Part 1

Flooding

1

Modelling Flooding in Edge- or Node-weighted Graphs

1.1. Summary of the chapter

This chapter opens Volume 2 of *Topographical Tools for Filtering and Segmentation*, which is, devoted to flooding. In the Introduction we stressed the importance of flooding for filtering and segmenting images. Flooding is the perfect companion for a watershed, as it is able to suppress regional minima by filling them with lakes. The catchment zone of a suppressed regional minimum is entirely absorbed by the catchment basin of a remaining regional minimum [MEY 02].

For the sake of generality, we model topographic surfaces as node- or edge-weighted graphs.

A node-weighted graph may represent a topographic surface at pixel level. The nodes represent the pixels, and the edges link neighboring pixels. The nodes have two weights: the lower weight represents the ground level, and the higher weight represents the flooding level, which cannot be below the ground level. The edges are unweighted.

An edge-weighted graph may represent a flooding at a larger scale. The nodes represent the catchment zones of a topographic surface, and the edges represent the pass points between neighboring basins. The node weights indicate the level of the lake covering the corresponding basin, equal to $-\infty$ if the basin is unflooded. The edge weights represent the altitude of the corresponding pass point.

Thus, we obtain two models of graphs that are able to represent the same flooding of the same topographic surface on two scales: a micro scale with the node-weighted graph and a macro scale with the edge-weighted graph. We will call them, respectively, micro and macro models of flooding.

We then propose two physical models ensuring that a particular flood distribution is in an equilibrium state, one for node-weighted graphs and the other for edge-weighted graphs. Before, finally, showing the coherence between these models.

1.2. The importance of flooding

1.2.1. *Flooding creates lakes*

In *Topographical Tools for Filtering and Segmentation 1*, we studied watersheds in node- and edge-weighted graphs. The watershed theory is based entirely on the trajectory of a drop of water deposited at a particular node. For node-weighted graphs, a drop of water never glides upwards. For edge-weighted graphs in which the nodes have no weights, a drop of water deposited on a node follows the lowest adjacent edge. This drop of water ultimately ends up in a regional minimum of the graph. A catchment zone is associated with each regional minimum of the graph.

Regional minima, however, do not all have the same importance. Some are at the bottom of large and deep catchment basins. Others simply are small fluctuations of the surface due to the presence of noise. A watershed will be largely useless if we are not able to regularize the fluctuations producing these minima. Flooding a topographic surface is an efficient way to simplify a topographic surface; indeed, flooding creates lakes that are flat surfaces, totally or partially filling the catchment basins.

The term “flooding” is borrowed from the physical situation in which a graph represents a topographic surface. A flooding of a topographic surface creates several lakes in the catchment basins, as shown in Figure 1.2(a). After flooding, this topographic surface has less minima than the initial topographic surface. Some catchment basins contain a lake which is full: adding a drop of water would provoke an overflow into a neighboring basin. In the flooded surface, such a catchment basin is no longer a regional minimum and does not appear in the watershed segmentation of the flooded surface. Other lakes completely cover several regional minima of the initial topographic surface; the catchment basin associated with such lakes are the union of several catchment basins of the unflooded surface. Only the unflooded catchment basins or the lakes of the flooded surface, which are still regional minima, give birth to a catchment basin of the flooded surface.

1.2.2. *Flooding for controlling watershed segmentation*

Watersheds are frequently used for segmenting images. A gradient image is derived for the image to be segmented. The contours of the object follow crest lines of the gradient image. The inside and outside of the object constitute regional minima, and the catchment basins of these minima constitute segmentation of the object. Unfortunately such an ideal situation almost never happens. Instead of obtaining a partition in which each region represents one object or the background, we obtain a tessellation of several small fragments, an oversegmentation of the image. Each regional minimum of the gradient image gives birth to a catchment basin. In order to reduce the oversegmentation, we have to reduce the number of minima. An appropriate flooding of the topographic surface will suppress unwanted minima, while preserving and merging others, such that the watershed partition associated with the flooded surface yields the desired segmentation.

1.2.3. *Flooding, razing, leveling and flattening*

More generally, flooding is a powerful filter for simplifying images. Razing is derived from flooding by duality. Duality is based on negation or complementation. If g is a topographic surface (Figure 1.1(a)), its negation will be $-g$ (Figure 1.1(b)). The peaks become wells and vice versa. The negated surface may then be flooded (Figure 1.1(c)). The flooded surface is then negated again (Figure 1.1(d)). The result is called razing, as some peaks have been razed. A peak of g which has been filled by a lake in the function $-g$ appears as a razed peak after the second negation.

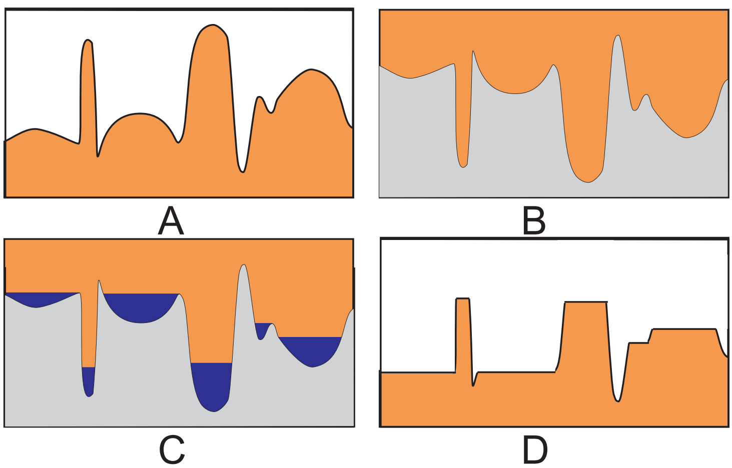

Figure 1.1. *a) A function g representing a topographic surface. b) The negated surface $-g$. c) The surface $-g$ is flooded and lakes are created. d) The flooded surface is inverted. The resulting surface constitutes a razing of the original surface g. For a color version of the figures in this chapter, see www.iste.co.uk/meyer/topographical2.zip*

Thus, flooding fills wells and valleys while peaks and crest lines are razed. When combined, they produce powerful filters for simplifying images or weighted graphs such as leveling and flattening (see [MEY 98]).

Razing is obtained by duality, it is sufficient to develop the flooding theory, as all the flooding properties are easily transposed to razing.

In this volume we develop the theory of floodings with the same level of abstraction used to cover watersheds in Volume 1, remaining within the framework of node- or edge-weighted graphs. We start from the elementary laws of physics which govern the equilibrium state of a flooding and derive from them the theoretical properties of flooding, including the algorithms for effectively constructing them.

1.3. Description of the flood covering a topographic surface

1.3.1. *Observing the same flooding on two levels of abstraction*

Consider flooding of a topographic surface as illustrated in Figure 1.2(a) and Figure 1.2(b). While Figure 1.2(a) shows all the topography details of the flooding and of the ground level, Figure 1.2(b) is a sketch retaining only the interconnections of the catchment basins and the level of the lakes covering them. The first describes the flooding at a micro scale, the second at a macro scale.

In Figure 1.2(b), each catchment basin of Figure 1.2(a) appears as a tank. Tanks are connected by a pipe if the corresponding catchment basins are neighbors, and the pipe level is equal to the level of the pass point making it possible to pass from one catchment basin to the other. If a catchment basin is flooded, as shown in Figure 1.2(a), then its corresponding tank, as shown in Figure 1.2(b), is also filled at an identical level. On the contrary, an unflooded catchment basin (as is the case for the catchment basin b) in Figure 1.2(a) corresponds to an empty tank in Figure 1.2(a).

A unique lake covers the regional minima e and f; adding a drop of water to this lake slightly increases its level, without provoking an overflow into another lake. This is true in both the topographic surface in Figure 1.2(a) and the tank network in Figure 1.2(b).

On the contrary, the catchment basin a in Figure 1.2(a) is covered by a full lake, which reaches the level of the lowest pass point between the catchment basins a and b. Adding a drop of water to this lake does not change its altitude but provokes an overflow into the neighboring catchment basin b. The same overflow occurs in Figure 1.2(b) if we add a drop of water. The overflow passes through the pipe connecting tanks a and b and flows into tank b.

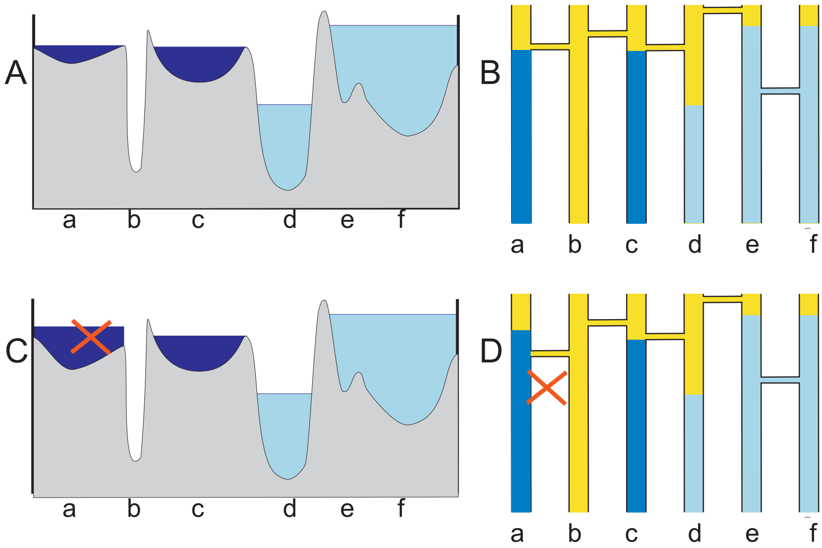

Figure 1.2. *a) Flooding of a topographic surface. b) Its modeling as a pipe network. c) A physically impossible flooding of a topographic surface. d) Its modeling as a pipe network*

Because the lakes covering the regional minima e and f have merged in Figure 1.2(a), they form a unique lake, with a uniform level. In Figure 1.2(b), both tanks also have the same level, as the pipe connecting them has a lower level. In the following, we present the equations governing the hydrostatic equilibrium of fluid on a topographic surface or a tank network.

1.3.2. *Modeling the two scales of flooding: at the pixel level or at the region level*

We have thus considered the same flooding of Figure 1.2(a) on two scales. We now model each scale of observation and establish the relationships between the two. The fine scale will be modeled as a node-weighted graph, at the pixel level. The coarse scale will be modeled by an edge-weighted graph, at the level of the catchment basins. We then derive from the laws of physics criteria characterizing an equilibrium state of flooding on each of these scales and show that both models are coherent, producing identical flooding levels on the fine and coarse scales.

1.3.2.1. *The fine scale modeled as a node-weighted graph*

A gray tone image defined on a grid of pixels may be modeled as a node-weighted graph. The pixels of the image become the nodes of the graph. A function ν represents the gray tones of the image and the weights of the nodes in the graph. If the image is considered as a topographic surface, the function ν represents its ground level.

Neighboring pixels in the image are connected by an unweighted edge. The second function $\tau \geq \nu$ defines the flooding level of each node. The nodes of the graph thus hold two weights: a lower weight ν equal to the altitude of the node and the second weight $\tau \geq \nu$ indicating the level of the flooding at each node. We say that the flooding τ is an **n-flooding** of the node-weighted graph. A node for which $\tau = \nu$ is unflooded or dry.

In the following, we establish the relationships between τ and ν on neighboring nodes, in order to model a physically realistic flooding, observed at the scale of the pixels. By physically realistic, we mean a flooding that behaves as a real flooding on a topographic surface.

1.3.2.2. *The coarse scale modeled as an edge-weighted graph*

On a coarse scale, we only consider the catchment basins of the topographic surface, classically modeled as a region adjacency graph. The nodes represent the catchment basins. Neighboring nodes are connected by an edge: the edge e_{pq} connecting the nodes p and q is weighted by the altitude η_{pq} of the pass point between the two basins. The nodes are weighted by the altitude τ of the lake inside the corresponding catchment basin or covering it. In the absence of any lake, this altitude is equal to $-\infty$. In the following sections, we establish the relationships between τ and η on neighboring nodes, in order to produce a flooding following the laws of hydrostatics.

This model correctly represents the flooding of a topographic surface seen on a coarse scale.

REMARK 1.1.– *We aim to model the flooding such that it behaves as a real flooding on a topographic surface. However, node- or edge-weighted graphs are able to model many types of networks. Flooding a weighted graph will, by creating lakes, highlight some salient features and hide others.*

1.3.3. *Modeling a flooded topographic surface as a node-weighted graph*

1.3.3.1. *The fine scale of flooding*

In this section, we describe the mechanisms governing the distribution of water on a flooded topographic surface at pixel level. We first observe what happens when a surface is flooded and then identify the laws governing the creation and distribution of lakes on the surface.

Figure 1.3(a) shows an unflooded surface ν defined on five consecutive pixels, and Figure 1.3(b) shows a valid flooding of this surface. The function ν represents the ground level of the surface before flooding. The flooded surface in Figure 1.3(b) is a function $\tau \geq \nu$. Some pixels (in yellow) for which $\tau > \nu$ are covered by water, and the others for which $\tau = \nu$ are dry. The distribution of water is at equilibrium.

We have to describe the relationships between the unflooded surface and the flooded surface. Let us start with the unflooded surface in Figure 1.3(a). The nodes a and d are regional minima of the function ν. The flooded surface τ in Figure 1.3(b) has only one regional minimum lake covering the pixels c and d. A drop of water falling on nodes c or d will slightly increase the level of the lake covering these nodes. A drop of water falling on the nodes a or b provokes an overflow of the lake on top of a into the lake (c, d) covering c and d. A drop of water falling on e will also glide down towards the lake (c, d).

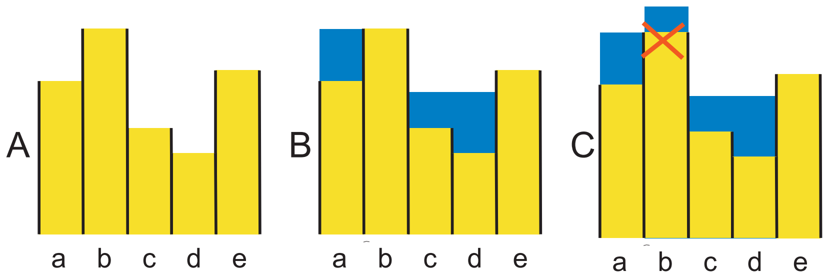

Figure 1.3. *a) A topographic surface at the pixel level. b) A physically possible flooding. c) A physically impossible flooding*

1.3.3.2. *The laws of gravity govern the flood*

We suppose that the laws of gravity apply to our topographic surface. A drop of water falling on a node p will glide downwards towards a neighboring node q, if the n-flooding level τ_p is higher than the n-flooding level τ_q. This is the case in Figure 1.3(b), where no drop of water may stay on top of the node b, as $\tau_b > \tau_c$; hence, the node b is necessarily dry: $\nu_b = \tau_b$.

Figure 1.3(c), in contrast, shows an example that violates the laws of gravity, as the n-flooding on top of node b should disappear and be absorbed by the lake covering the neighboring nodes c and d. Moreover, the nodes a, b, c and d are covered by some water and thus form a lake with an unequal surface, a phenomenon that never occurs in nature.

In order to avoid the occurrence of such a phenomenon, it is sufficient that the flooding verifies this property: if a flooded surface has two nodes whose flooding level is unequal, then the highest node is necessarily dry. This is expressed by the following criterion (first established in [MEY 98]).

CRITERION 1.– *The distribution τ of water on the nodes of the graph $G[N, E]$ with unweighted edges and ground level ν is an n-flooding of this graph, i.e. it is a stable distribution of fluid if it verifies the criterion: $\tau \geq \nu$ and for any couple of neighboring nodes (p, q), we have $\tau_p > \tau_q \Rightarrow \tau_p = \nu_p$. (criterion $n1$)*

Figure 1.4(a) presents a valid flooding, whereas the flooding in Figure 1.4(b) violates the criterion $n1$ at nodes p and q. The lake covering the leftmost regional minimum ends as a wall of water, not constrained by solid ground.

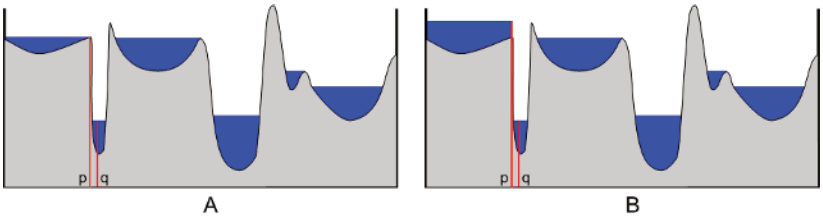

Figure 1.4. *a) A valid flooding, verifying criterion $n1$. b) This flooding is not valid, as the criterion $n1$ is violated for the neighboring pixels p and q: $\tau_p > \tau_q$ and $\tau_p > \nu_p$*

1.3.3.3. *Notations*

Given a graph $G[N, E]$ with node weights ν, we write: $\tau \uplus \nu$ when the function τ is an n-flooding of the graph $G[\nu, nil]$.

1.3.3.4. *Equivalent criteria*

As $\tau \geq \nu$, the relation $\tau_p = \nu_p$ is equivalent to $\tau_p \leq \nu_p$. We have the following equivalences:

$$\{\tau_p > \tau_q \Rightarrow \tau_p \leq \nu_p\} \Leftrightarrow \{\tau_p \leq \tau_q \text{ or } \tau_p \leq \nu_p\} \Leftrightarrow \{\tau_p \leq \tau_q \vee \nu_p\} \quad \text{(criterion } n2)$$

Applying this criterion to all neighbors of p yields $(\tau_p \leq \nu_p \vee \bigwedge_{(p,q) \text{ neighbors}} \tau_q)$, which is equivalent to $\tau_p = \tau_p \wedge (\nu_p \vee \bigwedge_{(p,q) \text{ neighbors}} \tau_q) = (\tau_p \wedge \nu_p) \vee (\tau_p \wedge \bigwedge_{(p,q) \text{ neighbors}} \tau_q) = (\nu \vee \varepsilon\tau)_p$, where $(\varepsilon\tau)_p = \tau_p \wedge \bigwedge_{(p,q) \text{ neighbors}} \tau_q$ is the erosion assigning to a node the minimum weight of itself and all its neighbors. This is a well-known criterion (see [MEY 98]) for recognizing and constructing n-floodings on images: $\tau = (\nu \vee \varepsilon\tau)$ (criterion n).

This relation is also the basis of the classical algorithm for closing reconstructions or n-floodings. If a function $\tau \geq \nu$ is not a flooding of ν, then it does not verify the relation $\tau = \nu \vee \varepsilon\tau$.

The following algorithm will progressively lower the value of τ until criterion n is satisfied. I proposed this algorithm in my master's thesis in 1975; it has then been applied in the first version of the texture analyzer, a hardwired image processing device developed by J.C. Klein at the CMM (see [KLE 76]):

– $\tau^{(0)} = \tau$;

– repeat until stability: $\tau^{(n+1)} = \nu \vee \varepsilon\tau^{(n)}$.

This algorithm necessarily converges, as ε is anti-extensive ($\varepsilon \leq Id$). Thus, $\nu \vee \varepsilon\tau^{(n)}$ decreases and is lower bounded by ν. At convergence, the limit $\tau^{(\infty)}$ verifies $\tau^{(\infty)} = \nu \vee \varepsilon\tau^{(\infty)}$, and we will see in the next chapters that it is the highest n-flooding of ν under the function τ, or dominated flooding of ν under τ.

1.3.3.5. *Flooding properties of a node-weighted graph*

Consider a graph $G = [\nu, nil]$ with a ground-level distribution on the nodes equal to a function ν, with unweighted edges, and $\tau \geq \nu$ as an n-flooding of this graph.

1.3.3.5.1. Wet and dry nodes

There are two types of nodes in an n-flooding:

– $\{p$ is a wet node$\} \Leftrightarrow \{\ \tau_p > \nu_p\}$

– $\{p$ is a dry node$\} \Leftrightarrow \{\ \tau_p = \nu_p\}$

1.3.3.5.2. Flooding is a connected operator

An n-flooding is a connected operator, i.e. each flat zone of the ground level ν is also flat for any n-flooding of the graph, which is expressed by the relation: for any two neighboring nodes p and q : $\nu_p = \nu_q \Rightarrow \tau_p = \tau_q$.

Indeed, criterion n-2 states $\tau_p \leq \tau_q \vee \nu_p = \tau_q \vee \nu_q = \tau_q$. Inverting p and q yields $\tau_q \leq \tau_p$. Hence, $\tau_p = \tau_q$.

1.3.3.5.3. The upstream of a dry node is dry

The following property is obviously verified on any topographic surface.

LEMMA 1.1.– *If p, q are two neighboring nodes verifying $\nu_q \geq \nu_p$, then if p is a dry node, so is q.*

PROOF.– p is a dry node, so $\tau_p = \nu_p$. As $\nu_q \geq \nu_p$, it follows $\nu_q \geq \tau_p$. On the contrary, the n-flooding level of q verifies $\tau_q \leq \tau_p \vee \nu_q \leq \nu_q$ as $\nu_q \geq \tau_p$. As an n-flooding cannot be lower than the ground level, we have $\tau_q = \nu_q$, showing that q is also dry. □

Applying the same lemma on all the nodes of a non-descending connected path, we get another lemma:

LEMMA 1.2.– *If a point p of a surface is dry, then all the nodes q linked with p by a non-descending path with p are also dry.*

These lemmas have important algorithmic consequences for speeding up the construction of n-floodings. The upstream of a flat zone containing a dry node is also dry. Fast algorithms able to construct these zones are derived from these properties.

1.3.3.6. *Properties of n-floodings*

In the following proofs, p and q are neighboring nodes on the graph.

1.3.3.6.1. The function ν is an n-flooding of $G[\nu, nil]$

The function ν itself is a valid n-flooding of $G[\nu, nil]$, in which all the nodes are dry.

Indeed, for two neighboring nodes p, q, the criterion $n2$ is verified: $\nu_p \leq \nu_q \vee \nu_p$.

1.3.3.6.2. The flooding of an n-flooding is an n-flooding

$\{\tau \uplus \nu$ and $\sigma \uplus \tau\} \Rightarrow \sigma \uplus \nu$

$\{\tau \uplus \nu\} \Rightarrow \{\tau_p \leq \tau_q \vee \nu_p\}$ and $\tau \geq \nu$

$\{\sigma \uplus \tau\} \Rightarrow \{\sigma_p \leq \sigma_q \vee \tau_p\}$ and $\sigma \geq \tau$

From which, we derive $\sigma_p \leq \sigma_q \vee \tau_p \leq \sigma_q \vee \tau_q \vee \nu_p = \sigma_q \vee \nu_p$ as $\sigma_q \geq \tau_q$

1.3.3.6.3. The algebra of n-floodings

$\{\tau \uplus \nu$ and $\tau' \uplus \nu\} \Rightarrow (\tau \vee \tau') \uplus \nu$ and $(\tau \wedge \tau') \uplus \nu$:

$\tau_p \leq \tau_q \vee \nu_p$ and $\tau'_p \leq \tau'_q \vee \nu_p$, which implies $\tau_p \vee \tau'_p \leq \tau_q \vee \tau'_q \vee \nu_p$ and $\tau_p \wedge \tau'_p \leq \left(\tau_q \wedge \tau'_q\right) \vee \nu_p$.

The flooding family of a node-weighted graph forms a complete lattice for the ordinary order relation $<$. The minimal flooding has the level ν, whereas the maximal flooding has the level Ω or ∞, the highest value taken by the node weights.

1.3.3.6.4. n-floodings of increasing topographic surfaces

$\{\tau \uplus \nu$ and $\nu \leq \nu' \leq \tau\} \Rightarrow \tau \uplus \nu'$:

If τ is an n-flooding of $G[\nu, nil]$, then it is possible to increase the ground level below the lakes without disturbing the flooding τ:

Indeed, $\{\tau_p \leq \tau_q \vee \nu_p\} \Rightarrow \left\{\tau_p \leq \tau_q \vee \nu_p \leq \tau_q \vee \nu'_p\right\} \Rightarrow \tau \uplus \nu'$

1.3.3.7. *The special role of $\varphi_n\nu$*

We have analyzed the closing $\varphi_n = \varepsilon_{ne}\delta_{en}$ of node-weighted graphs in *Topographical Tools for Filtering and Segmentation 1*.

Any closing is extensive: $\varphi_n\nu \geq \nu$. The only nodes for which $\varphi_n\nu > \nu$ are the regional minimum nodes. For all other nodes, $\varphi_n\nu = \nu$.

If s is an isolated regional minimum node and t its lowest neighbor, then $(\varphi_n\nu)_s = \nu_t$.

If $\tau \uplus \nu$ and $\tau_s < (\varphi_n\nu)_s = \nu_t$, then each neighboring node q of s verifies $\tau_q \geq \nu_q \geq \nu_t = (\varphi_n\nu)_s > \tau_s$. The criterion $n1$ then applies and $\tau_q > \tau_s \Rightarrow \tau_q = \nu_q$. This shows that regardless of the value $\tau_s < (\varphi_n\nu)_s$ taken by τ on the isolated regional minimum s, we have $\tau_q = \nu_q$ on all neighbors q of s.

1.3.3.7.1. The function $\varphi_n\nu$ is an n-flooding of $G[\nu, nil]$

$\varphi_n\nu \uplus \nu$ since:

– if p and q are not isolated regional minima, then $(\varphi_n\nu)_p = \nu_p \leq (\varphi_n\nu)_q \vee \nu_p$ and $(\varphi_n\nu)_q = \nu_q \leq (\varphi_n\nu)_p \vee \nu_q$.

– if p is an isolated regional minimum, then q is not an isolated regional minimum and $(\varphi_n\nu)_q = \nu_q \leq (\varphi_n\nu)_p \vee \nu_q$. On the other hand, $(\varphi_n\nu)_p$ is equal to the weight of the lowest neighboring node of p; hence, $(\varphi_n\nu)_p \leq \nu_q = (\varphi_n\nu)_q$. And $(\varphi_n\nu)_p \leq (\varphi_n\nu)_q \leq (\varphi_n\nu)_q \vee \nu_p$.

1.3.3.7.2. The function $\tau \vee \varphi_n\nu$

Let τ be a flooding of $G[\nu, nil]$. Since $\varphi_n\nu$ is a particular flooding of $G[\nu, nil]$, the supremum $\tau \vee \varphi_n\nu$ is also a flooding of $G[\nu, nil]$.

As $\nu \leq \varphi_n\nu \leq \tau \vee \varphi_n\nu$, the function $\tau \vee \varphi_n\nu$ is also a flooding of $G[\varphi_n\nu, nil]$.

Inversely, if $\tau \vee \varphi_n\nu$ is a flooding of $G[\varphi_n\nu, nil]$, then $\tau \vee \varphi_n\nu$ is a flooding of $G[\nu, nil]$, as $\varphi_n\nu$ is a flooding of $G[\nu, nil]$.

Summarizing

$$\{\tau \uplus \nu\} \Rightarrow \{(\tau \vee \varphi_n\nu) \uplus \nu\} \Leftrightarrow \{\tau \vee \varphi_n\nu \uplus \varphi_n\nu\}$$

Interpretation of $\tau \vee \varphi_n\nu$

If $\tau \geq \varphi_n\nu$, then $\tau \vee \varphi_n\nu = \tau$

Else, the inequality $\tau < \varphi_n\nu$ implies $\nu \leq \tau < \varphi_n\nu$, which only happens on isolated regional minima nodes of $G[\nu, nil]$. All other nodes verify $\nu = \varphi_n\nu$.

If s is an isolated regional minimum node on which $\tau_s < (\varphi_n\nu)_s$ and t is one of the lowest neighbors of s, then $(\tau \vee \varphi_n\nu)_s = (\varphi_n\nu)_s = \nu_t$. In other terms, if for each node s (or only some of them) verifying $\tau_s < (\varphi_n\nu)_s$ we replace τ_s by $(\varphi_n\nu)_s$, then the new function $\tau \vee \varphi_n\nu$ is still an n-flooding of ν.

Inversely

Let us show that for $\tau \geq \nu : \{(\tau \vee \varphi_n\nu) \uplus \nu\} \Rightarrow \{\tau \uplus \nu\}$

Outside the isolated regional minima of $G[\nu, nil]$, we have $\varphi_n\nu = \nu$ and $\tau \vee \varphi_n\nu = \tau$. If p is a regional minimum and $\tau_p \geq (\varphi_n\nu)_p$, then we also have $(\tau \vee \varphi_n\nu)_p = \tau_p$. In all such cases, the criterion $n2$: $(\tau \vee \varphi_n\nu)_p \leq (\tau \vee \varphi_n\nu)_q \vee \nu_p$ simply becomes $\tau_p \leq \tau_q \vee \nu_p$.

The last case to consider is when p is an isolated regional minimum and $\tau_p < (\varphi_n\nu)_p$. We then have $(\tau \vee \varphi_n\nu)_p = (\varphi_n\nu)_p = \nu_s$, where s is the smallest neighbor of p.

Any neighboring node q of p is then not a regional minimum and $(\varphi_n\nu)_q = \nu_q \geq \nu_s$. Hence, $\tau_p < (\varphi_n\nu)_p = \nu_s \leq \nu_q \leq \tau_q \leq \tau_q \vee \nu_p$.

On the one hand, $(\tau \vee \varphi_n\nu) \uplus \nu$ implies $(\tau \vee \varphi_n\nu)_q \leq (\tau \vee \varphi_n\nu)_p \vee \nu_q$. As $(\varphi_n\nu)_q = \nu_q$ and $\tau \geq \nu$, we have $(\tau \vee \varphi_n\nu)_q = \tau_q$. On the other hand, $(\tau \vee \varphi_n\nu)_p = \nu_s \leq \nu_q$ and $(\tau \vee \varphi_n\nu)_p \vee \nu_q = \nu_q \leq \tau_p \vee \nu_q$. Putting everything together, we have $\tau_q = (\tau \vee \varphi_n\nu)_q \leq (\tau \vee \varphi_n\nu)_p \vee \nu_q \leq \tau_p \vee \nu_q$. The function τ verifies the criterion $n2$ everywhere, showing that $\tau \uplus \nu$. QED

Application

If $\sigma \uplus \nu$, we may modify the function σ on some isolated regional minimum nodes of $G[\nu, nil]$ on which $\sigma = (\varphi_n\nu)$. If s is an isolated regional minimum node and if we replace the value σ_s by a value τ_s verifying $\nu_s \leq \tau_s < \sigma_s$, then $\tau \vee \varphi_n\nu = \sigma$, and applying the preceding implication, we have $\tau \uplus \nu$.

Summarizing

For $\tau \geq \nu : \{\tau \uplus \nu\} \Leftrightarrow \{(\tau \vee \varphi_n\nu) \uplus \nu\} \Leftrightarrow \{\tau \vee \varphi_n\nu \uplus \varphi_n\nu\}$

Illustration

Figure 1.5(a) shows a topographic surface ν. The nodes b and e are isolated regional minima. In Figure 1.5(b), these regional minima have been filled up to the lowest neighboring node, producing the function $\varphi_n\nu$. The increment from ν to $\varphi_n\nu$ is in orange color. This topographic surface $\varphi_n\nu$ is flooded by a flood σ, increasing the level of water on the nodes d and e. Thus, $\sigma = \sigma \vee \varphi_n \nu$ is a flooding of $G[\varphi_n \nu, nil]$ and as a result of $G[\nu, nil]$.

Figure 1.5(c) shows another distribution of water on the nodes of $G[\nu, nil]$. On the isolated regional minimum b, the flood τ takes a value $\nu_b < \tau_b \leq (\varphi_n \nu)_b$. The function τ verifies $\tau \vee (\varphi_n \nu) = \sigma \vee (\varphi_n \nu) = \sigma$. And, as σ is an n-flooding of $G[\nu, nil]$, τ is also an n-flooding of $G[\nu, nil]$.

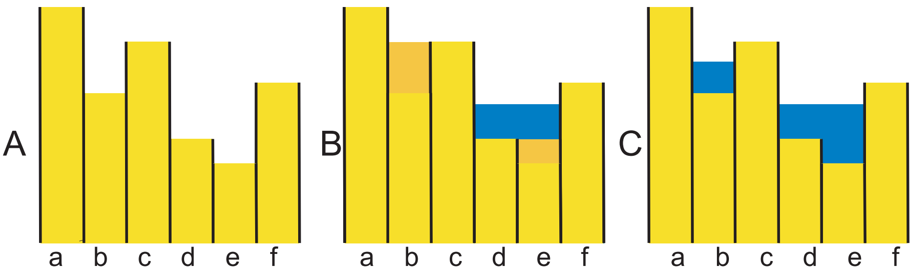

Figure 1.5. *a) A topographic surface $G[\nu, nil]$. b) The isolated regional minima have been filled up to their lowest neighbor, producing the surface $G[\varphi_n \nu, nil]$. This surface is then flooded by a flood $\sigma = \sigma \vee (\varphi_n \nu)$. Thus, σ is a flooding of $G[\varphi_n \nu, nil]$ and of $G[\nu, nil]$. c) The new distribution τ of water in $G[\nu, nil]$ verifies $\tau \vee (\varphi_n \nu) = \sigma \vee (\varphi_n \nu)$. And as σ is a flooding of $G[\nu, nil]$, τ is also a flooding of $G[\nu, nil]$*

1.3.4. *Modeling an edge-weighted graph as a tank network*

1.3.4.1. *Modeling the flooding distribution of a topographic surface*

Consider again the levels of the lakes in the flooding of Figure 1.2(a). Obviously, the depth and shape of the catchment basins below the lakes have no influence on the level τ of the flood. Only the neighborhood relations of the catchment basins play a role. The distribution of the flood depends on the levels of the pass points separating neighboring basins. It is through the pass points that the flood progresses, in the form of an overflow from one basin to the neighboring basin. On the contrary, a lake is a regional minimum of the flooded surface if all its outside neighbors have a higher altitude.

Since shape and depth of the catchment basin have no influence on the level of the flooding, each basin may be modeled as an infinitely deep tank: in other terms, there is no ground level. Two neighboring catchment basins i and j separated by a pass point of altitude η_{ij} are linked by a horizontal pipe at altitude η_{ij}, producing communication between the two tanks. The weight τ_i represents the level of water in the tank i; it is equal to $-\infty$ if the tank is empty. Figure 1.2(b) shows the tank model corresponding to the flooded topographic surface represented in Figure 1.2(a). For each catchment basin of a minimum in Figure 1.2(a) there is a corresponding tank in Figure 1.2(b). The empty catchment basin b corresponds to the empty tank b. For all other catchment basins, the levels of the lakes correspond to the levels of the tanks.

The tank network, in turn, may be modeled as an edge-weighted graph: each tank is represented by a node; if two tanks p and q are linked by a pipe at altitude η_{pq}, then the corresponding nodes are connected by an edge with this same weight η_{pq}.

1.3.4.2. *The hydrostatic equilibrium of a tank network*

The laws of hydrostatics explain the equilibrium state of water in a network of tanks linked together by pipes:

– If the level τ_i in the tank i is higher than the pipe e_{ij}, then the levels are the same in both tanks i and j: $\tau_i = \tau_j$. This is the case in Figure 1.2(b), where the level of flood in the tanks e and f is higher than the level of the pipe between them.

– The level τ_i in the tank i cannot be higher than the level τ_j unless $\eta_{ij} \geq \tau_i$. This is the case in Figure 1.2(b), where the level of flood in the tanks a is higher than the level of the tank b (which is empty); this is possible as $\eta_{ab} = \tau_a$. Conversely, Figure 1.2(c) and (d) shows a situation violating the laws of hydrostatics, as the level of tank a should go down, producing a flow through the pipe e_{ab} until reaching the level of the pipe η_{ab} as in Figure 1.2(a).

In fact, as we establish below, this second condition implies the first one. Hence, we use it as a criterion to ensure the hydrostatic equilibrium:

CRITERION 2.– *The distribution τ of water in the nodes of the graph $G[N, \mathcal{E}]$ is an e-flooding of this graph, i.e. it is a stable distribution of fluid if it verifies the criterion: for any couple of neighboring nodes (p, q), we have: $(\tau_p > \tau_q \Rightarrow \eta_{pq} \geq \tau_p)$ (criterion e1)*

Figure 1.6 shows several configurations compatible with the laws of hydrostatics and others which are not.

1.3.4.3. *Equivalent criteria*

The following criteria are equivalent to criterion $e1$:

$$(\tau_p > \tau_q \Rightarrow \eta_{pq} \geq \tau_p) \Leftrightarrow (\text{ not } (\tau_p > \tau_q) \text{ or } \eta_{pq} \geq \tau_p)$$

$$\Leftrightarrow (\tau_p \leq \tau_q \text{ or } \tau_p \leq \eta_{pq}) \Leftrightarrow (\tau_p \leq \tau_q \vee \eta_{pq}) \quad (\text{criterion } e2);$$

$$\Leftrightarrow (\tau_p \leq \bigwedge_{(p,q) \text{ neighbors}} (\tau_q \vee \eta_{pq}) \quad (\text{criterion } e3);$$

$$\Leftrightarrow (\tau_p = \tau_p \wedge \bigwedge_{(p,q) \text{ neighbors}} (\tau_q \vee \eta_{pq}) \quad (\text{criterion } e4).$$

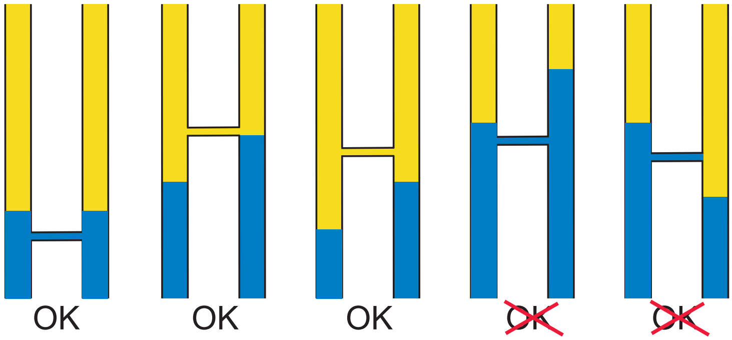

Figure 1.6. *The water distributions marked OK are compatible with the laws of hydrostatics; the others are not*

1.3.4.4. *Notations*

Given a graph $G[N, E]$ with edge weights ν, we write: $\tau \,\overline{\cup}\, \eta$ when the function τ is an e-flooding of the graph $G[nil, \eta]$.

1.3.4.5. *Properties of the e-flooding of an edge-weighted graph*

1.3.4.5.1. The algebra of e-floodings

$$\{\tau \,\overline{\cup}\, \eta \text{ and } \tau' \,\overline{\cup}\, \eta\} \Rightarrow (\tau \vee \tau') \,\overline{\cup}\, \eta \text{ and } (\tau \wedge \tau') \,\overline{\cup}\, \eta:$$

$\tau_p \leq \tau_q \vee \eta_{pq}$ and $\tau'_p \leq \tau'_q \vee \eta_{pq}$, which implies $\tau_p \vee \tau'_p \leq \tau_q \vee \tau'_q \vee \eta_{pq}$ and $\tau_p \wedge \tau'_p \leq \left(\tau_q \wedge \tau'_q\right) \vee \eta_{pq}$.

The flooding family of an edge-weighted graph forms a complete lattice for the ordinary order relation $<$. The minimal flooding has the level $-\infty$, whereas the maximal flooding has the level T.

1.3.4.5.2. e-floodings for increasing edge weights

$\{\tau \,\overline{\cup}\, \eta \text{ and } \eta \leq \eta'\} \Rightarrow \tau \,\overline{\cup}\, \eta'$ as:

$$\{\tau \,\overline{\cup}\, \eta\} \Rightarrow \{\tau_p \leq \tau_q \vee \eta_{pq}\} \Rightarrow \left\{\tau_p \leq \tau_q \vee \eta_{pq} \leq \tau_q \vee \eta'_{pq}\right\} \Rightarrow \{\tau \,\overline{\cup}\, \eta'\}.$$

1.3.4.6. *The special role of $\varepsilon_{ne}\eta$*

The function $(\varepsilon_{ne}\eta)_p$ is the weight of the lowest adjacent edge of p. Hence, if a flooding τ of $G[nil, \eta]$ verifies $\tau_p \leq (\varepsilon_{ne}\eta)_p$, then its level can vary below $(\varepsilon_{ne}\eta)_p$ without an influence on the flooding level of the neighboring node, as there is never an overflow into a neighboring pipe of p. Inversely, if q is a neighboring node of p, then $\tau_q \leq \tau_p \vee \eta_{pq} \leq (\varepsilon_{ne}\eta)_p \vee \eta_{pq} = \eta_{pq}$, showing that there is no overflow from q to p.

The function $\varepsilon_{ne}\eta$ is a kind of ground level. If $\tau_p \leq (\varepsilon_{ne}\eta)_p$, then any other flooding level $\tau'_p \leq (\varepsilon_{ne}\eta)_p$ has no effect on the flooding level on the other nodes.

1.3.4.6.1. The function $\varepsilon_{ne}\eta$ is an e-flooding of $G[nil, \eta]$

For two neighboring nodes p and q, we have $(\varepsilon_{ne}\eta)_p \leq \eta_{pq}$. Criterion e2 is verified: $(\varepsilon_{ne}\eta)_p \leq \eta_{pq} \leq (\varepsilon_{ne}\eta)_q \vee \eta_{pq}$. Hence, $\varepsilon_{ne}\eta \,\overline{\cup}\, \eta$.

1.3.4.6.2. The function $\tau \vee \varepsilon_{ne}\eta$

Since $\varepsilon_{ne}\eta \,\overline{\cup}\, \eta : \{\tau \,\overline{\cup}\, \eta\} \Rightarrow \{\tau \vee \varepsilon_{ne}\eta \,\overline{\cup}\, \eta\}$, as the supremum of two e-floodings is an e-flooding.

1.3.4.6.3. Inversely

We now prove the inverse implication: $\{\tau \vee \varepsilon_{ne}\eta \,\overline{\cup}\, \eta\} \Rightarrow \{\tau \,\overline{\cup}\, \eta\}$. We again consider two neighboring nodes p and q:

– the case where $\tau_p \geq (\varepsilon_{ne}\eta)_p$ and $\tau_q \geq (\varepsilon_{ne}\eta)_q$ As $\tau \vee \varepsilon_{ne}\eta \,\overline{\cup}\, \eta$, criterion e2 applies to $\tau_p \vee (\varepsilon_{ne}\eta)_p = \tau_p$ and $\tau_q \vee (\varepsilon_{ne}\eta)_q = \tau_q$;

– the case where $\tau_p < (\varepsilon_{ne}\eta)_p$ (the same reasoning applies if we exchange the roles of p and q). Let us show that the criterion $e2$ is still satisfied for p and $q.\tau_p < (\varepsilon_{ne}\eta)_p \leq \eta_{pq} \leq \tau_q \vee \eta_{pq}$. As $\tau \vee \varepsilon_{ne}\eta \,\overline{\cup}\, \eta$, the criterion $e2$ applies: $(\tau \vee \varepsilon_{ne}\eta)_q \leq (\tau \vee \varepsilon_{ne}\eta)_p \vee \eta_{pq}$ Then $\tau_q \leq (\tau \vee \varepsilon_{ne}\eta)_q \leq (\tau \vee \varepsilon_{ne}\eta)_p \vee \eta_{pq} = (\varepsilon_{ne}\eta)_p \vee \eta_{pq} = \eta_{pq} \leq \tau_p \vee \eta_{pq}$.

Criterion $e2$ is satisfied for all pairs of nodes, showing that indeed $\tau \,\overline{\cup} \eta$.

Summarizing

We thus have the equivalence $\{\tau \vee \varepsilon_{ne}\eta \,\overline{\cup}\, \eta\} \Leftrightarrow \{\tau \,\overline{\cup}\, \eta\}$

Illustration

This result may be interpreted as follows. If μ is a flooding of $G[nil, \eta]$, then the exact value taken by μ on all the nodes for which $\mu \leq \varepsilon_{ne}\eta$ has no importance and may be replaced by any other value $\mu' \leq \varepsilon_{ne}\eta$.

Such a situation is shown in Figure 1.7, which presents two valid floodings τ_A for figure A and τ_B for figure B. They verify $\tau_A \vee \varepsilon_{ne}\eta = \tau_B \vee \varepsilon_{ne}\eta$; they differ by the level of flooding in the tank e, which is equal to $(\varepsilon_{ne}\eta)_e$ in Figure 1.7(a) but below the lowest adjacent pipe in Figure 1.7(b). If one of them is an $e - flooding$ of $G[nil, \eta]$, so is the other.

This lemma implies that from any e-flooding θ of an edge-connected graph $G[nil, \eta]$, it is possible to generate other e-floodings by lowering the flood level in all tanks where $\theta_p \leq (\varepsilon_{ne}\eta)_p$. We thus obtain a new flooding level θ' verifying $\theta' = \theta \vee \varepsilon_{ne}\eta$.

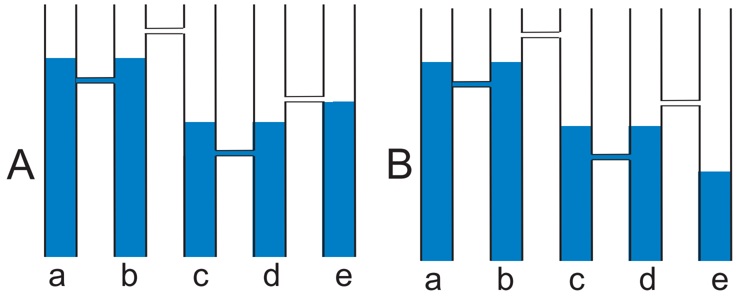

Figure 1.7. *Two distributions of water τ_A in A and τ_B in B of the same graph $G[nil, \eta]$. They verify the relation $\tau_A \vee \varepsilon_{ne}\eta = \tau_B \vee \varepsilon_{ne}\eta$; they differ by the level of flooding in the tank e, which is equal to $(\varepsilon_{ne}\eta)_e$ in A but is below the lowest adjacent pipe in B. If one is an n-flooding of $G[nil, \eta]$, so is the other*

1.4. The relations between n-floodings and e-floodings

1.4.1. *Modeling the flooding on two scales: equivalence of both models*

We have modeled the same flooding of a topographic surface on two scales. At the scale of the catchment basins, we have modeled it as an e-flooding of an edge-weighted graph or as a tank network submitted to the laws of hydrostatics. At the scale of the nodes, we have modeled it as an n-flooding of a node-weighted graph: the n-flooding following the laws of gravity, which governs the flowing from node to node. We thus have two models on two scales of the same phenomenon. Although the models are different, we now verify that they result in the same flooding, as stated in the following theorem.

THEOREM 1.1.– *For $\tau \geq \nu$: the function τ is an n-flooding of $G[\nu, nil]$ if and only if τ is an e-flooding of $G[nil, \delta_{en}\nu]$*

PROOF.– $\{\tau \geq \nu$ is an e-flooding of $G[nil, \delta_{en}\nu]\} \Leftrightarrow \{\tau \geq \nu$ and $\tau_p \leq \tau_q \vee \nu_p \vee \nu_q\}$ (criterion e2) $\Leftrightarrow$ $\{\tau \geq \nu$ and $\tau_p \leq \tau_q \vee \nu_p\}$ (criterion n-2) $\Leftrightarrow$ $\{\tau$ is an n-flooding of $G[nil, \delta_{en}\nu]\}$. The second equivalence is due to the fact that $\tau_q \vee \nu_q = \tau_q$, since $\tau \geq \nu$. □

We now have three possible representations of a graph. Figure 1.8(a) shows a topographic surface ν defined on five adjacent nodes, modeled as a node-weighted graph $G[\nu, nil]$. Figure 1.8(c) shows a tank network. The tanks corresponding to neighboring nodes p and q are linked by a horizontal pipe at a level $\eta_{pq} = \nu_p \vee \nu_q = (\delta_{en}\nu)_{pq}$; this model represents the edge-weighted graph $G[nil, \delta_{en}\nu]$. Figure 1.8(a) shows the function ν within the tank network. We remark that each pipe is the lowest pipe of one of its adjacent tanks. This property is linked, as was established in *Topographical Tools for Filtering and Segmentation 1*, with the fact that the dilated function $\delta_{en}\nu$ is invariant by the opening $\gamma_e : \gamma_e\delta_{en}\nu = \delta_{en}\varepsilon_{ne}\delta_{en}\nu = \delta_{en}\nu$.

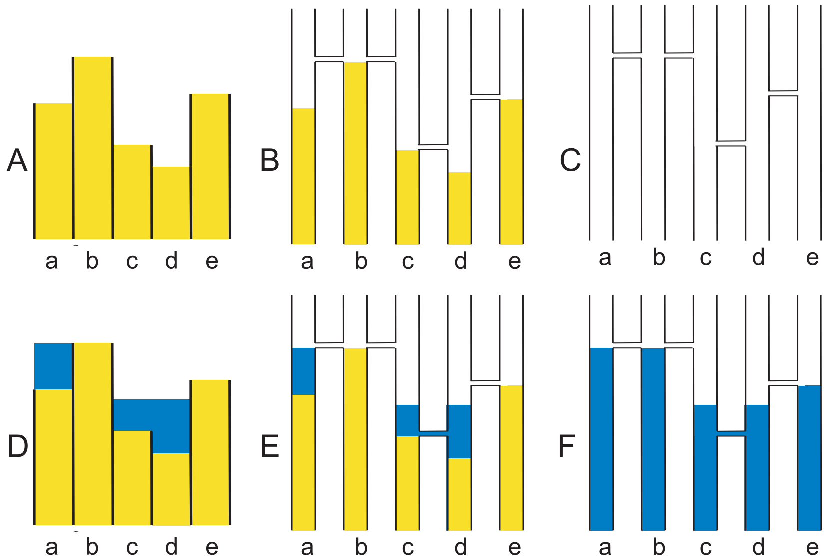

Figure 1.8. *Various modelings of the same topographic surface. a) The pixel level is modeled as a node-weighted graph $G[\nu, nil]$. b) The tank network in which a ground level ν is indicated in yellow and pipes at the level $\delta_{en}\nu$ connect the neighboring pipes. It corresponds to a node- and edge-weighted graph $G[\nu, \delta_{en}\nu]$. c) The associated tank network without ground level, which corresponds to an edge-weighted graph $G[nil, \delta_{en}\nu]$. d), e) and f) The same flooding distribution is displayed on these three graphs*

Figure 1.8(d) shows a flooding $\tau \geq \nu$ of $G[\nu, nil]$, and Figure 1.8(f) shows the same flooding on the tank network representing $G[nil, \delta_{en}\nu]$. Figure 1.8(e) regroups all functions on the nodes, i.e. in the tanks: ν for the ground level, τ the flooding, with the pipes having the level $\eta_{pq} = \nu_p \vee \nu_q = (\delta_{en}\nu)_{pq}$.

In Figure 1.8(d,e) and (f), we present the same flood distribution, following the laws of gravity in Figure 1.8(d) and the laws of hydrostatics in Figure 1.8(e) and (f). Conversely, Figure 1.9(c) and (d) shows a case where these laws are violated. In Figure 1.9(c), the flooding τ at nodes a and b verifies $\tau_b > \tau_a$, but the node b is not dry, as $\tau_b > \nu_b$. The criterion n1 is thus violated and the water distribution is not a valid n-flooding. The same distribution of water in the tank network of Figure 1.9(d) violates criterion e1 of e-floodings: $\tau_b > \tau_a$ should imply $\tau_b \leq \eta_{ab}$, which is not verified.

As a result, it is possible to compute a flood distribution of a node-weighted graph, indifferently with algorithms developed for n-floodings or developed for e-floodings.

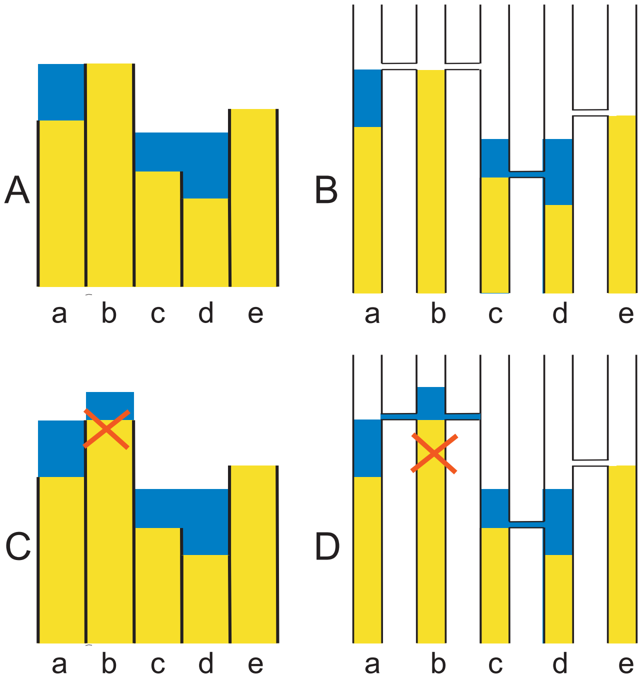

Figure 1.9. *a) an n-flooding of a topographic surface. b) the e-flooding of its equivalent pipe network. c) an impossible n-flooding. d) the equivalently impossible e-flooding*

1.5. Flooding a flowing graph

1.5.1. *Flowing graphs: reminder*

A graph $G[\nu, \eta]$ is a flowing graph if node and edge weights are coupled by the following relations: $\nu = \varepsilon_{ne}\eta$ and $\eta = \delta_{en}\nu$.

Replacing ν and η with their values, we get:

– $\nu = \varepsilon_{ne}\eta = \varepsilon_{ne}\delta_{en}\nu. = \varphi_n\nu$;

– $\eta = \delta_{en}\nu = \delta_{en}\varepsilon_{ne}\eta = \gamma_e\eta$.

1.5.2. *Starting from an edge-weighted graph* $G[nil, \eta]$

1.5.2.1. *The case where* $\gamma_e\eta = \eta$

We define $\nu = \varepsilon_{ne}\eta$. The graph $G[\nu, \eta]$ is then a flowing graph as $\eta = \gamma_e\eta = \delta_{en}\varepsilon_{ne}\eta = \delta_{en}\nu$. We then have

$$\{\tau \,\overline{\cup}\, \eta\} \Leftrightarrow \{\tau \vee \varepsilon_{ne}\eta \,\overline{\cup}\, \eta\} \Leftrightarrow \{(\tau \vee \nu\,) \,\overline{\cup}\, \eta\} \Leftrightarrow \{(\tau \vee \nu) \uplus \nu\}$$

The first equivalence is true for all e-floodings of $G[nil, \eta]$.

The second equivalence is obtained by replacing $\varepsilon_{ne}\eta$ with the value ν.

The last equivalence is obtained by applying theorem 1.1 to the function $\tau \vee \nu$.

REMARK 1.2.– *As* $G[\nu, \eta]$ *is a flowing graph:* $\nu = \varphi_n\nu$

1.5.2.2. *The case where* $\gamma_e\eta < \eta$

The edges of an edge-weighted graph verifying $\gamma_e\eta > \eta$ are not the lowest neighbors of any of their extremities.

We define $\nu = \varepsilon_{ne}\eta$ and $\eta' = \delta_{en}\nu = \delta_{en}\varepsilon_{ne}\eta = \gamma_e\eta$.

And $\gamma_e\eta' = \gamma_e\gamma_e\eta = \gamma_e\eta = \eta'$.

$\varepsilon_{ne}\eta' = \varepsilon_{ne}\delta_{en}\varepsilon_{ne}\eta = \varepsilon_{ne}\eta = \nu$.

The graph $G[\nu, \eta']$ is then a flowing graph and the results of the preceding sections apply:

$$\{\tau \,\overline{\cup}\, \eta'\} \Leftrightarrow \{\tau \vee \varepsilon_{ne}\eta' \,\overline{\cup}\, \eta'\} \Leftrightarrow \{(\tau \vee \nu\,) \,\overline{\cup}\, \eta'\} \Leftrightarrow \{(\tau \vee \nu) \uplus \nu\}$$

Conversely

As $\eta' = \gamma_e\eta \le \eta$, we have $\tau \,\overline{\cup}\, \eta' \Rightarrow \tau \,\overline{\cup}\, \eta$. The inverse implication, however, is not true. If η is not open by γ_e, there are edges e_{pq} for which $\eta_{pq} > (\gamma_e\eta)_{pq}$. In the section on watersheds, we have cut all these edges without disturbing the flowing paths, the regional minima and the catchment zones. By cutting the edges, we indeed obtained a flowing graph, but one which may be disconnected. Here, we deal with flooding, and we cannot cut such edges: if we cut the edge e_{pq}, then no lake can exist covering both nodes p and q.

The problem is due to the fact that the edges of an edge-weighted graph verifying $\gamma_e\eta > \eta$ are not the lowest neighbor of any of their extremities. Hence, n-floodings progressing from node to node do not cross this edge in the same way as an e-flooding crossing the same edge. This is illustrated in Figure 1.10. Figure 1.10(a) presents the tank network for an edge-weighted graph $G[nil, \eta]$, which does not verify $\gamma_e \eta = \eta$. Indeed, $(\gamma_e \eta)_{bc} < \eta_{bc}$, the edge e_{bc} of the graph is not the lowest edge of either of its extremities. In Figure 1.10(a), the ground level $\nu = \varepsilon_{ne}\eta$ shows that the pipe e_{bc} is strictly above the ground level of both adjacent tanks. However, the floodings in Figure 1.10(a) and (b) are perfectly correct, as they verify $\tau_b \leq \tau_c \vee \eta_{bc}$ and $\tau_c \leq \tau_b \vee \eta_{bc}$. However, the same flooding distribution in the associated topographic surface of Figure 1.10(c) is not correct, as $\tau_b > \tau_c \vee \nu_b$.

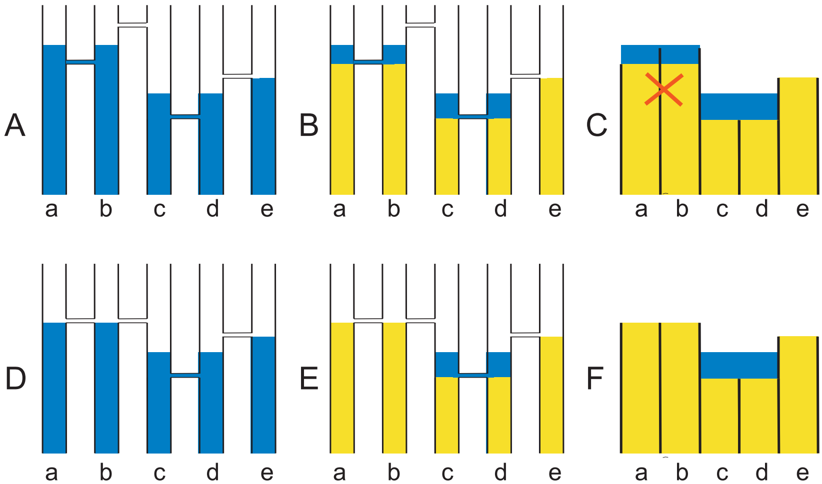

Figure 1.10. *a) The tank network associated with an edge-weighted graph $G[nil, \eta]$, which does not verify $\gamma_e \eta = \eta$. Indeed, $(\gamma_e \eta)_{bc} < \eta_{bc}$, the edge e_{bc} of the graph is not the lowest edge of one of its extremities. b) The ground level $\nu = \varepsilon_{ne}\eta$ is indicated in yellow. The pipe e_{bc} is strictly above the ground level of both adjacent tanks. However, the floodings in A and B are valid, as they verifiy $\tau_b \leq \tau_c \vee \eta_{bc}$ and $\tau_c \leq \tau_b \vee \eta_{bc}$. c) The same flooding of the topographic surface associated with the graph $G[\varepsilon_{ne}\eta, nil]$ is not correct, as $\tau_b > \tau_c \vee \nu_b$. d) The tank network associated with an edge-weighted graph $G[nil, \eta]$, which does verify $\gamma_e \eta = \eta$. e) The ground level $\nu = \varepsilon_{ne}\eta$ is indicated in yellow. The pipe e_{bc} is at the same level as the ground level of the tank b. The floodings in D, E and F are identical and valid. For the tank network in D, it verifies $\tau_b \leq \tau_c \vee \eta_{bc}$ and $\tau_c \leq \tau_b \vee \eta_{bc}$. For topographic surface of F, it verifies $\tau_b \leq \tau_c \vee \nu_b$ and $\tau_c \leq \tau_b \vee \nu_c$*

In contrast, Figure 1.10(d) presents the tank network for an edge-weighted graph $G[nil, \eta]$, which does verify $\gamma_e \eta = \eta$. Here, $(\gamma_e \eta)_{bc} = \eta_{bc}$, the edge e_{bc} of the graph is the lowest edge of the node b. In Figure 1.10(e), the ground level $\nu = \varepsilon_{ne}\eta$ (in yellow) clearly shows that the pipe e_{bc} is at the same level as the ground level of the tank b. In accordance with the lemma, the flooding of Figure 1.10(d), (e) and (f) is the same and perfectly correct. For the pipe network, it verifies $\tau_b \leq \tau_c \vee \eta_{bc}$ and $\tau_c \leq \tau_b \vee \eta_{bc}$. For the associated topographic surface of Figure 1.10(a), it verifies $\tau_b \leq \tau_c \vee \nu_b$ and $\tau_c \leq \tau_b \vee \nu_c$.

1.5.3. *Starting from a node-weighted graph* $G[\nu, nil]$

We define $\eta = \delta_{en}\nu$. As a result of a dilation, η is open: $\gamma_e \eta = \eta$. The results of the preceding section apply to the flowing graph $G(\nu', \eta)$, where $\nu' = \varepsilon_{ne}\eta = \varepsilon_{ne}\delta_{en}\nu = \varphi_n \nu$.

$$\{\tau \,\overline{\cup}\, \eta\} \Leftrightarrow \{\tau \vee \varepsilon_{ne}\eta \,\overline{\cup}\, \eta\} \Leftrightarrow \{(\tau \vee \nu') \,\overline{\cup}\, \eta\} \Leftrightarrow \{(\tau \vee \nu') \uplus \nu'\}$$

Replacing ν' by its value $\varphi_n \nu$, we get:

$$\{\tau \,\overline{\cup}\, \eta\} \Leftrightarrow \{(\tau \vee \varepsilon_{ne}\eta) \,\overline{\cup}\, \eta\} \Leftrightarrow \{(\tau \vee \varphi_n \nu) \,\overline{\cup}\, \eta\} \Leftrightarrow \{(\tau \vee \varphi_n \nu) \uplus \varphi_n \nu\}$$

In the section on n-floodings, we have established for any graph $G[\nu, nil]$:

for $\tau \geq \nu : \{\tau \uplus \nu\} \Leftrightarrow \{(\tau \vee \varphi_n \nu) \uplus \nu\} \Leftrightarrow \{\tau \vee \varphi_n \nu \uplus \varphi_n \nu\}$, which may be rephrased for an arbitrary function τ, not necessarily verifying $\tau \geq \nu$:

$$\{(\tau \vee \nu) \uplus \nu\} \Leftrightarrow \{(\tau \vee \varphi_n \nu) \uplus \nu\} \Leftrightarrow \{\tau \vee \varphi_n \nu \uplus \varphi_n \nu\}$$

Given a node-weighted graph $G[\nu, nil]$ and defining $\eta = \delta_{en}\nu$, we may put all equivalences together and obtain for an arbitrary function τ

$$\begin{aligned}\{\tau \,\overline{\cup}\, \eta\} &\Leftrightarrow \{(\tau \vee \varepsilon_{ne}\eta) \,\overline{\cup}\, \eta\} \Leftrightarrow \{(\tau \vee \varphi_n \nu) \,\overline{\cup}\, \eta\} \\ &\Leftrightarrow \{(\tau \vee \varphi_n \nu) \uplus \varphi_n \nu\} \Leftrightarrow \{(\tau \vee \varphi_n \nu) \uplus \nu\} \Leftrightarrow \{(\tau \vee \nu) \uplus \nu\} \\ &\Leftrightarrow \{(\tau \vee \nu) \,\overline{\cup}\, \eta\}\end{aligned}$$

the last equivalence by application of Theorem 1.1.

1.5.4. *Summarizing*

We have now multiple equivalences which may be used for flooding weighted graphs.

– For an edge weighted graph $G[nil, \eta]$

– Case where $\gamma_e \eta = \eta$ We define $\nu = \varepsilon_{ne}\eta$. We then have:

$$\{\tau \,\overline{\cup}\, \eta\} \Leftrightarrow \{\tau \vee \varepsilon_{ne}\eta \,\overline{\cup}\, \eta\} \Leftrightarrow \{(\tau \vee \nu) \,\overline{\cup}\, \eta\} \Leftrightarrow \{(\tau \vee \nu) \uplus \nu\}$$

– Case where $\gamma_e \eta < \eta$ We define $\nu = \varepsilon_{ne}\eta$ and $\eta' = \delta_{en}\nu = \delta_{en}\varepsilon_{ne}\eta = \gamma_e\eta$. We then have:

$$\{\tau \,\overline{\cup}\, \eta'\} \Leftrightarrow \{\tau \vee \varepsilon_{ne}\eta' \,\overline{\cup}\, \eta'\} \Leftrightarrow \{(\tau \vee \nu\,) \,\overline{\cup}\, \eta'\} \Leftrightarrow \{(\tau \vee \nu) \uplus \nu\}$$

– For a node weighted graph $G[\nu, nil]$

– We define $\eta = \delta_{en}\nu$ and we have:

$$\{\tau \,\overline{\cup}\, \eta\} \Leftrightarrow \{\tau \vee \varepsilon_{ne}\eta \,\overline{\cup}\, \eta\} \Leftrightarrow \{\tau \vee \varphi_n\nu \,\overline{\cup}\, \eta\} \Leftrightarrow \{(\tau \vee \varphi_n\nu) \uplus \varphi_n\nu\}$$
$$\Leftrightarrow \{(\tau \vee \varphi_n\nu) \uplus \nu\} \Leftrightarrow \{(\tau \vee \nu) \uplus \nu\} \Leftrightarrow \{(\tau \vee \nu\,) \,\overline{\cup}\, \eta\}$$