Part 1

Advanced Elements and Test Bench of Computer-aided Feedback Control

1

Canonical Discrete State Models of Dynamic Processes

1.1. Interest and construction of canonical state models

Even though a dynamic process can be described in the state space by an infinity of discrete state models, the types of discrete state models of greatest interest in practice are structurally canonical.

Indeed, the morphology of the parametric space of a canonical state model offers:

– maximum number of null terms, which substantially reduces the cost of numerical analysis, if needed;

– several apparent elements indicative for the fundamental dynamic properties of the model, such as: stability (imposed by the nature of eigenvalues), controllability, observability, etc.

A canonical discrete state model can be obtained from:

– canonical realization of the z-transfer function of the same process;

– canonical transformation of an existing discrete state model.

Canonical realizations and transformations presented in this chapter can be extended to the multivariable case [FOU 87].

1.2. Canonical realizations of a transfer function G(z)

Canonical realizations result from the transformation of a *z*-transfer function defined by [1.1]:

$$G(z)=\frac{y(z)}{u(z)}=\frac{b_1z^{n-1}+b_2z^{n-2}+\ldots+b_{n-1}\,z+b_n}{z^n+a_1z^{n-1}+a_2z^{n-2}+\ldots+a_{n-1}\,z+a_n} \qquad [1.1]$$

Three main types of canonical realizations can be distinguished as follows:

– Jordan canonical realization;

– controllable canonical realization;

– observable canonical realization.

1.2.1. *Jordan canonical realization*

The construction of the Jordan discrete state model requires the decomposition of G(*z*) into simple elements (see [1.2]).

1.2.1.1. *G(z) admits distinct real poles*

If all the poles of the transfer function G(*z*) defined by [1.1] are simple, the latter could be decomposed into [1.2], where a_i designates the pole *i* of [1.1] with $i = 1, 2,\ldots, n$, and k_i is the static gain associated with pole a_i:

$$G(z)=\sum_{i=1}^{n}\frac{k_i}{(z-a_i)}=\sum_{i=1}^{n}\left(\frac{\left(\frac{1}{z}\right)}{\left(1-\frac{a_i}{z}\right)}\ \mathrm{k_i}\right) \qquad [1.2]$$

The block diagram associated with this decomposition into simple elements corresponds to Figure 1.1, in which the fixed state vector corresponds to $x = [x_1 \ \ x_2 \ \ldots \ x_n]^T$.

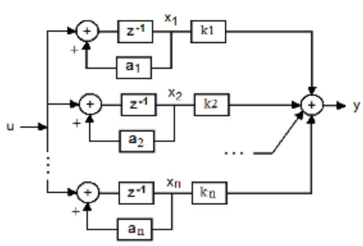

Figure 1.1. *Block diagram of a Jordan realization of G(z): case of distinct simple poles*

The specific choice of the elements of vector x in Figure 1.1 leads to the following Jordan discrete canonical state model:

$$\begin{cases} x(k+1) = \begin{bmatrix} a_1 & 0 & 0 & . & . & 0 & 0 \\ 0 & a_2 & 0 & . & . & 0 & 0 \\ . & 0 & . & . & . & 0 & . \\ 0 & 0 & 0 & . & . & a_i & 0 \\ 0 & 0 & 0 & . & . & 0 & a_n \end{bmatrix} \text{x(k)} + \begin{bmatrix} 1 \\ 1 \\ . \\ 1 \\ 1 \end{bmatrix} \text{u(k)} \\ y(k) = \begin{bmatrix} \text{k}_1 & \text{k}_2 & . & . & \text{k}_\text{n} \end{bmatrix} \text{x(k)} \end{cases} \quad [1.3]$$

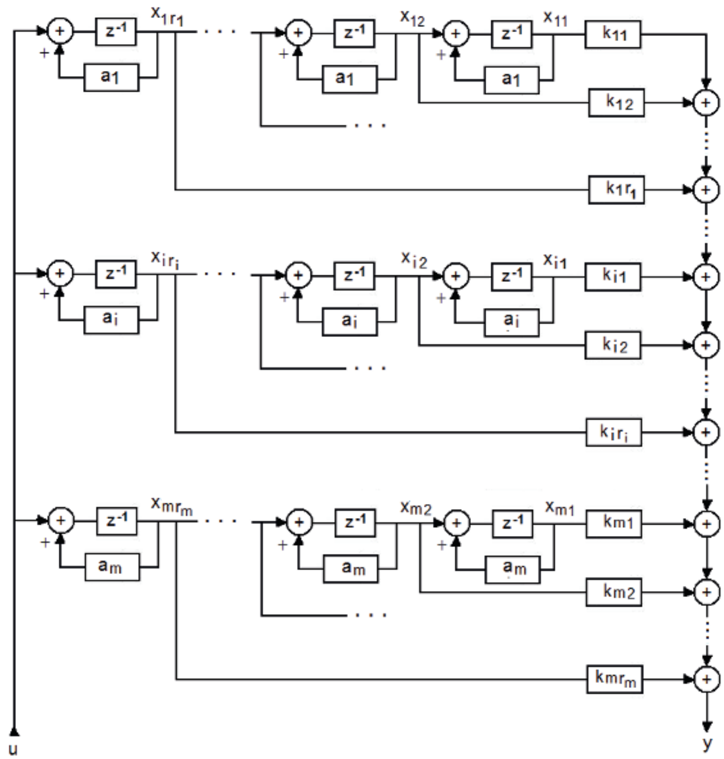

Figure 1.2. *Block diagram of a Jordan realization of G(z): case of multiple poles*

1.2.1.2. *G(z) admits multiple real poles*

Let us consider the specific case of a discrete transfer function G(z) that admits m poles with multiplicity orders $r_1, r_2,\ldots, r_m$, respectively, with $r_1 + r_2 + \ldots + r_m = n$, then G($z$) can be decomposed into the following form:

$$G(z) = \sum_{i=1}^{m}\left(\sum_{j=1}^{r_i}\left(\frac{k_{i\,j}}{(z-a_i)^{r_i-j+1}}\right)\right) \qquad [1.4]$$

This decomposition leads to the block diagram in Figure 1.2.

The specific choice of given state variables leads to the following Jordan canonical model:

$$\begin{cases} x(k) = \begin{pmatrix} A_1 & 0 & 0 & . & . & 0 & 0 \\ 0 & A_2 & 0 & . & . & 0 & 0 \\ . & 0 & . & . & . & 0 & . \\ 0 & 0 & 0 & . & . & A_{r-1} & 0 \\ 0 & 0 & 0 & . & . & 0 & A_r \end{pmatrix} x(k) + \begin{bmatrix} B_1 \\ B_2 \\ . \\ B_{m-1} \\ B_m \end{bmatrix} u(k) \\ y(k) = \begin{bmatrix} C_1 & C_2 & . & . & C_{m-1} & C_m \end{bmatrix} x(k) \end{cases} \qquad [1.5]$$

with:

$$A_i = \begin{bmatrix} a_i & 1 & 0 & . & . & 0 & 0 \\ 0 & a_i & 1 & . & . & 0 & 0 \\ . & 0 & . & . & . & 1 & . \\ 0 & 0 & 0 & . & . & a_i & 1 \\ 0 & 0 & 0 & . & . & 0 & a_i \end{bmatrix}_{(r_i \, x \, r_i)} ; \qquad B_i = \left.\begin{bmatrix} 0 \\ 0 \\ . \\ 0 \\ 1 \end{bmatrix}\right\} r_i \text{ lines} \qquad [1.6]$$

$$C_i = \begin{bmatrix} \underbrace{k_{i\,1} \quad k_{i\,2} \quad \ldots \quad k_{i\,ri}}_{r_i \text{ columns}} \end{bmatrix}$$

1.2.1.3. *Problems raised by Jordan realization*

Jordan canonical realization raises two practical problems. The first problem is posed by the difficulty in factorizing the denominator of the transfer function for a degree above 3. The second problem stems from the difficulty in implementing subsystems admitting complex poles.

In the first case, the solution involves factorization by means of an advanced numerical analysis tool, such as Matlab®, using, for example, the "roots" command. The solution to the second problem results from the properties of block diagrams. Indeed, given that complex poles of a dynamic model are necessarily present in conjugated pairs, then each pair of conjugated poles appears in the decomposed form of the transfer function in the following form:

$$G(z)=\frac{k}{(z-a)^2+b^2}=\frac{\frac{1}{(z-a)^2}}{1+\frac{b^2}{(z-a)^2}}k \qquad [1.7]$$

Relation [1.7] corresponds to two first-order systems in cascade, forming a closed-loop system with a negative feedback of b^2 and an output gain k. The resulting block diagram is presented in Figure 1.3.

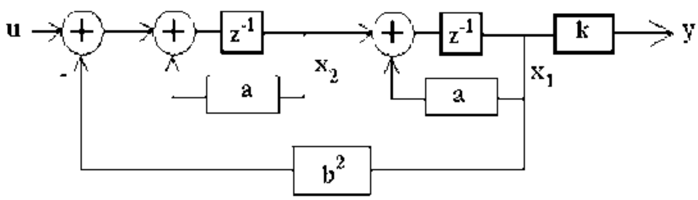

Figure 1.3. *Block diagram of a Jordan realization of G(z): case of complex poles*

1.2.2. *Controllable canonical realization*

The concept of controllability of dynamic systems will be clarified further. For the time being, let us consider a discrete transfer function G(z) given by [1.1]. It is easy to prove that it can be written as a z^{-1} function that has the form [1.8]:

$$G(z)=\frac{y(z)}{u(z)}=\frac{b_1z^{-1}+b_2z^{-2}+\ldots+b_{n-1}z^{-(n-1)}+b_nz^{-n}}{1+a_1z^{-1}+a_2z^{-2}+\ldots+a_{n-1}z^{-(n-1)}+a_nz^{-n}} \qquad \text{(a)}$$

$$=\frac{\left(b_1z^{-1}+b_2z^{-2}+\ldots+b_{n-1}z^{-(n-1)}+b_nz^{-n}\right)}{\left(1+a_1z^{-1}+a_2z^{-2}+\ldots+a_{n-1}z^{-(n-1)}+a_nz^{-n}\right)}\left(\frac{\mathrm{w(z)}}{\mathrm{w(z)}}\right) \qquad \text{(b)} \qquad [1.8]$$

where $w(z)$ is a fictitious function.

Relation [1.8] leads to the following equalities:

$$u(z) = (1 + a_1 z^{-1} + ... + a_{n-1} z^{-(n-1)} + a_n z^{-n})w(z) \quad [1.9]$$

$$y(z) = (b_n z^{-n} + b_{n-1} z^{-(n-1)} + ... + b_1 z^{-1})w(z) \quad [1.10]$$

Therefore:

$$w(z) = u(z) - (a_1 z^{-1} w(z) - ... - a_{n-1} z^{-(n-1)} w(z) - a_n z^{-n} w(z)) \quad [1.11]$$

and:

$$y(z) = b_n z^{-n} w(z) + b_{n-1} z^{-(n-1)} w(z) + ... + b_1 z^{-1} w(z) \quad [1.12]$$

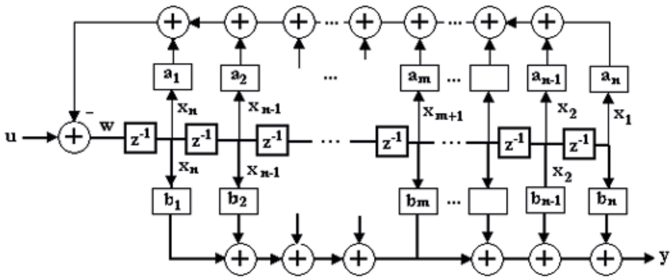

Figure 1.4. *Block diagram of the controllable realization of G(z)*

Expressions [1.9] to [1.12] lead to the block diagram of controllable realization of G(z), which is represented in Figure 1.4. The choice of the system of state variables:

$$\begin{aligned} x_n(z) &= z^{-1} w(z) \\ x_{n-1}(z) &= z^{-1} x_n(z) \\ &\ldots \\ x_1(z) &= z^{-1} x_2(z) \end{aligned} \quad [1.13]$$

leads to the controllable discrete state model given by [1.14]:

$$\begin{cases} x(k+1) = \begin{bmatrix} 0 & 1 & 0 & . & . & . & 0 \\ 0 & 0 & 1 & 0 & . & . & 0 \\ . & . & . & . & . & 1 & . \\ 0 & 0 & . & . & . & 0 & 1 \\ -a_n & -a_{n-1} & . & . & . & -a_2 & -a_1 \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 0 \\ . \\ 0 \\ 1 \end{bmatrix} u(k) \\ y(k) = \begin{bmatrix} b_n & b_{n-1} & . & . & b_m & ... & b_1 \end{bmatrix} x(k) \end{cases} \quad [1.14]$$

EXAMPLE.– Let us consider:

$$G(z) = \frac{y(z)}{u(z)} = \frac{0.5z^2 + 2.5z + 1}{z^3 + 6z^2 + 10z + 8} \quad [1.15]$$

In this case, the following relations are obtained:

$$\begin{aligned} x(k+1) &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -8 & -10 & -6 \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(k) \\ y(k) &= \begin{bmatrix} 1 & 2.5 & 0.5 \end{bmatrix} x(k) \end{aligned} \quad [1.16]$$

1.2.3. *Observable canonical realization*

The concept of observability of dynamic systems will be clarified further. Given a transfer function G(z), the following can be written as:

$$G(z) = \frac{y(z)}{u(z)} \frac{z^{-n}}{z^{-n}} = \frac{\left(b_1 z^{-1} + b_2 z^{-2} + \ldots + b_{n-1} z^{-(n-1)} + b_n z^{-n}\right)}{\left(1 + a_1 z^{-1} + a_{n-2} z^{-2} + \ldots + a_{n-1} z^{-(n-1)} + a_n z^{-n}\right)} \quad [1.17]$$

therefore:

$$y(z)\,(1 + a_1 z^{-1} + a_2 z^{-2} + \ldots + a_n z^{-n}) = u(z)\,(b_1 z^{-1} + \ldots b_{n-1} z^{-(n-1)} + b_n z^{-n}) \quad [1.18]$$

hence, the following relation:

$$\begin{aligned} y(z) &= (-a_n y(z) + b_n u(z)) z^{-n} \\ &= (-a_{n-1} y(z) + b_{n-1} u(z)) z^{-(n-1)} + \ldots + (-a_1 y(z) + b_1 u(z)) z^{-1} \end{aligned} \quad [1.19]$$

This relation leads to the block diagram of the observable realization of G(z), which is represented in Figure 1.5.

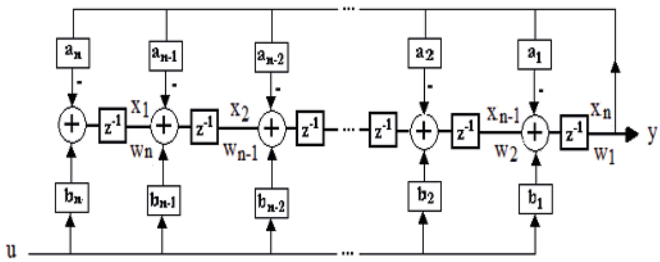

Figure 1.5. *Block diagram of the observable realization of G(z)*

The choice of the system of state variables $\{x_1, x_2, \ldots, x_n\}$ leads to equations [1.20]:

$$zx_1(z) = -a_n x_n(z) + b_n u(z)$$

$$zx_2(z) = x_1(z) - a_{n-1} xn(z) + bn_{-2} u(z) \qquad [1.20]$$

…

$$zx_n(z) = x_{n-1} - a_1 xn(z) + b_1 u(z)$$

that facilitate the writing in discrete time of the full observable state model in the following form:

$$\begin{cases} x(k) = \begin{bmatrix} 0 & 0 & 0 & . & . & 0 & -a_n \\ 1 & 0 & 0 & . & . & 0 & -a_{n-1} \\ . & 1 & 0. & . & . & 0 & . \\ 0 & 0 & 1 & . & . & 0 & -a_2 \\ 0 & 0 & 0 & . & . & 1 & -a_1 \end{bmatrix} x(k) + \begin{bmatrix} b_n \\ b_{n-1} \\ . \\ . \\ b_1 \end{bmatrix} u(k) \\ y(k) = \begin{bmatrix} 0 & 0 & . & . & 0 & 1 \end{bmatrix} \text{x(k)} \end{cases} \qquad [1.21]$$

On the contrary, the choice of the system of state variables $\{w_1 = xn, w_2 = xn - 1, \ldots, w_n = x_1\}$ leads to the following new relations:

$$zw_1(z) = -a_1 w_1(z) + w_2(z) + b_1 u(z))$$

$$zw_2(z) = -a_2 w_1(z) + w_3(z) + b_2 u(z)) \qquad [1.22]$$

…

$$zw_n(z) = -a_n w_1(z) + b_n u(z))$$

and in this case, the observable state model that results in discrete time is written as follows:

$$\begin{cases} w(k)=\begin{bmatrix} -a_1 & 1 & 0 & . & . & 0 & 0 \\ -a_2 & 0 & 1 & . & . & 0 & 0 \\ ... & ... & 0. & . & . & 1 & . \\ -a_{n-1} & 0 & 0 & . & . & 0 & 1 \\ -a_n & 0 & 0 & . & . & 0 & 0 \end{bmatrix} w(k)+\begin{bmatrix} b_1 \\ b_2 \\ . \\ b_{n-1} \\ b_n \end{bmatrix} u(k) \\ y(k)=\begin{bmatrix} 1 & 0 & .... & 0 & 0 \end{bmatrix} \mathrm{w(k)} \end{cases} \quad [1.23]$$

EXAMPLE.– Let us consider:

$$G(z)=\frac{y(z)}{u(z)}=\frac{0.5z^2+2.5z+1}{z^3+6z^2+10z+8} \quad [1.24]$$

The state model can thus be written in the following form:

$$\begin{cases} w(k+1)=\begin{bmatrix} -8 & 1 & 0 \\ -10 & 0 & 1 \\ -6 & 0 & 0 \end{bmatrix} w(k)+\begin{bmatrix} 1 \\ 2.5 \\ 0.5 \end{bmatrix} u(k) \\ y(k)=\begin{bmatrix} 1 & 0 & 0 \end{bmatrix} w(k) \end{cases} \quad [1.25]$$

1.3. Canonical transformations of discrete state models

Canonical transformations allow for the construction of discrete canonical state models (controllable, observable and Jordan realizations) based on arbitrary discrete state models.

Each type of transformation is based on an appropriate choice of reversible transformation matrix $\mathrm{P}=\mathrm{Q}^{-1}$, allowing the description of the same dynamic process by means of a new state vector $\overline{x}$, in such a way that:

$$\overline{x}(k)=P\,x(k)=Q^{-1}x(k) \quad [1.26]$$

The new representation obtained after expansion of [1.26] can be written as:

$$\begin{cases} \bar{x}(k+1) = \underbrace{\left(P\ A\ P^{-1}\right)}_{\bar{A}} \bar{x}(k) + \underbrace{P\ B}_{\bar{B}}\ x(k) \\ y(k) = \underbrace{C\ P^{-1}}_{\bar{C}}\ x(k) + \underbrace{D}_{\bar{D}}\ u(k) \end{cases} \equiv \left\{\bar{A}, \bar{B}, \bar{C}, \bar{D}\right\} \qquad [1.27]$$

It is worth noting here that:

– if the choice of P in [1.26] is not appropriate, then the resulting state model [1.27] would be just similar to the original state model, without offering any canonical structure;

– the fundamental properties of discrete state models are conserved during canonical transformations.

In practice, the most commonly used transformations are those of Luenberger (Jordan, controllable and observable) and those of Kalman. The difference between them resides in the structure of the transformation matrix P.

These properties are:

– stability;

– controllability;

– observability.

1.3.1. *Jordan canonical transformation*

In a Jordan canonical transformation, Q represents the matrix of eigen vectors and can be calculated with Matlab command:

[Q, D] = eig(A,B); [1.28]

Thus, considering $P = Q^{-1}$, then:

$$\begin{cases} \bar{x}(k+1) = \underbrace{\left(P\ A\ P^{-1}\right)}_{\bar{A}} \bar{x}(k) + \underbrace{P\ B}_{\bar{B}}\ x(k) \\ y(k) = \underbrace{C\ P^{-1}}_{\bar{C}}\ x(k) + \underbrace{D}_{\bar{D}}\ u(k) \end{cases} \equiv \left\{\bar{A}, \bar{B}, \bar{C}, \bar{D}\right\} \qquad [1.29]$$

EXAMPLE OF JORDAN TRANSFORMATION.– Let us consider:

$$x(k+1) = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u(k)$$
$$y(k) = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} x(k) \qquad [1.30]$$

In this case, it can be verified that:

$$Q = \begin{bmatrix} 0.8507 & -0.5257 & 0 \\ 0 & 0 & 0.7071 \\ 0.5257 & 0.8507 & -0.7071 \end{bmatrix} \qquad [1.31]$$

therefore:

$$P = Q^{-1} = \begin{bmatrix} 0.8507 & 0.5257 & 0.5257 \\ -0.5257 & 0.8507 & 0.8507 \\ 0 & 1.4142 & 0 \end{bmatrix} \qquad [1.32]$$

Thus, the controllable form can be written as follows:

$$\bar{A} = PAQ = \begin{bmatrix} 2.6180 & 0 & 0 \\ 0 & 0.3820 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \; \bar{B} = PB = P \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1.3764 \\ 0.3249 \\ 0 \end{bmatrix} \qquad [1.33]$$

$$\bar{C} = CQ = [0 \;\; 0 \;\; 1] \; Q = [0.5257 \quad 0.8507 \quad -0.707]$$

1.3.2. *Controllable canonical transformation*

In a controllable canonical transformation, Q can be built using the controllability matrix U_c defined by:

$$U_c = \begin{bmatrix} B \; AB \; AB^2 \ldots A^r B \ldots A^{n-1} B \end{bmatrix} \qquad [1.34]$$

If the rank of U_c is equal to n, then:

$$Q = U_c = \left[B \ AB \ AB^2 \ldots AB^r \ldots A^{n-1}B \right] \qquad [1.35]$$

Otherwise, if the rank of U_c is equal to $r < n$, then:

$$Q = \left[\underbrace{B \ AB \ldots A^{r-1}B}_{r \ columns} \ \underbrace{R_1 \ldots R_{n-r}}_{(n-r)} \right] \qquad [1.36]$$

which is composed of r linearly independent columns of U_c, selected from left to right, and of an arbitrary choice of $n-r$ remaining columns R_1, R_2, …, R_{n-r}, so that Q is regular (rank n). In this case, the resulting controllable canonical system $\{\bar{A}, \bar{B}, \bar{C}, \bar{D}\}$ takes the following form:

$$\begin{cases} \bar{x}(k+1) = \begin{bmatrix} \bar{A}_c & \bar{A}_{12} \\ 0 & \bar{A}_{\bar{c}} \end{bmatrix} \bar{x}(k) + \begin{bmatrix} \bar{B}_c \\ 0 \end{bmatrix} u(k) \\ y(k) = \begin{bmatrix} \bar{C}_c & \bar{C}_{\bar{c}} \end{bmatrix} \bar{x}(k) + \underbrace{D}_{\bar{D}} u(k) \end{cases} \qquad [1.37]$$

EXAMPLE OF TRANSFORMATION OF A CONTROLLABLE MODEL.– Let us consider:

$$\begin{aligned} x(k+1) &= \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u(k) \\ y(k) &= \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} x(k) \end{aligned} \qquad [1.38]$$

In this case, it can be verified that the rank of $U_c = \begin{bmatrix} 0 & 1 & 3 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ is 3, and it can be considered that:

$$Q = U_c = \begin{bmatrix} 0 & 1 & 3 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Therefore:

$$P = Q^{-1} = \begin{bmatrix} -1 & 1 & 2 \\ 1 & 0 & -3 \\ 0 & 0 & 1 \end{bmatrix} \qquad [1.39]$$

Thus, the controllable form can be written as follows:

$$\overline{A} = PAQ = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & -4 \\ 0 & 1 & 4 \end{bmatrix}, \ \overline{B} = PB = \begin{bmatrix} -1 & 1 & 2 \\ 1 & 0 & -3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad [1.40]$$

$$\overline{C} = CQ = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 3 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \qquad [1.41]$$

EXAMPLE OF TRANSFORMATION OF AN UNCONTROLLABLE MODEL.– Let us consider:

$$\begin{aligned} x(k+1) &= \begin{bmatrix} 2 & 1 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 1 \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} u(k) \\ y(k) &= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x(k) \end{aligned} \qquad [1.42]$$

In this case, it can be verified that the rank of $U_c = \begin{bmatrix} 1 & 3 & 8 \\ 0 & 0 & 0 \\ 1 & 2 & 5 \end{bmatrix}$ is 2. The first two columns are chosen (they are linearly independent), together with an arbitrary vector $[1\ 1\ 0]^T$ in order to form:

$$Q = \begin{bmatrix} 0 & 3 & 1 \\ 1 & 0 & 1 \\ 0 & 2 & 0 \end{bmatrix}$$

or:

$$P = Q^{-1} = \begin{bmatrix} -2 & 2 & 3 \\ 1 & -1 & -1 \\ 0 & 1 & 0 \end{bmatrix} \qquad [1.43]$$

which yields:

$$\bar{A} = PAQ = \begin{bmatrix} 0 & -1 & 5 \\ 1 & 3 & 3 \\ 0 & 0 & -1 \end{bmatrix}, \ \bar{B} = PB = \begin{bmatrix} -2 & 2 & 3 \\ 1 & -1 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad [1.44]$$

$$\bar{C} = CQ = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 3 & 1 \\ 1 & 0 & 1 \\ 0 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 1 \end{bmatrix} \qquad [1.45]$$

and it can be verified that the subsystem defined by:

$$\bar{A}_c = \begin{bmatrix} 0 & -1 \\ 1 & 3 \end{bmatrix}, \ \bar{B}_c = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad [1.46]$$

is controllable.

1.3.3. *Observable canonical transformation*

In an observable canonical transformation, Q can be built using the observability matrix U_0 defined by:

$$U_o = \begin{bmatrix} C \\ CA \\ \cdots \\ CA^r \\ \cdots \\ CA^{n-1} \end{bmatrix} \qquad [1.47]$$

If the rank of U_0 is equal to n, it can be considered that $P = U_0$, or $Q = (U_0)^{-1}$, and otherwise, if the rank of U_0 is equal to $r < n$, matrix [1.48] is considered:

$$P = \begin{bmatrix} \left.\begin{matrix} C \\ C\,A \\ \cdots \\ C\,A^{r-1} \end{matrix}\right\} r \text{ lines} \\ \left.\begin{matrix} R_1 \\ \cdots \\ R_{n-r} \end{matrix}\right\} (n-r) \text{ lines} \end{bmatrix} \qquad [1.48]$$

It is composed of r linearly independent lines of U_0 selected from up to down and the remaining $n - r$ lines are arbitrarily chosen, so that the rank of P is n. In this case, the resulting observable canonical system $\{\overline{A}, \overline{B}, \overline{C}, \overline{D}\}$ can be written in the following form:

$$\begin{cases} \overline{x}(k+1) = \begin{bmatrix} \overline{A}_o & 0 \\ \overline{A}_{21} & \overline{A}_{\bar{o}} \end{bmatrix} \overline{x}(k) + \begin{bmatrix} \overline{B}_o \\ \overline{B}_{\bar{o}} \end{bmatrix} u(k) \\ y(k) = \begin{bmatrix} \overline{C}_o & 0 \end{bmatrix} x(k) + \underbrace{D}_{\overline{D}}\, u(k) \end{cases} \qquad [1.49]$$

For example, let us consider:

$$\begin{aligned} x(k+1) &= \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} u(k) \\ y(k) &= \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} x(k) \end{aligned} \qquad [1.50]$$

In this case, it can be verified that the rank of $U_o = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 3 & 1 & 2 \end{bmatrix}$ is 3, and it can be considered that $P = U_o = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 3 & 1 & 2 \end{bmatrix}$ and therefore $Q = P^{-1} = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -3 & 1 \\ 1 & 0 & 0 \end{bmatrix}$.

Thus, the observable canonical form can be written as:

$$\bar{A} = PA\,Q = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -4 & 4 \end{bmatrix},\ \bar{B} = P\,B = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix} \qquad [1.51]$$

$$\bar{C} = C\,P^{-1} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

Let us also consider the following discrete observable model:

$$x(k+1) = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} u(k) \qquad [1.52]$$

$$y(k) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x(k)$$

Once again, it can be verified that the rank of $U_o = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 1 \\ 5 & 3 & 3 \end{bmatrix}$ is 2, and matrix P can be chosen, so that $P = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$ composed of the first two lines of U_0 and of a third arbitrary line [1 0 1]. In this case, it can be readily verified that P is regular. Therefore, $Q = P^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix}$. Thus, the observable canonical form sought for can be written as:

$$\bar{A} = PA\,Q = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 3 & 0 \\ 0 & 1 & 1 \end{bmatrix},\ \bar{B} = P\,B = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} \qquad [1.53]$$

$$\bar{C} = C\,P^{-1} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

1.3.4. *Kalman canonical transformation*

Kalman canonical transformation relies on an orthogonal matrix H (in the sense of Householder) of sequential decomposition of A, which means:

$$A = HR \qquad [1.54]$$

with $H^{-1} = H^T$ (orthogonality condition), R being an upper triangular matrix. In this case, the required orthogonal transformation matrix is $Q = P^{-1} = H^T$. The calculation of $H = Q^T$ can be done with Matlab command:

```
>> [Q , R] = qr(A)                                                    [1.55]
```

For example, let us consider:

$$x(k+1) = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u(k)$$
$$y(k) = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} x(k) \qquad [1.56]$$

Under these conditions, using Matlab to calculate H = Q and R yields:

$$Q = \begin{bmatrix} -0.8944 & -0.1826 & -0.4082 \\ 0 & -0.9129 & 0.4082 \\ -4472 & 0.3651 & 0.8165 \end{bmatrix}, R = \begin{bmatrix} -2.2361 & -0.8944 & -1.3416 \\ 0 & -1.0954 & 0.1826 \\ 0 & 0 & 0.4082 \end{bmatrix} \qquad [1.57]$$

Therefore:

$$\overline{A} = PA\,Q = \begin{bmatrix} 2.6000 & 0.7348 & -0.5477 \\ -0.0816 & 1.0667 & -0.2981 \\ -0.1826 & 0.1491 & 0.3333 \end{bmatrix}, \ \overline{B} = P\,B = \begin{bmatrix} 0 \\ -0.9129 \\ 0.4082 \end{bmatrix} \qquad [1.58]$$

$$\overline{C} = C\,P^{-1} = \begin{bmatrix} -0.4472 & 0.3651 & 0.8165 \end{bmatrix}$$

1.4. Canonical decomposition diagram

The previous review of canonical structures of discrete state models of dynamic processes leads to the diagram in Figure 1.6 which represents the canonical decomposition of the state model of a discrete process.

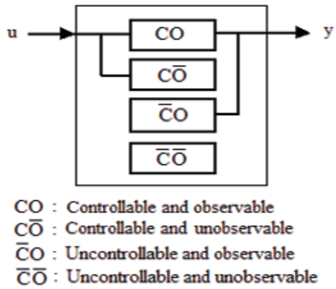

Figure 1.6. *Diagram of canonical decomposition*

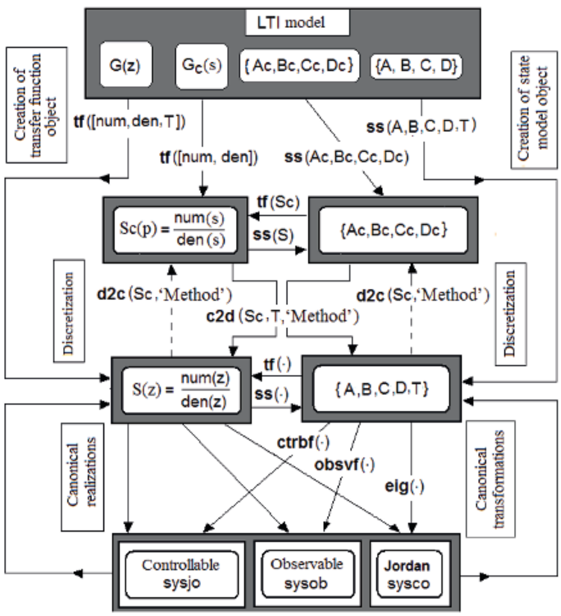

Figure 1.7. *Discretization and canonical transformations of dynamic models using Matlab*

1.5. Discretization and canonical transformations using Matlab

The diagram in Figure 1.7 presents the examples of Matlab commands for discretization and canonical transformation of models of linear and time invariant dynamic processes. Each arrow indicates the direction of creation of a new structure of dynamic model based on the corresponding initial model.

It is worth remembering that “Sc” and “S” denote “continuous object model” and “discrete object model”, respectively. On the contrary, the models created with “ss” are “structures of state model objects”.

1.6. Exercises and solutions

Exercise 1.1.

A dynamic process is described by the following transfer function in z:

$$G(z) = \frac{Y(z)}{U(z)} = \frac{z^2 + 2z + 2}{10\,z^3 + z^2 + 3z + 1}$$

Find:

a) a controllable state representation;

b) an observable state representation.

Solution – Exercise 1.1.

The first step is to set G(z) in the form [1.1], then in the form [1.8a] and deduce a controllable structure corresponding to [1.14].

a) A controllable state representation is given by:

$$\begin{cases} x(k) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -0.1 & -0.3 & -0.1 \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(k) \\ y(k) = \begin{bmatrix} 0.2 & 0.2 & 0.1 \end{bmatrix} \mathrm{x(k)} \end{cases}$$

b) An observable state representation is given by:

$$\begin{cases} x(k+1) = \begin{bmatrix} 0 & 0 & -0.1 \\ 1 & 0 & -0.3 \\ 0 & 1 & -0.1 \end{bmatrix} x(k) + \begin{bmatrix} 0.2 \\ 0.2 \\ 0.1 \end{bmatrix} u(k) \\ y(k) = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \mathrm{x(k)} \end{cases}$$

Exercise 1.2.

A process is described by the z-transfer function:

$$G(z) = \frac{Y(z)}{U(z)} = \frac{1}{(z+1)(z+2)}$$

Find a Jordan canonical state realization.

Solution – Exercise 1.2

$$G(z) = \frac{1}{(z+1)(z+2)} = G(z) = \frac{1}{z+1} - \frac{1}{z+2} = \frac{\frac{1}{z}}{1+\frac{1}{z}} - \frac{\frac{1}{z}}{1+\frac{2}{z}}$$

Therefore, a Jordan state representation is:

$$\begin{cases} x(k+1) = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(k) \\ y(k) = \begin{bmatrix} 1 & 1 \end{bmatrix} x(k) \end{cases}$$

Exercise 1.3.

A process is described by the z-transfer function:

$$G(z) = \frac{Y(z)}{U(z)} = \frac{1}{(z+1)^2(z+2)}$$

Find a Jordan canonical state realization.

Solution – Exercise 1.3.

$$G(z) = \frac{Y(z)}{U(z)} = \frac{1}{(z+1)^2(z+2)} = \frac{1}{(z+1)^2} - \frac{1}{z+1} + \frac{1}{z+2}$$

A Jordan canonical state realization leads to:

$$\begin{cases} x(k+1) = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} u(k) \\ y(k) = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \mathrm{x(k)} \end{cases}$$

Exercise 1.4.

The block diagram of a servomechanism that is digitally controlled by a discrete PI controller corresponds to Figure 1.8, where K_p and b designate the parameters of the transfer function D(z) of the controller, K_m and a being the parameters of the z-transfer function of the dynamic process (with $0 < a < 1$).

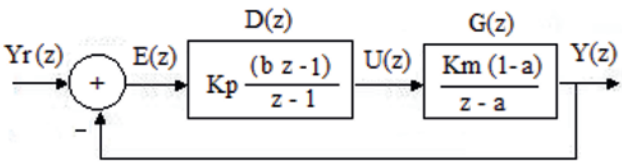

Figure 1.8. *Block diagram of a servomechanism controlled by a discrete PI controller*

a) Find and represent an equivalent block diagram of this servomechanism in the discrete state space based on the respective (Jordan) canonical realizations of G(z) and D(z).

b) Find the discrete state feedback control law, as well as the discrete state equation of this control system.

c) Knowing that $\{K_m = 1.1;\ a = 0.8065;\ K_p = 0.7;\ T = 0.2\ s\}$, use Matlab to generate the simulation results of the unit step response of the complete discrete state feedback control system. Then, proceed to the interpretation of the graphical results obtained.

Solution – Exercise 1.4.

a) A simple expansion leads to new expressions $G(z) = K_m\,(1-a)\,\dfrac{1/z}{1-1/z}$ and $D(z) = b\,K_p\,(1 + \dfrac{(1-1/b)\,(1/z)}{1-1/z})$. Jordan block diagrams of G(z) and D(z) are presented in Figure 1.9.

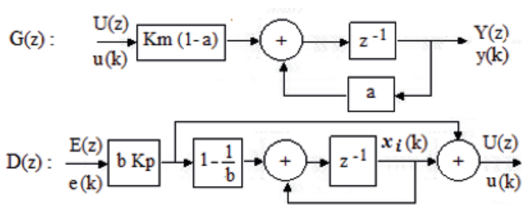

Figure 1.9. *Jordan block diagrams of G(z) and D(z)*

The combined control block diagram that results in the discrete state space is presented in Figure 1.10.

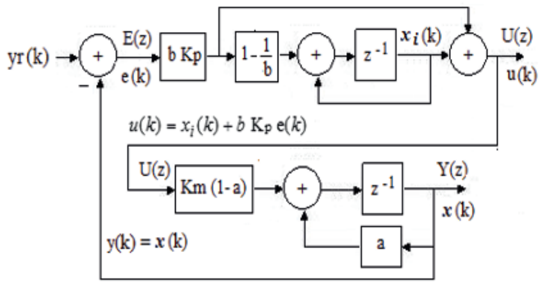

Figure 1.10. *Block diagram of discrete state space control*

b) Given the state variables x and x_i chosen in Figure 1.10, the discrete state feedback control law can be written as:

$$u(k) = \begin{bmatrix} -b\,Kp & 1 \end{bmatrix} \begin{bmatrix} x(k) \\ x_i(k) \end{bmatrix} + b\,Kp\; y_r(k)$$

Then, the discrete state equation of the control system is given by:

$$\begin{bmatrix} x(k+1) \\ x_i(k+1) \end{bmatrix} = \begin{bmatrix} a - Km\;(1-a)\;b\;Kp & Km\;(1-a) \\ -\;(1-(1/b))\;Kp & 1 \end{bmatrix} \begin{bmatrix} x(k) \\ x_i(k) \end{bmatrix} + \begin{bmatrix} Km\;(1\text{-a})\;b\;Kp \\ (1-(1/b))\;\;\mathrm{b}\;\;Kp \end{bmatrix} y_r(k)$$

$$y(k) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x(k) \\ x_i(k) \end{bmatrix}$$

c) Figure 1.11 presents the obtained simulation results. The Matlab program in Figure 1.12 can then be used to replicate these results. From a numerical point of view, the closed-loop results are:

$$\begin{cases} \begin{bmatrix} x(k+1) \\ x_i(k+1) \end{bmatrix} = \begin{bmatrix} 0.6157 & 0.2129 \\ -0.1965 & 1.0000 \end{bmatrix} \begin{bmatrix} x(k) \\ x_i(k) \end{bmatrix} + \begin{bmatrix} 0.1908 \\ 0.1965 \end{bmatrix} yr(k) \\ y(k) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x(k) \\ x_i(k) \end{bmatrix} \end{cases}$$

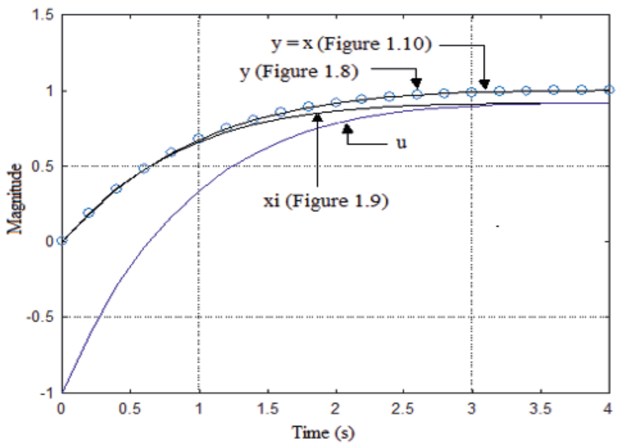

Figure 1.11. *Result of the Matlab-based simulation*

It can be noted that the block diagrams described in the frequency domain (Figure 1.8) and in the discrete state space (Figure 1.10) are equivalent from the input/output point of view, but they are not similar at the internal level, where the control of state x_i is explicit in the state space. Moreover, knowing that $u(\infty) = \begin{bmatrix} -b\,K_P & 1 \end{bmatrix} \begin{bmatrix} x(\infty) \\ x_i(\infty) \end{bmatrix} + b\,K_P\,y_r(\infty)$, if $y(\infty) = x(\infty) = y_r(\infty)$, then $u(\infty) = x_i(\infty)$ (see Figure 1.11).

```
                 % EXERCISE_I4.m
Km = 1.1     a = 0.8065;   b = 1.2807;     Kp = 0.7;     T= 0.2;
                  t = 0:T:4;    N = length(t);     z = tf('z');
               Dz = Kp*(b*z-1)/(z-1);          Gz = Km*(1-a)/(z-a);
 [numF, denF] = tfdata(feedback(series(Dz, Gz),1));

   Ytf = dstep(numF,denF, N);             %  Simulation Figure 1.8
 A = [a - Km*(1-a)*b*Kp    Km*(1-a);    -(1-(1/b))*b*Kp    1];
 B = [Km*(1-a)*Kp*b ;      b*Kp*(1-(1/b))];
 C = [1     0];    D = 0;
 [Yss, X] = dstep(A,B,C,D,1,N);          %  Simulation  Figure 1.10
        Yr = ones(N,1);  Er = Yr-Yss;  U = X(:,2) - Er;   % {u(k}
 plot(t, Ytf,'o', t,X(:,1),'k',t,X(:,2),'k',t,U,'b');  grid
```

Figure 1.12. *Example of Matlab program for the simulation of the control system in the discrete state space*