1

Modeling of Heat Transfer

This chapter presents a reminder of courses on heat transfer limited to what is necessary to understand and master the methods of measuring the thermal properties of materials which will be described in the rest of this book.

1.1. The different modes of heat transfer

1.1.1. *Introduction and definitions*

We will first define the main quantities involved in solving a heat transfer problem.

1.1.1.1. *Temperature field*

Energy transfers are determined from the evolution of the temperature in space and time: $T = f(x, y, z, t)$. The instantaneous value of the temperature at any point of space is a scalar quantity called a temperature field. We will distinguish two cases:

– time-independent temperature field: the regime is called steady state or stationary;

– evolution of the temperature field over time: the regime is called variable, unsteady or transient.

1.1.1.2. *Temperature gradient*

If all the points of space which have the same temperature are combined, an isothermal surface is obtained. The temperature variation per unit length is maximal in the direction normal to the isothermal surface. This variation is characterized by the temperature gradient:

$$\overrightarrow{grad}(T) = \vec{n}\,\frac{\partial T}{\partial n} \qquad [1.1]$$

where: $\vec{n}$ is the normal unit vector;

$\frac{\partial T}{\partial n}$ is the derivative of the temperature along the normal direction.

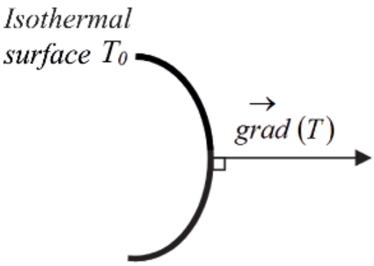

Figure 1.1. *Isothermal surface and thermal gradient*

1.1.1.3. *Heat flux*

Heat flows under the influence of a temperature gradient from high to low temperatures. The quantity of heat transmitted per unit time and per unit area of the isothermal surface is called the heat flux ϕ (W m^{-2}):

$$\phi = \frac{1}{S}\frac{dQ}{dt} \qquad [1.2]$$

where S is the surface area (m^2).

φ (W) is called the heat flow rate and is the quantity of heat transmitted to the surface S per unit time:

$$\varphi = \frac{dQ}{dt} \qquad [1.3]$$

1.1.1.4. *Energy balance*

The determination of the temperature field involves the writing of one or more energy balances. First, a system (S) must be defined by its limits in space and the different heat flow rates that influence the state of the system must be established and they can be:

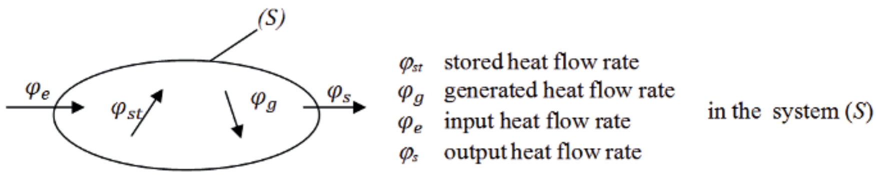

Figure 1.2. *System and energy balance*

The first principle of thermodynamics is then applied to establish the energy balance of the system (S):

$$\varphi_e + \varphi_g = \varphi_s + \varphi_{st} \tag{1.4}$$

After having replaced each of the terms by its expression as a function of the temperature, we obtain a differential equation whose resolution, taking into account the boundary conditions of the system, makes it possible to establish the temperature field. We will first give the possible expressions of the heat flow rates that can enter or exit a system by conduction, convection or radiation before giving an expression of the flux stored by sensible heat.

1.1.2. *Conduction*

Conduction is the transfer of heat within an opaque medium, without displacement of matter, under the influence of a temperature difference. The transfer of heat via conduction within a body takes place according to two distinct mechanisms: transmission via atomic or molecular vibrations and transmission via free electrons.

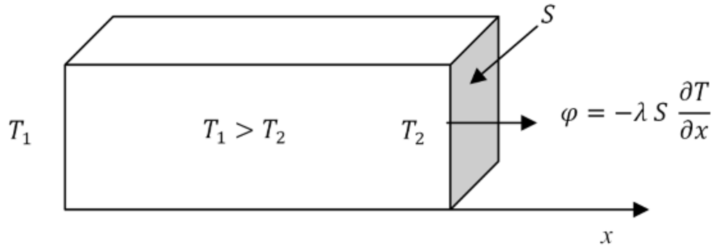

Figure 1.3. *Conductive heat transfer scheme*

The theory of conduction is based on the Fourier hypothesis: the heat flux ϕ is proportional to the temperature gradient:

$$\vec{\phi} = -\lambda \, \overrightarrow{grad}(T) \qquad [1.5]$$

The heat flow rate transmitted by conduction in the direction *Ox* can therefore be written in algebraic form:

$$\varphi = -\lambda \, S \, \frac{\partial T}{\partial x} \qquad [1.6]$$

where: ϕ is the conductive heat flux (W m^{-2});

φ is the conductive heat flow rate (W);

λ is the thermal conductivity of the medium (W m^{-1} K^{-1});

x is the space variable in the heat flow's direction (m);

S is the surface area of the passage of the heat flux (m^2).

The values of the thermal conductivity λ of some of the most common materials are given in Table 1.1. A more complete table is given in Appendices A.1 and A.2.

Material	λ (W m^{-1} K^{-1})	Material	λ (W m^{-1} K^{-1})
Silver	419	Plaster	0.48
Copper	386	Asbestos	0.16
Aluminum	204	Wood (hard, soft wood)	0.12–0.23
Mild steel	45	Cork	0.044–0.049
Stainless steel	15	Stone wool	0.038–0.041
Ice	1.9	Glass wool	0.035–0.051
Concrete	1.4	Expanded polystyrene	0.036–0.047
Clay brick	1.1	Polyurethane (foam)	0.030–0.045
Glass	1.0	Extruded polystyrene	0.028
Water	0.60	Air	0.026

Table 1.1. *Thermal conductivity of certain materials at room temperature*

1.1.3. *Convection*

Here, we will only consider the heat transfer between a solid and a fluid, the energy being transmitted by the fluid's displacement. A good representation of this transfer mechanism is given by Newton's law:

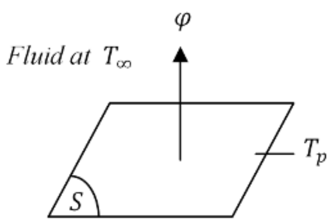

Figure 1.4. *Convective heat transfer scheme*

$$\varphi = h_c\, S\left(T_p - T_\infty\right) \qquad [1.7]$$

where: φ is the heat flow rate transmitted by convection (W);

h_c is the convective heat transfer coefficient (W m^{-2} K^{-1});

T_p is the solid's surface temperature (K);

T_∞ is the temperature of fluid away from solid surface (K);

S is the area of solid/fluid contact surface (m^2).

The value of the convective heat transfer coefficient h_c is a function of the fluid's nature, temperature, velocity or the temperature difference and the geometrical characteristics of the solid/fluid contact surface. The correlations in the most common cases of natural convection are given in Appendix 3, i.e. when the fluid's movement is due to temperature differences (no pump or fan).

Thermal characterization aims to measure the conductive and diffusing properties of a material. Convection most often occurs as a mode of "parasitic" transfer on the boundaries of the system by influencing the internal temperature field. We therefore have to take this into account. The correlations presented in Appendix 3 show that the coefficient of heat transfer by natural convection depends on the temperature difference between the surface and the surrounding fluid. Most often this difference is not perfectly uniform on surfaces and varies over time. It is therefore not possible to calculate it precisely and it will most often have to be estimated.

In natural convection, its value is generally between 2 and 5 W m^{-2} K^{-1}. The radiation heat transfer coefficient that will be defined below is of the same order of magnitude. It will therefore be noted that placing a device under vacuum makes it possible to reduce losses by decreasing convective transfers but not canceling them, because radiation transfer is not affected by pressure.

1.1.4. *Radiation*

Radiation is a transfer of energy by electromagnetic waves (it does not need material support and even exists in a vacuum). We will only focus here on the transfer between two surfaces. In conduction problems, we take into account the radiation between a solid (whose surface is assumed to be gray and diffusing) and the surrounding environment (of large dimensions). In this case, we have the equation:

$$\varphi = \sigma\, \varepsilon_p\, S\left(T_p{}^4 - T_\infty{}^4\right) \quad [1.8]$$

where: φ is the radiation heat flow rate (W);

σ is Stefan's constant ($5{,}67.10^{-8}$ W m^{-2} K^{-4});

ε_p is the surface emission factor;

T_p is the surface temperature (K);

T_∞ is the temperature of the medium surrounding the surface (K);

S is the area of surface (m^2).

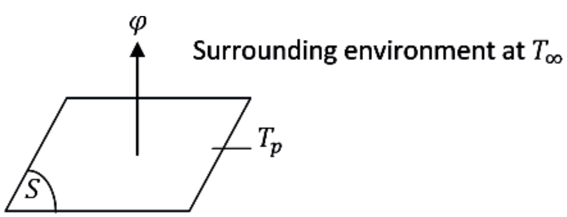

Figure 1.5. *Radiation heat transfer scheme*

NOTE.– In equations [1.6] and [1.7], temperatures can be expressed either in °C or K because they appear only in the form of differences. On the contrary, in equation [1.8], the temperature must be expressed in K.

1.1.4.1. *Linearization of the radiation flux*

In the case where the fluid in contact with the surface is a gas and where the convection is natural, the radiation heat transfer with the walls (at the average temperature T_r) surrounding the surface can become of the same order of magnitude as the convective heat transfer with the gas (at temperature T_f) at the contact with the surface and can no longer be neglected. The heat flow rate transferred by radiation is written according to equation [1.8]:

$$\varphi_r = \sigma \varepsilon S\left({T_p}^4 - {T_\infty}^4\right)$$

It can take the form: $\varphi_r = h_r S\left(T_p - T_\infty\right)$

where h_r is called the radiation transfer coefficient:

$$h_r = \sigma\varepsilon\left({T_p}^2 + {T_\infty}^2\right)\left(T_p + T_\infty\right) \qquad [1.9]$$

The radiation transfer coefficient h_r varies very slightly for limited variations of the temperatures T_p and T_∞ and can be regarded as constant for a first simplified calculation. For example, with $\varepsilon = 0.9$, $T_p = 60°C$ and $T_\infty = 20°C$, the exact value is $h_r = 6.28$ W m^{-2} K^{-1}. If T_p becomes equal to 50°C (instead of 60°C), the value of h_r becomes equal to 5.98 W m^{-2} K^{-1}, we get a variation of only 5%. When T_p is close to T_∞, we can consider: $h_r \approx 4\sigma\varepsilon\, {T_\infty}^3$. It is also noted that the calculated values are of the order of magnitude of a natural convection coefficient in air.

It is to be remembered that when the convection exchange of a surface with its environment takes place by natural convection, we write the global heat flow rate (convection + radiation) exchanged by the surface in the form of:

$$\varphi = hS\left(T_p - T_\infty\right)$$

where $h = h_c + h_r$

1.1.4.2. *Case of a high-temperature source*

The radiation transfer also occurs in the exchange of a wall with temperature T_p with a high-temperature heat source T_s, for example, the Sun (≈ black body at 5760 K). In this case, equation [1.8] becomes:

$$\varphi = K\, S\,\left({T_p}^4 - {T_s}^4\right) \qquad [1.10]$$

where K is a constant taking into account the surface and source emissivities as well as the geometric shape factor between the surface and the source.

Then equation [1.10] becomes: $\varphi = -K\,S\,T_s^4 = -\varphi_0$ and the radiation source is modeled by a constant heat flow rate φ_0 imposed on the wall (this is, for example, the case of a thermal sensor exposed to the Sun).

1.1.5. *Heat storage*

The storage of energy in sensitive form in a body corresponds to an increase of its enthalpy in the course of time from which (at constant pressure and in the absence of change of state):

$$\varphi_{st} = \rho V c \frac{\partial T}{\partial t} \tag{1.11}$$

where: φ_{st} is the stored heat flow rate (W);

ρ is density (kg m^{-3});

V is volume (m^3);

c is specific heat ($\text{J kg}^{-1}\ \text{K}^{-1}$);

T is temperature (K);

t is time (s).

The product $\rho V c$ (J K^{-1}) is called the thermal capacitance of the body.

1.2. Modeling heat transfer by conduction

1.2.1. *The heat equation*

In its mono-dimensional form, this equation describes the one-directional transfer of heat through a flat wall (see Figure 1.6).

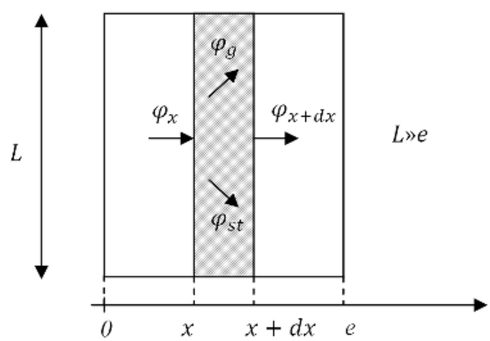

Figure 1.6. *Thermal balance of an elementary system*

Consider a system of thickness dx in the direction x and of section of area S normal to direction Ox. The energy balance of this system is written as:

$$\varphi_x + \varphi_g = \varphi_{x+dx} + \varphi_{st}$$

where: $\varphi_x = -\left(\lambda\, S\, \frac{\partial T}{\partial x}\right)_x$;

$$\varphi_{x+dx} = -\left(\lambda\, S\, \frac{\partial T}{\partial x}\right)_{x+dx};$$

$$\varphi_g = \dot{q} S dx;$$

$$\varphi_{st} = \rho c S\, dx \frac{\partial T}{\partial x}.$$

By plotting in the energy balance and dividing by dx, we obtain:

$$\frac{\left(\lambda\, S \frac{\partial T}{\partial x}\right)_{x+dx} - \left(\lambda\, S \frac{\partial T}{\partial x}\right)_x}{dx} + \dot{q} S = \rho c S\, \frac{\partial T}{\partial t}$$

or: $$\frac{\partial}{\partial x}\left(\lambda\, S\, \frac{\partial T}{\partial x}\right) + \dot{q} S = \rho c S\, \frac{\partial T}{\partial t}$$

and in the three-dimensional case, we obtain the heat equation in the most general case:

$$\frac{\partial}{\partial x}\left(\lambda_x \frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(\lambda_y \frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(\lambda_z \frac{\partial T}{\partial z}\right) + \dot{q} = \rho c \frac{\partial T}{\partial t} \qquad [1.12]$$

This equation can be simplified in a number of cases:

a) if the medium is isotropic: $\lambda_x = \lambda_y = \lambda_z = \lambda$;

b) if there is no generation of energy inside the system: $\dot{q} = 0$;

c) if the medium is homogeneous, λ is only a function of T.

The hypotheses a) + b) + c) make it possible to write:

$$\lambda\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right) + \frac{d\lambda}{dT}\left[\left(\frac{\partial T}{\partial x}\right)^2 + \left(\frac{\partial T}{\partial y}\right)^2 + \left(\frac{\partial T}{\partial z}\right)^2\right] = \rho c \frac{\partial T}{\partial t} \qquad [1.13]$$

d) If λ is constant (moderate temperature deviation), we obtain the Poisson equation:

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}=\frac{1}{a}\frac{\partial T}{\partial t} \quad [1.14]$$

The ratio $a=\frac{\lambda}{\rho c}$ is called thermal diffusivity (m^2 s^{-1}), it characterizes the propagation velocity of a heat flux through a material. Values can be found in Appendix 1.

e) In steady state, we obtain Laplace's equation:

$$\nabla^2 T=\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}=0 \quad [1.15]$$

Moreover, hypotheses a), c) and d) make it possible to write:

– heat equation in cylindrical coordinates (r, θ, z):

$$\frac{\partial^2 T}{\partial r^2}+\frac{1}{r}\frac{\partial T}{\partial r}+\frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2}+\frac{\partial^2 T}{\partial z^2}+\frac{\dot{q}}{\lambda}=\frac{1}{a}\frac{\partial T}{\partial t} \quad [1.16]$$

In the case of a cylindrical symmetry problem where the temperature depends only on r and t, equation [1.16] can be written in simplified form:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)+\frac{\dot{q}}{\lambda}=\frac{1}{a}\frac{\partial T}{\partial t}.$$

– heat equation in spherical coordinates (r,θ,φ):

$$\frac{1}{r}\frac{\partial^2}{\partial r^2}(rT)+\frac{1}{r^2 \sin(\theta)}\frac{\partial}{\partial \theta}\left[\sin(\theta)\frac{\partial T}{\partial \theta}\right]+\frac{1}{r^2 \sin^2(\theta)}\frac{\partial^2 T}{\partial \varphi^2}+\frac{\dot{q}}{\lambda}=\frac{1}{a}\frac{\partial T}{\partial t} \quad [1.17]$$

1.2.2. *Steady-state conduction*

1.2.2.1. *Simple wall*

We will place ourselves in the situation where the heat transfer is one directional and where there is no energy generation or storage.

We consider a wall of thickness e, thermal conductivity λ and large transverse dimensions whose extreme faces are at temperatures T_1 and T_2 (see Figure 1.7).

By carrying out a thermal balance on the system (S) constituted by the wall slice comprised between the abscissae x and $x + dx$, we obtain:

$$\varphi_x=\varphi_{x+dx} \Rightarrow \left(\lambda S \frac{dT}{dx}\right)_x=\left(\lambda S \frac{dT}{dx}\right)_{x+dx}$$

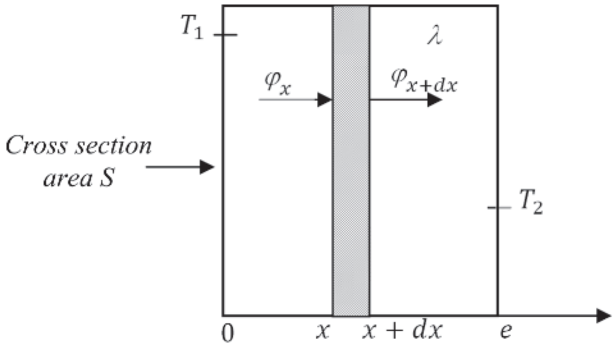

Figure 1.7. *Basic thermal balance on a simple wall*

where: $\frac{dT}{dx} = A$ and $T(x) = A\,x + B$

with boundary conditions: $T(x = 0) = T_1$ et $T(x = e) = T_2$

Thus: $T(x) = T_1 - \frac{x}{e}\,(T_1 - T_2\,)$ [1.18]

The temperature profile is therefore linear. The heat flux passing through the wall is deduced by equation: $\phi = -\lambda\,\frac{\partial T}{\partial x}$

Thus: $\phi = \frac{\lambda}{e}\,(T_1 - T_2\,)$ [1.19]

Equation [1.19] can also be written as: $\varphi = \frac{T_1 - T_2}{\frac{e}{\lambda S}}$, this equation is analogous to Ohm's law in electricity which defines the intensity of the current as the ratio of the electrical potential difference on the electrical resistance. The temperature T thus appears as a thermal potential and the term $\frac{e}{\lambda S}$ appears as the thermal resistance of a plane wall of thickness e, thermal conductivity λ and lateral surface S. We thus get the equivalent network represented in Figure 1.8.

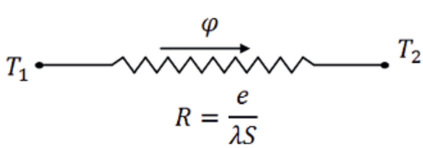

Figure 1.8. *Equivalent electrical network of a single wall*

NOTE.– The flux is constant, it is a general result for any tube of flux in steady state (system with conservative flux).

1.2.2.2. *Multilayer wall*

This is the case for real walls (described in Figure 1.9) made up of several layers of different materials and where only the temperatures T_{f1} and T_{f2} of the fluids that are in contact with the two faces of the lateral surface wall S are known.

In steady state, the heat flow rate is preserved when the wall is crossed and is written as:

$$\varphi = h_1 S\left(T_{f1} - T_1\right) = \frac{\lambda_A S(T_1 - T_2)}{e_A} = \frac{\lambda_B S(T_2 - T_3)}{e_B} = \frac{\lambda_C S(T_3 - T_4)}{e_C} = h_2 S\left(T_4 - T_{f2}\right)$$

where: $$\varphi = \frac{T_{f1} - T_{f2}}{\frac{1}{h_1 S} + \frac{e_A}{\lambda_A S} + \frac{e_B}{\lambda_B S} + \frac{e_C}{\lambda_C S} + \frac{1}{h_2 S}} \quad [1.20]$$

It was considered that the contacts between the layers of different natures were perfect and that there was no temperature discontinuity at the interfaces. In reality, given the roughness of the surfaces, a micro-layer of air exists between the surface hollows contributing to the creation of a thermal resistance (the air is an insulator) called thermal contact resistance. The previous formula is then written as:

$$\varphi = \frac{T_{f1} - T_{f2}}{\frac{1}{h_1 S} + \frac{e_A}{\lambda_A S} + R_{AB} + \frac{e_B}{\lambda_B S} + R_{BC} + \frac{e_C}{\lambda_C S} + \frac{1}{h_2 S}} \quad [1.21]$$

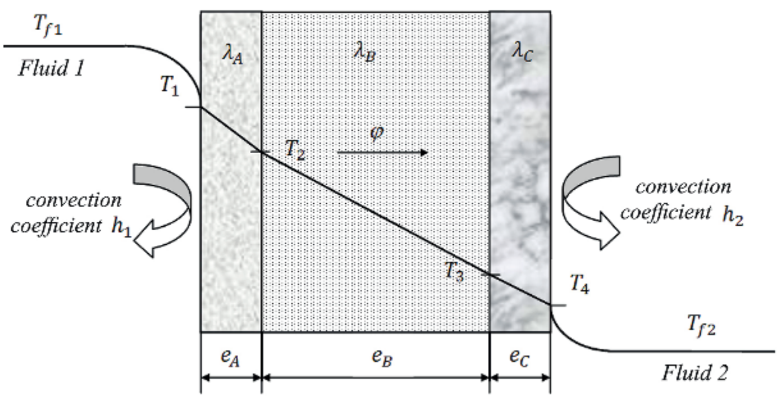

Figure 1.9. *Schematic representation of heat flow and temperatures in a multilayer wall*

The equivalent electrical diagram is shown in Figure 1.10.

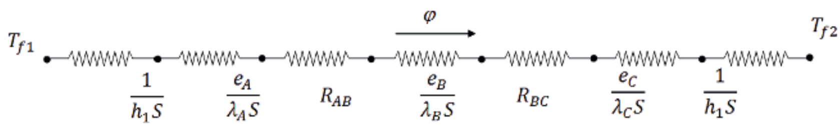

Figure 1.10. *Equivalent electrical network of a multilayer wall*

NOTES.–

– Thermal resistance can only be defined on a flux tube.

– The thermal contact resistance between two layers is neglected if one layer is an insulating material or if the layers are joined by welding.

1.2.2.3. *Composite wall*

This is the case most commonly encountered in reality where the walls are not homogeneous. Let us consider, by way of example, a wall of width L consisting of hollow agglomerates (Figure 1.11).

Considering the symmetries, the calculation of the wall's thermal resistance can be reduced to that of a unit cell defined by the diagram in Figure 2.6. This unit cell is a flux tube (isotherm at $x = 0$ and $x = e_1 + e_2 + e_3$, and adiabatic at $y = 0$ and $y = \ell_1 + \ell_2 + \ell_3$) and can therefore be represented by a resistance R.

The exact calculation of this resistance is complex because each medium does not constitute a flux tube. If there is no need for high accuracy, several approximate calculations are possible by making assumptions on isothermal and adiabatic surfaces.

For example, assuming the surfaces at $x = e_1$ and $x = e_1 + e_2$ to be isothermal and the surfaces at $y = \ell_3$ and $y = \ell_2 + \ell_3$ adiabatic and by using the laws of association of the resistors in series and in parallel, we obtain:

$$R = R_1 + R_2 + \frac{1}{\frac{1}{R_3}+\frac{1}{R_4}+\frac{1}{R_5}} + R_6 + R_7$$

where:

$$R_1 = \frac{1}{h_1 \ell L}; R_2 = \frac{e_1}{\lambda_1 \ell L}; R_3 = \frac{e_2}{\lambda_2 \ell_1 L}; R_4 = \frac{e_2}{\lambda_1 \ell_2 L}; R_5 = \frac{e_2}{\lambda_1 \ell_3 L}; R_6 = \frac{e_3}{\lambda_1 \ell L}; R_7 = \frac{1}{h_2 \ell L}$$

which can be represented by the equivalent electrical network shown in Figure 1.12.

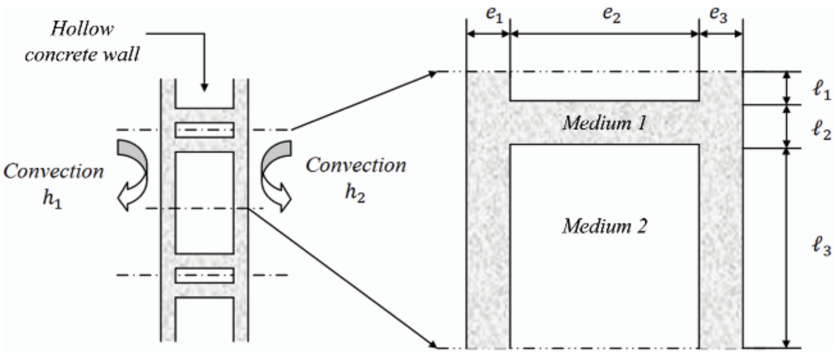

Figure 1.11. *Diagram of a composite wall*

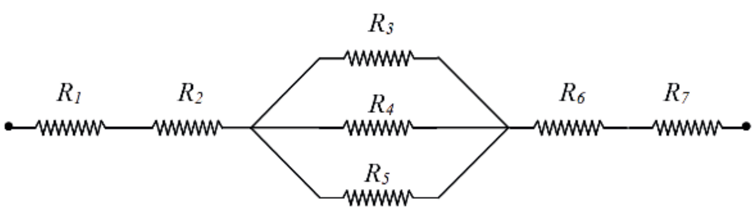

Figure 1.12. *Electrical equivalent network of a composite wall*

It will be noted that this solution is only approximate since the real transfer is 2D because of the differences between the thermal conductivities λ_1 and λ_2.

1.2.2.4. *Long hollow cylinder (tube)*

We consider a hollow cylinder with a thermal conductivity λ, internal radius r_1, external radius r_2, length L, and with internal and external face temperatures being T_1 and T_2, respectively (see Figure 1.13). It is assumed that the longitudinal temperature gradient is negligible compared to the radial gradient.

Let us carry out the thermal balance of the system constituted by the part of the cylinder comprised between radii r and $r + dr$: $\varphi_r = \varphi_{r+dr}$

where: $\varphi_r = -2\pi r L\lambda\left(\frac{dT}{dr}\right)_r$ and: $\varphi_{r+dr} = -2\pi(r+dr)L\lambda\left(\frac{dT}{dr}\right)_{r+dr}$

Thus: $-2\pi r L\lambda\left(\frac{dT}{dr}\right)_r = -2\pi(r+dr)L\lambda\left(\frac{dT}{dr}\right)_{r+dr}$ where: $r\frac{dT}{dr} = C$

with boundary conditions: $T(r_1) = T_1$ and $T(r_2) = T_2$

where: $\frac{T(r)-T_1}{T_2-T_1} = \frac{ln\left(\frac{r}{r_1}\right)}{ln\left(\frac{r_2}{r_1}\right)}$ [1.22]

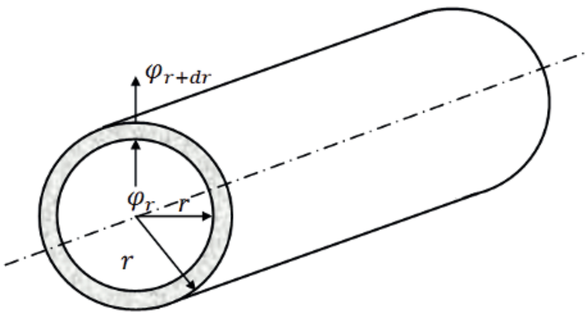

Figure 1.13. *Diagram of transfers in a hollow cylinder*

and by applying equation $\varphi = -2\pi r L\lambda\left(\frac{dT}{dr}\right)$, we obtain:

$$\varphi = \frac{2\pi\lambda L(T_1-T_2)}{ln\left(\frac{r_2}{r_1}\right)} \quad [1.23]$$

This equation can also be written as: $\varphi = \frac{T_1-T_2}{R_{12}}$ with $R_{12} = \frac{ln\left(\frac{r_2}{r_1}\right)}{2\pi\lambda L}$ and can be represented by the equivalent electrical network of Figure 1.14.

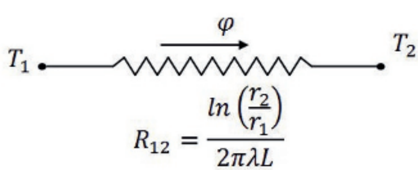

Figure 1.14. *Equivalent electrical network of a hollow cylinder*

1.2.2.5. *Multilayer hollow cylinder*

This is the practical case of a tube covered with one or more layers of different materials and where only temperatures T_{f1} and T_{f2} of the fluids in contact with the inner and outer faces of the cylinder are known; h_1 and h_2 are the global (convection + radiation) heat transfer coefficients between the fluids and the internal and external faces (see Figure 1.15).

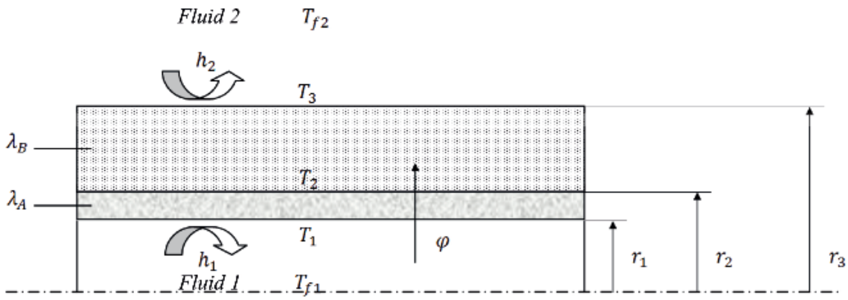

Figure 1.15. *Diagram of heat transfers in a multilayer hollow cylinder*

In steady-state conditions, the heat flow rate φ is conserved when passing through the different layers and is written as:

$$\varphi = h_1 2\pi r_1 L\left(T_{f1} - T_1\right) = \frac{2\pi\lambda_A L(T_1 - T_2)}{ln\left(\frac{r_2}{r_1}\right)} = \frac{2\pi\lambda_B L(T_2 - T_3)}{ln\left(\frac{r_3}{r_2}\right)} = h_2 2\pi r_3 L\left(T_3 - T_{f2}\right)$$

Hence: $$\varphi = \frac{T_{f1} - T_{f2}}{\frac{1}{h_1 2\pi r_1 L} + \frac{ln\left(\frac{r_2}{r_1}\right)}{2\pi\lambda_A L} + \frac{ln\left(\frac{r_3}{r_2}\right)}{2\pi\lambda_B L} + \frac{1}{h_2 2\pi r_3 L}} \qquad [1.24]$$

which can be represented by the equivalent electrical network of Figure 1.16.

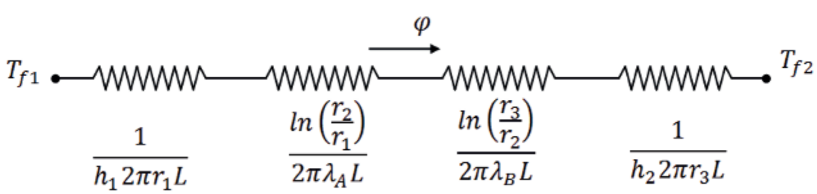

Figure 1.16. *Equivalent electrical network of a multilayer hollow cylinder*

1.2.2.6. *General case*

We have just given the value of the thermal resistance for some particular cases of practical interest. More generally, the thermal resistance of a flux tube (see Figure 1.17) is written as:

$$R = \int_{r_1}^{r_2} \frac{dr}{\lambda(r)\, S(r)} \qquad [1.25]$$

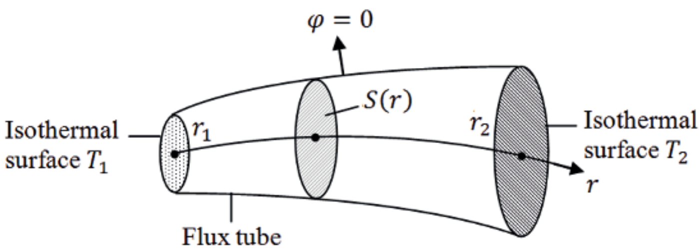

Figure 1.17. *Diagram of a flux tube*

1.2.3. *Conduction in unsteady state*

1.2.3.1. *Medium at uniform temperature*

We will study the transfer of heat to a medium at a uniform temperature ("small body" hypothesis), which is *a priori* contradictory because it is necessary that there is a thermal gradient for heat transfer to occur. This approximation of the medium at uniform temperature may nevertheless be justified in certain cases which will be specified. For example, the quenching of a metal ball which consists of immersing a ball initially at the temperature T_i in a bath of constant temperature T_0. Assuming that the temperature inside the ball is uniform, which will be all the more true as its size is small and its thermal conductivity is high, we can write the thermal balance of this ball between two points in time as t and $t + dt$:

$$-hS(T - T_0) = \rho c V \frac{dT}{dt} \quad \text{or} \quad \frac{dT}{T - T_0} = -\frac{hS}{\rho c V}\, dt$$

Hence: $\dfrac{T - T_0}{T_i - T_0} = exp\left(-\dfrac{hS}{\rho c V}\, t\right)$ [1.26]

This solution is shown in Figure 1.18. We note that the grouping $\frac{\rho cV}{hS}$ is homogeneous at a time which will be called τ and is the system's time constant:

$$\tau = \frac{\rho cV}{hS} \qquad [1.27]$$

This quantity is fundamental insofar as it gives the order of magnitude of the physical phenomenon at that point in time; we have in fact: $\frac{T-T_0}{T_i-T_0} = exp\left(-\frac{t}{\tau}\right)$

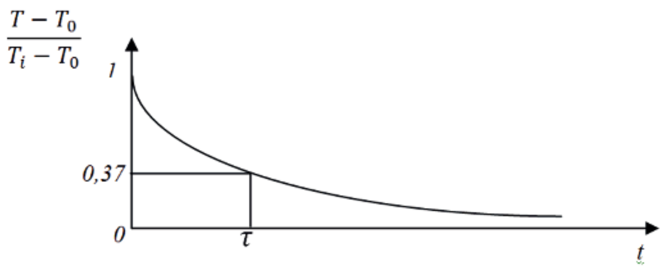

Figure 1.18. *Evolution of the temperature of a medium at uniform temperature*

It is always interesting in physics to present the results in dimensionless form. Two dimensionless numbers are particularly important in variable regimes:

– The Biot number: $Bi = \frac{Internal\ thermal\ resistance}{External\ thermal\ resistance} = \frac{\frac{\ell}{\lambda S}}{\frac{1}{hS}}$, ℓ is the medium's characteristic dimension, $\ell = R$ for a sphere.

Or: $Bi = \frac{h\ell}{\lambda}$ [1.28]

– The Fourier number:

$$Fo = \frac{at}{\ell^2} \qquad [1.29]$$

The Fourier number characterizes heat penetration in a variable regime.

The definition of these two dimensionless numbers makes it possible to write the equation of the temperature of the ball in the form:

$$\frac{T-T_0}{T_i-T_0} = exp(-Bi\ Fo) \qquad [1.30]$$

Knowing the product of the Biot and Fourier numbers makes it possible to determine the evolution of the sphere's temperature. It is generally considered that a system such as $Bi < 0.1$ can be considered to be at a uniform temperature; the criterion $Bi < 0.1$ is called the criterion of "thermal accommodation".

1.2.3.2. *Semi-infinite medium*

A semi-infinite medium is a wall of sufficiently large thickness so that the perturbation applied to one face is not felt by the other face. Such a system represents the evolution of a wall of finite thickness for a sufficiently short time so that the disturbance created on one face has not reached the other side (true for the point in time that the temperature of the other face does not vary). We will consider, for example, a layer of thickness 1 µm as a semi-infinite medium in the 3ω method!

EXAMPLE.– *Imposed surface temperature.*

Method: integral Laplace transform in time and inversion by tables.

The semi-infinite medium is initially at the uniform temperature T_i (see Figure 1.19). The surface temperature is suddenly maintained at a temperature T_0; this boundary condition is called the Dirichlet condition.

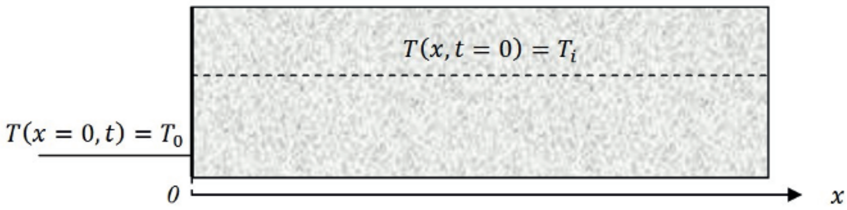

Figure 1.19. *Diagram of semi-infinite medium with imposed surface temperature*

The heat equation is written as: $\dfrac{\partial^2 T}{\partial x^2} = \dfrac{1}{a}\dfrac{\partial T}{\partial t}$ [1.31]

$$T(x, 0) = T_i \quad [1.32]$$

with boundary conditions: $T(0, t) = T_0$ [1.33]

$$T(x, t) \to T_i \text{ when } x \to \infty \quad [1.34]$$

The following variable change is made: $\bar{T} = T - T_i$

Hence: $\frac{\partial \bar{T}}{\partial x} = \frac{\partial T}{\partial x}, \frac{\partial^2 \bar{T}}{\partial x^2} = \frac{\partial^2 T}{\partial x^2}$ and $\frac{\partial \bar{T}}{\partial t} = \frac{\partial T}{\partial t}$

The equation [1.30] can thus be written as: $\frac{\partial^2 \bar{T}}{\partial x^2} = \frac{1}{a}\frac{\partial \bar{T}}{\partial t}$ [1.35]

$\bar{T}(x, 0) = 0$ [1.36]

The boundary conditions become: $\bar{T}(0, t) = T_0 - T_i$ [1.37]

$\bar{T}(x, t) \to 0$ when $x \to \infty$ [1.38]

The Laplace transform of $\bar{T}(x, t)$ with respect to time is written as (see Appendix 4 on integral transformations): $\theta(x, p) = L[\bar{T}(x, t)] = \int_0^\infty exp(-pt)\bar{T}(x, t)\, dt$.

The Laplace transform of equation [1.35] leads to: $\frac{d^2\theta}{dx^2} - \frac{1}{a}\,[p\theta - \bar{T}(x, 0)] = 0$

with: $\bar{T}(x, 0) = 0$ (which justifies the change of variable).

This equation is therefore of the form: $\frac{d^2\theta}{dx^2} - q^2\theta = 0$ with: $q = \sqrt{\frac{p}{a}}$.

Hence: $\theta(x, p) = A\, exp(-qx) + B\, exp(qx)$.

The temperature keeps a finite value when $x \to \infty$, thus $B = 0$ and:

$\theta(x, p) = A\, exp(-qx)$.

The Laplace transform of equation [1.37] leads to: $(0, p) = \frac{T_0 - T_i}{p}$,

hence: $A = \frac{T_0 - T_i}{p}$

and: $\theta(x, p) = \frac{T_0 - T_i}{p}\, exp(-qx)$.

Using the inverse Laplace transform tables presented in Appendix 5 leads to the following result:

$\frac{T(x,t) - T_0}{T_i - T_0} = erf\left(\frac{x}{2\sqrt{at}}\right)$ [1.39]

where: $erf(u) = \frac{2}{\sqrt{\pi}}\int_0^u exp(-t^2)dt$.

The function erf is called the error function (see values in Appendix 6).

EXAMPLE.– *Imposed heat flux at the surface.*

Method: integral Laplace transform in time and inversion by tables.

Consider the same configuration but instead by brutally imposing a heat flux density on the surface of the semi-infinite medium (see Figure 1.20); this boundary condition is called the Neumann condition.

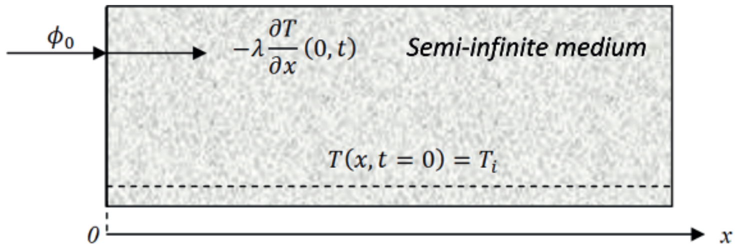

Figure 1.20. *Diagram of the semi-infinite medium with imposed surface flux*

The heat equation is written as: $\frac{\partial^2 T}{\partial x^2} = \frac{1}{a}\frac{\partial T}{\partial t}$ [1.40]

$T(x,0) = T_i$ [1.41]

with the boundary conditions: $-\lambda \frac{\partial T}{\partial x}(0,t) = \phi_0$ [1.42]

$T(x,t) \to T_i$ when $x \to \infty$ [1.43]

Condition [1.42] expresses the conservation of heat flux at the surface of the semi-infinite medium.

The following change of variable is made: $\bar{T} = T - T_i$.

Equation [1.40] can then be written as: $\frac{\partial^2 \bar{T}}{\partial x^2} = \frac{1}{a}\frac{\partial \bar{T}}{\partial t}$ [1.44]

$\bar{T}(x,0) = 0$ [1.45]

The boundary conditions become: $-\lambda \frac{\partial \bar{T}}{\partial x}(0,t) = \phi_0$ [1.46]

$\bar{T}(x,t) \to 0$ when $x \to \infty$ [1.47]

The Laplace transform of equation [1.44] leads to: $\frac{d^2\theta}{dx^2} - q^2\theta = 0$ with: $q = \sqrt{\frac{p}{a}}$. hence: $\theta(x,p) = Aexp(-qx) + Bexp(qx)$.

The temperature keeps a finite value when $x \to \infty$, thus $B = 0$ and $\theta(x,p) = Aexp(-qx)$.

The Laplace transform of equation [1.46] leads to: $\frac{\phi_0}{p} = -\lambda \frac{d\theta}{dx}(0,p)$.

Hence: $A = \frac{\phi_0}{\lambda p q}$ and $\theta(x,p) = \frac{\phi_0}{\lambda}\frac{exp(-qx)}{pq}$.

Using the inverse Laplace transform tables presented in Appendix 5 leads to the following result:

$$T(x,t) - T_i = \frac{2\phi_0}{\lambda}\sqrt{at}\; ierfc\left(\frac{x}{2\sqrt{at}}\right) \qquad [1.48]$$

where: $ierfc(u) = \frac{1}{\sqrt{\pi}} exp(-u^2) - u[1 - erf(u)]$, this function is tabulated in Appendix 6.

EXAMPLE.– *Imposed heat transfer coefficient.*

Method: integral Laplace transform in time and inversion by tables.

We consider the case where the convective heat transfer coefficient h between the semi-infinite medium and the ambient medium is imposed (see Figure 1.21). This boundary condition is called Newton's condition:

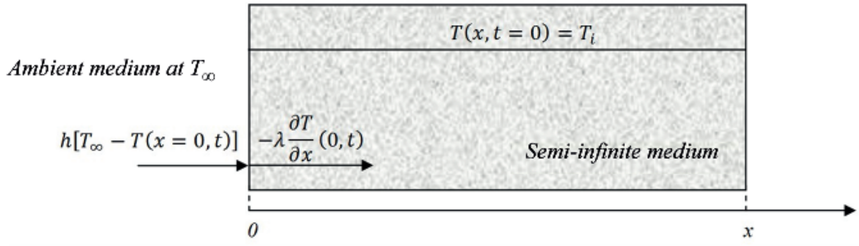

Figure 1.21. *Diagram of the semi-infinite medium with imposed convective transfer coefficient*

The heat equation is written as: $\frac{\partial^2 T}{\partial x^2} = \frac{1}{a}\frac{\partial T}{\partial t}$ [1.49]

$T(x,0) = T_i$ [1.50]

with the boundary conditions: $-\lambda \frac{\partial T}{\partial x}(0,t) = h[T_\infty - T(x=0,t)]$ [1.51]

$T(x,t) \to T_i$ when $x \to \infty$ [1.52]

The following change of variable is made: $\bar{T} = T - T_i$.

Equation [1.49] can then be written as: $\frac{\partial^2 \bar{T}}{\partial x^2} = \frac{1}{a}\frac{\partial \bar{T}}{\partial t}$ [1.53]

$\bar{T}(x,0) = 0$ [1.54]

The boundary conditions become: $\lambda \frac{\partial \bar{T}}{\partial x}(0,t) = h[\bar{T}(x=0,t) - (T_\infty - T_i)]$ [1.55]

$\bar{T}(x,t) \to 0$ when $x \to \infty$ [1.56]

The Laplace transform of equation [1.53] leads to: $\frac{d^2\theta}{dx^2} - q^2\theta = 0$ with $q = \sqrt{\frac{p}{a}}$.

Hence: $\theta(x,p) = A\,exp(-qx) + B\,exp(qx)$.

The temperature keeps a finite value when $x \to \infty$, thus $B = 0$ and:

$\theta(x,p) = A\,exp(-qx)$.

The Laplace transform of equation [1.55] is written as:

$\lambda \frac{d\theta}{dx}(0,p) = h\theta(0,p) + h\frac{T_i - T_\infty}{p}$

or: $-\lambda\,Aq = hA + h\frac{T_i - T_\infty}{p}$ hence: $A = \frac{\frac{h}{\lambda}(T_\infty - T_i)}{p\left(\frac{h}{\lambda}+q\right)}$

and: $\theta(x,p) = \frac{h}{\lambda}(T_\infty - T_i)\frac{exp(-qx)}{p\left(q+\frac{h}{\lambda}\right)}$.

Using the inverse Laplace transform tables presented in Appendix 5 leads to the following result:

$$\frac{T(x,t)-T_\infty}{T_i - T_\infty} = erf\left(\frac{x}{2\sqrt{at}}\right) + exp\left(\frac{hx}{\lambda} + \frac{ah^2 t}{\lambda^2}\right) erfc\left(\frac{x}{2\sqrt{at}} + \frac{h\sqrt{at}}{\lambda}\right) \quad [1.57]$$

EXAMPLE.– *Sinusoidal surface temperature, established periodic regime (see Figure 1.22).*

Method: search for a solution with the same frequency as the excitation

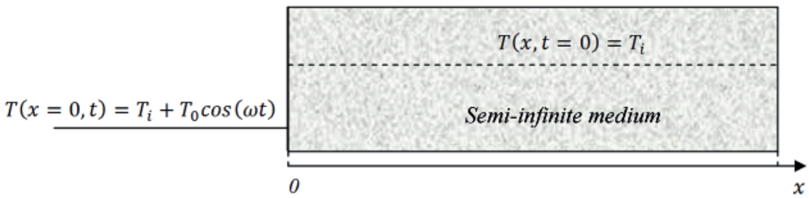

Figure 1.22. *Diagram of a semi-infinite medium with surface-imposed sinusoidal temperature*

The heat equation is written as: $\frac{\partial^2 T}{\partial x^2} = \frac{1}{a}\frac{\partial T}{\partial t}$

with the boundary conditions: $T(x = 0, t) = T_i + T_0 cos(\omega t)$

$T(x, t) \to T_i$ when $x \to \infty$

A steady-state solution is sought for which the temperature field of the medium evolves as follows:

$$T(x, t) = T_i + f(x)exp(i\omega t)$$

The problem, being linear, makes us consider either the real part or the imaginary part of the solution depending on whether the temperature varies as $cos(\omega t)$ or $sin(\omega t)$. The complex function f is a solution of:

$$\frac{d^2 f}{dx^2} - i\frac{\omega}{a} f = 0 \text{ with } f(0) = T_0$$

$$f(x) = Aexp\left(-\sqrt{\frac{i\omega}{a}}x\right) + Bexp\left(\sqrt{\frac{i\omega}{a}}x\right) \text{ with: } \sqrt{\frac{i\omega}{a}} = \sqrt{\frac{\omega}{2a}}(1 + i)$$

The function f must remain finite when $x \to \infty$, thus $B = 0$ and $f(0) = T_0$ leading to $A = T_0$.

Hence:

$$T(x, t) - T_i = T_0 exp\left[-\sqrt{\frac{\omega}{2a}}(1 + i)x + i\omega t\right] = T_0 exp\left(-\sqrt{\frac{\omega}{2a}}x\right) exp\left[i\left(\omega t - \sqrt{\frac{\omega}{2a}}x\right)\right]$$

or by taking the real part of the solution:

$$T(x,t) = T_i + T_0 \, exp\left(-\sqrt{\frac{\omega}{2a}}x\right) cos\left(\omega t - \sqrt{\frac{\omega}{2a}}x\right) \quad [1.58]$$

NOTES.–

– The amplitude of the oscillations decreases rapidly when moving away from the interface.

– The amplitude of the oscillations also decreases rapidly when the excitation frequency increases: a high frequency excitation applied to the surface of a solid will only change its temperature at a shallow depth.

– Between temperatures T_1 and T_2 of two points with a distance respectively of x_1 and x_2 from the surface, there exists a phase shift equal to $\sqrt{\frac{\omega}{2a}}\,(x_1 - x_2)$. The knowledge of ω and the temperature measurement within the medium at two points located at known distances x_1 and x_2 from the surface can make it possible to evaluate the thermal diffusivity a.

1.2.4. *The quadrupole method*

In the following, we will note:

– $\theta(x,p)$ the Laplace transform of the temperature $T(x,t)$ (see Appendix 4).

– $\Phi(x,p)$ the Laplace transform of the heat flow rate $\varphi(x,t)$.

In the following, we will explain the principle of quadrupole modeling and demonstrate the expressions of the most common quadrupole matrices. The expressions of these matrices for other configurations of interest are given in Appendix 7 and [MAI 00].

1.2.4.1. *Unidirectional transfer in plane walls*

1.2.4.1.1. Single wall

We consider the case of a unidirectional heat transfer in a wall of thickness e, with zero initial temperature: $T(x,0) = 0$.

The temperature $T(x,t)$ within the wall satisfied the equation: $\frac{\partial^2 T}{\partial x^2} = \frac{1}{a}\frac{\partial T}{\partial t}$ [1.59]

By applying the Laplace transformation to equation [1.59], we obtain:

$$\frac{d^2\theta}{dx^2} = \frac{p}{a}\theta \qquad [1.60]$$

Equation [1.60] takes a solution in the form:

$$\theta(x,p) = k_1\, cosh(qx) + k_2\, sinh(qx) \qquad [1.61]$$

where: $q = \sqrt{\frac{p}{a}}$.

The Laplace transform of the heat flow rate at any point of the wall is written as:

$$\Phi(x,p) = L\left[-\lambda S \frac{\partial T}{\partial x}(x,t)\right] = -\lambda S\, L\left[\frac{\partial T}{\partial x}(x,t)\right] = -\lambda S\, \frac{d\theta}{dx}(x,p)$$

This equation makes it possible to express $\Phi(x,p)$ as a function of k_1, k_2 and x:

$$\Phi(x,p) = -\lambda S q k_1\, sinh(qx) - \lambda S q k_2\, cosh(qx) \qquad [1.62]$$

Equations [1.61] and [1.62] can be written for $x = 0$ and $x = e$, in which case we obtain:

$$\theta(0,p) = k_1 \qquad \theta(e,p) = k_1\, cosh(qe) + k_2\, sinh(qe)$$
$$\Phi(0,p) = -\lambda S q k_2 \qquad \Phi(e,p) = -\lambda S q k_1\, sinh(qe) - \lambda S q k_2\, cosh(qe)$$

It is possible to eliminate k_1 and k_2 from these four equations, which amounts, for example, to expressing $(\theta(0), \Phi(0))$ as a function of $(\theta(e), \Phi(\mathrm{e}))$, in which case we obtain:

$$\begin{bmatrix}\theta(0,p)\\ \Phi(0,p)\end{bmatrix} = \begin{bmatrix} cosh(qe) & \frac{1}{\lambda q S}\, sinh(qe) \\ \lambda q S\, sinh(qe) & cosh(qe)\end{bmatrix}\begin{bmatrix}\theta(e,p)\\ \Phi(e,p)\end{bmatrix} \qquad [1.63]$$

$M = \begin{bmatrix} cosh(qe) & \frac{1}{\lambda q S}\, sinh(qe) \\ \lambda q S\, sinh(qe) & cosh(qe)\end{bmatrix}$ is called the quadrupole matrix.

We have the property: $det(M) = 1$, which makes it possible to establish the reciprocal relation:

$$\begin{bmatrix}\theta(e,p)\\ \Phi(e,p)\end{bmatrix} = \begin{bmatrix} cosh(qe) & -\frac{1}{\lambda q S}\, sinh(qe) \\ -\lambda q S\, sinh(qe) & cosh(qe)\end{bmatrix}\begin{bmatrix}\theta(0,p)\\ \Phi(0,p)\end{bmatrix}$$

An analogy can also be made between the propagation of an unsteady state current and the unidirectional thermal transfer in transitory mode:

Intensity of the electric current I	$\rightarrow$	Heat flow rate in Laplace space $\Phi(x,p)$
Electrical potential U	$\rightarrow$	Temperature in Laplace space $\theta(x,p)$
Electrical impedance Z	$\rightarrow$	Thermal impedance Z

Ohm's law: $U_1 - U_2 = ZI$ is analogous to: $\theta_1 - \theta_2 = Z\Phi$

Kirchhoff's law: $\sum I = 0$ is analogous to: $\sum \Phi = 0$

By means of these notations, the quadrupole relation [1.63] can be represented by the equivalent electrical network of Figure 1.23.

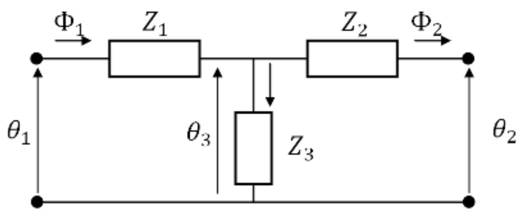

Figure 1.23. *Equivalent electrical network to a single wall in unsteady state*

In the case of a plane wall:

$$Z_1 = Z_2 = \frac{cosh(qe) - 1}{\lambda qS\, sinh(qe)} \quad \text{and} \quad Z_3 = \frac{1}{\lambda qS\, sinh(qe)}$$

1.2.4.1.2. Wall with a convective exchange

We consider the case of a wall exchanging heat by convection with a fluid (see Figure 1.24).

Equation $\varphi = hS[T_\infty - T(0,t)]$ can also be written as: $T_\infty = \frac{\varphi}{hS} + T(0,t)$ which can be translated in the Laplace space by: $\theta_\infty = \frac{\Phi}{hS} + \theta(x=0)$.

We can therefore write in quadrupole matrix form:

$$\begin{bmatrix} \theta_\infty \\ \Phi_\infty \end{bmatrix} = \begin{bmatrix} 1 & \frac{1}{hS} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \theta(0,p) \\ \Phi(0,p) \end{bmatrix} \qquad [1.64]$$

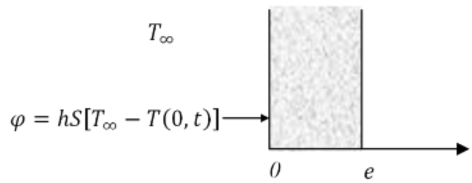

Figure 1.24. *Diagram of a single wall with convective transfer*

The quadrupole relation [1.64] can be represented by the equivalent electrical diagram of Figure 1.25.

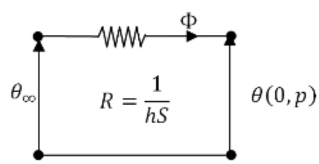

Figure 1.25. *Equivalent electrical network to a convective transfer in unsteady state*

1.2.4.1.3. Contact resistance between two walls

Consider now the case of the heat transfer through a contact resistance R at the interface between two solid media as shown in Figure 1.26.

The heat flow rate is written as $\varphi = \frac{T_{1(x=0)} - T_{2(x=0)}}{R}$ and can also be written as: $T_{1(x=0)} = T_{2(x=0)} + R\varphi$ that we can translate in the Laplace space by:

$$\theta_{1(x=0)} = \theta_{2(x=0)} + R\Phi.$$

We can therefore write in quadrupole matrix form:

$$\begin{bmatrix} \theta_1 \\ \Phi_1 \end{bmatrix} = \begin{bmatrix} 1 & R \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \theta_2 \\ \Phi_2 \end{bmatrix} \qquad [1.65]$$

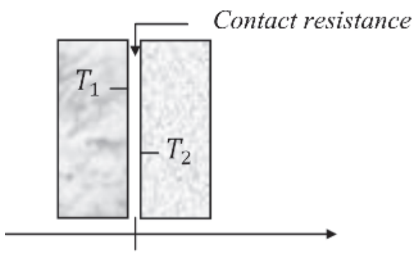

Figure 1.26: *Diagram of two walls with contact resistance*

This expression is analogous to equation [1.64], and the equivalent electrical network is therefore of the same type as that shown in Figure 1.25.

1.2.4.1.4. Multilayer wall with convection and contact resistance

The previously established quadrupole matrix equations allow us to write for the system under consideration (see Figure 1.27):

$$\begin{bmatrix}\theta_{f1}\\ \Phi_1\end{bmatrix}=\begin{bmatrix}1 & \frac{1}{h_1 S}\\ 0 & 1\end{bmatrix}\begin{bmatrix}A_1 & B_1\\ C_1 & D_1\end{bmatrix}\begin{bmatrix}1 & R_{12}\\ 0 & 1\end{bmatrix}\begin{bmatrix}A_2 & B_2\\ C_2 & D_2\end{bmatrix}\begin{bmatrix}1 & R_{23}\\ 0 & 1\end{bmatrix}\begin{bmatrix}A_3 & B_3\\ C_3 & D_3\end{bmatrix}\begin{bmatrix}1 & \frac{1}{h_2 S}\\ 0 & 1\end{bmatrix}\begin{bmatrix}\theta_{f2}\\ \Phi_2\end{bmatrix}$$

with: $A_i = D_i = cosh(q_i e_i)$; $C_i = \lambda_i q_i S\, sinh(q_i e_i)$; $B_i = \frac{sinh(q_i e_i)}{\lambda_i q_i S}$ and: $q_i = \sqrt{\frac{p}{a_i}}$

The description of the problem in matrix form makes it possible to obtain a very simple formulation, which shows the advantage of the quadrupole method.

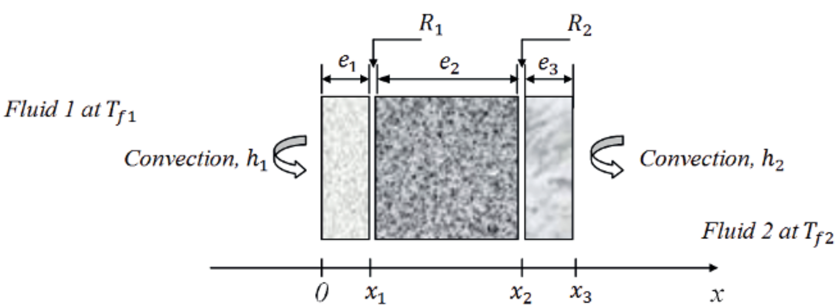

Figure 1.27. *Diagram of a multilayer wall with convection and contact resistance*

1.2.4.1.5. Semi-infinite medium

It has been demonstrated in the preceding section that the temperature in Laplace space of a semi-infinite medium is written as:

$$\theta(x,p) = A\,e^{-qx} \quad \text{where: } q = \sqrt{\frac{p}{a}} = \sqrt{\frac{\rho c p}{\lambda}}.$$

We deduce the value of the Laplace transform of the heat flow rate at a point of the semi-infinite medium as:

$$\Phi(x,p) = -\lambda S \frac{d\theta}{dx}(x,p) = \lambda q S A\,e^{-qx} = \lambda q S \theta(x,p)$$

Φ can therefore also be written as:

$$\Phi = \lambda q S \theta = \lambda \sqrt{\frac{\rho c p}{\lambda}}\, S\theta = \sqrt{\lambda \rho c}\, S\,\sqrt{p}\theta = ES\sqrt{p}\theta$$

where: E is the thermal effusivity.

We can therefore write at every point of a semi-infinite medium:

$$\begin{bmatrix} \theta \\ \Phi \end{bmatrix} = \begin{bmatrix} 1 & \frac{1}{ES\sqrt{p}} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ \Phi \end{bmatrix} = \begin{bmatrix} \theta \\ ES\sqrt{p}\theta \end{bmatrix} \qquad [1.66]$$

The quadrupole equation [1.66] can be represented by the equivalent electrical diagram of Figure 1.28.

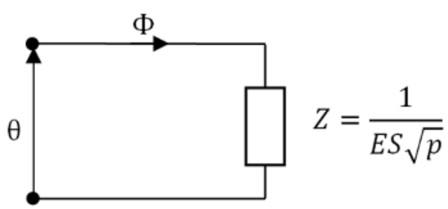

Figure 1.28. *Equivalent electrical network to a semi-infinite medium in unsteady state*

1.2.4.1.6. Wall at uniform temperature

In the case of a "thin system", a wall whose thickness and/or thermal conductivity make it possible to consider its temperature as uniform ($Bi < 0.1$, small body hypothesis), the difference between the incoming heat flow rate and the outgoing heat flow rate leaving the system is simply written as:

$\varphi_1 - \varphi_2 = \rho c V \frac{dT}{dt}$ or by applying the Laplace transform: $\Phi_1 - \Phi_2 = \rho c V p \theta$.

This equation can be expressed in quadrupole form as:

$$\begin{bmatrix} \theta_1 \\ \Phi_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \rho c V p & 1 \end{bmatrix} \begin{bmatrix} \theta_2 \\ \Phi_2 \end{bmatrix} \qquad [1.67]$$

The quadrupole equation [1.67] can be represented by the equivalent electrical network of Figure 1.29.

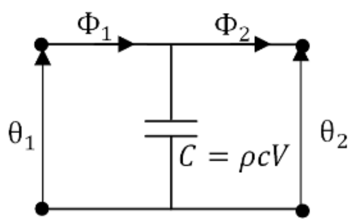

Figure 1.29. *Equivalent electrical network to a medium at uniform temperature in unsteady state*

1.2.4.2. *Radial transfer*

1.2.4.2.1. Hollow cylinder

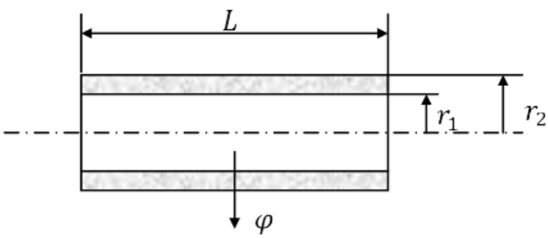

Figure 1.30. *Diagram of a hollow cylinder*

It is shown in the same manner as previously demonstrated [MAI 00] that the temperatures and fluxes in the Laplace space can be connected by a quadrupole equation:

$$\begin{bmatrix}\theta(r_1,p)\\ \Phi(r_1,p)\end{bmatrix}=\begin{bmatrix}A & B\\ C & D\end{bmatrix}\begin{bmatrix}\theta(r_2,p)\\ \Phi(r_2,p)\end{bmatrix}$$

$$A = qr_2[K_1(qr_2)I_0(qr_1) + K_0(qr_1)I_1(qr_2)]$$

$$B = \frac{1}{2\pi\lambda L}[K_0(qr_1)I_0(qr_2) - K_0(qr_2)I_0(qr_1)] \qquad [1.68]$$

$$C = 2\pi L\rho c r_1 r_2 p[K_1(qr_1)I_1(qr_2) - K_1(qr_2)I_1(qr_1)]$$

$$D = qr_1[K_1(qr_1)I_0(qr_2) + K_0(qr_2)I_1(qr_1)]$$

I_0, I_1, K_0 and K_1 are Bessel's functions (see Appendix 8). The determinant of the quadrupole matrix is equal to 1.

1.2.4.2.2. Semi-infinite hollow cylinder

As in the case of the plane wall, we show that we can write at any point of a semi-infinite hollow cylinder ($r_2 \to \infty$) [MAI 00]:

$$\begin{bmatrix}\theta\\ \Phi\end{bmatrix}=\begin{bmatrix}1 & Z\\ 0 & 1\end{bmatrix}\begin{bmatrix}0\\ \Phi\end{bmatrix}=\begin{bmatrix}\theta\\ 2\pi\lambda L\,\frac{qr_1K_1(qr_1)}{K_0(qr_1)}\,\theta\end{bmatrix} \qquad [1.69]$$

where $Z = \frac{K_0(qr_1)}{2\pi\lambda L\, qr_1K_1(qr_1)}$

The quadrupole equation [1.69] can be represented by the equivalent electrical network of Figure 1.31.

APPLICATION.– *Modeling of the hot wire method.*

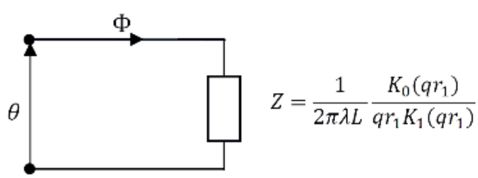

Figure 1.31. *Equivalent electrical network to a semi-infinite medium in unsteady state*

1.3. The thermal properties of a material

In the following, we will recapitulate the various characteristic quantities of a material that have appeared in the heat transfer equations.

1.3.1. *Thermal conductivity*

The thermal conductivity λ characterizes the resistance to the passage of heat; it is expressed in W m^{-1} K^{-1}. It is the only thermal property involved in steady-state equations. The values for the most common materials can be found in Appendix 1. In the case of solids, it varies from $0.014\,\mathrm{W\,m^{-1}\,K^{-1}}$ for the superinsulators to $400\,\mathrm{W\,m^{-1}\,K^{-1}}$ for copper ($1500\,\mathrm{W\,m^{-1}\,K^{-1}}$for diamond). It should be noted that electrical insulators may have higher thermal conductivities than electrical conductors (we have ceramics that have a value of $50\,\mathrm{W\,m^{-1}\,K^{-1}}$ and some stainless steels with a value of $15\,\mathrm{W\,m^{-1}\,K^{-1}}$). For multi-constituent media, it must be known that thermal conductivity is not an additive quantity. There exist a very large number of models of equivalent conductivity to predict the thermal conductivity of a multi-constituent medium as a function of the thermal conductivities of each of the constituents. We will simply present the main ones here. We start with the parallel and series models, since the lowest possible value of the thermal conductivity is given by the series model and the highest by the parallel model [WIE 12]. We will assume in the following that the medium consists of *N* components of conductivity λ_i and each occupying the volume fraction ε_i.

1.3.1.1. *Parallel model*

In this model, the different constituents are assumed to be arranged in parallel layers, and the heat flux is parallel to the layers as shown in Figure 1.32.

The equivalent thermal conductivity is given by:

$$\lambda = \sum_{i=1}^{N} \varepsilon_i \lambda_i \qquad [1.70]$$

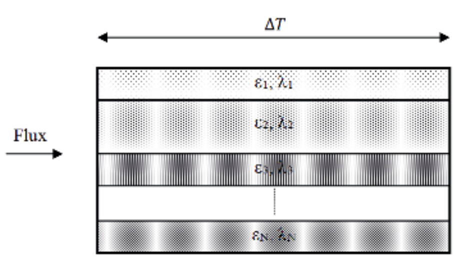

Figure 1.32. *Diagram representing the parallel model*

The parallel model represents the thermal conductivity of light insulating materials.

1.3.1.2. *Series model*

In this model, the different constituents are assumed to be arranged in parallel layers, and the heat flux is perpendicular to the layers as shown in Figure 1.33.

The equivalent thermal conductivity is given by:

$$\lambda = \frac{1}{\sum_{i=1}^{N} \frac{\varepsilon_i}{\lambda_i}} \qquad [1.71]$$

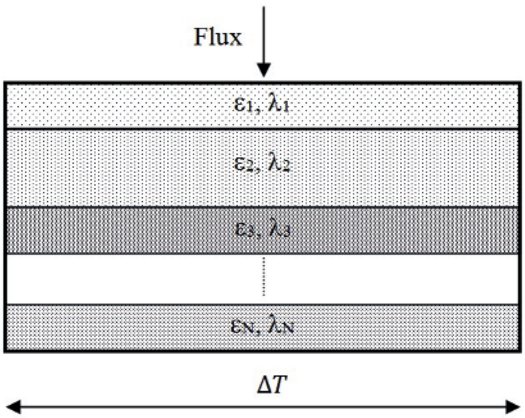

Figure 1.33. *Diagram representing the series model*

1.3.1.3. *Maxwell's model*

This simple model is an exact representation of the thermal conductivity of a continuous medium (*0*) containing spherical particles (*1*) sufficiently far apart to be able to neglect their mutual interactions [MAX 54]. The thermal conductivity is then given by:

$$\lambda = \frac{\varepsilon_0 \lambda_0 (2\lambda_0 + \lambda_1) + 3\varepsilon_1 \lambda_0 \lambda_1}{\varepsilon_0 (2\lambda_0 + \lambda_1) + 3\varepsilon_1 \lambda_0} \qquad [1.72]$$

This model represents well the thermal conductivity of a medium containing inclusions such as lightened concretes, for example.

1.3.1.4. *Bruggeman's model*

Bruggeman's model [DEV 52] is constructed by adding "progressively" ellipsoidal inclusions of the dispersed phase (1) to the continuous medium (0). The equivalent thermal conductivity λ is deduced from:

$$\left(\frac{\lambda}{\lambda_0}\right)^{A_0}\left(\frac{\lambda_1-\lambda_0}{\lambda_1-\lambda}\right)=(1-\varepsilon_1)^{\frac{1}{3}} \qquad [1.73]$$

where ε_1 is the volume fraction of the inclusions and $A_0=\frac{1-e^2}{e^3}\left[\frac{1}{2}\,ln\left(\frac{1+e}{1-e}\right)-e\right]$ with: $e=\sqrt{1-\frac{b^2}{a^2}}$, a being the semi-major axis and b the semi-minor axis of the ellipsoids.

This model makes it possible to correctly represent the conductivity of a medium containing inclusions (solid or gaseous), for volume fraction values of inclusions higher than those acceptable by Maxwell's model.

1.3.1.5. *Special case of a porous medium*

A porous medium consists of a hollow solid matrix delimiting pores filled with a gas. The pores are most often assimilated to cylinders of diameter D_i. The conductivity λ_{D_i} of the air contained in the pores depends on the diameter of the pore and the gas pressure according to the equation [LIT 96]:

$$\lambda_{D_i}=\frac{\lambda_0}{1+C\,\frac{T}{PD_i}} \qquad [1.74]$$

where: λ_0 is the air conductivity (0.026 W m^{-1} K^{-1} at 300 K);

T is the air temperature (K);

P is the air pressure (Pa);

C is the constant (6.8×10^{-5} Pa m K^{-1}).

The conductivity of the air at atmospheric pressure in pores with a diameter of 1 µm is equal to 0.021 W m^{-1} K^{-1}, and only 0.008 W m^{-1} K^{-1} in pores of diameter

100 nm. A material containing a high proportion of nanopores can therefore be superinsulating, i.e. have a thermal conductivity that is lower than that of air at atmospheric pressure (or $\lambda < 0.026\ \mathrm{W\,m^{-1}\,K^{-1}}$).

1.3.2. *Thermal diffusivity*

The thermal diffusivity a is expressed in $\mathrm{m^2\,s^{-1}}$. It characterizes the rate at which a heat flux diffuses into a material. For solids, it varies from $5 \times 10^{-8}\ \mathrm{m^2 s^{-1}}$ for certain rubbers to $3 \times 10^{-4}\ \mathrm{m^2 s^{-1}}$ for diamond.

It is important to note that it is not correlated with the thermal conductivity: a low conductive material can be very diffusive. For example, air with a low thermal conductivity: $\lambda = 0.026\ \mathrm{W\,m^{-1}\,K^{-1}}$ has a thermal diffusivity $a = 2{,}1.\,10^{-5}\ \mathrm{m^2 s^{-1}}$ which is identical to that of iron with a much higher thermal conductivity: $\lambda = 73\ \mathrm{W\,m^{-1}\,K^{-1}}$. The thermal diffusivity makes it possible to estimate the time constant of a material of thickness e with:

$$\tau = \frac{e^2}{a} \qquad [1.75]$$

The time constant is the time taken to reach a steady state when it exists.

1.3.3. *Volumetric heat capacity*

It is noted that in all the equations for the heat transfer in solids, the quantities ρ and c never appear separately but only in the form of their product ρc volumetric heat capacity expressed in $\mathrm{J\,m^{-3}\,K^{-1}}$. This quantity represents the ability of a material to store heat. For solids, it varies from $10^4\ \mathrm{J\,m^{-3}\,K^{-1}}$ for low density aerogels to $4 \times 10^6\ \mathrm{J\,m^{-3}\,K^{-1}}$ for certain steel.

This magnitude is additive for multi-constituent materials:

$$\rho c = \sum_{i=1}^{N} \varepsilon_i \rho_i c_i \qquad [1.76]$$

1.3.4. *Thermal effusivity*

The thermal effusivity E characterizes the transitory variation in the temperature rise of a surface subjected to a heat flux. It is expressed in $\mathrm{W\,m^{-2}\,K^{-1} s^{1/2}}$. For solids, it varies from $35\ \mathrm{W\,m^{-2}\,K^{-1} s^{1/2}}$ for low density aerogels to $3.6 \times 10^4\ \mathrm{W\,m^{-2}\,K^{-1} s^{1/2}}$ for certain steels.

1.3.5. *Conclusion*

Although four parameters characterizing the heat transfer within a material have been defined, it is sufficient to measure two parameters, since these parameters are linked by the following equations:

$$a = \frac{\lambda}{\rho c} \quad [1.77]$$

and

$$E = \sqrt{\lambda \rho c} \quad [1.78]$$