1

Introduction to a Universal Model: the Vlasov Equation

1.1. A historical point of view

Halfway between the N body model and the usual hydrodynamics, the Vlasov equation, or better the Vlasov model, describes different media going from nuclear matter to the expanding universe (via semiconductors, plasmas and stellar dynamics problems and the introduction of a quantum counterpart, the so-called *Wigner equation*). The aim of this book is to provide the reader with a good knowledge of the Vlasov model, which offers a specific mathematical description for different parts of physics or astrophysics.

Figure 1.1. *Sir James Hopwood Jeans (1877–1946)*

About 100 years ago, Jeans (1915) for the first time (to our knowledge) considered this equation to study the behavior of an infinite number of interacting masses (galaxies)[1]. But due to analytical difficulties, he considered only time-independent problems.

But when Vlasov (1945) and Landau (1946) gave the first time-dependent solution for the plasma case, the plasma physicists took the leadership in the study of this equation and called it .... the Vlasov equation.

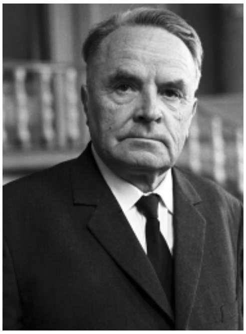

Figure 1.2. *Anatoly Alexandrovitch Vlasov (1908–1975)*

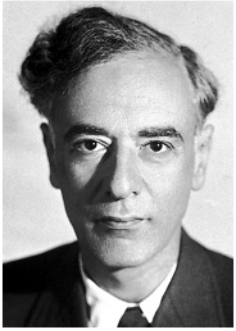

Figure 1.3. *Lev Davidovitch Landau (1908–1968) (Nobel prize 1962)*

1 See also the classical textbook by Chandrasekhar (1942).

For a historical review on the Vlasov equation, the reader is also referred to an interesting paper by M. Hénon (1982).

The aim of this book is to present an overview of the Vlasov model in its various aspects:

1) From a physical point of view, what situations and which physical systems are described by this model?

2) From a mathematical point of view, what problems can be solved analytically?

3) Another concern is the link with other models, especially hydrodynamic or more specifically magnetohydrodynamic (MHD).

4) A further point deals with the numerical aspect of the problem and is connected with the huge field of computer simulation with hundreds of papers published each year. We will concentrate on what, in our opinion, is the central problem, putting aside technical and often non-trivial aspects.

The word "plasma" was introduced by I. Langmuir (1881–1957) around 1920.

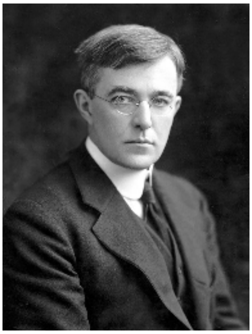

Figure 1.4. *Irving Langmuir (1881–1957) (Nobel prize, Chemistry, 1932)*

Together with Lewis Tonks, he discovered the phenomenon of electronic oscillations in electrical discharges and for the first time introduced the idea of collective phenomena. Considering a homogeneous neutral distribution of ions and electron (with equal electron n_e and ion density n_i), if an electron (e, m_e) is moved from its equilibrium position, the other particles create a net charge and our test electron further oscillates around its equilibrium position with a "frequency" (or rather a pulsation ω_p) given by the well-known formula

$$\omega_p = \sqrt{\frac{n_e e^2}{\epsilon_0 m_e}}$$

This collective behavior of charged particles was also observed at the same time by Peter Debye (1884–1966) together with Erich Huckel in electrolytes.

Figure 1.5. *Peter Debye (1884–1966) (Nobel prize, Chemistry, 1936)*

Considering a test ion, the canonical equilibrium distribution of surrounding electrons at temperature T_e is given by the Laplace equation for the so-called Debye screening

$$\triangle(e\phi/\kappa T_e) = \frac{1}{\lambda_D^2}\frac{n_e - n_i}{n_e}$$

where λ_D is the well-known Debye length given by

$$\lambda_D = \sqrt{\frac{\epsilon_0 \kappa T_e}{n_e e^2}}$$

As a matter of fact, Debye length λ_D and plasma frequency ω_p are fundamental quantities in plasma physics and further details can be found in elementary textbooks.

In a usual weakly ionized or even neutral gas, binary interactions involve only a limited number of particles and we have to cope with individual phenomena. On the other hand, in a high-temperature tokamak plasma ($T_e = 10^8$ K) particles are quite ionized, and the electromagnetic interactions, involving a large number of charges, point to the idea of collective phenomena. Let us clarify this concept.

1.2. Individual and collective effects in plasmas

A more rigorous derivation of long-range versus short-range interactions can be obtained by considering a simplified plasma model consisting of discrete charged particles: electrons and massive (immobile) ions. The electron gas (charge e, mass m_e, density n_e and temperature T_e) is immersed in a fixed neutralizing homogeneous ion background.

The discrete (individual) character implies the existence of Coulomb collisions. Suppose a large deflection angle (see Figure1.6).

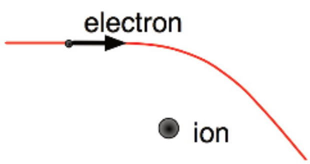

Figure 1.6. *Large angle collision*

The impact parameter P_{impact} is defined as the distance between the electron and the attractive ion when the thermal energy of the electron κT_e and the potential energy are of the same order of magnitude

$$\kappa T_e \sim \frac{e^2}{\epsilon_0 P_{impact}}$$

giving an estimate of the mean free path ℓ_{mfp}

$$\ell_{mfp} \sim \frac{1}{n_e} \left(\frac{\epsilon_0 \kappa T_e}{e^2} \right)^2$$

and the collision frequency

$$\nu_{coll} \sim \frac{\sqrt{\kappa T_e / m_e}}{\ell_{mfp}}$$

Obviously, particles experience large angle collisions if the distance between them is smaller than the Debye length. Above λ_D the Debye screening is the dominant phenomenon. Introducing the interparticle distance

$$\ell_{inter} \sim n_e^{-3}$$

this condition can be written as

$$\lambda_D \gg \ell_{inter}$$

or equivalently

$$n_e \lambda_D^3 \gg 1 \qquad [1.1]$$

The *dimensionless parameter* $n_e \lambda_D^3$ appears to play a fundamental role. It is a measure of the number of electrons in a Debye cube.

Actually, in our model, we have to deal with two characteristic lengths: the mean free path ℓ_{mfp} and the Debye length λ_D. In the same way, we have two characteristic frequencies: the collision frequency ν_{coll} and the plasma frequency ω_p.

Individual phenomena are characterized by a typical length ℓ_{mfp} and a typical frequency ν_{coll}.

Collective phenomena are characterized by a typical length D and a typical frequency ω_p.

The relations between these characteristic quantities are summarized in Table 1.1. The parameter $n_e \lambda_D^3$ appears to play a fundamental role; it gives the scaling between collective and individual phenomena.

A *thought experiment* as suggested by Rostoker and Rosenbluth (1960) (see also papers cited therein) will help to understand the role of this parameter.

Let us imagine a *dichotomy* process in which each electron is cut into two parts, each "half-electron" being cut into two parts again and so on:

$$\textit{dichotomy}\text{: } (e, m_e) \to 2(e/2, m_e/2) \to 4(e/4, m_e/4) \to \cdots$$

giving the mathematical limit $e \to 0$, $m_e \to 0$, $n_e \to \infty$ and $T_e \to 0$ but leaving *invariant* the following quantities $e/m_e = \text{const}$, $T_e/m_e = \text{const}$ and $n_e e = \text{const}$.

	Individual phenomena	Collective phenomena
Characteristic lengths	Mean free path $\ell_{mfp} \sim n_e^{-1}\left(\epsilon_0 \kappa T_e/e^2\right)^2$	Debye length $\lambda_D = \sqrt{\epsilon_0 \kappa T_e/n_e e^2}$
	$\ell_{mfp} \sim (\mathbf{n_e}\lambda_\mathbf{D}^\mathbf{3})\lambda_D$	
Characteristic frequencies	Collision frequency $\nu_{coll} \sim \sqrt{\kappa T_e/m_e}/\ell_{mfp}$	Plasma frequency $\omega_p^2 = ne^2/\epsilon_0 m_e$
	$\nu_{coll} \sim (\mathbf{n_e}\lambda_\mathbf{D}^\mathbf{3})^{-1}\omega_p$	

Table 1.1. *Individual and collective characteristic quantities*

Therefore, the collective characteristic quantities remain invariant

$$\omega_p^2 = \varepsilon_0^{-1}(n_e e)(e/m_e) = \text{const}$$

$$\lambda_D^2 = (\kappa T_e/m)/\omega_p^2 = \text{const}$$

while the individual characteristic quantities are modified at each step of the dichotomy to reach the following limits:

$$\nu_{coll} \approx \left(n_e \lambda_D^3\right)^{-1} \omega_p \to 0$$

$$\ell_{mfp} = \left(n_e \lambda_D^3\right) \lambda_D \to \infty$$

1.3. Graininess parameter

We have seen above the important role played by the parameter $n_e\lambda_D^3$ in allowing the distinction between collective and individual phenomena. Let us consider the dimensionless parameter

$$g = \frac{1}{n_e \lambda_D^3} \qquad [1.2]$$

The following properties are obvious:

– in the dichotomy experiment, g is divided by a factor 2 at each step;

– g is clearly a measure of the discrete (or granular) character of the plasma;

– collective effects are described in the limit $g \to 0$.

In terms of characteristic lengths and times, we have

– $\lambda_D \approx g\,\ell_{mfp}$ with the collective limit $\ell_{coll} \gg \lambda_D$;

– $\omega_p^{-1} \approx g\,\nu_{coll}^{-1}$ with the collective limit $\nu_{coll}^{-1} \gg \omega_p^{-1}$.

Since the Debye length λ_D in [1.2] depends on the electron density n_e and the electron temperature T_e, it is interesting to rewrite g in practical units

$$g = 3.10^{-6}\ \frac{n_e^{1/2}}{T_e^{3/2}}, \qquad [1.3]$$

where n_e is expressed in m^{-3} and T_e in kelvin.

For some typical plasmas, the corresponding values for the parameter g are summarized in Table 1.2.

	$n_e\ [\mathrm{cm}^{-3}]$	T_e [K]	ℓ_{mfp} [m]	D [m]	g
Discharges	10^{14}	10^4	2.10^{-5}	7.10^{-7}	3.10^{-2}
Tokamak	10^{14}	10^8	2.10^{+3}	7.10^{-5}	3.10^{-8}
Solar corona	10^6	10^6	2.10^{+3}	7.10^{-7}	3.10^{-9}
Interstellar plasma	1	10^4	2.10^{+9}	7	3.10^{-9}

Table 1.2. *The graininess parameter for typical plasmas*

Both small density and high temperature are needed to get a collective plasma with small g. Such a plasma appears to be a completely different medium than a usual gas (g *not* small) for which collisions are the dominant process. For fusion or astrophysical plasma, collective effects are so important that a usual fluid description cannot be used.

On the other hand, it is interesting to consider the electron gas in a metal like copper. Although a full quantum treatment would be necessary, an estimate for g with our formulas would give $g = 2.10^{+6}$, which is largely beyond the scope of a Vlasov description.

1.4. The collective description of a Coulomb gas: an intuitive approach

To build a Vlasov model, we must keep in mind the limit $g \to 0$ and derive an equation, which is invariant with respect to g in this limit.

As described above, we shall consider a fully ionized electron gas in a fixed homogeneous neutralizing ion background. To further simplify the problem, we deal with an electron–proton plasma (with an ion density n_0). No individual atomic physics processes are present and only Coulomb interactions will be taken into account (without any magnetic effects). These simplifying hypotheses are needed to get a comprehensive knowledge of the collective behavior. Generalization to more complex plasmas will be carried out in the following chapters.

Due to the huge value of the number of particles N, the motion of each electron is hard to derive even with the most powerful supercomputers available nowadays. Therefore, we introduce the concept of a distribution function $f(\boldsymbol{r}, \boldsymbol{v}, t)$ such as: $f\, d\boldsymbol{r}\, d\boldsymbol{v}$ is the probability to find an electron at the spatial coordinate $\boldsymbol{r}$ having a velocity $\boldsymbol{v}$ inside a phase space elementary volume $d\boldsymbol{r}\, d\boldsymbol{v}$. Obviously, we have the normalization condition:

$$\iint f(\boldsymbol{r}, \boldsymbol{v}, t)\, d\boldsymbol{r}\, d\boldsymbol{v} = 1$$

The electron charge density is given by

$$\rho_e = en_0(\int f\, d\boldsymbol{v} - 1)$$

and it is important to note that both f and ρ are invariant in the limit $g \to 0$.

In the Hamiltonian formalism, the distribution function is theoretically a function of canonically conjugated variables (position $\boldsymbol{r}$ and momentum $\boldsymbol{p}$). But it is a widespread practice to consider position and velocity as variables of the distribution function. In a general way, impulsion and linear momentum are not always proportional, for example in the relativistic regime or in the case of particles in a magnetic field. We will use $\boldsymbol{p}$ when needed.

Let us consider the motion of N particles in phase space $(\boldsymbol{r}, \boldsymbol{p})$, interacting through self-consistent Coulomb forces. To simplify the notations and without any loss of generality, we can deal with particles moving in a one-dimensional (1D) space (and consequently a two-dimensional (2D) phase space). A rigorous demonstration will be given in section 1.5, but here we do not want to cope with too much complicated mathematical details. Thus, at time t_0 we draw a closed "contour" $(\mathbb{C}_0)$ in phase space defining a "surface"

$$dS_0 = \oint d\boldsymbol{r}\, d\boldsymbol{p}$$

At a later time $t > t_0$, the particles on the initial contour have moved and define a new contour $(\mathbb{C})$ delineating a new surface

$$\oint d\boldsymbol{r}\, d\boldsymbol{p} = dS$$

Since particles evolve through Coulomb forces, the motion is *Hamiltonian* and obviously

$$dS = dS_0 \qquad [1.4]$$

Furthermore, individual collisions are neglected in the $g \to 0$ limit. Consequently, all particles which at time t_0 are inside the contour $(\mathbb{C}_0)$ are exactly the same as those which, at time t, are inside the contour $(\mathbb{C})$. No particles have left or entered the moving contour. Now, letting $dS_0 \to 0$ and $dS \to 0$ allows us to define the phase space densities $f(\boldsymbol{r}_0, \boldsymbol{p}_0, t_0)$ and $f(\boldsymbol{r}, \boldsymbol{p}, t)$. From the conservation relation

$$f(\boldsymbol{r}, \boldsymbol{p}, t)dS = f(\boldsymbol{r}_0, \boldsymbol{v}_0, t_0)dS_0 \qquad [1.5]$$

together with [1.4] we easily deduce

$$f(\boldsymbol{r}, \boldsymbol{p}, t) = f(\boldsymbol{r}_0, \boldsymbol{p}_0, t_0) \quad \text{which means} \quad \frac{\mathrm{d}f}{\mathrm{d}t} = 0 \qquad [1.6]$$

In [1.6], $\mathrm{d}/\mathrm{d}t$ stands for the total derivative

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + \frac{d\boldsymbol{r}}{dt} \cdot \frac{\partial}{\partial \boldsymbol{p}} + \frac{d\boldsymbol{p}}{dt} \cdot \frac{\partial}{\partial \boldsymbol{p}}$$

Taking into account

$$\frac{d\boldsymbol{r}}{dt} = \boldsymbol{v} \quad \text{(velocity)} \quad \text{and} \quad \frac{d\boldsymbol{p}}{dt} = \boldsymbol{F} \quad \text{(force)}$$

equation [1.6] can be explicitly written as

$$\frac{\partial f}{\partial t} + \boldsymbol{v} \cdot \frac{\partial f}{\partial \boldsymbol{r}} + \boldsymbol{F} \cdot \frac{\partial f}{\partial \boldsymbol{p}} = 0 \qquad [1.7]$$

In the non-relativistic case, $\boldsymbol{p} = m_e \boldsymbol{v}$. The electrostatic force is $\boldsymbol{F} = e\boldsymbol{E}$. Equation [1.7] can be reformulated in terms of $f(\boldsymbol{r}, \boldsymbol{v}, t)$

$$\frac{\partial f}{\partial t} + \boldsymbol{v} \cdot \frac{\partial f}{\partial \boldsymbol{r}} + \frac{e}{m_e}\boldsymbol{E} \cdot \frac{\partial f}{\partial \boldsymbol{v}} = 0 \qquad [1.8]$$

[1.8] is just the Vlasov equation introduced by Vlasov (1945) for a plasma.

Now we have to compute the field $\boldsymbol{E}$ in [1.8]. As far as collective phenomena are our concern, in the limit $g \to 0$, Coulomb interaction between charged particles must indeed be replaced by a *mean field* calculated from Poisson (or Maxwell equations) using charge density (and/or current density) where the microscopic fluctuations are averaged over the Debye length. In other words, it means that we stay at a space scale where fluctuations can be ignored, although the plasma is not a continuum but is formed of discrete particles.

In our plasma model, the electric field $\boldsymbol{E}$ is self-consistently computed using Poisson's equation

$$\mathrm{div}\boldsymbol{E} = \rho/\epsilon_0 \quad \text{with} \quad \rho = en_0 \int f \, d\boldsymbol{v} - en_0 \qquad [1.9]$$

Remembering that f and ρ are invariant through the dichotomy process described in section 1.2, it is clear that the Vlasov equation (or more precisely the Vlasov–Poisson system [1.8]–[1.9] is the pertinent model to describe an unmagnetized electronic plasma in the limit $g \to 0$.

More generally for more complex plasmas, Maxwell's equations have to be used to compute the self-consistent electromagnetic fields $\boldsymbol{E}, B$ (see section 1.9).

In the original work by Jeans in the case of interacting masses m, the Vlasov equation writes

$$\frac{\partial f}{\partial t} + \boldsymbol{v} \cdot \frac{\partial f}{\partial \boldsymbol{r}} + \frac{\boldsymbol{F}}{m} \cdot \frac{\partial f}{\partial \boldsymbol{v}} = 0 \qquad [1.10]$$

where the interacting force $\boldsymbol{F} = m\boldsymbol{E}$ (where this time $\boldsymbol{E}$ is the gravitational field) is obtained self-consistently using Poisson's equation

$$\mathrm{div}\boldsymbol{E} = -4\pi G\rho \qquad [1.11]$$

where G is the gravitational constant and ρ is the mass density given by

$$\rho = m \int f(\boldsymbol{r}, \boldsymbol{v}, t) \, d\boldsymbol{v}$$

1.5. From N-body to Vlasov

A more rigorous approach using the Hamiltonian formalism for the N-body system of interacting particles will now be outlined.

Let us consider again our plasma model consisting of a collection of N electrons embedded in a fixed homogeneous neutralizing ion background. Position and momentum coordinates of the electrons are, respectively

$$\big(\boldsymbol{r}_i(t)\big)_{i=1,\ldots,N} \quad \text{and} \quad \big(\boldsymbol{p}_i(t)\big)_{i=1,\ldots,N}$$

Using the Hamiltonian formalism, these conjugated phase space coordinates obey the Hamilton equations

$$\frac{\partial H}{\partial \boldsymbol{p}_i} = \frac{d\boldsymbol{r}_i}{dt} \tag{1.12}$$

$$\frac{\partial H}{\partial \boldsymbol{r}_i} = -\frac{d\boldsymbol{p}_i}{dt} \tag{1.13}$$

where H is the Hamiltonian of our system.

Since the forces acting on each particle i are supposed to derive from scalar potentials (no magnetic terms), the Hamiltonian can be written as

$$H = \sum_{i=1}^{N} \frac{\boldsymbol{p}_i^2}{2m_e} + \sum_{i=1}^{N} \varphi_{ext}(\boldsymbol{r}_i) + \sum_i \sum_{j>i} \phi_{ij}\big(|\boldsymbol{r}_i - \boldsymbol{r}_j|\big) \tag{1.14}$$

In [1.14], two different potentials have been introduced:

1) an external potential $\varphi_{ext}(\boldsymbol{r}_i)$ corresponding to external forces acting on the electron i and forces due to the ion background as well;

2) the interaction Coulomb potential $\phi_{ij}\big(|\boldsymbol{r}_i - \boldsymbol{r}_j|\big)$ between electrons i and j given by

$$\phi_{ij} = \frac{1}{4\pi\epsilon_0} \frac{e^2}{|\boldsymbol{r}_i - \boldsymbol{r}_j|} \tag{1.15}$$

Equations [1.12]–[1.13] form a set of $6N$-coupled equations, which are impossible to analytically solve since N is usually huge and statistical methods are necessary. A set of possible realizations must be considered, each of them being represented by a point in the $6N$-grand phase space and the statistical set represented by a cloud. Some configurations can be more probable than others, and it is convenient to introduce the distribution probability

$$D_N(\boldsymbol{r}_1, \ldots, \boldsymbol{r}_n, \boldsymbol{p}_1, \ldots, \boldsymbol{p}_n, t)$$

Now the Liouville theorem tells us that the cloud behaves like an incompressible fluid in the grand phase space. Consequently, D_N obeys Liouville's conservative equation

$$\frac{\mathrm{d}D_N}{\mathrm{d}t} = \frac{\partial D_N}{\partial t} + \big\{D_N; H\big\} = 0 \qquad [1.16]$$

where $\big\{\cdot;\cdot\big\}$ denotes the usual Poisson brackets. Using the Hamiltonian [1.14] Liouville's equation [1.16] now writes

$$\begin{aligned} \frac{\partial D_N}{\partial t} + \sum_{i=1}^{N} \frac{\boldsymbol{p}_i}{m_e} \cdot \frac{\partial D_N}{\partial \boldsymbol{r}_i} - \sum_{i=1}^{N} \frac{\partial \varphi_{ext}}{\partial \boldsymbol{r}_i} \cdot \frac{\partial D_N}{\partial \boldsymbol{p}_i} \\ - \sum_{i=1}^{N} \sum_{j>i} \frac{\partial \phi_{ij}}{\partial \boldsymbol{r}_i} \cdot \frac{\partial D_N}{\partial \boldsymbol{p}_i} = 0 \end{aligned} \qquad [1.17]$$

Actually, we have replaced the $6N$ time differential equations [1.12]–[1.13] by one partial differential equation, but involving $6N$ phase space variables plus time. The microscopic information is the same in equations [1.12]–[1.13] or equation [1.16]. This *microscopic* information is clearly oversized.

On the contrary, the *macroscopic* information available in an experiment is not sufficient and involves only some quantities like particle density $n(\boldsymbol{r}, t)$ or pressure $(P\boldsymbol{r}, t)$ at a given point $\boldsymbol{r}$ at time t in the plasma.

To reduce the unnecessary information, let us introduce a series of reduced distributions

$$\Big(D_n(\boldsymbol{r}_1, \ldots, \boldsymbol{r}_n, \boldsymbol{p}_1, \ldots, \boldsymbol{p}_n, t)\Big)_{n=1,\ldots,N-1}$$

by averaging over particles running from $n+1$ to N

$$\begin{aligned} &D_n(\boldsymbol{r}_1, \ldots, \boldsymbol{r}_n, \boldsymbol{p}_1, \ldots, \boldsymbol{p}_n, t) \\ &\qquad = \frac{1}{(N-n)!} \int D_N \, d\boldsymbol{r}_{n+1} \cdots d\boldsymbol{r}_N \, d\boldsymbol{p}_{n+1} \cdots d\boldsymbol{p}_N \end{aligned}$$

Note that the factor $1/(N-n)!$ has been introduced to take into account the electrons identity since D_N must be symmetrical by exchanging $(\boldsymbol{r}_i, \boldsymbol{p}_i)$ and $(\boldsymbol{r}_j, \boldsymbol{p}_j)$ with the corresponding normalization condition

$$\frac{1}{N!}\int D_N \, d\boldsymbol{r}_1 \cdots d\boldsymbol{r}_N \, d\boldsymbol{p}_1 \cdots d\boldsymbol{p}_N = 1$$

The nature of the first reduced distribution $D_1(\boldsymbol{r}_1, \boldsymbol{p}_1, t)$ is particularly interesting:

$D_1(\boldsymbol{r}_1, \boldsymbol{p}_1, t)\, d\boldsymbol{r}_1\, d\boldsymbol{p}_1$ is a measure of the number of electrons in the ordinary phase space volume $d\boldsymbol{r}_1\, d\boldsymbol{p}_1$ around $(\boldsymbol{r}_1, \boldsymbol{p}_1)$. The knowledge of $D_1(\boldsymbol{r}_1, \boldsymbol{p}_1, t)$ allows a straightforward calculation of the macroscopic (*fluid*) quantities:

– the local density of particles:

$$n(\boldsymbol{r}_1, t) = \int D_1(\boldsymbol{r}_1, \boldsymbol{p}_1, t)\, d\boldsymbol{p}_1$$

– the mean *fluid* velocity:

$$\boldsymbol{u}(\boldsymbol{r}_1, t) = \frac{1}{m_e n(\boldsymbol{r}_1, t)} \int \boldsymbol{p}_1\, D_1(\boldsymbol{r}_1, \boldsymbol{p}_1, t)\, d\boldsymbol{p}_1$$

– the kinetic pressure tensor:

$$\boldsymbol{P}(\boldsymbol{r}_1, t) = \frac{1}{m_e} \int (\boldsymbol{p}_1 - m_e \boldsymbol{u}) \otimes (\boldsymbol{p}_1 - m_e \boldsymbol{u})\, D_1(\boldsymbol{r}_1, \boldsymbol{p}_1, t)\, d\boldsymbol{p}_1$$

from which we get the kinetic temperature $n\kappa T = tr(\boldsymbol{P})$.

Obviously, $D_1(\boldsymbol{r}_1, \boldsymbol{p}_1, t)$ is just the distribution function $f(\boldsymbol{r}, \boldsymbol{p}, t)$ already introduced in section 1.4. It can be interpreted as a *one particle distribution function* describing a particle (which we have labeled as particle 1), which statistically represents all the particles. Therefore, $\boldsymbol{r}$ is identified as $\boldsymbol{r}_1$ and $\boldsymbol{p}$ as $\boldsymbol{p}_1$.

From the Liouville equation [1.17], it is easy to obtain an equation for $D_1(\boldsymbol{r}_1, \boldsymbol{p}_1, t)$ (and thus for $f(\boldsymbol{r}, \boldsymbol{p}, t)$): we have to integrate [1.17] over $d\boldsymbol{r}_2 \cdots d\boldsymbol{r}_N\, d\boldsymbol{p}_2 \cdots d\boldsymbol{p}_N$ to get

$$\frac{\partial D_1}{\partial t} + \frac{\boldsymbol{p}_1}{m_e} \cdot \frac{\partial D_1}{\partial \boldsymbol{r}_1} - \frac{\partial \varphi_{ext}}{\partial \boldsymbol{r}_1} \cdot \frac{\partial D_1}{\partial \boldsymbol{p}_1} = \int \frac{\partial \phi_{12}}{\partial \boldsymbol{r}_1} \cdot \frac{\partial D_2}{\partial \boldsymbol{p}_1}\, d\boldsymbol{r}_2\, d\boldsymbol{p}_2 \quad [1.18]$$

Unfortunately, this equation for D_1 depends on D_2. Repeating this process gives an equation for D_2, which depends on D_3 and so on. Actually, we get a hierarchy of equations that is equivalent to solving the Liouville equation alone and is known as the BBGKY hierarchy (Born, Bogoliubov, Greene, Kirkwood, Yvon). For further explanations about the BBGKY, see, for instance, the book by Montgomery and Tidman (1964).

Again we are faced with the same problem of cutting a hierarchy of equations and it seems that no advantage has been obtained. But remember our discussion about collective effects and the limit $g \to 0$. Let us therefore introduce an expansion with respect to the small parameter g.

First remember that two random variables x_1 and x_2 are said *uncorrelated* if the 2-probability $P_2(X_1, X_2)$ to have $x_1 = X_1$ *and* $x_2 = X_2$ can be factorized in terms of the 1-probability P_1 namely

$$P_2(X_1, X_2) = P_1(X_1)\, P_1(X_2)$$

Coming back to our electrons, the limit $g \to 0$ is equivalent to consider particles i and j ($\forall i, j,\ i \neq j$) as uncorrelated. Let us introduce the following ordering in powers of g:

$$D_1(\boldsymbol{r}_1, \boldsymbol{p}_1, t) = \mathbb{O}(g^0)$$

$$D_2(\boldsymbol{r}_1, \boldsymbol{p}_1, \boldsymbol{r}_2, \boldsymbol{p}_2, t) = D_1(\boldsymbol{r}_1, \boldsymbol{p}_1, t)\, D_1(\boldsymbol{r}_2, \boldsymbol{p}_2, t) + \mathbb{O}(g^1)$$

Reporting this expansion into [1.18] and keeping only *first-order* terms in g yields

$$\frac{\partial f}{\partial t} + \frac{\boldsymbol{p}}{m_e} \cdot \frac{\partial f}{\partial \boldsymbol{r}} - \frac{\partial \varphi_{ext}}{\partial \boldsymbol{r}} \cdot \frac{\partial f}{\partial \boldsymbol{p}} = \int \frac{\partial \phi_{12}}{\partial \boldsymbol{r}} \cdot \frac{\partial f(\boldsymbol{r}, \boldsymbol{p}, t)}{\partial \boldsymbol{p}}\, f(\boldsymbol{r}_2, \boldsymbol{p}_2, t)\, d\boldsymbol{r}_2\, d\boldsymbol{p}_2$$

where D_1 has been identified as f and index 1 has been dropped. This equation is nothing else but the Vlasov equation that can be written in the more simple form as

$$\frac{\partial f}{\partial t} + \frac{\boldsymbol{p}}{m_e} \cdot \frac{\partial f}{\partial \boldsymbol{r}} + \boldsymbol{F} \cdot \frac{\partial f}{\partial \boldsymbol{p}} = 0 \qquad [1.19]$$

where the force term $\boldsymbol{F} = \boldsymbol{F}_{ext} + \boldsymbol{F}_{sc}$ is split into two parts. Referring to the Hamiltonian [1.14], we can distinguish:

– external forces $\boldsymbol{F}_{ext}$ (i.e. external forces applied to the whole plasma, and forces due to the ion background):

$$\boldsymbol{F}_{ext} = -\frac{\partial}{\partial \boldsymbol{r}} \varphi_{ext}(\boldsymbol{r}) \tag{1.20}$$

– self-consistent forces $\boldsymbol{F}_{sc}$ due to Coulomb interactions between electrons in the mean field approximation (i.e. smoothed using f):

$$\boldsymbol{F}_{sc} = -\frac{\partial}{\partial \boldsymbol{r}} \int \phi_{12}\big(|\boldsymbol{r} - \boldsymbol{r}_2|\big)\ f(\boldsymbol{r}_2, \boldsymbol{p}_2, t)\, d\boldsymbol{r}_2\, d\boldsymbol{p}_2 \tag{1.21}$$

Introducing the electric field in the plasma $\boldsymbol{E} = \boldsymbol{F}/m_e$ and taking the divergence of [1.20] and [1.21] allow us to recover Poisson's equation [1.9].

1.6. The graininess parameter and 1D, 2D or 3D models

The Vlasov–Poisson system we have derived is a self-consistent model that describes the time evolution of density perturbations. If these perturbations are along a given axis (say the x-axis) and if no external field is applied, then the self-consistent electric field $\boldsymbol{E}$ has only one component along this axis. In that case, the model is often referred to as 1D electrostatic. This 1D plasma model invites to go back to the 1D meaning of the graininess parameter.

Let us consider again the microscopic plasma model described in section 1.5, and imagine an initial situation where the charged particles are distributed on parallel infinite sheaths perpendicular to the x-axis. The electric forces take the form of plane wave. Between sheaths the electric field is constant and the sheaths experience a uniformly accelerated motion. To advance the system, we have only to compute the crossing times, find the smallest and rearrange the sheaths (see Figure 1.7).

This is the basis of an algorithm to solve exactly the N-body problem in one dimension. It is very easy to handle in 1D because the field computation is straightforward, the motion of each sheath being uniformly accelerated.

For higher dimensions, for instance 2D, where charges are now rods as seen in Figure 1.8, or in three-dimensions (3D) (points), the problem becomes much more complicated because the interacting forces now depend on the distance between particles. As introduced by Rostoker and Rosenbluth (1960), the analysis we have developed in section 1.2 can be extended to any dimensionality d of the plasma ($d = 1, 2$ or 3). The reader is referred to an interesting paper by Lotte and Feix (1984) for more details we have not developed here. Their analysis is based again on the search of length and time scales and on a dimensionless parameter g used under the "virtual" dichotomy experiment ($\mathbb{D}$) described in sections 1.2 and 1.3: each particle (e,m_e) is divided into two (charge $e/2$, mass $m_e/2$, kinetic energy divided by two but the velocity is consequently left unchanged).

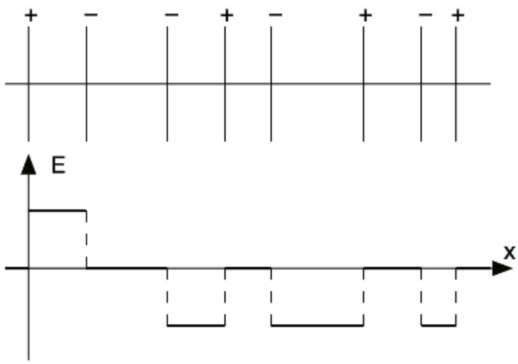

Figure 1.7. *The 1D N-body model: charged sheaths (above) and corresponding electric field (below)*

Now we look for length and time scales invariant under this dichotomy ($\mathbb{D}$) while the parameter g is divided by two (under $\mathbb{D}$). It can be shown that λ_D and ω_p^{-1} are just the length and time scale invariant under $\mathbb{D}$. Furthermore, the Vlasov equation also remains invariant. The general expression for g becomes

$$g = \frac{1}{n_e \lambda_D^d}$$

For $d = 3$ (respectively, 2,1), i.e. a 3D plasma (respectively, a 2D and a 1D plasma), n_e is the density per cubic meter (respectively, square meter and meter), e and m_e the charge and mass (respectively, charge and mass per meter or per square meter).

It is interesting to point out that the graininess factor $g = 1/n_e\lambda_D^d$ decreases with n_e in a 3D plasma, it is independent of n_e in a 2D plasma and decreases when n_e increases in a 1D plasma. In this last case, rather than speaking of *collisions* it is more convenient to introduce the notion of *fluctuations* connected to the discrete nature of the *phase space fluid* while both concepts (collisions and field fluctuations) agree in 3D.

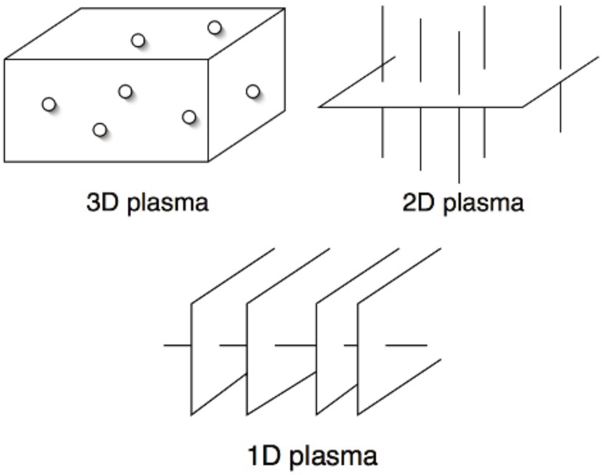

Figure 1.8. *The three possible plasma models*

In a plasma, the size of the system can be arbitrarily large. In the *gravitational* case, the quasi-totality of the steady states has a size of order $L \sim \lambda_D = v_T/\omega_J$ where ω_J is the Jeans frequency defined by

$$\omega_J^2 = 4\pi G\rho$$

and ρ is the typical mass density and G is the gravitational constant.

Consequently, $n\lambda_D^d \sim nL^d \sim N$, where N is the total number of particles.

As shown in Table 1.2 (section 1.3), the Vlasov equation is an excellent approximation in many cases. *Fusion plasmas* exhibit a graininess factor g of order 10^{-7} to 10^{-8} and space plasmas can have still smaller values of g. The situation is similar in the *gravitational case* with $g \sim N^{-1}$ (from 10^{-5} in a cluster to 10^{-9}-10^{-10} for galaxies).

1.7. The Vlasov equation at the microscopic fluctuations level

The Vlasov model describes the time evolution of a collective plasma; it is obvious that time-dependent solutions are valid up to a time of order $(\omega_p g)^{-1}$ (or $(\omega_J g)^{-1}$). Now it turns out that for $g \to 0$ but not strictly 0, there is a correcting term for D_2 in [1.18], which is of order g and should be evaluated.

Before discussing this problem, it should be pointed out that when g is *not small*, the plasma behaves like a classical gas where binary interactions between particles are the dominant processes. The right-hand side of [1.18] can be evaluated using the Boltzmann formalism without having to cope with D_2. This RHS is usually written in the general form of a collision operator $(\partial f/\partial t)_{collision}$ and [1.18] becomes Boltzmann's equation:

$$\frac{\partial f}{\partial t} + \frac{\boldsymbol{p}}{m_e} \cdot \frac{\partial f}{\partial \boldsymbol{r}} - \frac{\partial \varphi_{ext}}{\partial \boldsymbol{r}} \cdot \frac{\partial f}{\partial \boldsymbol{p}} = \left(\frac{\partial f}{\partial t}\right)_{collision} \qquad [1.22]$$

Now coming back to the case $g \to 0$, it turns out that the correcting term (of order g) can be computed by extending the application of the Vlasov equation at this microscopic level through one of the most fruitful concepts in plasma physics: the *test particle* picture, introduced in the early 1960s by Rostoker and Rosenbluth (1960) and Rostoker (1961).

The idea is the following: a particle with velocity $\boldsymbol{v}_0$ is launched into a homogeneous stable plasma, and this *test particle* is considered as a perturbation of the Vlasov distribution function while all the others (the *field* particles) are treated as a continuum. After some algebra, drag and diffusion coefficients can be calculated.

This resembles the Langevin formalism for Brownian motion. Thus, in a similar way, a *Fokker–Planck*-like equation can be developed. It is interesting to note that the same result has been obtained by Balescu (1960) and Lenard (1960) using complex diagram methods of statistical mechanics. For more details, the reader is referred to the textbook by R. Balescu (1963).

This equation is known as the so-called *Lenard–Balescu equation* (see also Lenard (1960)) and written as

$$\begin{aligned}\frac{\partial f_d}{\partial t} = {} & 16\pi^3 e^4 m_e \int d\boldsymbol{k} \int d\boldsymbol{v}_t\, \boldsymbol{k} \cdot \frac{\partial}{\partial \boldsymbol{v}_d} k^{-4} \left|\epsilon(\boldsymbol{k}, \omega = \boldsymbol{k} \cdot (\boldsymbol{v}_t - \boldsymbol{v}_d)\right|^{-2} \\ & \delta\big(\boldsymbol{k} \cdot (\boldsymbol{v}_d - \boldsymbol{v}_t)\big) \left(\frac{\partial}{\partial \boldsymbol{v}_d} - \frac{\partial}{\partial \boldsymbol{v}_t}\right) f_d(\boldsymbol{v}_d)\, f_t(\boldsymbol{v}_t) \qquad [1.23]\end{aligned}$$

Actually, the RHS of [1.23] gives the first-order $\mathbb{O}(g)$ correction to the Vlasov equation. This rather complicated equation is quite subtle. It gives the evolution of a distribution function f_d of *distinguished* particles when the *total* distribution is f_t. Note that the total distribution includes the *distinguished* distribution and that quantifying the evolution of the distinguished distribution is only possible in a *computer* experiment. Finally, $\epsilon(k, \omega)$ is the plasma dispersion function we shall discuss in Chapter 2 (section 2.2).

We have written the Lenard–Balescu equation using the form [1.23] to exhibit the fundamental differences between 1D and 3D plasmas and between the evolution of the *distinguished* and global distributions (the global distribution is obtained with $f_d = f_t$.) It is clear from equation [1.23] that we just have to consider the *kernel*

$$\delta\big(\boldsymbol{k}\cdot(\boldsymbol{v}_d - \boldsymbol{v}_t)\big)\left(\frac{\partial}{\partial \boldsymbol{v}_d} - \frac{\partial}{\partial \boldsymbol{v}_t}\right) f_d(\boldsymbol{v}_d)\, f_t(\boldsymbol{v}_t) \qquad [1.24]$$

Because of the presence of the δ distribution in [1.24], contribution to [1.23] is non-zero only in the three following cases:

1) $\boldsymbol{k} = 0$;

2) $\boldsymbol{k} \perp (\boldsymbol{v}_d - \boldsymbol{v}_t)$;

3) $\boldsymbol{v}_d = \boldsymbol{v}_t$.

The first case does not give any contribution if we assume global neutrality. For cases 2 and 3, the situation for 1D and 3D plasmas and for *distinguished* and *global* distribution functions is quite different as pointed out by Rouet and Feix (1991). Table 1.3 shows the contribution of these terms.

Distribution	Dimensionality	$\boldsymbol{k}\perp(\boldsymbol{v}_d - \boldsymbol{v}_t)$	$\boldsymbol{v}_d = \boldsymbol{v}_t$
Global	1D	Non-existing term	$\left(\frac{\partial}{\partial \boldsymbol{v}_d} - \frac{\partial}{\partial \boldsymbol{v}_t}\right) f_d\, f_t = 0$
Global	3D	Non-zero contribution	$\left(\frac{\partial}{\partial \boldsymbol{v}_d} - \frac{\partial}{\partial \boldsymbol{v}_t}\right) f_d\, f_t = 0$
Distinguished	1D	Non-existing term	$\left(\frac{\partial}{\partial \boldsymbol{v}_d} - \frac{\partial}{\partial \boldsymbol{v}_t}\right) f_d\, f_t \neq 0$
Distinguished	3D	Non-zero contribution	$\left(\frac{\partial}{\partial \boldsymbol{v}_d} - \frac{\partial}{\partial \boldsymbol{v}_t}\right) f_d\, f_t \neq 0$

Table 1.3. *Contributions to the Lenard–Balescu equation*

Consequently, for 3D plasma the contribution is not zero both for $f_d = f_t$ and $f_d \neq f_t$. Remembering that these calculations are of the first order in the graininess parameter g, we can state that for time up to $(\omega_p g)^{-1}$, namely $\omega_p^{-1}\left(n_e \lambda_D^3\right)$, both *distinguished* and global distribution functions relax toward a thermodynamic equilibrium.

On the other hand, for 1D plasma the global distribution function does not change while the *distinguished* distribution relaxes toward this global distribution. This result is *a priori* strange because this is exactly what happens for very short range potential between two charged sheets, which either pass each other without changing their velocities or bounce off each other by exchanging velocities; in both cases, the total distribution function is not changed whereas for the bounce case, a *distinguished* particle changes its velocity. The result remains true for a 1D plasma.

The above results, very briefly detailed in this section, involve complex and sophisticated treatments. Computer experiments in one dimension were ideally suited to check these results and from the very beginning of computer simulations, "experimental results" have been compared with theory by different groups (see Dawson (1962), Eldridge and Feix (1962), Eldridge and Feix (1963), Dawson (1964)). In all cases, the agreement was well inside error amplitudes.

The last result concerning the relaxation of the *distinguished* distribution toward the global one (which does not change at first order in g) had to wait for the possibility of treating systems with typical values of $n\lambda_D = 200$ and $L/\lambda_D = 200$ (i.e. 4.10^4 particles) over times of order 1,000 ω_p^{-1} using the exact N-body code sketched in Figure 1.7 (see, for instance, the papers by Rouet and Feix (1996) or Ricci and Lapenta (2002)).

This is one of the finest example of checking a theory by computer experiments. These theories are largely based on the use of the Vlasov model to help to understand the behavior of the microscopic fluctuations in a plasma. For further studies, the reader will find a very detailed discussion in the book by Elskens and Escande (2003).

1.8. The Wigner equation (Vlasov equation for quantum systems)

Introducing quantum effects in Vlasov plasmas is a complex and difficult task. The quantum treatment of the N-body quantum problem and its domains of validity are beyond the scope of this book. This subject is addressed in the papers by Clérouin *et al.* (1990), Clérouin *et al.* (1992), Yalabik *et al.* (1989) and Remler and Madden (1990), for instance.

Nevertheless, it is very interesting to focus on a paper by Wigner (1932) which allows us to discuss similarities and differences with the classical Vlasov equation: discussing the properties of Wigner equation will hopefully shed some light on the more general quantum statistical problem.

The simplest way to introduce the Wigner equation is to consider the wave function $\psi(x,t)$ describing the motion of one particle (i.e. an electron in a 1D plasma so as to simplify the problem). Following Wigner (1932), let us introduce the *Wigner* function $f_W(x,p,t)$ in the 2D (x,p) phase space through the relation

$$f_W(x,p,t) = \frac{1}{2\pi\hbar}\int_{-\infty}^{+\infty} \psi\left(x+\frac{\Delta}{2}\right)\psi^*\left(x-\frac{\Delta}{2}\right)\exp\left(-\frac{ip\Delta}{\hbar}\right)d\Delta$$

It is easy to see that the respective integrations of f_W with respect to p and x lead to the so-called marginal distributions $n(x)$ and $N(p)$ in agreement with the well-known quantum interpretations

$$n(x) = \psi(x)\psi^*(x) \text{ and } N(p) = \hbar^{-1}\Phi(p/\hbar)\Phi^*(p/\hbar)$$

where Φ is the Fourier transform of ψ.

Unfortunately, this Wigner function can exhibit not only positive but also negative values, which prevents f_W from being a *bona fide* distribution in phase space. It is nevertheless a useful mathematical tool allowing us to understand the transition from classical to quantum mechanics.

Since ψ obeys the usual Schrödinger equation

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m_e}\frac{\partial^2\psi}{\partial x^2} + V\psi,$$

where $V(x,t)$ is the potential seen by our electron, it is easy to find the corresponding *Wigner equation* for f_W.

$$\frac{\partial f_W}{\partial t} + \frac{p}{m_e}\frac{\partial f_W}{\partial x} = \qquad [1.25]$$

$$\frac{i}{2\pi\hbar^2}\iint\left(V\left(x+\frac{\lambda}{2}\right)-V\left(x-\frac{\lambda}{2}\right)\right)f_W(x,p',t)\exp\left(\frac{i\lambda}{\hbar}(p-p')\right)d\lambda dp'$$

Detailed calculations can be found in the works by Wigner (1932) or Moyal (1949).

The form of this equation is very similar to the Vlasov equation. Nevertheless, the field term exhibits a more complicated expression involving convolution products. But it is easy to demonstrate that the Wigner equation [1.25] coincides with the classical Vlasov one in only three cases, as explained in the paper by Bertrand *et al.* (1980):

1) free particle motion $V = 0$;

2) uniform field $V = -xE_0$;

3) harmonic oscillator $V(x) = (A/2)x^2$.

The first case is obvious: for free particle motion, we have $V = 0$ and the second member of [1.25] is zero.

For the two other cases, it is obvious that

$$V\left(x - \frac{\lambda}{2}\right) - V\left(x + \frac{\lambda}{2}\right) = -\lambda E(x)$$

where

$$E(x) = -\partial V/\partial x$$

is the electric field deriving from the potential V. Then the λ-integration is easily performed which brings in a Dirac delta function $\delta(p - p')$, allowing the p'-integration by part. Therefore, the right hand side of [1.25] is just equal to

$$-E(x)\frac{\partial f_W}{\partial p}$$

i.e. the classical result. At the first sight, this result is surprising since we know that the quantum treatment of the harmonic oscillator brings quantified levels of energy. As a matter of fact, we must pay attention to the initial conditions introduced in the Wigner equation [1.25]: the conditions corresponding to a given $\psi(x, 0)$ describe what is called *a pure state*. Following the usual explanation of Von Neumann, the Wigner equation also describes *mixtures,* i.e. situations described by different ψ_i with probability P_i and a Wigner distribution given by

$$f_W = \sum_i P_i f_{Wi}$$

where f_{Wi} is the Wigner distribution associated with ψ_i.

This very delicate aspect of quantum statistical mechanics needs the introduction of the random phase approximation between the different ψ_i's. Details can be found in the reference textbook by Huang (1963).

Coming back to plasma, this *mixture* interpretation is needed to deal with the quantum treatment of any basic plasma problems. Let us consider the well-known linear analysis of a small perturbation in a homogeneous plasma. This is the well-known Landau (1946) problem. We shall discuss it more extensively in Chapter 2 (section 2.2). Let us briefly outline the main idea.

Starting with the Vlasov–Poisson system [1.8]–[1.9] describing a 1D (say x-axis) electron plasma in a fixed homogeneous ion background

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} + \frac{e}{m_e}E\frac{\partial f}{\partial v} = 0 \quad \text{and} \quad \frac{\partial E}{\partial x} = \frac{en_0}{\epsilon_0}\left(\int_{-\infty}^{+\infty} f\, dv - 1\right) \qquad [1.26]$$

we look at a small perturbation around an electronic homogeneous equilibrium solution (with a normalized homogeneous equilibrium distribution function $F_0(v)$)

$$f(x,v,t) = F_0(v) + \varepsilon f_1(x,v,t) \quad \text{and} \quad E(x,t) = 0 + \varepsilon E_1(x,t)$$

with $\varepsilon \ll 1$. Keeping only first order ε terms, [1.26] becomes linear

$$\frac{\partial f_1}{\partial t} + v\frac{\partial f_1}{\partial x} + \frac{e}{m_e}E_1\frac{dF_0}{dv} = 0 \quad \text{and} \quad \frac{\partial E_1}{\partial x} = \frac{en_0}{\epsilon_0}\int_{-\infty}^{+\infty} f_1\, dv \qquad [1.27]$$

allowing us to seek harmonic solutions for f_1 and E_1 of the form $\exp\left(i(kx - \omega t)\right)$ so that equation [1.27] becomes

$$\epsilon(k,\omega)\, E_1(k,\omega) = 0$$

where $\epsilon(k,\omega)$ is the plasma dispersion function

$$\epsilon(k,\omega) = 1 + \frac{\omega_p^2}{k}\int_{-\infty}^{+\infty} \frac{dF_0/dv}{\omega - kv}\, dv \qquad [1.28]$$

We recognize the plasma frequency $\omega_p^2 = n_e e^2/\epsilon_0 m_e$.

Repeating the same treatment on the Wigner equation [1.25], we get

$$\epsilon(k,\omega) = 1 + \frac{\omega_p^2}{k}\int_{-\infty}^{+\infty} \frac{F_0\left(v + \hbar k/2m_e\right) - F_0\left(v - \hbar k/2m_e\right)}{(\hbar k/m_e)(\omega - kv)}\, dv \qquad [1.29]$$

Comparing relations [1.29] and [1.28], we see that the quantum treatment replaces the derivative dF_0/dv by its *centered finite difference* using a step $\hbar k/m$. Obviously, long wavelengths are unaffected by the quantum treatment (of course an expected result). More precisely, wavenumbers such that $k\overline{\lambda} \ll 1$ have negligible corrections. Here, $\overline{\lambda} = \hbar/mv_T$ is the characteristic de Broglie wavelength of the plasma and v_T is a characteristic velocity of the plasma (for example the thermal velocity).

An interesting difference can be pointed out in studying the behavior of a cold plasma with $F_0(v) = \delta(v)$. The well-known classical result does not exhibit any k-dispersion (we have $\omega^2 = \omega_p^2$). On the other hand, using [1.29] and solving $\epsilon = 0$ yields the corresponding quantum result

$$\omega^2 = \omega_p^2 + \frac{\hbar^2 k^4}{4m_e^2} \qquad [1.30]$$

with a k^4 correction as pointed by Bertrand *et al*. (1980).

A useful model to study the oscillations and the stability properties of a quantum plasma is given by the so-called *multistream model* introduced by Haas *et al*. (2000)

$$F_W = \sum_{i=1}^{N} \delta\left(v - a_i\right) \qquad [1.31]$$

This Wigner distribution corresponds to a mixture of N pure states. The state i is characterized by a wavefunction

$$\psi_i = \sqrt{n_i}\exp\left(-ik_i x\right) \quad \text{with} \quad k_i = \frac{m_e a_i}{\hbar} \qquad [1.32]$$

Plugging [1.32] into [1.29] and introducing the plasma frequency of beam (i) defined by $\omega_i^2 = n_i e^2/m_e \epsilon_0$ yields

$$\epsilon(k,\omega) = 1 - \sum_{i=1}^{N} \frac{\omega_i^2}{(\omega - ka_i)^2 - \hbar^2 k^4/4m_e^2} \qquad [1.33]$$

Note that [1.33] generalizes to the quantum case the well-known dispersion relation of the classical case. Obviously, with a sufficiently large number of streams we can modelize any homogeneous plasma. The form of [1.33] confirms for this general case the k^4 quantum correction of [1.30]. Details and applications can be found in the paper by Haas *et al*. (2000).

As a final remark, a similarity has to be pointed out between the classical Vlasov and the quantum Wigner equations even for the general nonlinear problem. As noticed in [1.29] for the linear case, the potential term in the Wigner equation [1.25] exhibits a replacement of a derivative by its finite difference counterpart. This similarity is somewhat surprising since the classical equation is entirely based on a trajectory concept and the incompressibility of the phase space fluid. At least, it points out the usefulness of this phase space treatment of quantum problems. But *a priori* this was not suspected.

1.9. The relativistic Vlasov–Maxwell model

Up to now, we have derived the Vlasov equation as the fundamental paradigm for collective effects in the case of an electrostatic model of a one species (electrons) plasma embedded in a fixed ion background taking into account only electrostatic forces in classical mechanics. The generalization to a fully electromagnetic relativistic multispecies plasma needs further discussion but involves at least the following steps.

For each species α, we define a distribution function $f_\alpha(\boldsymbol{r}, \boldsymbol{p}_\alpha, t)$ and add to the force term the Lorentz force $(\boldsymbol{p}_\alpha/m_\alpha\gamma_\alpha) \times \boldsymbol{B}$, where $m_\alpha\gamma_\alpha$ is the relativistic mass and γ_α is the Lorentz factor

$$\gamma_\alpha^2 = \left(1 + \boldsymbol{p}_\alpha^2/m_\alpha^2 c^2\right)$$

In laser-plasma experiments, some electrons may reach velocities close to the light velocity c.

Now for each species, the Vlasov equation is written

$$\frac{\partial f_\alpha}{\partial t} + \frac{\boldsymbol{p}_\alpha}{m_\alpha\gamma_\alpha} \cdot \frac{\partial f_\alpha}{\partial \boldsymbol{r}} + q_\alpha \left(\boldsymbol{E} + \frac{\boldsymbol{p}_\alpha}{m_\alpha\gamma_\alpha} \times \boldsymbol{B}\right) \cdot \frac{\partial f_\alpha}{\partial \boldsymbol{p}_\alpha} = 0 \qquad [1.34]$$

where the electromagnetic fields obey Maxwell's equations

$$\frac{\partial}{\partial \boldsymbol{r}} \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \tag{1.35}$$

$$\frac{\partial}{\partial \boldsymbol{r}} \times \boldsymbol{B} = \mu_0 \left(\boldsymbol{J} + \varepsilon_0 \frac{\partial \boldsymbol{E}}{\partial t} \right) \tag{1.36}$$

$$\frac{\partial}{\partial \boldsymbol{r}} \cdot \boldsymbol{E} = \frac{\rho}{\varepsilon_0} \tag{1.37}$$

$$\frac{\partial}{\partial \boldsymbol{r}} \cdot \boldsymbol{B} = 0 \tag{1.38}$$

Finally, to ensure self-consistency, charge and current densities are to be expressed in terms of distribution functions $f_\alpha(\boldsymbol{r}, \boldsymbol{p}_\alpha, t)$

$$\rho = \sum_\alpha q_\alpha \int f_\alpha \, d\boldsymbol{p}_\alpha \tag{1.39}$$

$$\boldsymbol{J} = \sum_\alpha q_\alpha \int \frac{\boldsymbol{p}_\alpha}{m_\alpha \gamma_\alpha} f_\alpha \, d\boldsymbol{p}_\alpha \tag{1.40}$$

The set of equations [1.34]–[1.40] forms the basis of the relativistic Vlasov–Maxwell model. Two specific points deserve a few comments.

First, the formal derivation of the Vlasov equation from the N-body distribution function in the presence of a magnetic field is a little more delicate than the electrostatic example that we have seen in section 1.5. The reason can be ascribed to the complications introduced by the need to extend the mean field theory to $\boldsymbol{B}$: similarly to the way that the redistribution of charges in a plasma modifies the effective electric field felt by an electron (or ion) in a certain point of space and time, the collective motion of the same charges corresponds to electric currents, which in turn contribute to a mean magnetic field inside of the plasma. The fact that the magnetic induction $\boldsymbol{B}$ appearing in the set of equations above corresponds to the self-consistent field, which is obtained inside of the plasma, requires some further hypotheses, especially in relativistic regimes. These hypotheses are in the end related to those that justify the use of the well-known minimal coupling assumption,

$$\boldsymbol{P}_\alpha = \boldsymbol{p}_\alpha + q_\alpha \boldsymbol{A},$$

between the particle kinetic momentum $\boldsymbol{p}_\alpha$ and the canonical momentum $\boldsymbol{P}_\alpha$, which defines the "appropriate" conjugate coordinate to spatial position $\boldsymbol{r}$ in the phase space of a magnetized system. In view of the BBGKY cluster expansion, some approximations are indeed required already at the level of the N-body Hamiltonian of a charged system in order to write it by isolating the kinetic component of the energy from higher order interparticle interactions of magnetic nature. Because of these hypotheses, the relativistic Lorentz factor of each particle in the plasma can be expressed in terms of the canonical momentum by relying on the minimal coupling contribution only, and the subsequent integrations in the velocity space lead to the Vlasov–Maxwell system in the form it is written above. A more detailed discussion on this topic, toghether with some bibliographical references, can be found, for example, in the PhD thesis of Sarrat (2017).

The second comment is more like a note, aimed at drawing the attention of the reader to a point: the procedure with which we have obtained the Vlasov equation in the electrostatic regime (see section 1.5) authorized us to write it in terms of the self-consistent electric field $\boldsymbol{E}$ defined by means of equation [1.9]. This definition in terms of the *free* charges inside of the plasma makes it possible to write the electrostatic forces in terms of the Poisson equation for the electric field $\boldsymbol{E}$, by adopting for it the same definition used in vacuum, without any need to introduce a "dielectric induction" vector $\boldsymbol{D}$. For similar reasons, when including the magnetic field, in the end we can couple the Vlasov equation with the set of Maxwell's equations used for electromagnetic fields in a vacuum. In Chapter 3, we will discuss the response of the plasma to the electromagnetic perturbations and we will see why and in which context in collisionless plasmas we speak of polarization, finite conductivity, dielectric tensors and other features, which are normally related to the physics of electromagnetic fields in materials of dielectric nature.

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