PART 1

Basic Methods

1

Preliminary Concepts

1.1. Introduction to random evolutions

Evolution in a random environment means that the system depends on the state of the environment, and this occurs in many real systems in nature. Similarly, if the evolution of the system does not affect the random environment, but the environment is described by a random process, say a Markov renewal process (MRP), then such systems are called stochastic. Some stochastic systems change their states abruptly; that is, in every state, the system spends a random holding time and then immediately transfers to another state. Such systems are called discrete-state systems. The simulation of discrete stochastic systems often uses jumping Markov and semi-Markov processes. However, in many applications stochastic systems are changing their states continuously or in a combination of continuous and discrete, with the result of it being ineffective to model such systems with jumping Markov or semi-Markov processes. A better modeling strategy for such systems is the notion of semi-Markov (Markov) evolution, which is given by two processes: the switching MRP, describing the random environment, and the switched process that describes the evolution of the system.

This book is aimed at studying both discrete systems, the models for which are Markov and semi-Markov processes, and continuous systems, which are simulated by random evolutions (RE).

One of the relevant areas in the study of semi-Markov processes is the theory of large deviations, in particular, the asymptotic analysis of some functionals associated with the hitting time of a "hard-to-achieve" level. Such results are useful in queuing theory and reliability theory (Korolyuk and Turbin 1982).

One of the earliest asymptotic analyses of semi-Markov processes was done by Korolyuk, where he generalized the Vishyk–Lyusternik algorithm (Vasileva and

Butuzov 1973, 1990; Vishik and Lusternik 1960) to study the distributions of large deviations. Basically, the method consists of representing a distribution as an asymptotic series with two types of terms. One type of these terms regularly depends on a small parameter of perturbations of embedded (in the semi-Markov process) Markov chain, and the other type of terms is of the kind of boundary layer functions. This method allows us to study the asymptotic behavior of the residence time of a semi-Markov process in a fixed set of states. By using the approach of Korolyuk, the asymptotic analysis of hitting time of a "hard-to-achieve" level by semi-Markov processes was carried out by Korolyuk *et al.* (1973), Korolyuk and Tadjiev (1977), Korolyuk and Borovskikh (1981), Korolyuk and Turbin (1993, 1982), and others. An alternative method for the asymptotic analysis of singularly perturbed semigroups of operators, which makes it possible to write the asymptotic expansion coefficients explicitly, bypassing the recursive calculations, is presented in the papers by Turbin (1981) and Pogorui (1990, 1992, 1994).

The research to find the limiting distributions and the asymptotic analysis of semi-Markov processes continues to be an active area in recent times. For instance, we should mention in this line the works of Foss (2007), Kabanov and Pergamenshchiskov (2003), Semenov (2008), Silvestrov (2004, 2007a,b), Soloviev (1993), and Yeleyko and Zhernovyi (2002). Limiting distributions for stationary processes and their application in the assessment of financial risks are studied by Novak (2007, 2011).

The adapted Vishyk–Lyusternik algorithm also finds its application in the study of asymptotic expansions for functionals of random evolution in the phase averaging and diffusion approximation. This topic is studied in the works of Albeverio *et al.* (2009), Korolyuk and Limnios (2009, 2005), Samoilenko (2005), Pogorui (2009a), and Pogorui and Rodríguez-Dagnino (2010a).

Another important approach in the study of REs is the convergence rate of random walks by using limit theorems based on the martingale nature of RE. This has been the topic of various books and articles (Pinsky 1991; Korolyuk 1993; Swishchuk 1989; Iksanov 2006; Iksanov and Résler 2006; Sviridenko 1989).

Stochastic processes with Markov and semi-Markov switching to simulate the motion of a particle in a finite-dimensional space are called RE. One of the first works along this line was a random evolution model describing the motion of a particle on a line at a constant absolute speed whose directions are switched by the Poisson process. Such an analysis was made by Goldstein (1951) and Kac (1951, 1974). This model was later called the telegraph process because the distribution of the motion of such particles is a solution of the telegraph equation. However, the telegraph process $\{x(t)\}$ does not have some properties, which makes it difficult to study it. For example, the telegraph process $x(t)$, in contrast to the Wiener process, is neither Markov nor a martingale. In particular, these "defects" complicate the study of systems of interacting particles, whose trajectories are described by the telegraph processes. For example, after a hard collision of a particle with another particle, its whole trajectory is not described by a telegraph process. Pogorui (2012b) studied a system of particles and specified a manner to overcome problems arising from the non-Markov trajectories of particles.

The connections of the telegraph process with the random motion of particles have raised to new research areas:

1) The generalization of telegraph processes on multidimensional spaces, especially on a plane with a finite number of symmetrical directions of a particle movement, which changes at random Poisson events. These studies were performed in the works of Orsingher (1985), Orsingher and Somella (2004) and Lachal (2006).

2) The telegraph processes with reflecting and partially reflecting boundaries and their distribution. This problem was studied by Masoliver *et al.* (1993). The stationary distribution of Markov and semi-Markov RE with delays in boundaries (or sticky boundaries) and their applications to evaluate the effectiveness of the multiphase inventory control systems with feedback was studied by Pogorui and Turbin (2002), Pogorui (2003, 2004a), and Pogorui and Rodríguez-Dagnino (2009b, 2010b).

3) The generalizations of telegraph processes on heterogeneous cases in which the absolute speed is variable or the distribution switching processes change over time. For example, the work of Stadje and Zacks (2004) is devoted to the generalization of the telegraph process to the case where the particle velocity is an independent and identically distributed random variable that changes at Poisson events. Di Crescenzo and Martinucci (2010) studied a generalized telegraph process with increasing parameters of the switching process. Samoilenko (2002), Pogorui (2009b, 2010b), and Pogorui and Rodríguez-Dagnino (2009b, 2005b) studied fading evolution, where speed goes to zero at an infinite increase in the number of switches. In the paper by Pogorui and Rodríguez-Dagnino (2005a), the authors study a generalization of the telegraph process on the case of general Erlang sojourn times of the switching process and obtain the hyperbolic partial differential equation for the distribution of this process. In addition, in Pogorui (2011a) and Pogorui *et al.* (2014) a method for solving such kind of differential equation was developed.

4) Multidimensional random motion with an infinite number of directions of motion, which changes its direction at renewal epochs of the switching process, has been developed in some recent works. Apparently, one of the first papers in this field was that by Stadje (2007), where motion in the plane with Poisson switching process is studied. Further works in this direction are by Orsingher and De Gregorio (2007), and Franceschetti (2007). Most of these works use the apparatus of characteristic functions to study the distribution of a particle in multidimensional spaces with constant absolute speed when at Poisson events the particle changes its direction to another uniformly distributed on a unit sphere. In these papers,

the authors obtained the distribution of the particle position in explicit form for dimensions $n = 2, 4, 6$.

Pogorui and Rodríguez-Dagnino (2012) study an isotropic random motion in multidimensional space with a random velocity that is more natural from the physics point of view. When we have random velocity, then an "explosive effect" for some distributions is observed.

Pogorui and Rodríguez-Dagnino (2011b, 2013) studied an isotropic random motion with gamma steps in higher dimensions. Le Caér (2010, 2011), De Gregorio and Orsingher (2012), Beghin and Orsingher (2010a), De Gregorio (2014), and Letac and Piccioni (2014) considered a random motion in $\mathbb{R}^n$ by suggesting that switching points $0 < \tau_1 < \tau_2 < \cdots < \tau_n < t$ have the Dirichlet distribution on the interval $[0, t]$ and changes of direction have the uniform distribution on a sphere, which change at epochs τ_i.

In all of the papers mentioned above, the authors study the conditional distributions of the particle position at renewal epochs of the switching direction process.

5) Recently, there has been an increasing interest in studying stochastic flows in a system of interacting particles, the so-called Arratia flow. The first results in this field were found by Arratia (1979), where the author investigated a system of Wiener particles on a line that coalesce in the collision and continue to move as one particle. Many authors have continued this line of research and they have made important contributions (Dorogovtsev 2007b,a, 2010; Dawson 1993; Le and Raimond 2004; Yu 1999; Konarovskii 2011).

Pogorui (2012b) investigates a system of interacting particles with Markov switching. In particular, the author finds the distribution of the first time collision of two telegraph particles that started simultaneously from different points on the line, and he found the limit of the distribution under the Kac condition. Based on this result, the author investigates the time of free path for a family of particles with elastic collision, and he studies the distributions of particles both with reflecting boundaries and without them, and the limiting properties of these distributions.

Since the semi-Markov REs appear as abstract models for a wide class of real stochastic systems (Korolyuk 1987, 1993; Korolyuk *et al.* 1973; Korolyuk and Swishchuk 1986; Korolyuk and Turbin 1982), tthe study of their properties is important not only from a theoretical point of view but also in practical problems. In papers by Pogorui and Turbin (2002), Pogorui (2003, 2004b), Rodríguez-Said *et al.* (2007, 2008), and Pogorui *et al.* (2006), the authors study the stationary distributions of random evolution and their application to calculate the efficiency of inventory (or reservoir) control systems with feedback. Semi-Markov evolutions also find a wide use in the modeling and study of stochastic processes in finance and queueing systems, for example, in the works of Korolyuk (1993), Korolyuk and Swishchuk (1995b), Mitra (1998), Pogorui and Rodríguez-Dagnino (2009a), Pogorui and

Rodríguez-Dagnino (2008), (Ratanov 2007), López and Ratanov (2012), Swishchuk and Burdeinyi (1996), Swishchuk (2004), Maglaras and Zeevi (2004), and others.

1.2. Abstract potential operators

In the study of the asymptotic distribution of probability of reaching a "hard-to-reach domain" by semi-Markov processes, the theory of perturbation for linear operators is systematically used. As basic tools for this theory, it is necessary to introduce the notion of the projector and the generalized inverse operator or potential operator.

Let $\mathcal{B}$ be a Banach space. Two closed linear manifolds M_1 and M_2 in $\mathcal{B}$ are called supplemented if $M_1 \cap M_2 = 0$, where 0 is the zero element in $\mathcal{B}$, and $M_1 + M_2 = \{x : x = x_1 + x_2,\ \ x_1 \in M_1,\ x_2 \in M_2\} = \mathcal{B}$. Then, we say that $\mathcal{B}$ is a direct sum of M_1 and M_2 and we use the usual notation $\mathcal{B} = M_1 \oplus M_2$. If M_1 and M_2 are closed subspaces in $\mathcal{B}$, then to each image in the form of a direct sum corresponds a limited projector P such that $P\mathcal{B} = M_1$, $(I - P)\,\mathcal{B} = M_2$.

Let $A : \mathcal{B} \to \mathcal{B}$ be a linear operator, and we have the following definitions:

DEFINITION 1.1.– (Kato 1980) *A linear operator A, defined on some linear manifold $\mathcal{D}(A)$, is called closed if from $x_n \to x$ $(x_n \in \mathcal{D}(A))$ and $Ax_n \to y$ it follows that $x \in \mathcal{D}(A)$ and $y = Ax$.*

DEFINITION 1.2.– (Kato 1980) *A closed densely defined operator A $\left(\overline{\mathcal{D}(A)} = \mathcal{B}\right)$ is called normally solvable if its range $R(A)$ is a subspace of $\mathcal{B}$.*

The following theorem characterizes normally solvable operators:

THEOREM 1.1.– (Kato 1980) A closed densely defined operator A, with $R(A) \neq \mathcal{B}$, is normally solvable if and only if (iff)

$$N(A^*)^{\perp} = R(A),$$

where $N(A^*)^{\perp}$ is a subset of $\mathcal{B}$ such that $f(x) = 0$ for all $f \in N(A^*)$.

The following lemma states another important condition of normal solvability of operators:

LEMMA 1.1.– (Kato 1980; Krein 1971) Suppose A is a closed operator and there exists a closed linear subspace M in $\mathcal{B}$ such that $\mathcal{B} = M \oplus R(A)$. Then the operator A is normally solvable.

The following characteristic for normal solvability of an operator is well known (Cox 1976). A bounded linear operator A is a normally solvable operator if and only if its restriction $\overline{A}$ on $R(A)$ has a bounded inverse operator $\overline{A}^{-1}$.

DEFINITION 1.3.– (Kato 1980) *A normally solvable operator* A *is called a Fredholm operator if* $\dim N\left(A^{*}\right)=\dim N\left(A\right)=r<\infty$.

DEFINITION 1.4.– (Kato 1980; Krein 1971) *A closed densely defined operator* A *on* $\mathcal{B}$ *is called reducible-invertible if* $\mathcal{B}$ *can be represented as follows:*

$$\mathcal{B}=N\left(A\right)\oplus R\left(A\right). \tag{1.1}$$

Since $R\left(A\right)$ is closed, then a reducible-invertible operator is normally solvable. Hence, for a reducible-invertible operator A, we have $R\left(A\right)\cap N\left(A\right)=\{0\}$ and $\forall b\in\mathcal{B}$ can be represented in the form $b=f+u$, where $f\in R\left(A\right)$, $u\in N\left(A\right)$ and $N\left(A\right)$ and $R\left(A\right)$ are subspaces of $\mathcal{B}$.

The decomposition in equation [1.1] generates a projective operator Π on $N\left(A\right)$ such that

$$\Pi N\left(A\right)=N\left(A\right),\qquad\left(I-\Pi\right)R\left(A\right)=R\left(A\right). \tag{1.2}$$

DEFINITION 1.5.– (Kato 1980) *A projective operator* Π, *which satisfies properties of equation* [1.2] *for some reducible-invertible operator* A, *is said to be the proper projector of* A.

The proper projector Π *satisfies the following conditions* $\forall\,\varphi\in B$:

1) $\Pi^{2}\varphi=\Pi\varphi$;

2) $\Pi A\varphi=A\Pi\varphi=0$.

LEMMA 1.2.– (Kato 1980; Korolyuk and Turbin 1993) If A is a reducible-invertible operator, then there exists a bounded inverse operator $\left(A+\Pi\right)^{-1}$.

Let us consider an example of a proper projective operator of a reducible-invertible operator A. Let E be a countable set. Denote by l_{∞} the real space (Banach space) of vectors $\mathbf{u}=\{u_i,\ i\in E\}$ with the norm $\|\mathbf{u}\|=\max_{i\in E}|u_i|$ and by l_1 the real space of vectors $\boldsymbol{\rho}=\{\rho_i,\ i\in E\}$ with the norm $\|\boldsymbol{\rho}\|=\sum_{i\in E}|\rho_i|$. For any $\boldsymbol{\rho}\in l_1$ and $\mathbf{u}\in l_{\infty}$, define the scalar product

$$(\boldsymbol{\rho},\mathbf{u})=\sum_{i\in E}\rho_i u_i.$$

The tensor product of $\mathbf{u}\in l_{\infty}$ and $\boldsymbol{\rho}\in l_1$ is defined as follows:

$$[\mathbf{u}\otimes\boldsymbol{\rho}]=\{u_i\rho_j,\ i,j\in E\}.$$

It is easy to see that for $\mathbf{f}\in l_{\infty}$, we have

$$[\mathbf{u}\otimes\boldsymbol{\rho}]\,\mathbf{f}=(\boldsymbol{\rho},\mathbf{f})\,\mathbf{u}.$$

Now, suppose that an operator $A = \{a_{ij},\, i,j \in E\}$ with real a_{ij} satisfies the condition

$$\sup_{i\in E} \sum_{j\in E} |a_{ij}| \; = \|A\| < \infty. \qquad [1.3]$$

Consider

$$A\mathbf{u} = \sum_{j\in E} a_{ij} u_j.$$

It follows from equation [1.3] that $A : l_\infty \to l_\infty$. In addition, A is bounded on l_∞ with the norm $\|A\|$.

The adjoint operator A^* of A is defined on l_1 as follows:

$$A^* \boldsymbol{\rho} = \sum_{i\in E} a_{ij} \rho_i.$$

It is easily verified that A^* is bounded on l_1 and $\|A^*\| = \|A\|$.

Suppose A is Fredholm, i.e. $\dim N(A) = \dim N(A^*) = r < \infty$, where $N(A)$ and $N(A^*)$ are the kernels of operators A and A^*, respectively.

Let $u^{(1)}, u^{(2)}, \ldots, u^{(r)}$ be a basis of $N(A)$ and $\rho^{(1)}, \rho^{(2)}, \ldots, \rho^{(r)}$ be a basis of $N(A^*)$. Suppose that

$$\left(\rho^{(i)}, u^{(j)}\right) = \delta_{ij},$$

where δ_{ij} is the Kronecker symbol.

The operator $\Pi = \sum_{k=1}^{r} \left[u^{(k)} \otimes \rho^{(k)}\right]$ is the proper operator of A and it satisfies the following properties:

1) $\Pi^2 = \Pi$;

2) $\Pi u = u$, $u \in N(A)$;

3) $\rho\Pi = \rho$, $\rho \in N(A^*)$;

4) $\Pi f = 0$, $f \in R(A)$;

5) $\varphi\Pi = 0$, $\varphi \in R(A^*)$;

6) $\Pi A\varphi = A\Pi\varphi$, $\varphi \in R(A)$.

LEMMA 1.3.– (Kato 1980; Korolyuk and Turbin 1993) Suppose that an operator A is reducible-invertible and Π is its proper projector. Then, there exists the bounded inverse operator $(A + \Pi)^{-1}$.

It follows from lemma 1.3 that there exists $(A+\Pi)^{-1}-\Pi$ and it is bounded.

DEFINITION 1.6.– (Korolyuk and Turbin 1976, 1993; Sato 1971) *The operator* $R_0 = (A+\Pi)^{-1}-\Pi$ *is called the generalized inverse operator or potential operator of* A.

The operator R_0 *is also called the abstract potential operator of the reducible-invertible operator* A.

LEMMA 1.4.– (Kato 1980; Korolyuk and Turbin 1993) The operator R_0 satisfies the following properties:

$$AR_0 = R_0A = I - \Pi;$$

$$R_0\Pi = \Pi R_0 = 0;$$

$$\|R_0\| = \left\|\overline{A}^{-1}\right\|,$$

where $\overline{A}$ is the restriction of A to $R(A)$.

The notion of a generalized inverse operator plays an important role in the study of Markov processes (Gikhman and Skorokhod 1975; Korolyuk and Turbin 1993, 1982). In particular, for a Markov chain with transition probabilities matrix P, the generalized inverse operator R_0 of $I-P$ is called the potential of this Markov chain.

EXAMPLE 1.1.– Consider a Markov chain $\{\xi_n,\, n \geq 0\}$ with transition probabilities matrix

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{4} & \frac{3}{4} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

Denote by A the following matrix

$$A = I - P = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & \frac{3}{4} & -\frac{3}{4} \\ 0 & 0 & -\frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

The matrix A is called infinitesimal for the Markov chain ξ_n. It is easily verified that $u^{(1)} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$, $u^{(2)} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}$ is a basis of $N(A)$ and $\rho^{(1)} = \begin{pmatrix} 1, & 0, & 0, & 0 \end{pmatrix}$, $\rho^{(2)} = \begin{pmatrix} 0, & 0, & \frac{2}{5}, & \frac{3}{5} \end{pmatrix}$ is a basis of $N(A^*)$, where A^* is the transpose matrix of A. Thus, $\dim N(A) = \dim N(A^*)$.

Let us calculate the following scalar and tensor products $\left(\rho^{(i)}, u^{(j)}\right) = \delta_{ij}$,

$$\left[u^{(1)} \otimes \rho^{(1)}\right] = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \left(1,\ 0,\ 0,\ 0,\right) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\left[u^{(2)} \otimes \rho^{(2)}\right] = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \left(0,\ 0,\ \tfrac{2}{5},\ \tfrac{3}{5}\right) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{2}{5} & \frac{3}{5} \\ 0 & 0 & \frac{2}{5} & \frac{3}{5} \end{pmatrix}.$$

Thus, the proper projector of A is as follows:

$$\Pi = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{2}{5} & \frac{3}{5} \\ 0 & 0 & \frac{2}{5} & \frac{3}{5} \end{pmatrix}.$$

Then,

$$A + \Pi = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{23}{20} & -\frac{3}{20} \\ 0 & 0 & -\frac{1}{10} & \frac{11}{10} \end{pmatrix}.$$

Therefore,

$$(A + \Pi)^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{22}{25} & \frac{3}{25} \\ 0 & 0 & \frac{2}{25} & \frac{23}{25} \end{pmatrix}.$$

From this we have the potential of Markov chain ξ_n in the form of the generalized inverse operator A:

$$R_0 = (A + \Pi)^{-1} - \Pi = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & \frac{12}{25} & -\frac{12}{25} \\ 0 & 0 & -\frac{8}{25} & \frac{8}{25} \end{pmatrix}.$$

1.3. Markov processes: operator semigroups

In this section, we study elementary concepts of semigroup of operators generated by Markov process. We also define the infinitesimal operator, and the stationary distribution of Markov processes. These notions will be used for the asymptotic analysis (large deviations) of semi-Markov processes.

Denote by $(\Omega, \mathcal{F}, \mathcal{P})$ a probability space and by (X, Σ) a complete separable metric space, where Σ is a σ-algebra of Borel subsets of X.

Let $\{\xi(t),\, t \geq 0\}$ be a homogeneous Markov process on $(\Omega, \mathcal{F}, \mathcal{P})$ on the phase space (X, Σ). Denote by $P\,(t, x, B) = P\,\{\xi\,(t) \in B |\, \xi\,(0) = x\}$ the transition probability of $\xi\,(t)$. It is well known that for every $t \in T$ the probability $P(t, x, B)$ is a stochastic kernel, i.e. for a fixed $x \in X$, $P(t, x, B)$ is a measure on Σ and $P\,(t, x, X) = 1$, and for a fixed B, $P(t, x, B)$ is a measurable function with respect to x.

Denote by $B\,(\Sigma)$ a set of bounded and Σ-measurable functions on X. Consider the family of operators generated by the transition probabilities: for a function $f \in B\,(\Sigma)$, we define

$$T_t f\,(x) = \int_X f\,(y)\,P\,(t, x, dy).$$

It is easy to see that $T_t : B\,(\Sigma) \to B\,(\Sigma)$, and it follows from the Chapman–Kolmogorov equation that

$$T_{s+t} = T_s\,T_t.$$

Thus, the family of operators T_t, $t \in T$ is a semigroup.

Let D be a subset of $B\,(\Sigma)$ such that there exist the following limits for any $\varphi \in D$

$$A\varphi = \lim_{\Delta t \to 0^+} \frac{T_{\Delta t}\varphi - \varphi}{\Delta t},$$

$$\lim_{\Delta t \to 0^+} T_{\Delta t}\varphi = \varphi.$$

The operator A is called the infinitesimal operator of the semigroup of operators T_t and of the Markov process $\xi\,(t)$ (Gikhman and Skorokhod 1975; Kovalenko 1980; Kovalenko *et al.* 1983), respectively.

DEFINITION 1.7.– *It is said that $\xi(t)$ has the stationary distribution π if for any $B \in \Sigma$*

$$\pi\,(B) = \int_X \pi\,(dx) P\,(t, x, B)\,,\; t \geq 0.$$

The stationary projector Π of $\xi(t)$ is defined as follows:

$$\Pi f\left(x\right)=\int_{X}\pi\left(ds\right)f\left(s\right)\mathbb{I}(x),$$

where $\mathbb{I}\left(x\right)=1$ for all $x\in X$ and 0 otherwise.

DEFINITION 1.8.– *A Markov process $\xi(t)$ is said to be uniformly ergodic if $\left\|T_t-\Pi\right\|\to 0$ as $t\to\infty$.*

It is well known (Korolyuk and Turbin 1976, 1993) that the infinitesimal operator A of a uniformly ergodic Markov process $\xi(t)$, $t\ \geq\ 0$ is reducible-invertible. Thus, there exists the inverse operator $\left(A+\Pi\right)^{-1}$. Moreover, the operator $R_0=\left(A+\Pi\right)^{-1}-\Pi$ is called the potential of the stochastic process $\xi(t)$.

Let us define $Q=I-\Pi$. Then, the potential R_0 satisfies the following properties (Korolyuk and Turbin 1976, 1993):

1) $R_0\Pi=\Pi R_0=0$;

2) $R_0Q=QR_0=R_0$;

3) $R_0A=AR_0=Q$.

DEFINITION 1.9.– *The number $\varpi\ =\ \lim_{t\to\infty}\frac{1}{t}\ln\left\|T_t\right\|$ is called the type of a semigroup T_t.*

The following property of the resolvent $R\left(\lambda,A\right)$ of operator A is important (Korolyuk and Turbin 1993):

THEOREM 1.2.– If semigroup T_t is of type ϖ, then for any λ such that $\operatorname{Re}\lambda>\varpi$ we have

$$R\left(\lambda,A\right)f:=\left(\lambda I-A\right)^{-1}f=\int_0^{\infty}e^{-\lambda t}T\left(t\right)f dt,\quad f\in B\left(\Sigma\right).$$

The resolvent of a reducible-invertible operator A has the following representation for $0<\left|\lambda\right|<\frac{1}{R_0}$ (Korolyuk and Turbin 1993, 1982):

$$R\left(\lambda,A\right)=\frac{1}{\lambda}\Pi+R_0(\lambda R_0-I)^{-1}.$$

Hence, it follows that for a reducible-invertible operator A for $0<\left|\lambda\right|<\frac{1}{R_0}$

$$\Pi R\left(\lambda,A\right)=\frac{1}{\lambda}\Pi,$$

$$QR\left(\lambda,A\right)=R_0(\lambda R_0-I)^{-1}.$$

Now, let us define $T_t - \Pi = \mathfrak{H}\,(t)$. For a uniformly ergodic Markov process, the integral $\int_0^\infty \mathfrak{H}\,(t)dt$ converges and (Sato 1971):

$$\lim_{\lambda\downarrow 0}\int_0^\infty e^{-\lambda t}\mathfrak{H}\,(t)dt = \int_0^\infty \mathfrak{H}\,(t)dt.$$

It can be proved (Kato 1980; Korolyuk and Turbin 1993) that if the following two conditions are fulfilled, i.e. $\forall f \in D\,(R_0)$, $\int_0^{+\infty}\|(T_t - \Pi)\,f\|\,dt < +\infty$, then the following equality holds:

$$\int_0^{+\infty}[T_t - \Pi]\,f dt = \int_0^\infty \mathfrak{H}\,(t)\,f dt = -R_0 f.$$

1.4. Semi-Markov processes

This section is devoted to providing a constructive definition of a semi-Markov process, which is based upon the notion of an MRP, and a semi-Markov kernel, which is a crucial part of the definition of a semi-Markov process. In addition, we introduce an auxiliary process useful in reducing a semi-Markov process to the Markov with the standard phase space extension. For this Markov process, the infinitesimal operator is presented.

DEFINITION 1.10.– *A semi-Markov kernel in a measurable space* (X, Σ) *is defined by a function* (x, B, t) $x \in X$, $B \in \Sigma$, $t \in [0, +\infty]$, *which satisfies the following conditions:*

1) $Q\,(x, B, t)$ *is non-decreasing and right-continuous with respect to* $t \geq 0$ *for fixed* $x \in X$ *and* $B \in \Sigma$, and $Q\,(x, B, 0) = 0$ *for* $x \notin B$;

2) *for a fixed* $t \geq 0$, $Q\,(x, B, t)$ *is a semi-stochastic kernel, i.e.* $\forall x \in X$ $Q\,(x, X, t) \leq 1$;

3) $Q\,(x, B, +\infty)$ *is a stochastic kernel with respect to* x *and* B, *i.e.* $Q\,(x, X, +\infty) = 1$.

DEFINITION 1.11.– *A two-component Markov chain* $\{\xi_n, \theta_n;\ n \geq 0\}$ *on the phase space* $X \times [0, +\infty)$ *is said to be an MRP if its transition probabilities depend only on the value of the first component* ξ_n *and is defined by a semi-Markov kernel* $Q\,(x, B, t)$ *as follows:*

$$P\left\{\xi_{n+1} \in B, \theta_{n+1} \leq t \middle|\, \xi_n = x\right\} = Q\,(x, B, t)\,.$$

The first component $\{\xi_n,\ n \geq 0\}$ of $\{\xi_n, \theta_n;\ n \geq 0\}$ is a Markov chain with the transition probabilities $P\,(x, B) = Q\,(x, B, +\infty)$ and it is called the embedded Markov chain.

The non-negative random variables $\theta_n,\ n \geq 0$ in $\{\xi_n, \theta_n;\ n \geq 0\}$ are the sojourn times of the MRP

$$\tau_n = \sum_{k=1}^{n} \theta_k,\ \ n \geq 1,\ \ \tau_0 = \theta_0 = 0.$$

The distribution function of θ_{n+1} depends on the state ξ_n as follows:

$$G_x\left(t\right) = P\left\{\theta_{n+1} \leq t \middle| \xi_n = x\right\} = Q\left(x, X, t\right).$$

A random variable θ_{n+1} with the distribution function $G_x\left(t\right)$ is interpreted as the holding time of the MRP in a state $\xi_n = x$ and it is convenient to denote it by θ_x.

Thus, to complete the constructive definition of the MRP, it is necessary to define a semi-Markov kernel.

Since for all $t \geq 0$ and $x \in X$, $Q\left(x, B, t\right) \leq P\left(x, B\right) \forall B \in \Sigma$, the measure Q is absolutely continuous with respect to P. Consequently, there exists a measurable function $G\left(x, B, t\right)$ such that

$$Q\left(x, B, t\right) = \int_B G\left(x, y, t\right) P\left(x, dy\right).$$

The function $G\left(x, y, t\right)$ is a conditional distribution of θ_{n+1} assuming that the embedded Markov chain transits from x to y:

$$G\left(x, y, t\right) = P\left\{\theta_{n+1} \leq t \middle| \xi_n = x, \xi_{n+1} = y\right\}.$$

For a fixed trajectory of the Markov chain $\{\xi_n,\ n \geq 0\}$, the random variables $\theta_n,\ n \geq 0$ are independent and it can be said that they are conditionally independent. Indeed, by using the formula of total probability, we have

$$\begin{aligned} &P\left\{\theta_1 \leq t_1, \cdots, \theta_{n+1} \leq t_{n+1} \middle| \xi_1 = x_1, \cdots, \xi_n = x_n, \xi_{n+1} = x_{n+1}\right\} \\ &\qquad = \prod_{k=0}^{n} G\left(x_k, x_{k+1}, t_{k+1}\right). \qquad [1.4] \end{aligned}$$

The conditional independence of $\theta_n,\ n \geq 0$ for a fixed trajectory of the Markov chain $\{\xi_n,\ n \geq 0\}$ makes it possible to give another definition of the MRP $\{\xi_n,\ \theta_n,\ n \geq 0\}$, which is based on a sequence of non-negative independent random variables $\{\theta_n,\ n \geq 0\}$. These random variables are defined on the Markov chain $\{\xi_n,\ n \geq 0\}$ by the joint distribution [1.4].

It was shown in Korolyuk and Turbin (1976) and Silvestrov (1980) that if X is a countable set then, without loss of generality, we may assume that $G\left(x, y, t\right)$ does not depend on y, i.e. the semi-Markov kernel is of the following form:

$$Q\left(x, B, t\right) = P\left(x, B\right) G_x\left(t\right),$$

where $G_x\left(t\right) = P(\theta_x \leq t | \xi_n = x)$ and θ_x is a sojourn time of the MRP at state x.

Let us introduce a counting process $\{\nu(t)\}$ as follows:

$$\nu(t) = \sup\{n : \tau_n \le t\}.$$

The process $\nu(t)$ counts the number of renewal occurrences in the interval $[0, t]$.

LEMMA 1.5.– The process $\xi(t) := \xi_{\nu(t)}$, $t \ge 0$, is semi-Markov.

If, in addition, for every $t \ge 0$, $P\{\nu(t) < \infty\} = 1$, then the process $\xi_{\nu(t)}$ is called regular.

For a semi-Markov process $\xi(t)$ define

$$\tau(t) = t - \sup\{u \le t : \xi(u) \ne \xi(t)\}.$$

The process $\tau(t)$ is called an auxiliary process. Let us consider the bivariate process $\zeta(t) = (\xi(t), \tau(t))$ on the phase space $X \times [0, +\infty)$. It is well known that the process $\zeta(t)$ is a Markov process (Gikhman and Skorokhod 1975; Corlat *et al.* 1991; Korolyuk and Turbin 1982).

Let $\varphi(x, t)$, $x \in X$, $t \in [0, +\infty)$ be a $\Sigma \times \mathcal{R}_+$-measurable function differentiable with respect to t. Then the infinitesimal operator of $\zeta(t)$ is of the following form (Corlat *et al.* 1991; Korolyuk and Turbin 1982):

$$A\varphi(x,t) = \frac{d}{dt}\varphi(x,t) + \frac{g_x(t)}{1 - G_x(t)}\left[P\varphi(x,0) - \varphi(x,t)\right],$$

where $G_x(t) = P(\theta_x < t)$, $g_x(t) = \frac{d}{dt}G_x(t)$ and $P(x, dy)$ is the transition probability of the Markov chain $\{\xi_n,\ n \ge 0\}$. In addition,

$$P\varphi(x,0) = \int_X P(x,dy)\varphi(y,0).$$

A jumping Markov process is a particular case of a semi-Markov process with the following semi-Markov kernel:

$$Q(x,B,t) = P(x,B)\left(1 - e^{-q(x)t}\right),\ q(x) > 0,\ t \ge 0.$$

The function $q(x)$, $x \in X$ determines the intensity of staying at a particular state x. The infinitesimal operator of the jumping Markov process is given as (Korolyuk 1993; Korolyuk and Korolyuk 1999):

$$A\varphi(x) = q(x)\left[\int_X \varphi(y)P(x,dy) - \varphi(x)\right],\ x \in X,\ B \in \Sigma,$$

where φ is a real Σ-measurable function, which is bounded on X.

1.5. Lumped Markov chains

We will describe a phase merging or state lumping scheme for Markov chains. This scheme applies to the investigation of the probability distribution of reaching a “hard-to-reach domain” by semi-Markov processes. This state lumping scheme was developed and introduced in seminal works by Korolyuk and Turbin (1976, 1982). In Korolyuk *et al.* (1979a,b), lumping states of the phase space of the Markov chain are described, and we will describe the basic issues of it in order to use it as an approach for the investigation of the probability distribution of reaching a level, which goes to infinity.

We will describe the state lumping algorithm for Markov chains.

Let $\{\xi_n,\ n \geq 0\}$ be a homogeneous Markov chain with the phase or state space $E = \{1, 2, \ldots, M\}$ and the matrix of transition probabilities $P = \{p_{ij},\ i, j \in E\}$. We assume that all states of the Markov chain communicate, i.e. they are accessible to each other or $\forall\, i, j \in E$, $\exists\, l \geq 1$, such that $p_{ij}^{(l)} > 0$.

Let us establish that $E_0 = \{1, 2, \ldots, m\}$, $m < M$, $E_1 = E \setminus E_0$. Consider the following random time instants:

$$\begin{cases} \nu_0 = 0, \\ \nu_1 = \min\{n > 0 :\ \xi_n \in E_0\} \\ \vdots \\ \nu_k = \min\{n > \nu_{k-1} :\ \xi_n \in E_0\}. \end{cases}$$

Now, we introduce the sequence

$$\xi_k^{E_0} = \xi_{\nu_k},\ k \geq 0,\ \xi_0^{E_0} = \xi_0 \in E_1.$$

It is easily verified that $\xi_k^{E_0}$, $k \geq 0$ is a homogeneous Markov chain with the state space $E_0 = \{1, 2, \ldots, m\}$ and transition probabilities matrix $P^{E_0} = \left\{p_{ij}^{E_0},\ i, j \in E_0\right\}$. Moreover

$$P^{E_0} = P_{00} + P_{01}(I - P_{11})^{-1} P_{10},$$

where P_{kl}, k, $l = 0, 1$, are matrices with elements $p_{ij},\ i \in E_k,\ j \in E_l$.

Since the matrix P_{11} is not stochastic by assumption, then there exists the inverse matrix $(I - P_{11})^{-1} = \sum_{k=0}^{\infty} P_{11}^k < \infty$ (Kemeny and Snell 1960).

DEFINITION 1.12.– *A Markov chain* $\xi_k^{E_0} = \xi_{\nu_k}$ *with transition probabilities matrix* P^{E_0} *will be called the lumped Markov chain, and it is obtained from the chain* ξ_n *by state lumping from* E *to* E_0.

Let $\{\xi_n,\, n \geq 0\}$ be a homogeneous Markov chain with the state space $E = \{1, 2, \cdots\}$, which satisfies the condition:

C1. *A Markov chain* $\{\xi_k\}$ *is irreducible (Meyn and Tweedie 1993) and it has the stationary distribution* $\boldsymbol{\rho} = \{\rho_i, i \in E\}$. *In addition,*

$$\varepsilon_n = \sum_{i>n} \rho_i > 0.$$

Denote by e_n the new state obtained after merging the set of states $E^n = \{n+1, n+2, \ldots\}$ (Korolyuk *et al.* 1979a,b), i.e. the transition probabilities of the lumped Markov chain $\left\{\overline{\xi}_m, m \in \mathbb{N}\right\}$ on the state space $E_n \cup \{e_n\}$, where $E_n = \{1, 2, \ldots, n\}$, are as follows:

$$\overline{p}_{ij} = \begin{cases} p_{ij}, & i, j \in E_n, \\ \sum_{k \in E^n} p_{ik}, & i \in E_n,\ j = e_n, \\ \frac{1}{\varepsilon_n} \sum_{k \in E^n} \rho_k p_{kj}, & i = e_n,\ j \in E_n, \\ 1 - \sum_{l \in E_n} \overline{p}_{e_n l}, & i = e_n,\ j = e_n. \end{cases}$$

It is easily seen that the stationary distribution $\boldsymbol{\rho}^{(n)}$ of the chain $\left\{\overline{\xi}_m\right\}$ is of the form $\boldsymbol{\rho}^{(n)} = (\rho_1, \rho_1, \cdots, \rho_n, \varepsilon_n)$.

Consider a Markov process $\{\xi_\varepsilon(t),\, t \geq 0\}$, on the state space (X, Σ), where Σ is a σ-algebra of subsets of X. We introduce the following finite partition of the state space $X = \cup_{i=1}^r X_i,\ X_i \in \Sigma$, where $X_i \cap X_j = \emptyset,\ i \neq j$. The process $\xi_\varepsilon(t)$ depends on $\varepsilon > 0$ (where ε is the small series parameter) and its infinitesimal operator A_ε is of the form

$$A_\varepsilon = A + \varepsilon A_1,$$

where A is the infinitesimal operator of the support Markov process $\xi_0(t),\ t \geq 0$, which is obtained from $\xi_\varepsilon(t)$ by letting $\varepsilon = 0$. We assume that $\xi_0(t)$ is uniformly ergodic in X with the transition probabilities $P(x, B)$, $x \in X$, $B \in \Sigma$, which satisfy the following condition:

$$P(x, X_i) = \mathbb{I}_i(x) = \begin{cases} 1, & x \in X_i, \\ 0, & x \notin X_i. \end{cases}$$

We also assume that for all $i = 1, 2, \ldots, r$, there is not a proper subset $B_i \subset X_i$ such that $P(x, B_i) = 1\ \forall x \in B_i$ and $(x, X_i \backslash B_i) = 1\ \forall x \in X_i \backslash B_i$.

Denote by $\pi_i(B)$, $B \in \Sigma$, $\pi_i(X_i) = 1$, $1 \leq i \leq r$ the stationary distribution of the support process $\xi_0(t)$. As mentioned above, the infinitesimal operator A of $\xi_0(t)$ is reducible-invertible (Korolyuk and Turbin 1993).

The kernel of A, say $ker\,(A)$, has dimension r and its proper projector Π is as follows:

$$\Pi f\,(x) = \sum_{i=1}^{r} \int_{X_i} f\,(y)\,\pi_i(dy)\mathbb{I}_i\,(x)\,.$$

The contracted operator $\hat{A}_1$ is defined in the following form:

$$\Pi A_1 \Pi = \hat{A}_1 \Pi.$$

It was shown in Korolyuk and Turbin (1976) that the operator $\hat{A}_1$ can be represented in the matrix form $\hat{A}_1 = [a_{ij},\ i,j \in \{1,2,\ldots,r\}]$, and it is the infinitesimal operator of the new merged or lumped Markov process $\widehat{\xi}(t)$, $t \geq 0$, with the state space $\hat{X} = \{1,2,\ldots,r\}$.

Now, we define the following merging function $k\,(x) = i$ for $x \in X_i$.

It was proved in Korolyuk and Turbin (1976) and Korolyuk and Turbin (1982) that $k\,(\xi_\varepsilon\,(t/\varepsilon))$ weakly converges to $\widehat{\xi}\,(t)$ following weak convergence to $\widehat{\xi}\,(t)$ as $\varepsilon \to 0$ for all $t > 0$, i.e. for all $t > 0$:

$$P\left\{k\,(\xi_\varepsilon\,(t/\varepsilon)) < x\right\} \to P\left\{\widehat{\xi}\,(t) < x\right\} \ \text{ as } \varepsilon \to 0, \ \ x \in \mathbb{R}.$$

1.6. Switched processes in Markov and semi-Markov media

In this section, we introduce the switched process driven by Markov or semi-Markov process, and it represents the stochastic media, where the switched process evolves. The semigroup of operators associated with a switched process is defined and its infinitesimal operator is represented. In addition, we consider the superposition of independent semi-Markov processes.

Denote by $\{\xi\,(t)\,,\, t \geq 0\}$, a semi-Markov process in a standard state or phase space (X,Σ) given by the semi-Markov kernel

$$Q\,(x,B,t) = P\,(x,B)\,G_x(t).$$

The evolutionary switched process $V(t)$ in a semi-Markov media $\xi\,(t)$ is defined as a solution of the evolution equation (Korolyuk 1993; Korolyuk and Swishchuk 1995b; Anisimov 1977):

$$\frac{dV(t)}{dt} = C\,(V(t),\xi(t))\,, \qquad V(0) = v \tag{1.5}$$

where $C(v,x) : \mathbb{R}{\times}X \to \mathbb{R}$ satisfies the unique valued solvability condition of equation [1.5]. A sufficient condition for this is that the function $C(v,x)$ satisfies a Lipschitz condition with respect to the variable v uniformly in $x \in X$.

Such a stochastic process with reflecting boundaries is a good model for a multiphase supplying system with feedback, which will be studied in Volume 1, Chapter 3. For instance, the estimation of effectiveness of a supplying system with feedback is reduced to the calculations of the stationary distribution of a switched process that models the system.

Denote by τ_k, $k = 1, 2, \ldots$, the renewal instants of a semi-Markov process $\xi(t)$. Then, equation [1.5] can be solved sequentially in each of the segments $[\tau_n, \tau_{n+1})$, where it is deterministic, i.e.

$$\frac{dV(t;\xi_n)}{dt} = C\left(V(t;\xi_n), \xi_n\right), \quad t \in [\tau_n, \tau_{n+1}),$$

with boundary condition $\xi_n = \xi(\tau_n)$, where $V(\tau_n;\xi_n) = V(\tau_{n-1} - 0; \xi_{n-1})$, $n \geq 2$.

The following recurrence relation represents a solution of equation [1.5]

$$V(t) = V(t;\xi_n), \quad t \in [\tau_n, \tau_{n+1}), \, n \geq 0.$$

Since a solution $V(t)$ of equation [1.5] depends on $x = \xi(t)$ and v, i.e. $V(t) = V(t, x, v)$.

On the space of boundary functions $C_b(R^n)$, we introduce a family of operators T_t, which is defined as follows:

$$T_t(x) f(v) = f(V(t)) = f(V(t, x, v)). \tag{1.6}$$

LEMMA 1.6.– (Korolyuk 1993; Korolyuk and Swishchuk 1995b) The family of operators $T_t(x)$, $t \geq 0$, introduced in equation [1.6], satisfies the semigroup property, namely for any t, $t' \geq 0$,

$$T_t(x) T_{t'}(x) = T_{t+t'}(x).$$

We should mention that the process $V(t)$ is not Markov even if $\xi(t)$ is Markov. In the case of the Markov switching process $\xi(t)$, it is found that the couple of processes $(V(t), \xi(t))$ is Markov and its infinitesimal operator is defined by the following lemma:

LEMMA 1.7.– (Korolyuk 1993; Korolyuk and Korolyuk 1999) The infinitesimal operator $G(x)$ of the couple of processes $(V(t), \xi(t))$ is of the following form:

$$Gf(v, x) = Qf(v, x) + C(v, x) \frac{\partial}{\partial v} f(v, x),$$

where $f(v, x)$ is a bounded function differentiable with respect to v, and $Qf(v, x) = \int_E Q(x, dy) f(v, y) - \lambda_x f(v, x)$ is the infinitesimal operator of the Markov process $\xi(t)$.

As we mentioned above, the process $V(t)$ with delaying barriers models a multiphase system with feedback. In order to investigate the efficiency of such systems, it is enough to calculate the stationary distribution of the couple of processes $(V(t), \xi(t))$. For such a purpose, we obtain the operator G^* adjoint to the infinitesimal operator G of $(V(t), \xi(t))$ with the same boundary conditions as the process $V(t)$, and we should find a solution of the following equation:

$$G^* \rho = 0.$$

The normalized solution ρ of this equation is the stationary distribution of $(V(t), \xi(t))$.

Another important result used in the study of multiphase systems is the superposition of independent semi-Markov processes. Assuming the case when the semi-Markov process is completely described by its corresponding MRP, we define a superposition of semi-Markov processes by using the superposition of MRPs.

Consider N independent MRPs $\left\{\xi_n^{(i)}, \tau_n^{(i)}\right\}$, where $\tau_n^{(i)}$ *are renewal instants* of the ith MRP. Let us introduce an auxiliary process as follows:

$$\gamma^{(i)}(t) = \inf\left\{s > t : \xi_n^{(i)}(s) \neq \xi_n^{(i)}(t)\right\}.$$

Consider the processes

$$\gamma(t) = \min\left\{\gamma^{(1)}(t), \gamma^{(2)}(t), \cdots, \gamma^{(N)}(t)\right\};$$

$$\zeta^{(i)}(t) = \gamma^{(i)}(t) - \gamma(t).$$

Then the renewal instants $\{\tau_n,\ n \geq 1\}$ of the superposition of MRPs are defined by the following relations:

$$\gamma(\tau_n - 0) = 0,\ n \geq 1.$$

Define

$$\zeta_n^{(i)} = \gamma^{(i)}(\tau_{n-1}) - \gamma(\tau_{n-1}),\ \ n \geq 1.$$

DEFINITION 1.13.– (Korolyuk and Turbin 1982) *The Markov renewal process* $\{\xi_n, \zeta_n\}$ *with components* $\xi_n = \left\{\xi_n^{(1)}, \xi_n^{(2)}, \cdots, \xi_n^{(N)}\right\}$ *and* $\zeta_n = \left\{\zeta_n^{(1)}, \zeta_n^{(2)}, \cdots, \zeta_n^{(N)}\right\}$ *is called the superposition of Markov renewal processes* $\left\{\xi_n^{(i)}, \tau_n^{(i)}\right\}$, $i = 1, 2, \cdots, N$.

The superposition of processes models a multiphase system with feedback, which consists of several consecutive aggregates such that in between any two successive aggregates there is a reservoir. This type of system will be studied in Volume 1, Chapter 3.