFINITION.– *The characteristic polynomial of A, written as $P_A(s)$, is the polynomial:*

$$P_A(s) = det(sI - A) = s^n + \mu_1 s^{n-1} + \cdots + \mu_{n-1} s + \mu_n. \qquad [1.10]$$

THEOREM.– *By d'Alembert's theorem on the roots of a polynomial, every $n \times n$ square matrix has n eigenvalues (counted with multiplicity). By writing $\lambda_1, \ldots, \lambda_p$ for the p distinct roots of the polynomial and n_i for the multiplicity of λ_i, it follows that:*

$$P_A(s) = (s - \lambda_1)^{n_1}(s - \lambda_2)^{n_2} \ldots (s - \lambda_p)^{n_p}. \qquad [1.11]$$

Note that the characteristic polynomial is fully determined by the eigenvalues of A, which are also the eigenvalues of the linear mapping represented by this matrix. Therefore:

DEFINITION.– *Let $u \in \mathcal{L}(E, E)$. The characteristic polynomial of u, written as P_u, is defined as the characteristic polynomial of any matrix representing u with respect to any given basis of E.*

THEOREM.– *Let $u \in \mathcal{L}(E, E)$. The trace of u, written as tr(u), is defined by either of the following two equivalent conditions:*

– *tr(u) = tr(A), where A is an arbitrary matrix representing u;*

– *$tr(u) = \sum_{i=1}^{n} \lambda_i$, where λ_i, $i = 1, \ldots, n$ are the n (possibly non-distinct) eigenvalues of u.*

THEOREM.– *Let y be an eigenvector of A with the eigenvalue λ. Then, equivalently:*

– λ is an eigenvalue of A^T (however, the corresponding eigenvector is not y in general);

– $\bar{\lambda}$ is an eigenvalue of $\bar{A}$ and $\bar{y}$ is the corresponding eigenvector;

– if $R \in \mathcal{M}_{n,n}(K)$ is invertible, then λ is an eigenvalue of $R^{-1}AR$.

1.7.2. *Matrix diagonalization*

Let $u \in \mathcal{L}(E, E)$.

THEOREM.– *Let A be a matrix, and suppose that $\mathcal{Y} = \{Y_1, \ldots, Y_n\}$ is a basis of eigenvectors of A with possibly non-distinct eigenvalues $\lambda_1, \ldots, \lambda_n$. Let T be the change-of-basis matrix from the canonical basis of $\mathbb{C}^n$ to $\mathcal{Y}$. Consider the matrix D defined by:*

$$D = T^{-1}AT. \quad [1.12]$$

Then, D is a diagonal matrix with eigenvalues along the diagonal. We write that:

$$D = diag(\lambda_1, \ldots, \lambda_n). \quad [1.13]$$

THEOREM.– *If the eigenvalues of A are all distinct, then their eigenvectors form a basis.*

DEFINITION.– *Two square matrices A and B are said to be similar if there exists an invertible matrix U such that:*

$$B = U^{-1}AU.$$

A matrix is said to be diagonalizable if it is similar to a diagonal matrix, or triangularizable if it is similar to a triangular matrix.

1.7.3. *Triangularization of matrices*

In the case where there does not exist a basis of eigenvectors, we can still write the matrix of u in a triangular form in $\mathbb{C}$ by performing a change of basis.

THEOREM.– *For every $u \in \mathcal{L}(E, E)$, there exists a basis of E with respect to which the matrix T of u is triangular, with the eigenvalues of u along the diagonal.*

The above theorem states that we can use a change of basis to express every matrix in a triangular form in $\mathbb{C}$. This does not hold in general in $\mathbb{R}$.

1.8. Using *Matlab*

Matlab allows us to define arrays or matrices of complex numbers in the same way as arrays or matrices of real numbers:

```
>>Z=[1+2i 0 z1;1 z2 z1+z2;z1*z2 z1-z2 z1/z2]
Z =
    1.0000 + 2.0000i    0                   3.5000 - 1.2500i
    1.0000              1.3140 - 0.0948i    4.8140 - 1.3448i
    4.4805 - 1.9744i    2.1860 - 1.1552i    2.7181 - 0.7551i
```

We can also perform arithmetic operations on matrices:

```
>>A=Z^(-1)
A =
     0.2279 - 0.1872i    -0.2475 + 0.1665i   0.1564 - 0.0282i
    -0.6237 + 0.3429i     0.4146 - 0.4741i   0.0537 + 0.3314i
     0.1251 - 0.0321i     0.1174 + 0.1358i  -0.0282 - 0.0914i
>>A*Z
ans =
     1.0000 + 0.0000i    0.0000 + 0.0000i    0.0000 - 0.0000i
    -0.0000 - 0.0000i    1.0000              0.0000 + 0.0000i
     0.0000 - 0.0000i   -0.0000              1.0000
>>A/Z
ans =
     0.1328 - 0.2826i     -0.0267 + 0.2886i    -0.0451 - 0.1223i
    -0.1566 + 0.6724i      0.0056 - 0.5356i     0.1202 + 0.1689i
    -0.1037 - 0.0857i      0.0965 + 0.0148i    -0.0276 + 0.0428i
```

A matrix of order 0 is a scalar, and a matrix of order 1 is a vector. This vector can be either a row vector or a column vector.

```
>>x=[1,2,3,4]
x =
    1 2 3 4
```

Alternatively:

```
>>x=[0 1 2 3]
```

also returns a row vector. For example, $x(2) = 1$.

```
>>x=[1 ;2 ;3 ;4]
x =
    1
    2
    3
    4
```

```
>>x=[0 ;1 ;2 ;3]'
```

returns a row vector.

```
>>x=[0 1 2 3]'
```

returns a column vector.

REMARK.– In *Matlab*, the row and column indices always start at 1 rather than 0, unlike in other programming languages. Thus, $x(0)$ is not defined.

A matrix of order greater than 1 is a two-dimensional matrix. For example:

```
>>x=[0 :2 ;4 :6]
x =
    0 1 2
    4 5 6
```

This is a matrix with 2 rows and 3 columns.

```
>>y=[0 :2 ;4 :6]'
y =
    0   4
    1   5
    2   6
```

The size of the matrix y is returned by the *size* function:

```
>>size(y)
ans =
    3   2
```

The answer returned is 3 rows and 2 columns. The j-th column of the matrix x is returned by *y(:,j)*. For example, when $j = 2$, we have *y(:,2)= 4 5 6*. The i-th row of the matrix x is returned by *y(i,:)*. For $i = 2$, we have *y(2,:)= 1 5*.

Given an $n \times n$ square matrix A, the corresponding identity matrix is returned by the *eye* function. For example, for $n = 3$, we have:

```
>>A=[1:3;4:6;7:9]
A =
     1    2    3
     4    5    6
     7    8    9
>>eye(size(A))
ans =
     1    0    0
     0    1    0
     0    0    1
```

eye(size(A)) returns the identity matrix with the same dimensions as A.

Consider the row matrix (row vector) *x=[2 15 0]*. The function *sort(x)* returns a row matrix whose elements have been sorted into increasing order:

```
>>sort(x)
ans =
     0    2   15
```

sort(x') returns a column matrix whose elements have been sorted into increasing order:

```
>>sort(x')
ans =
     0
     2
     15
```

sum(x) calculates the sum of the column elements of the matrix x:

```
>>sum(x)
ans =
     17
>>sum([4 2 1 0;8 9 3 12])
ans =
     12 11 4 12
```

To find the maximum and minimum elements of the vector x, we can use the functions *max* and *min*:

```
>>max(x)
ans =
    15
>>min(x)
ans =
    0
```

The *max* and *min* commands can also be applied to any arbitrary matrix y.