1

Fundamental Equations of Conduction

1.1. Introduction

In this chapter we explore the general equations that reflect energy balances in conduction. These equations will then be used to solve a number of physical problems of interest to the engineer.

Equations governing conduction are extremely useful, both for quantifying energy losses in industrial installations (steam pipes, furnaces, reactors, etc.) and for the application of thermal building regulations. In each case, a comprehensive implementation of these equations makes it possible to calculate the fluxes of energy losses if no action was taken. The analysis methods presented in the following sections offer the possibility of quantifying energy flows, as well as indicating the actions to implement in order to limit energy loss.

For each of the situations that are explored, special attention is paid to the practical applications of the equations established. Thus, problems relating to the thermal insulation of industrial installations are approached both technically, by assessing the fluxes of energy losses, and with regard to the profitability of the investments to be made in order to achieve thermal insulation. For each of the situations explored, special attention is paid to the practical applications of the equations established. Thus, thermal insulation problems are concretely dealt with by means of technical and economic analyses, which demonstrate the benefit in terms of reducing production costs.

In this chapter, the problems relating to conduction in the presence of energy generation are also examined. They correspond to practical situations encountered in exothermic reactors or in nuclear-reactor fuel bars.

1.2. General equations of conduction

Consider a solid, of arbitrary shape, in which we assume that there is conduction heat transfer.

In order for the development to be general, we will assume that there is heat generation in the solid, and that P is the generation power per unit volume.

Let us establish an energy balance on a microscopic element of this solid with sides dx, dy and dz (see Figure 1.1).

The general balance equation is as follows:

Input – Output + Generation = Accumulation

which can be written simply as:

$I - O + G = A$

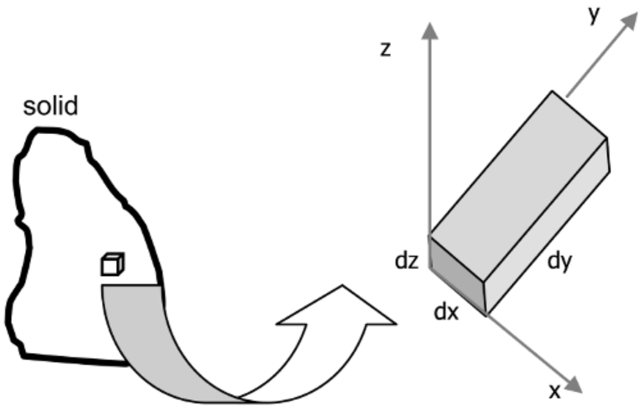

Figure 1.1. *Differential energy balance*

1.2.1. *Expressing the term* (I – O)

In the most general case, energy flows in the three directions, ox, oy and oz. Consequently, its flux entering or exiting the solid element considered will have three components: one component for each direction of propagation. These three components will be indicated as follows:

ϕ_x: flux in direction x

ϕ_y: flux in direction y

ϕ_z: flux in direction z

The flux vector is then written: $\Phi = \begin{pmatrix} \phi_x \\ \phi_y \\ \phi_z \end{pmatrix}$.

1.2.1.1. *Expressing ϕ_x*

Conduction heat input is the energy flux in the direction x, expressed at abscissa x. It is given by: $\phi_{x_{\downarrow x}} = -\lambda_x \left(\mathrm{dydz}\right)\left(\frac{\partial T}{\partial x}\right)_{\downarrow x}$.

Similarly, the output corresponds to the energy flux in the direction x, expressed at abscissa $x + dx$: $\phi_{x_{\downarrow x+dx}} = \phi_{x_{\downarrow x}} + \left(\frac{\partial \phi_x}{\partial x}\right)_{\downarrow x} \mathrm{dx}$.

Hence: $\left(\mathrm{I}-\mathrm{O}\right)_x = \phi_{x_{\downarrow x}} - \phi_{x_{\downarrow x+dx}} = -\left(\frac{\partial \phi_x}{\partial x}\right)_{\downarrow x} \mathrm{dx}$.

Therefore: $\left(\mathrm{I}-\mathrm{O}\right)_x = \frac{\partial}{\partial x}\left[\lambda_x \frac{\partial T}{\partial x}\right]\mathrm{dxdydz}$.

1.2.1.2. *Expressing ϕ_y*

Using a similar development we obtain: $\left(\mathrm{I}-\mathrm{O}\right)_y = \frac{\partial}{\partial y}\left[\lambda_y \frac{\partial T}{\partial y}\right]\mathrm{dxdydz}$.

1.2.1.3. *In direction z*

We have: $\left(\mathrm{I}-\mathrm{O}\right)_z = \frac{\partial}{\partial z}\left[\lambda_z \frac{\partial T}{\partial z}\right]\mathrm{dxdydz}$.

1.2.1.4. *For the three directions*

When we consider the flow of energy in the three propagation directions, the term (*I – O)* is then given by: $I - O = (I - O)_x + (I - O)_y + (I - O)_z$.

Therefore: $I - O = \left[\frac{\partial}{\partial x}\left(\lambda_x \frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(\lambda_y \frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(\lambda_z \frac{\partial T}{\partial z}\right)\right] dxdydz$.

> NOTE.– The equation giving the term (I – O), established above, is valid in the general case where the conduction behavior of the solid is different along the three directions of propagation; i.e., the heat conductivity is different for each direction of the space, $\lambda x \neq \lambda y \neq \lambda z$. This is the case for orthotropic materials.
>
> In practice, we often encounter materials where conduction is identical in the three directions of propagation: the heat conductivity λ is the same in the three directions. This is the case for *isotropic* materials. Thus, the equation giving the term *I – O* for isotropic conduction is given by:
>
> $$I - O = \left[\frac{\partial}{\partial x}\left(\lambda \frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(\lambda \frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(\lambda \frac{\partial T}{\partial z}\right)\right] dxdydz$$
>
> For the sake of clarity, we will continue developing the balance equation for the isotropy case; the development is identical in the general case.

1.2.2. *The term "generation"*

P being the volumetric generation power, the energy generated in the volume element dxdydz is given by: $\phi_G = P\, dx\, dy\, dz$.

1.2.3. *The term "accumulation"*

If *m designates the mass* of the volume element considered and if C_p is the sensible heat of the material constituting it, the energy accumulated in this element is given by: $\phi_A = mC_p \frac{\partial T}{\partial t}$

The mass, *m*, can be expressed as follows: $m = \rho\, dx\, dy\, dz$.

Therefore: $\phi_A = \rho C_p \frac{\partial T}{\partial t} dxdydz$.

1.2.4. *Energy balance equation*

The balance equation can then be written as follows for isotropic materials:

$$\rho C_p \frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left(\lambda \frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(\lambda \frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(\lambda \frac{\partial T}{\partial z}\right) + P$$

This energy balance equation is often referred to as the *general conduction equation*.

NOTE.–

i) In the case of isotropy: λ is a scalar value.

We then have: $$\varphi = \begin{pmatrix} \varphi_x \\ \varphi_y \\ \varphi_z \end{pmatrix} = \begin{pmatrix} -\lambda \frac{\partial T}{\partial x} \\ -\lambda \frac{\partial T}{\partial y} \\ -\lambda \frac{\partial T}{\partial z} \end{pmatrix} = -\lambda \nabla T$$

Thus, for isotropic materials with constant λ, the conduction equation becomes:

$$\left(\frac{\rho C_p}{\lambda}\right) \frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} + \frac{P}{\lambda}$$

This equation is usually written as follows, using the Laplace notation:

$$\frac{1}{\alpha}\frac{\partial T}{\partial t} = \nabla^2 T + \frac{P}{\lambda} \quad \text{where} \quad \alpha = \frac{\lambda}{\rho C_p}$$

α is called the thermal diffusivity of the solid considered. It is a physical property of the material considered. It is easily verified that the dimensions of α correspond to a surface per unit time:

$[\alpha] = [L]^2 [T]^{-1}$

The thermal diffusivity could then be interpreted as a surface propagation speed of heat.

The thermal diffusivity values for the usual materials are presented in the Appendix. Some orders of magnitude of α:

α Cu = 0.4 m^2/hr

α Steel = 0.05 m^2/hr

α Glass = 1.96 10^{-3} m^2/hr

α Wood = 3.5 10^{-4} m^2/hr

It should be noted that the thermal diffusivities of copper and steel are two orders of magnitude greater than those of glass or cork.

ii) In the case of anisotropic (non-isotropic) materials, directional thermal conductivities are usually grouped into a matrix (Λ). The conduction equations can be obtained from the general equation presented in section 1.2.4, taking into consideration the variability of λ or, where applicable, using the thermal conductivity matrix corresponding to the case considered (McAdams, 1954; Sherwood and Reed, 1957).

iii) In the case of orthotropy

Orthotropic materials have different conductivities for each of the different directions. However, the conductivity is the same in a given direction with a rotational symmetry around each axis.

Mathematically, the thermal conductivity of an orthotropic material is a diagonal matrix Λ. The energy flux density is then as follows:

$$\varphi = \begin{pmatrix} \varphi_x \\ \varphi_y \\ \varphi_z \end{pmatrix} = -\begin{pmatrix} \lambda_x & 0 & 0 \\ 0 & \lambda_y & 0 \\ 0 & 0 & \lambda_z \end{pmatrix} \begin{pmatrix} \frac{\partial T}{\partial x} \\ \frac{\partial T}{\partial y} \\ \frac{\partial T}{\partial z} \end{pmatrix} = -\Lambda \bullet \nabla T$$

Consequently, the conduction equation is given by:

$$\rho C_p \frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left(\lambda_x \frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(\lambda_y \frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(\lambda_z \frac{\partial T}{\partial z}\right) + P$$

iv) In the most general case of anisotropic materials, thermal conductivity is an arbitrary matrix (Λ). We then have:

$$\varphi = \begin{pmatrix} \varphi_x \\ \varphi_y \\ \varphi_z \end{pmatrix} = -\begin{pmatrix} \lambda_{xx} & \lambda_{xy} & \lambda_{xz} \\ \lambda_{yx} & \lambda_{yy} & \lambda_{yz} \\ \lambda_{zx} & \lambda_{zy} & \lambda_{zz} \end{pmatrix} \bullet \nabla T$$

In this context:

$$\rho C_p \frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left(\lambda_{xx}\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(\lambda_{xy}\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(\lambda_{xz}\frac{\partial T}{\partial z}\right) + \frac{\partial}{\partial y}\left(\lambda_{yx}\frac{\partial T}{\partial x}\right)$$

$$+\frac{\partial}{\partial y}\left(\lambda_{yy}\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(\lambda_{yz}\frac{\partial T}{\partial z}\right) + \frac{\partial}{\partial z}\left(\lambda_{zx}\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(\lambda_{zy}\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(\lambda_{zz}\frac{\partial T}{\partial z}\right) + P$$

This case is rarely encountered in practice.

However, in most cases of interest to the engineer, the materials implemented can be considered isotropic, with thermal conductivities, λ*, variable or constant* (Brown and Marco, 1958). This is why the following section is dedicated to presenting the equation of conduction in different coordinate systems, for the isotropic case.

1.3. Equations of conduction in different coordinate systems

Anisotropic or orthotropic materials (variable thermal conductivity λ depending on the direction) are sometimes encountered in engineering calculations. This is particularly true when energy is called upon to propagate in wood or in salts of non-homogeneous crystalline structures. For such situations, the forms of the conduction equation that take account of this variability in thermal conductivity are to be used.

Yet in practice, we mainly encounter isotropic materials, that is, materials whose thermal conductivity, λ, is the same in all directions. Indeed, this is often the case for metals and alloys usually encountered in industrial equipment and devices. However, even in the isotropic case, λ is not necessarily constant. It is identical in all directions, but can vary with the temperature, for example. For this reason, this section is dedicated to presenting equations of conduction for the isotropic case, with either variable or constant thermal conductivities.

Moreover, the geometry of the solid determines which coordinate system to use. Thus, when the solid studied is a parallelepiped shape, for example, it makes sense to choose the Cartesian coordinates for analysis.

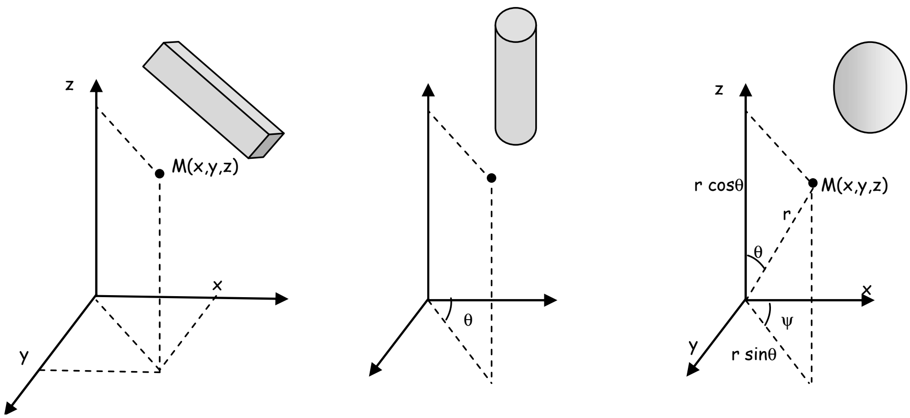

Figure 1.2. *The coordinate system depends on the shape of the solid considered*

Likewise, when the system studied is a pipe or a ball, we will opt for cylindrical or spherical coordinates, respectively.

The following sections present the equations of conduction in different coordinate systems for isotropic cases, but whose thermal conductivities can be either variable or constant.

1.3.1. *When λ is not constant*

– Rectangular coordinates:

$$\rho C_p \frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left(\lambda_x \frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(\lambda_y \frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(\lambda_z \frac{\partial T}{\partial z}\right) + P$$

– Cylindrical coordinates (r, θ, z):

$$\rho C_p \frac{\partial T}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left(\lambda r \frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial \theta}\left(\lambda \frac{\partial T}{\partial \theta}\right) + \frac{\partial}{\partial z}\left(\lambda \frac{\partial T}{\partial z}\right) + P$$

– Spherical coordinates (r, θ, ψ):

$$\rho C_p \frac{\partial T}{\partial t} = \frac{1}{r^2}\frac{\partial}{r}\left(\lambda r^2 \frac{\partial T}{\partial r}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial}{\partial\psi}\left(\lambda \frac{\partial T}{\partial \psi}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\lambda \sin\theta \frac{\partial T}{\partial \theta}\right) + P$$

1.3.2. *When λ is constant*

– Rectangular coordinates:

$$\frac{1}{\alpha}\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} + \frac{P}{\lambda}$$

– Cylindrical coordinates (r, θ, z):

$$\frac{1}{\alpha}\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} + \frac{\partial^2 T}{\partial z^2} + \frac{P}{\lambda}$$

– Spherical coordinates (r, θ, ψ):

$$\frac{1}{\alpha}\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial r^2} + \frac{2}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 T}{\partial \psi^2} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} + \frac{1}{r^2 \mathrm{tg}\theta}\frac{\partial T}{\partial \theta} + \frac{P}{\lambda}$$

1.3.3. *Simplified cylindrical and spherical coordinates*

Very often, temperature variations are confined to only two directions, and sometimes even only one. The latter cases are frequently encountered when the heat propagates from the center of a pipe to the outside under the action of a large temperature gradient in the r direction, while only a small gradient exists in the longitudinal direction. This also happens in the case of a sphere that is in a medium or bath of homogeneous temperature: it is obvious that, in this context, the existence of a temperature gradient between the center of the sphere and the bath will result in heat propagation along r only; due to the symmetry of the problem and the homogeneity of the bath, the other directions will not be involved in the transfer.

In these situations, simplified coordinates r and z are used for cylindrical systems, and r and θ, for spherical systems.

– Simplified cylindrical coordinates (*r, z*):

$$\frac{1}{\alpha}\frac{\partial T}{\partial t}=\frac{\partial^2 T}{\partial r^2}+\frac{1}{r}\frac{\partial T}{\partial r}+\frac{\partial^2 T}{\partial z^2}+\frac{P}{\lambda}$$

– Simplified spherical coordinates (*r, θ*):

$$\frac{1}{\alpha}\frac{\partial T}{\partial t}=\frac{\partial^2 T}{\partial r^2}+\frac{2}{r}\frac{\partial T}{\partial r}+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial T}{\partial\theta}\right)+\frac{P}{\lambda}$$

1.3.4. *One-dimensional conduction*

In most problems of interest to the engineer, using a single heat-propagation direction is sufficient to be able to describe fairly accurately what happens in practice (Jakob and Hawkins, 1957).

Thus, the equations of conduction presented previously in Cartesian, cylindrical and spherical coordinates can be further simplified when a single propagation direction is retained. This direction is generally *x* for Cartesian coordinates and *r* for cylindrical and spherical coordinates, in which case we will speak of one-dimensional or one-directional problems. This type of problem is commonly encountered as a simplification in engineering calculations. In this case, the equations are:

– In rectangular coordinates: $\frac{1}{\alpha}\frac{\partial T}{\partial t}=\frac{d^2 T}{dx^2}+\frac{P}{\lambda}$

– In simplified cylindrical coordinates: $\frac{1}{\alpha}\frac{\partial T}{\partial t}=\frac{d^2 T}{dr^2}+\frac{1}{r}\frac{dT}{dr}+\frac{P}{\lambda}$

– In simplified spherical coordinates: $\frac{1}{\alpha}\frac{\partial T}{\partial t}=\frac{d^2 T}{dr^2}+\frac{2}{r}\frac{dT}{dr}+\frac{P}{\lambda}$

1.4. Reading: metal tempering

Tempering is a heat treatment used in metallurgy to improve the mechanical properties of metals. It enables the hardness or the overall resistance of a given metal to be increased. This treatment generally consists of heating the metal to raise its temperature, then cooling it at a given speed. The thermal stresses thus imposed lead to changes in the metal’s crystalline structure.

As a result, with each metal having a crystalline form within which atoms of a so-called "interstitial" compound can be inserted, the solubility of these atoms in the mesh depends on the structure of the latter and its size. If a metal's temperature variations are conducted so that the phase change point is approached, they can then lead to changes in the crystalline structure, in particular with respect to its interstitial distances. We will thus pass from a mesh having important sites to a tighter mesh imprisoning interstitial atoms in the new crystalline structure.

Bearing in mind that the mechanical properties are closely related to the geometries of the crystalline meshes, it follows that the different temperature variations induce a stress in these meshes, and thus a change in the mechanical properties of the metal considered.

The origin of metal tempering dates back to the early 20th Century, when blacksmiths subjected the sharpened parts of a tool to this type of heat treatment, in order to increase their mechanical strength. The metal was heated to red before being immersed in water.

The temperature to which the metal is heated before cooling is significant. It is called the "phase change temperature". It must be chosen in such a way that it allows the "suspension" of the interstitial chemical compounds dissolved in the metal's crystalline structure.

The heating time, or the time during which the metal is subjected to high temperatures, is also significant. It must be long enough to ensure the transformation of the entire heated mass, but it must not be too long either.

The cooling time, for its part, is also a decisive parameter. Indeed, the latter must allow cooling of the entire mass considered at a great enough speed to lead to the imprisonment of interstitial chemical elements having diffused into the crystalline structure of the solid during the heating phase. This cooling operation is generally performed by soaking the metal part in a cooling fluid. The fluids used are either water or salt water to generate faster cooling.

As we will see in Chapter 6, the cooling speed will depend on the following three elements:

– The heat transfer in the metal considered, and therefore its thermal conductivity.

– The nature of the fluid used.

– The heat transfer between the tempering fluid and the solid, and therefore the convection heat transfer coefficient, h.

In Chapter 6 of this volume we will see how the conduction equations are used to accurately calculate the time required for a certain degree of heat penetration into a solid.

Knowledge of this time period will permit heating the material to a sufficiently high temperature for just long enough and, next, cooling it at an appropriate speed.

In this manner, chemical elements will be imprisoned in the crystalline structure creating, after cooling, tensions in the meshes which will contribute to improving the mechanical resistance of the tempered part.

Although mechanical resistance of metals can be augmented through tempering, it should be noted, however, that this thermal treatment often causes metals to lose their elasticity. Indeed, the modifications induced by tempering in the meshes of the crystalline structure constrain movements of the crystalline irregularities (vacancy defects or dislocations) which are important for elasticity. Thus, tempering results in a loss of the plastic deformation mechanism that allows a metal to return to its resting state following a deformation. Actually, tempering brings a metal's elastic resistance closer to its failure resistance.