1

Homogeneity of Relationships and Conversion of Units

1.1. Introduction

The creation of a system of units requires the definition of basic units, their values, and the units derived from them. In mechanics, the units used are length, mass and time, but other options are possible as well, such as length and time, force and time, or mass, speed and time.

The first general conference on weights and measures was held in 1889 at the headquarters of the BPIM (*Bureau international des poids et mesures* or International Office of Weights and Measures), at the Breteuil pavilion in Sèvres (in the suburbs of Paris). In this conference, new international prototypes of the meter and kilogram were officially adopted and filed with the Office.

In 1960, the 11th General Conference of Weights and Measures established the International System (SI), in which the rules for prefixes, derived units and other indications were established. The SI is based on a choice of seven well-defined base units that the convention considered dimensionally independent: the meter, the kilogram, the second, the ampere, the kelvin, the mole and the candela. Derived units are formed by combining the base units according to the algebraic relationships connecting the corresponding quantities. The names and symbols of some of these units may be replaced by special names and symbols, which may be used to express the names and symbols of other derived units.

In November 2018, the International System of units later underwent a significant revision at the 26th General Conference of weights and measures, the culmination of nearly 250 years of consideration given to finding the best way to define a system of units of measurement that would best reflect the natural world.

The General Conference of weights and measures, established at the end of the 19th Century, meets every four to six years to discuss, and possibly modify, the SI, standardizing the units on a global scale. The new changes were applied as of May 20, 2019.

Quantity	Name	Symbol	Dimension
Length	Meter	m	L
Mass	Kilogram	kg	M
Time	Second	s	T
Intensity of electric current	Ampere	A	I
Thermodynamic temperature	Kelvin	K	Q
Quantity of matter	Mole	mol	N
Intensity of light	Candela	cd	J

Table 1.1. *Basic SI units*

1.2. Definitions of the basic SI units

1.2.1. *Definition of the meter as adopted in 1983*

The “meter” is the “length” of the distance light travels in a vacuum for a period of 1/299792458ths of a second. From this, it can be determined that the speed of light in a vacuum is exactly: $c_0 = 299{,}792{,}458$ m/s.

1.2.2. *Definition of the kilogram*

The “kilogram” is the unit of mass. It is equal to the mass of the international prototype of the kilogram. The term “weight” refers to a quantity that is of the same nature as a force; the weight of a body is the product of the mass of that body multiplied by the acceleration due to gravity. Specifically, the normal weight of a body is the product of the mass of this body multiplied by the normal acceleration from gravity. The number adopted by the International Service of Weights and Measures for the value of normal acceleration from gravity is 980.665 m/s^2.

The kilogram is currently defined as the mass of an Iridium platinum cylinder (90% platinum and 10% iridium) 39 mm in diameter and 39 mm high declared the SI unit of mass in 1889 by the BIPM. This unit of measurement is the last SI unit to be defined using a man-made physical standard. It is stored under three sealed glass bells, and it is only removed from this covering for calibrations (an operation that has taken place only three times since its creation). As of May 20, 2019, the kilogram has been defined on the basis of Planck's constant (h)[1] from quantum physics, measured on the Kibble[2,3] scale at $6.626069934\times 10^{-34}$ $kg.m^2.s^{-1}$.

1.2.3. *Definition of the second adopted in 1967*

The "second" has the duration of 9,192,631,770 periods of radiation from the transition between the two hyperfine levels of the base state of a cesium 133 atom. As a result, the frequency of the hyperfine transition of the base state of the cesium atom is equal to 9,192,631,770 hertz (Hz). Thus, we obtain exactly ν (hfs[4] Cs) = 9,192,631,770 Hz. At its 1997 session, the International Committee on Weights and Measures confirmed that this definition refers to a cesium atom at rest, at a temperature of 0 K (Kelvin[4]).

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1 Max Planck, born Max Karl Ernst Ludwig Plank, on April 23, 1858, in Kiel, in the Duchy of Schleswig (Germany), living until October 4, 1947, was a physicist. Planck's constant h plays a central role in quantum mechanics: it links the energy of a photon to a photon's frequency.

2 The scale created by Bryan Peter Kibble is a device that allows for the conversion, with nine significant figures, of mechanical power into electrical power and vice versa. Kibble, born on August 10, 1938, in Letcombe Regis (England), who lived until April 28, 2016, was a physicist and an expert in metrology

3 Hyperfine splitting of cesium.

4 William Thomson, better known as Lord Kelvin, born on June 26, 1824, in Belfast (Northern Ireland), who lived until December 17, 1907, was a physicist best known for his works on thermodynamics.

1.2.4. *Definition of the ampere adopted in 1948*

The "ampere"[5] is the intensity of a constant current that is produced between two parallel conductors following straight lines of infinite length and of a negligible circular cross-section placed at a distance of 1 meter from each other in a vacuum between these conductors, a force equal to 2.10^{-7} newtons[6] per meter of length. As a result, the magnetic constant, also known as vacuum permeability, is equal to exactly $4\pi.10^{-7}$ Henrys[7] per meter ($4\pi.10^{-7}$ H/m). The Ampere is linked to the elementary charge (e), the electric charge of a proton. The mole, the unit for quantities of matter, used mainly in chemistry, is defined directly through the determining of Avogadro's number (AN)[8].

1.2.5. *Definition of Kelvin adopted in 1967*

The Kelvin, a unit of thermodynamic temperature, is the fraction (1/273.16) of the thermodynamic temperature of the triple point of water. As a result, the thermodynamic temperature of the triple point of water is exactly 273.16 Kelvin (273.16 K). Other changes that were made: the Kelvin scale, measured based on water, was redefined based on the Boltzmann[9] constant (k), related to the measurement of the thermal agitation of the fundamental constituents of a body.

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5 André-Marie Ampère, born in Lyon (France) on January 20, 1775, and who lived until June 10, 1836, was a mathematician, physicist, chemist and philosopher. He contributed to the development of mathematics by introducing it into physics. He made important discoveries in the field of electromagnetism.

6 Isaac Newton, born on January 4, 1643, in Woolsthorpe (England), and who lived until March 31, 1727, was a philosopher, mathematician, physicist, alchemist, astronomer and theologian. His book, *Philosophiæ Naturalis Principia Mathematica*, published in 1687, is considered one of the greatest works in the history of science.

7 Joseph Henry, born on December 17, 1797, in Albany (New York) and who lived until May 13, 1878, was a physicist who discovered self-induction and the principle of electromagnetic induction from induced currents.

8 Lorenzo Romano Amedeo Carlo Avogadro, Count of Quaregna and Cerreto, known as Amedeo Avogadro, was a physician and chemist, born in Turin (Italy) on August 9, 1776, and who lived until July 9, 1856.

9 Ludwig Eduard Boltzmann, born on February 20, 1844, in Vienna (Austria) and who lived until September 5, 1906, was a physician and philosopher, considered to be the father of statistical physics, and an avid proponent of the existence of atoms.

1.2.6. *Definition of a mole*

The mole[10] is the amount of matter in a system containing as many indivisible entities as there are atoms in 0.012 kilograms of carbon 12; the symbol for this unit is "mol". When using the mole, the individual entities must be specified, and may be atoms, molecules, ions, electrons, other particles, or specified groups of such particles. In this definition, it is understood that this refers to unlinked carbon 12 atoms, at rest, and in their base state. As a result, the molar mass of carbon 12 is equal to 0.012 kilograms per mole (0.012 g/mol).

1.2.7. *Definition of the candela adopted in 1979*

The Candela[11] measures the luminous intensity in a given direction of a source that emits energy in monochromatic rays in that direction; it is 1/6, 831ths of a watt[12] per steradian[13]. From this, it can be determined that the spectral luminous efficiency of monochromatic rays of a frequency of $540 x 10^{12}$ hertz is equal to 683 lumens[14] per watt (683 lm/W = 683 Sr/W).

1.3. Additional quantities and SI derived quantities

Table 1.2 gives the two additional quantities that have been introduced to ensure the coherence of the system.

Table 1.3 shows the derived quantities in the International System generally used in fluid mechanics and heat transfer.

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10 The name of the unit of quantities of matter, the mole, originates from the abbreviation of the German word *Molekül*: "Mole" (with a capital "m", and followed by a rapidly abandoned point) was proposed in 1893 by chemist Wilhem Ostwald to indicate "the weight in grams numerically equal to the molecular weight of a given substance". The term "Mole" first appeared in 1902 as the English equivalent of the German term.

11 The *candela* (abbreviated as "cd", from the Latin word meaning "candle") is one of the seven basic SI units.

12 James Watt, born on January 19, 1736, in Greenock (Scotland) and who lived until August 25, 1819, in Heathfield Hall (in his house in Handsworth, England), was an engineer whose improvements to the steam engine were one of the key advances in the industrial revolution.

13 The Steradian (symbol: "sr") is the SI derived unit for the measurement of solid angles.

14 The *lumen* (from the Latin word for "light") is the SI derived unit for luminous flux.

Quantity	Name	Symbol	Dimension
Angle of a plane	Radian	Rad	–
Solid angle	Steradian	sr	(Ω)

Table 1.2. *SI derived quantities*

Quantities	Unit	Name	Dimension
Speed	m/s	Meter per second	LT^{-1}
Angular velocity	rd/s	Radian per second	T^{-1}
Acceleration	m/s^2	Meter per second squared (meter per second per second)	LT^{-2}
Force	N	Newton	MLT^{-2}
Moment of force	N. m	Newton-meter	ML^2T^{-2}
Surface tension	N/m	Newton per meter	MT^{-2}
Work, energy	J	Joule	ML^2T^{-2}
Power	W	Watt	ML^2T^{-3}
Pressure	Pa	Pascal	$ML^{-1}T^{-2}$
Quantity of movement	N.s	Newton-second	MLT^{-1}
Dynamic viscosity	Pa.s	Pascal-second	$ML^{-1}T^{-2}$
Kinematic viscosity	m^2/s	Square meter per second	L^2T^{-1}
Mass	kg	Kilogram	M
Density	Kg/m^3	Kilogram per cubic meter	ML^{-3}
Volumetric flow	m^3/s	Cubic meter per second	L^3T^{-1}
Mass flow rate	Kg/s	Kilogram per second	ML^{-1}
Time	s	Second	T
Surface area	m^2	Square meter	L^2
Volume	m^3	Cubic meter	L^3
Temperature	K	Kelvin	Q
Thermal capacity	J/K	Joule per kelvin	$L^2T^2 Q^{-1}$
Thermal conductivity	W/m.K	Watt per meter-kelvin	$MLT^3 Q^{-1}$
Thermal convection	W/m^2K	Watt per square meter-kelvin	$MT^{-3} Q^{-1}$

Table 1.3. *Some of the SI derived quantities used in fluid mechanics and heat transfer*

1.4. Rules for the use of units

1.4.1. *Unit name*

All names of units, even those derived from the names of noteworthy figures, are considered as common nouns: pascal, newton, hertz, volt[15], ampere, henry, weber[16], watt, joule, pascal[17], newton, hertz, coulomb[18], etc.

The initials are written in lowercase, and are given in plural if they apply to a number greater than or equal to 2. For example: 1.9 volt, 3 amps, 1.4 newton, 5 watts, 3 henrys, etc.

The names of the units derived from the names of notable scientific figures are masculine in gender (one joule, one Ampere, one henry).

No qualifiers should be added to a unit name (the term “linear meter” is not used).

When one quantity is the product of two others, and neither of them is a quotient, the name of the unit is obtained by joining the two corresponding units with a hyphen (making sure not to use a forward slash, which indicates a quotient). For example, electrical energy that is the product of power and time can be expressed as a “watt-hour”. In the case of very common units, the two names can be combined: a “watthour” or a watt-hour. The plural mark is to be added to the two component names in the case of a hyphen and to the last one in the case of contiguous nouns: watt-hours or watthours, meters-newtons.

When one quantity is the quotient of two others which are not quotients themselves, the name is obtained by placing the word “per” (and not the

15 The name “Volt” was created in tribute to Alessandro Volta, an Italian inventor who invented the voltaic pile (battery) in 1800.

16 Wilhelm Eduard Weber, born on October 24, 1804, in Wittemberg (Germany) and who lived until June 23, 1891, was a physicist who developed an original theory of electromagnetic interaction.

17 Blaise Pascal, born on June 19, 1623, in Clermont-Ferrand, and who lived until 1662, was a mathematician, physicist, inventor, philosopher, moral thinker and theologian.

18 Charles-Augustin Coulomb, born on June 14, 1736, in Angoulême, and who lived until August 23, 1806, was an officer, engineer and physicist who is the namesake for the unit for electric charge in SI.

symbol for division) between the units of the dividend and those of the divisor: kilometer per hour, meter per second.

1.4.2. *Unit symbols*

Following a number, a unit name can be replaced by its symbol: we may write 5 meters or 5 m, but it is necessary to write five meters (and not "five m").

A unit symbol should not be changed; in particular, it should never be given in plural: 15 kg (and not "15 kgs") and it should not be followed by a dot, except at the end of a sentence. A symbol must be placed after the numerical results in the case of decimal units: 26.3m (not "26m3"). This rule does not apply to units that are not decimal units: 12hrs, 15m, 30s.

1.4.3. *Compound symbols*

In the case of compound units, the symbol is represented by an algebraic expression in which each of the symbols plays the same role as the magnitude defining the equation; thus in the case of a product (vector or scalar) or quotient, the symbol is the product (vector or scalar) or the quotient of the symbols of the compound units. For example, the watt-hour is represented by the symbol (Wh) for the unit of electrical energy.

A velocity, the quotient of length and time, can be expressed with a unit whose symbol is m÷s (horizontal bar) or m/s (forward slash) or ($m.s^{-1}$).

Since the moment of a force is numerically equal to the vector product of the intensity of a force and a length, the symbol of the unit of moment can be written (m∧N or N.m). It should be noted that, according to ISO 31, the point between the two symbols should be at the midpoint of the height of the line of text: N·m.

The expression obtained can be transformed by applying the rules of algebra, with electrical resistance being defined by the expression (RS/l) where R is a resistance, S a surface and l a length, and the corresponding unit has the symbol ($W.m^2/m$) or (W.m) after simplification. Positive or negative exponents can also be used in these expressions: density is the quotient of a mass and a volume; the unit uses the symbol (kg/m^3 or $kg.m^{-3}$).

1.5. Exercises

1.5.1. *Exercise 1: calculation of dimensions*

1) Write the dimension of force.

2) Write the dimension of energy.

3) Write the dimension of a pressure (force per unit area).

4) Show that a measurement of pressure is one of volumetric energy. Can you find a formula that translates this result?

5) Write the dimension of the expression $(\rho V^2/2)$ where ρ is a density and V is the velocity of a fluid. Compare with the dimension of a pressure.

6) Write the dimension of ρgz with g as the acceleration from gravity z as the side. What do you conclude from this?

7) Can you give a physical formula that translates questions 4), 5) and 6)?

8) Check the homogeneity of the equation $V = \sqrt{2gh}$, where h is a height. What do you think the result expresses?

9) In the following relationship $\tau = \mu \frac{dV}{dy}$, τ is the shear stress, μ is the dynamic viscosity and $\frac{dV}{dy}$ is the speed gradient. Determine the dimension of the dynamic viscosity μ.

1.5.1.1. *Solutions*

1) The force is given by $F = m\gamma$, and its dimension is $[F] = [m][\gamma]$, with m as mass and γ as acceleration.

$$\begin{cases} [\gamma] = \left[\dfrac{\text{speed}}{\text{time}}\right] = \dfrac{LT^{-}}{T} = LT^{-2} \\ [\text{mass}] = M \end{cases} \Rightarrow [F] = MLT^{-2}$$

2) The dimension of a measure of energy E is:

$$E = \underbrace{F}_{\text{force}} \times \underbrace{l}_{\text{lenght}}$$

$$[E]=[F][l]$$

$$\begin{cases}[F]=MLT^{-2}\\ [l]=L\end{cases} \Rightarrow [F]=ML^2T^{-2}$$

3) The dimension of pressure P is:

$$P=\frac{\text{force}}{\text{unit area}}=\frac{F}{S}$$

$$\begin{cases}[F]=MLT^{-2}\\ [S]=L^2\end{cases} \Rightarrow [P]=\frac{[F]}{[S]}=\frac{MLT^{-2}}{L^2}=ML^{-1}T^{-2}$$

4) the dimension of the relationship $\dfrac{\text{Energy}}{\text{volume}}=\dfrac{E}{v}$ is:

$$\left[\frac{E}{v}\right]=\frac{[E]}{[v]}=\frac{ML^2T^{-2}}{L^3}=ML^{-1}T^{-2}$$

COMMENT 1.1.– *The dimension of the relationship between energy and volume is that of a pressure. Pressure therefore represents a measure of energy per unit of volume, which represents one of the three forms of the Bernoulli equation expressed in joules per unit of volume. We recall that the Bernoulli equation expresses the conservation of total mechanical energy (kinetic energy + potential energy + pressure energy).*

For formulas translating the same result, we can cite the following: the manometric (or effective) pressure P_{man} *at a depth with respect to a free surface subjected to atmospheric pressure (taken as equal to zero) is* $P_{man}=\rho g h$.

With $[\rho] = ML^{-3}$, $[g] = LT^{-2}$ *and* $[h] = L$, *the dimension of the term* $P_{man} = \rho gh$ *is:*

$$[\rho gh] = ML^{-1}T^{-2}$$

The unit for this is:

$$\frac{kg}{ms^2} = \frac{kgm^2}{m^3s^2} = kg\frac{m}{s^2}\frac{m}{m^2} = \frac{Nm}{m^3} = \frac{\text{Joule}}{\text{unit volume}}$$

We can also give the expression of the amount of work exchanged with a perfect gas.

5) The term $\rho\frac{V^2}{2}$ has the dimension.

We recall that the dimension of a constant is equal to the unit.

The dimension $\rho\frac{V^2}{2}$ of the relationship is that of a pressure, and is called dynamic pressure.

6) The dimension of the term $\rho g z$, with z as the side.

$$\begin{cases} [\rho] = \dfrac{M}{L^3} \\ [g] = \dfrac{L}{T^2} \\ [z] = [\text{lenght}] \end{cases} \Rightarrow [\rho gz] = [\rho][g][z] = ML^{-1}T^{-2}$$

The dimension of the term $\rho g z$ is that of a pressure; it is referred to as potential energy of gravity or static pressure.

7) The relationship that translates questions 4), 5) and 6) is the total conservation of mechanical energy or Daniel Bernoulli's equation[19]:

$$P_{Total} = \underbrace{P_{static}}_{\substack{\text{energy from}\\ \text{pressure forces}}} + \underbrace{\rho g z}_{\substack{\text{potenial energy}\\ \text{of gravity}}} + \underbrace{\rho \frac{V^2}{2}}_{\substack{\text{kinetic energy or}\\ \text{dynamic pressure}}} = \text{constant}$$

This relationship represents the total pressure P_{Total}.

8) The homogeneity of the equation is $V = \sqrt{2gh}$.

The dimension of speed is $[V] = LT^{-1}$.

The dimension is $\sqrt{2gh}$.

$$\left[\sqrt{2gh}\right] = [2gh]^{1/2} = \underbrace{[2]^{1/2}}_{=1} [g]^{1/2} [h]^{1/2} = (LT^{-2})^{1/2} (L)^{1/2} = LT^{-1}$$

The equation $V = \sqrt{2gh}$ is therefore dimensionally homogeneous.

9) The dimension of dynamic viscosity is:

$$[\mu] = \left[\frac{\tau}{dV/dy}\right] = \frac{[\tau]}{[dV/dy]} = \frac{[\tau]}{[dV]/[dy]}$$

$$[\tau] = \frac{[\text{tangential force}]}{[\text{unit area}]} = \frac{MLT^{-2}}{L^2} = ML^{-1}T^{-1}$$

$$[dV] = [\text{speed}] = LT^{-1}$$

$$[dy] = [\text{length}] = L$$

19 Daniel Bernoulli, born in Groningen (Netherlands) on February 8, 1700, and who lived until March 17, 1782, was a doctor, physicist and mathematician. He formulated the fundamental theorem of fluid mechanics which bears his name, Bernoulli's theorem, in his book *Hydrodynamica* (1738).

And therefore:

$$[\mu] = ML^{-1}T^{-1}$$

COMMENT 1.2.– *From questions 5), 6) and 7), assuming that the side is zero, we obtain the relationship giving the dynamic pressure*:

$$\rho\frac{V^2}{2} = P_{Total} - P_{static}$$

which determines the velocity of the flow, or:

$$V = \sqrt{2\frac{P_{Total} - P_{static}}{\rho}}$$

In practice, the total pressure and the static pressure are measured using a Pitot tube[20]*, or separately using piezometric tubes.*

In question 8), the relationship $V = \sqrt{2gh}$ *is called Torricelli relationship*[21]*, and is obtained by applying the Bernoulli equation (with respect to a horizontal reference plane HRP) between the point (1) belonging to the free surface of a liquid contained in a reservoir and the point (2) belonging to an orifice located at a given depth below the free surface (see Figure 1.1), which is*:

$$\underbrace{\frac{V_1^2}{2g}}_{=\,0\text{, since the reservoir has large dimensions}} + \underbrace{\frac{P_1}{\rho g}}_{P_1 = P_{atm}} + \underbrace{z_1}_{=\,h_1} = \frac{V_2^2}{2g} + \underbrace{\frac{P_2}{\rho g}}_{P_2 = P_{atm}} + \underbrace{z_2}_{=\,0\text{, since the horizontal references passes through point }(2)}$$

The orifice is either at the bottom of the container (see Figure 1.1) or on its side on the surface. The drop in pressure is negligible between points (1) and (2). We obtain $V = \sqrt{2gh}$.

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20 Henri Pitot, born in Aramon on May 29, 1695, and who lived until December 27, 1771, was a hydraulic engineer and inventor of the Pitot tube, used to measure the speed of fluids.

21 Evangelista Torricelli, born on October 15, 1608, in Faenza (Italy) and who lived until October 25, 1647, was a physicist and mathematician known in particular for inventing the barometer.

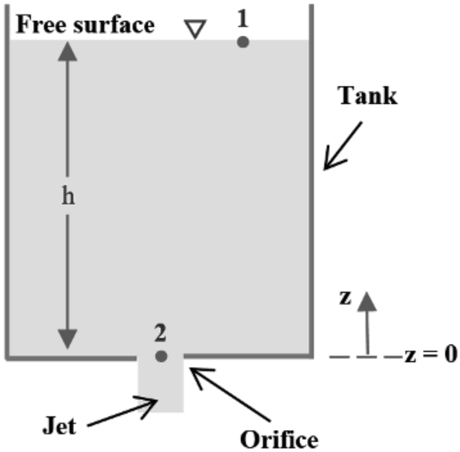

Figure 1.1. *Flow through an orifice, establishing of the Torricelli relationship; HRP: horizontal reference plane. For a color version of this figure, see www.iste.co.uk/sadchemloul/mechanics.zip*

COMMENT 1.3.– *For question 9), relationship* $\tau = \mu \frac{dV}{dr}$ *or Newton's relationship is obtained from the experiment known as "movable wall experiment" given in Figure 1.2.*

Or a fluid located between two horizontal planes separated by a height h. The lower plane is set and the upper plane is movable (in uniform movement) and moves at the speed V_0 *thereby causing the fluid to move.*

Several experiments were carried out using different geometries and fluids of different natures. The results of these experiments show that:

– *the relationship* $\frac{F}{S}$ *is proportional to the relationship* $\frac{V_0}{h}$*;*

– *the proportionality factor depends only on the nature of the fluid*;

– *the relationship between the different physical properties involved in this experiment is written as* $\frac{F}{S} \propto \frac{V_0}{h}$ or $\frac{F}{S} = k\frac{V_0}{h}$;

– *the relationship* $\frac{F}{S}$ *represents the tangential shear stress* $\tau = \frac{F}{S}$. *The triangle formed by* V_0 *and h, and the one formed by dV and dy, are similar right-angled triangles, which makes it possible to write* $\frac{V_0}{h} = \frac{dV}{dy}$ *;*

– *the proportionality factor* k *depends only on the nature of the fluid and represents dynamic viscosity, which is a thermo-physical property of the fluid; it depends more on temperature than on pressure. The relationship* $\frac{dV}{dy}$ *is called the speed gradient.*

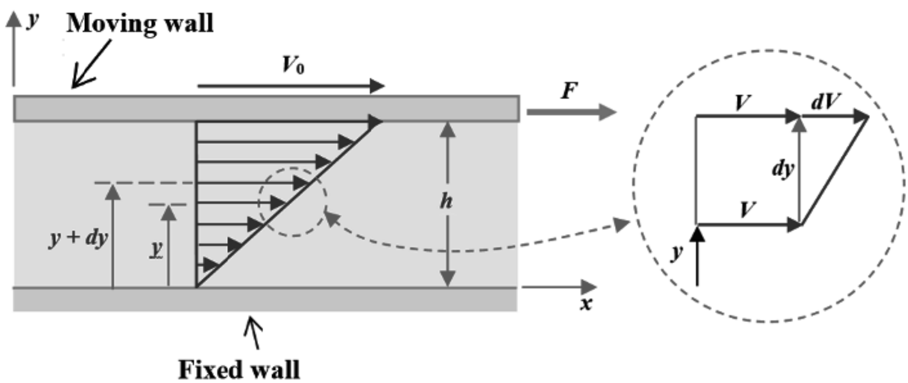

Figure 1.2. *Newton's experiment known as the "moving wall". For a color version of this figure, see www.iste.co.uk/sadchemloul/mechanics.zip*

1.5.2. *Exercise 2: homogeneity of relationships*

1) Check the homogeneity of the relationship $f = \frac{1}{2\pi}\sqrt{\frac{k_r}{m}}$ representing the oscillation frequency f of a solid-spring system, with m the mass of the solid and k_r the spring stiffness constant. The restoring force $\vec{F}$ relates to the extension $\overrightarrow{\Delta l}$ with the relationship $\vec{F} = k_r.\overrightarrow{\Delta l}$.

2) Are the following relationships valid dimensionally? Make a dimensional analysis to confirm or correct this.

a) $F = \frac{Gm}{r}$ like F is a force, G is a constant expressed in $\frac{m^3}{kgs^2}$, m is a mass and r is a distance.

b) $P = \rho g h_1 + h_2 F$ like P is a pressure, g is the acceleration due to gravity, h_1 and h_2 are heights and F force.

c) $\theta = \frac{b\sin(a)}{l\sin(c)}$, l, and c represent lengths.

3) Which of these relationships are homogeneous?

a) $t = 2\pi\sqrt{\frac{l}{g}}$, $t = 2\pi\sqrt{\frac{g}{l}}$, $t = \frac{1}{2\pi}\sqrt{\frac{l}{g}}$, $t = \frac{1}{2\pi}\sqrt{\frac{l+g}{lg}}$

with t as time, l the length and g acceleration from gravity.

b) $E = \sqrt{q_{mv}c^2 + mc^4}$, $E^2 - \frac{q_{mv}^2 c^2}{m} = m^4$, $E^2 = q_{mv}^2 c^2 + m^2 c^4$

with E as energy, q_{mv} the amount of motion, c the speed of light and m the mass.

c) $ma\left(\frac{d\alpha}{dt}\right)^2 = F_N + mg(\sin\alpha) + k_r a(\cos^2\alpha)$

d) $a\left(\frac{d\alpha}{dt}\right)^2 = F_N + g(\sin\alpha) + k_r a(\cos^2\alpha)$

e) $ma\left(\frac{d^2\alpha}{dt^2}\right) = F_N + mg(\sin\alpha) + k_r a(\cos^2\alpha)$

f) $ma\left(\frac{d\alpha}{dt}\right)^2 = F_N + mg(\sin\alpha) + k_r a(\cos\alpha)$

with a being the length, F_N the force, k_r spring stiffness constant, m the mass, t the time and α the angle.

1.5.2.1. *Solutions*

1) Ratio $f = \frac{1}{2\pi}\sqrt{\frac{k_r}{m}}$.

The dimension of the frequency f is $[f] = \left[\frac{1}{\text{time}}\right] = T^{-1}$ and the dimension of the mass is $[m] = M$.

The dimension of the spring stiffness constant k_r is therefore:

$$[k_r] = \frac{[\text{force}]}{[\text{spring extension}]} = \frac{[F]}{[\Delta l]} = MT^{-2}$$

The ratio $\sqrt{\frac{k_r}{m}}$ has the dimension:

$$\left[\frac{k_r}{m}\right]^{1/2} = \frac{[k_r]^{1/2}}{[m]^{1/2}} = \frac{(MT^{-2})^{1/2}}{M^{1/2}} = T^{-1}$$

We recall that the dimension of the constant 2π is equal to the unit.

The relationship $f = \frac{1}{2\pi}\sqrt{\frac{k_r}{m}}$ is therefore homogeneous from the point of view of the dimension.

2) a) The relationship $F = \frac{Gm}{r}$.

The dimension of the force F is $[F] = MLT^{-2}$.

The dimension of G is $[G] = \frac{[\text{lenght}]^3}{[\text{mass}][\text{time}]^2} = M^{-1}L^3T^{-2}$.

The dimension of the relationship $\frac{Gm}{r}$ is:

$$\left[\frac{Gm}{r}\right]=\frac{[G][m]}{[r]}=\frac{M^{-1}L^3T^{-2}M}{L}=L^2T^{-2}$$

which corresponds to the dimension of the square of the speed.

The formula $F=\frac{Gm}{r}$ is not valid dimensionally.

b) relationship $P=\rho gh_1+h_2F$.

The dimension of the pressure is $[P]=ML^{-1}T^{-2}$.

The dimension of the term ρgh_1 is:

$$\left[\rho gh_1\right]=[\rho][g]\left[h_1\right]=ML^{-3}LT^{-2}L=ML^{-1}T^{-2}$$

The dimension of the term h_2F is:

$$\left[h_2F\right]=\left[h_2\right][F]=LMLT^{-2}=ML^2T^{-2}$$

We note here that the terms of the sum ρgh_1+h_2F do not have the same dimension, and therefore the relationship of $P=\rho gh_1+h_2F$ is not valid dimensionally.

c) Ratio $\theta=\frac{b\sin(a)}{l\sin(c)}$.

Since the constants b and l represent lengths, the relationship $\frac{b\sin(a)}{l\sin(c)}$ has no dimension, and since the angle θ has no dimension, then the relationship $\theta=\frac{b\sin(a)}{l\sin(c)}$ is valid dimensionally.

3) a) Ratios $t = 2\pi\sqrt{\frac{l}{g}}$, $t = 2\pi\sqrt{\frac{g}{l}}$, $t = \frac{1}{2\pi}\sqrt{\frac{l}{g}}$, $t = \frac{1}{2\pi}\sqrt{\frac{l+g}{lg}}$.

$$\begin{cases} [t] = T \\ \left[2\pi\sqrt{\frac{l}{g}}\right] = T \end{cases}$$

The relationship $t = 2\pi\sqrt{\frac{l}{g}}$ is homogeneous.

$$\begin{cases} [t] = T \\ \left[2\pi\sqrt{\frac{g}{l}}\right] = T^{-1} \end{cases}$$

The relationship $t = 2\pi\sqrt{\frac{g}{l}}$ is not homogeneous.

$$\begin{cases} [t] = T \\ \left[\frac{1}{2\pi}\sqrt{\frac{l}{g}}\right] = T \end{cases}$$

The relationship $t = \frac{1}{2\pi}\sqrt{\frac{l}{g}}$ is homogeneous.

$$t = \frac{1}{2\pi}\sqrt{\frac{l+g}{lg}}$$

This relationship is not homogeneous, because the terms of a sum must be homogeneous, however l and g do not have the same dimension.

b) $E = \sqrt{q_{mv}c^2 + mc^4}$, $E^2 - \frac{q_{mv}^2 c^2}{m} = m^4$, $E^2 = q_{mv}^2 c^2 + m^2 c^4$.

relationship $E=\sqrt{q_{mv}c^2+mc^4}$:

$$\begin{cases}\left[q_{mv}c^2\right]=\left[q_{mv}\right]\left[c^2\right]=\left[mV\right]\left[c^2\right]=[m][V][c]^2=M(LT^{-1})^3\\ \left[mc^4\right]=[m]\left[c^4\right]=[m][c]^4=M(LT^{-1})^4\end{cases}$$

with V being the speed of the mass m.

Since the terms of the sum $q_{mv}c^2+mc^4$ do not have the same dimension, the relationship $E=\sqrt{q_{mv}c^2+mc^4}$ is not homogeneous.

Relationship $E^2-\dfrac{q_{mv}^2c^2}{m}=m^4$:

$$\left[\frac{q_{mv}^2c^2}{m}\right]=\frac{\left[q_{mv}^2c^2\right]}{[m]}=\frac{\left[q_{mv}\right]^2[c]^2}{[m]}=\frac{\left[mV\right]^2[c]^2}{[m]}=\frac{(MLT^{-1})^2(LT^{-1})^2}{M}=ML^4T^{-4}$$

$$\left[E^2\right]=M^2LT^{-4}$$

$$\left[m^4\right]=M^4$$

with V being the speed of the mass m. The components of this relationship do not have the same dimension, the relationship $E^2-\dfrac{q_{mv}^2c^2}{m}=m^4$ is not homogeneous.

Relationship $E^2=q_{mv}^2c^2+m^2c^4$:

$$\begin{cases}\left[E^2\right]=M^2L^4T^{-4}\\ \left[q_{mv}^2c^2\right]=M^2L^4T^{-4}\\ \left[m^2c^4\right]=M^2L^4T^{-4}\end{cases}$$

with V being the speed of the mass m.

The relationship $E^2 = q_{mv}^2 c^2 + m^2 c^4$ is homogeneous, because its terms have the same dimension.

c) Relationship $ma\left(\frac{d\alpha}{dt}\right)^2 = F_N + mg(sin\alpha) + ka(cos^2\alpha)$:

$$\begin{cases} \left[ma\left(\frac{d\alpha}{dt}\right)^2\right] = [ma]\left[\left(\frac{d\alpha}{dt}\right)^2\right] = [m][a]\left[\frac{d\alpha}{dt}\right] = MLT^{-2} \\ [F_N] = MLT^{-2} \\ [mg(sin\alpha)] = [m][g] = MLT^{-2} \\ ka(cos^2\alpha) = [k][a] = MT^{-2}L = MLT^{-2} \end{cases}$$

The relationship $ma\left(\frac{d\alpha}{dt}\right)^2 = F_N + mg(sin\alpha) + ka(cos^2\alpha)$ is homogeneous, because all its terms have the same dimension.

d) $a\left(\frac{d\alpha}{dt}\right)^2 = F_N + g(sin\alpha) + ka(cos^2\alpha)$:

$$\begin{cases} \left[a\left(\frac{d\alpha}{dt}\right)^2\right] = [a]\left[\frac{d\alpha}{dt}\right]^2 = LT^{-2} \\ [F_N] = MLT^{-2} \\ [g(sin\alpha)] = LT^{-2} \\ \left[ka(cos^2\alpha)\right] = MLT^{-2} \end{cases}$$

The relationship $a\left(\frac{d\alpha}{dt}\right)^2 = F_N + g(sin\alpha) + ka(\cos^2\alpha)$ is not homogeneous, because its terms do not have the same dimension.

e) $ma\left(\dfrac{d^2\alpha}{dt^2}\right) = F_N + mg(sin\alpha) + ka(cos^2\alpha)$:

$$\begin{cases} \left[ma\dfrac{d^2\alpha}{dt^2}\right] = MLT^{-2} \\ [F_N] = MLT^{-2} \\ [mg(sin\alpha)] = MLT^{-2} \\ \left[ka(cos^2\alpha)\right] = MLT^{-2} \end{cases}$$

The relationship $ma\left(\dfrac{d^2\alpha}{dt^2}\right) = F_N + mg(\sin\alpha) + ka(cos^2\alpha)$ is homogeneous, because its terms have the same dimension.

f) With regard to case 3c), it can easily be verified that the relationship:

$$ma\left(\frac{d\alpha}{dt}\right)^2 = F_N + mg(sin\alpha) + ka(cos\alpha)$$

is homogeneous.

For the four ratios 3c) to 3f), it must be remembered that the dimension of an angle is equal to the unit.

1.5.3. *Exercise 3: dimension of the constants of an equation*

We express the speed of a body by the equation $V = At^3 - Bt$ where t represents time.

1) What are the SI units of A and B ?

2) Give the SI units of A, B and C of the equation $V = At^2 - Bt + \sqrt{C}$, where V is the speed and t is the time.

1.5.3.1. *Solutions*

1) For the equation to be dimensionally valid, the terms V , At^3 , and Bt have the same dimension, which must be that of a velocity.

The dimension of A is:

$$[A] = \left[\frac{\text{speed}}{t^3}\right] = \frac{[\text{speed}]}{\left[t^3\right]} = LT^{-4}$$

and its unit is $m.s^{-4}$.

The dimension of B is:

$$[B] = \left[\frac{\text{speed}}{t}\right] = \frac{[\text{speed}]}{[t]} = LT^{-2}$$

and its unit is $m.s^{-2}$, which is that of acceleration.

2) In the same way as 1), we find:

– $[A] = LT^{-3}$ and its unit is $m.s^{-3}$;

– $[B] = LT^{-2}$ and its unit is $m.s^{-2}$;

– $\left[C^{1/2}\right] = [\text{speed}]$, where $[C] = [\text{speed}]^2 = L^2T^{-2}$ and its unit is $m.s^{-2}$.

1.5.4. *Exercise 4: equation for perfect gases*

The equation for perfect gases applying to moles is written as $Pv_m = RT$, where P is the pressure, v_m is the molar volume, R is the constant for perfect gases is T and the thermodynamic temperature of the gases.

1) Give the equation for the dimensions of the molar constant of the perfect gases.

2) Knowing that the normal molar volume value is $v_m = 22{,}414\ liters/mole$, calculate R using the International System of units.

1.5.4.1. *Solutions*

1) The equation of perfect gases is $Pv_m = RT$, with n being the number of moles, which gives us $v_m = \frac{v}{n}$, which is expressed in $liters/minute$ or $m^3/mole$.

With $[R]$ being the size of the constant of perfect gases R and θ the dimension of the temperature T, thus $[P] = ML^{-1}T^{-2}$, $[v_m] = L^3 mole^{-1}$, and $[T] = \theta$.

Therefore, we can write:

$$[R] = [P][v_m][T]^{-1} = ML^2T^{-2}mole^{-1}\theta^{-1}$$

and the unit of the perfect Gas Constant R is therefore $kg.m^2s^{-2}mole^{-1}K^{-1}$.

2) The value of the perfect gas constant is:

$$R = \frac{Pv_m}{T} = \frac{101325 \times 22.414 \times 10^{-3}}{273.5} = 8.314473\ JK^{-1}mole^{-1}$$

1.5.5. ***Exercise 5: unit conversions***

1) Convert a force of $1\,N$, an acceleration of $1\,ms^{-2}$ and a dynamic viscosity of $1\,kgm^{-1}s^{-1}$ to CGS units.

2) Convert a pressure of 10 hPa into dynes (CGS units).

1.5.5.1. *Solutions*

1) The dimension of a force F is:

$$[F] = [\text{mass} \times \text{acceleration}] = [\text{mass}] \times [\text{acceleration}] = MLT^{-2}$$

If we consider both systems of units (SI for index 1 and CGS for index 2), we can write:

$$\frac{[F_1]}{[F_2]} = \frac{M_1}{M_2}\frac{L_1}{L_2}\left(\frac{T_1}{T_2}\right)^{-2}$$

The relationship between the units of force taken in two systems is:

$$\frac{\text{unit } SI}{\text{unit } CGS} : \frac{F_1}{F_2} = \frac{1N}{1gcms^{-2}} = \frac{1kg}{1g}\frac{1m}{1cm}\frac{1s^{-2}}{1s^{-2}} = 10^3 \times 10^2 \times 1 = 10^5$$

So, we obtain:

$$1\,\text{N} = 10^5\,\text{gcms}^{-2} = 10\,\text{dyne}$$

The dyne is defined as the force required to accelerate a mass of one gram by $1\,gal$[22] (that is, $1\,cms^{-2}$), or $1\,gcms^{-2}$. A dyne has the value of exactly 10^{-5} newtons.

The dimension of an acceleration value is:

$$[\text{acceleration}] = [\gamma] = LT^{-2}$$

As for acceleration, we can write it as:

$$\frac{[\gamma_1]}{[\gamma_2]} = \frac{L_1}{L_2}\left(\frac{T_1}{T_2}\right)^{-2}$$

The relationship between the units of acceleration taken in two systems is:

$$\frac{\text{unit } SI}{\text{unit } CGS} : \frac{[\gamma_1]}{[\gamma_2]} = \frac{1\,ms^{-2}}{1\,cms^{-2}} = \frac{1m}{1cm}\frac{1s^{-2}}{1s^{-2}} = \times 10^2 \times 1 = 10^2$$

22 The gal (symbol: Gal) is a CGS unit of acceleration equal to 1 cm/s^2 = 0.01 m/s^2, used to express the acceleration from gravity in geodesy and geophysics.

So, we obtain:

$$1\ ms^{-2} = 10^2\ cms^{-2} = 1\ gal$$

The dimension of dynamic viscosity is:

$$[\mu] = \frac{[\text{tangential stress}]}{[\text{velocity gradient}]} = \frac{[\tau]}{[dV/dy]}$$

$$[\tau] = \frac{[\text{force}]}{[\text{unit area}]} = \frac{MLT^{-2}}{L^2} = ML^{-1}T^{-2}$$

$$\left[\frac{dV}{dy}\right] = \frac{[dV]}{[dy]} = \frac{LT^{-1}}{L} = T^{-1}$$

$$[\mu] = \frac{[\tau]}{[dV/dr]} = ML^{-1}T^{-1}$$

And for force, we can write it as:

$$\frac{[\mu_1]}{[\mu_2]} = \frac{M_1}{M_2}\left(\frac{L_1}{L_2}\right)^{-1}\left(\frac{T_1}{T_2}\right)^{-1}$$

The relationship between the units of viscosity used in two systems is:

$$\frac{\text{unit } SI}{\text{unit } CGS} : \frac{[\mu_1]}{[\mu_2]} = \frac{1\,kgms^{-1}}{1gcms^{-1}} = \frac{1kg}{1g}\,\frac{1m^{-1}}{1cm^{-1}}\,\frac{1s^{-1}}{1s^{-1}} = 10^3 \times 10^{-2} \times 1 = 10$$

So, we obtain:

$$1\,kgms^{-1} = 1\text{Poiseuille} = 10\,gcms^{-1} = 10\,Po = 10\text{Poises}$$ [23]

$$1\,PI\ \text{(Poiseuille)} = 1\,Pa.s\ \text{(Pascal.second)}$$

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23 Jean-Léonard-Marie Poiseuille, born on April 22, 1797, in Paris and who lived until December 26, 1869, was a physicist and doctor who graduated from the École polytechnique (France's Polytechnic University) in 1815. He published several of his findings on the heart and the circulation of blood in vessels.

Poise is the CGS unit of dynamic viscosity, defined as the viscosity of a fluid for which a tangential stress of one dyne/cm^2 allows it to maintain a speed of 1 cm/s between two parallel planes separated by 1 cm of this liquid; a poise has the value of one dyne-second per square centimeter or a tenth of a Poiseuille.

2) The dimension of a measure of pressure is:

$$[P] = \frac{[\text{normal force}]}{[\text{unit area}]} = \frac{MLT^{-2}}{L^2} = ML^{-1}T^{-2}$$

If we consider both systems of units (SI for index 1 and CGS for index 2), we can write:

$$\frac{[P_1]}{[P_2]} = \frac{M_1}{M_2}\left(\frac{L_1}{L_2}\right)^{-1}\left(\frac{T_1}{T_2}\right)^{-2}$$

The relationship between the units of pressure taken in two systems is:

$$\frac{\text{unit } SI}{\text{unit } CGS} : \frac{[P_1]}{[P_2]} = \frac{1\,kgm^{-1}s^{-2}}{1\,gcm^{-1}s^{-2}} = \frac{1kg}{1g}\,\frac{1\,m^{-1}}{1\,cm^{-1}}\,\frac{1\,s^{-2}}{1\,s^{-2}} = 10^3 \times 10^{-2} \times 1 = 10$$

So, we obtain:

$$1\,Pa = 10\,gcm^{-1}s^{-2} = 10\underbrace{\frac{gcms^{-2}}{cm^2}}_{=1\,dyne/cm^2} = 10\frac{dyne}{cm^2}$$

$$1\,hPa = 10^2(10\,gcm^{-1}s^{-2}) = 10^2\left(10\frac{dyne}{cm^2}\right) = 10^3\frac{dyne}{cm^2}$$

The pascal (Pa) is an SI unit, but non-SI units may be used, such as $\frac{\text{dyne}}{\text{cm}^2}$ *which is called the barye*:

– $1 \text{ barye (ba)} = 1 \text{ dyne/cm}^2$ *and* $1 \text{ dyne/cm}^2 = 0.1\text{Pa}$;

– *The millimeter of mercury (mmHg) or centimeter of water (cmH_2O)*;

– *The torr*[24]: $1\ \text{torr} = 1\ \text{mmHg} = 1.98066 \times 13595 = 133.322\ \text{Pa}$;

– *Normal atmosphere*[25]*:* 1 atm = 101325 Pa;

– *The technical atmosphere*[26]*:* 1 atm = 98066 Pa;

– *The bar* : $1\ bar = 10^5\ Pa$[27].

The PSI is the pound (force per square inch), or pound-force per square inch, abbreviated as "PSI" or "lbf/in^2". In some cases, this unit is wrongly referred to as "pound per square inch" which is a unit of measurement of stress and pressure used in English speaking countries:

$$1psi = \frac{1\ lbf}{(1\ in)^2} = \frac{4.4482\ N}{(0.0254\ m)^2} = 6894.76 Pa$$

1 lbf = 1 pound-force, an English unit of force (in the International System of units, strength is measured in newtons) and 1 pound = 453.49237g. The value of the pound-force, with respect to the SI, is defined as the product of $0.45349237 \times 9.80665$.

24 The torr (symbol: Torr) or millimeter of mercury is a unit of measure for pressure. This unit is not part of the International System of units, in which the unit of pressure is the pascal. The Torr is an abbreviation of the name of Italian physicist and mathematician Evangelista Torricelli, inventor of the mercury column barometer.

25 The atmosphere, according to its primary definition, is the layer of gas that surrounds a planet. However, the term is used with several other meanings and expressions. The atmosphere contains different gases such as pollution or nitrogen. To measure pressure, for instance, we can measure Earth's atmosphere, which is the gaseous layer surrounding the Earth which is known as "air". Dry air consists of 78.08% nitrogen molecules, 20.95% oxygen molecules, 0.93% argon, 0.04% carbon dioxide and traces of other gases. The atmosphere protects life on Earth by absorbing ultraviolet solar radiation, warming the surface by retaining heat (known as the greenhouse effect) and reducing temperature differences between day and night.

26 The technical atmosphere (symbol: at) is an older unit of pressure, defined as the force exerted by a kilogram on a surface of one square centimeter: 1 normalized atmosphere = 1 kilogram-force per square centimeter (kgf/cm^2).

27 The term "Bar" has its origins in ancient Greek, referring to gravity. The bar has the advantage of being close to the atmosphere (average atmospheric pressure on the surface of the sea). Outside of the SI, this unit derives from the barye (1 bar = 10^6 baryes).