1

Measurement Systems Using Polarized Light

The Maxwell equations represent the physical phenomenon of light as an electromagnetic field which includes an electric field component and a magnetic field component oscillating in phase quadrature in a plane perpendicular to the direction of the light wave propagation. Photon wave–particle duality makes it possible to present both particle-like and wave-like behaviors. Interference is a phenomenon which is explained by the wave-like behavior of light. The wave-like properties of light and interference make it possible to study a number of physical phenomena, such as diffraction, interferometry, ellipsometry or holography. This approach has led to the development of innovative instruments using light–matter interaction and to various measuring systems. Mathematical approaches based on matrices are often used to calculate the effects of interference. These approaches use software such as MATLAB, MAPLE or computer programs written in Python, C language or Fortran. The finite element method can also be applied to design the instruments and simulate the expected observations.

1.1. Introduction

Chapters 3–6 of [DAH 16] present the theoretical characteristics of light. Light has dual properties; it is both a particle and a wave which is true also in the context of light–matter interaction. This chapter concentrates on applications through exercises or analysis of results of studies on the use of techniques using interference to study matter and materials.

The wave–particle duality in the case of a given particle (electron diffraction, neutron diffraction, etc.) is linked to the equation of L. de Broglie (1924) which relates the wavelength to the quantity of movement: $\lambda = h/p$, where p = mv, v being the speed and m the mass at rest. For a relativistic particle, with a speed close to that of light, the energy (E) of the particle is given by: $E^2 = p^2c^2+m^2c^4$. In the case of the photon which is a massless particle, $E = pc = hc/\lambda = h\nu$, because $c = \lambda\nu$, which leads to the equation used by Einstein (1905) to explain the photoelectric effect. When analyzed from quantum mechanics, the wave (p representation) or particle (E representation) behavior of the photon depends on the measurement or on the method used to detect light.

Optical instruments are essentially centered systems, the reference axis being the optical axis. To approach the homodyne and heterodyne interferometry techniques, the Handbury–Brown–Twiss experiment, the study of the Lyot Filter, Lidar, ellipsometry, spectroscopy based on polarized light and holography, the matrix method, presented in section 1.2 in the context of geometric optics and wave optics can be used. In wave optics, considering the two states of light polarization, Jones formalism makes it possible to model the effects of transmission plates using 2×2 matrices (Chapter 3 of [DAH 16]). The matrix formalism makes it possible to use digital tools, either by programming in advanced languages (Fortran, C, C ++, Visual Basic, Python, etc.) or by using, for example, software like MATLAB, SCILAB, MAPLE, COMSOL, ANSYS, NASTRAN and many others.

1.2. Matrix optics

In Chapter 3 of [DAH 16], the conditions for applying geometric optics to model the propagation of light are specified with respect to a characteristic length which is the wavelength related to the color of light. Under these conditions, the Snell–Descartes laws can be used (Figure 1.1, $n_1 \sin i_1 = n_2 \sin i_2$ and the same planes of incidence, reflection and refraction) to obtain the light ray propagation when it crosses optical elements: the diopter, the prism, convex and concave lenses and mirrors.

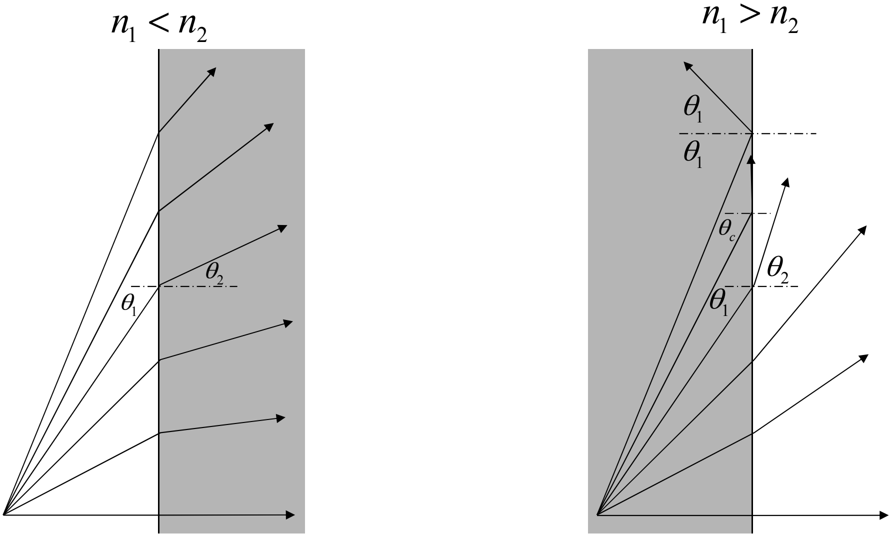


Figure 1.1. *Crossing of a plane diopter for $n_1<n_2$ and $n_1>n_2$. For a color version of this figure, see www.iste.co.uk/dahoo/metrology2.zip*

In Figure 1.1, the angles of incidence θ_1 and θ_2 verify the Snell–Descartes equation. When $n_1 > n_2$, there is the possibility of internal reflection when the angle of incidence is greater than the critical angle θ_c ($\sin \theta_c = n_2/n_1$).

A matrix approach can be applied to determine the trajectory of a light beam in the context of geometric optics. Usually, the optical systems used in instrumentation are axially symmetrical and only paraxial rays are considered which remain in the vicinity of the optical axis. Transfer matrices are used to locate the position of a ray as it propagates from one point to another. A ray of light can be defined by two coordinates: its position or height, r and its slope, $\theta = \partial r/\partial z$, relative to the optical axis, as shown in Figure 1.2.

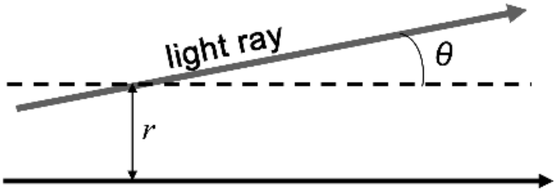


Figure 1.2. *Light radiation or vector radius of parameters r and θ. For a color version of this figure, see www.iste.co.uk/dahoo/metrology2.zip*

The transfer matrices, which represent the effect of translation, refraction or reflection, are used to locate the radius. The propagation of a ray in a homogeneous medium is shown diagrammatically in Figure 1.3.

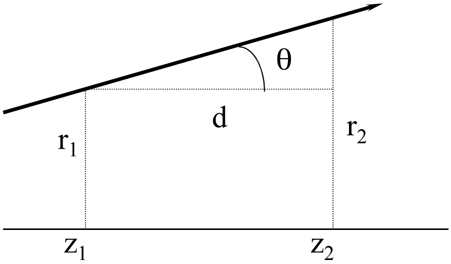


Figure 1.3. *Transfer matrix in an isotropic and homogeneous medium*

The light ray is identified by its distance r_i from the optical axis and the angle θ made by the tangent to the trajectory with the optical axis. For a homogeneous and isotropic medium, the propagation being rectilinear, the angle θ being a constant and having equation $r_2 = r_1 + \theta d$, the following matrix equation is obtained:

$$\begin{pmatrix} r_2 \\ \theta_2 \end{pmatrix} = \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix} \begin{pmatrix} r_1 \\ \theta_1 \end{pmatrix} \qquad [1.1]$$

where d is the distance between the two points parallel to the optical axis.

Similarly, the transfer matrix on a refractive surface from the characteristics, inclination of the rays, refractive indices of the media, distance from the optical axis is established as shown in Figure 1.4.

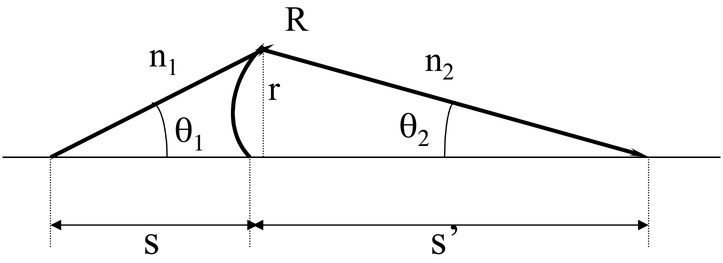


Figure 1.4. *Transfer matrix at the crossing of a diopter*

$r = r_1 = r_2$ is here a constant. In the Gaussian approximation of paraxial rays, the laws of geometric optics lead to: $-n_1/s + n_2/s' = (n_2-n_1)/R$, where R is the radius of curvature of the diopter. The relationship: $\theta_2 = \theta_1\, n_1/n_2 - (1- n_1/n_2)\,(r_1/R)$ is obtained, hence the matrix equation:

$$\begin{pmatrix} r_2 \\ \theta_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \dfrac{n_2-n_1}{n_2 R} & \dfrac{n_1}{n_2} \end{pmatrix} \begin{pmatrix} r_1 \\ \theta_1 \end{pmatrix} \tag{1.2}$$

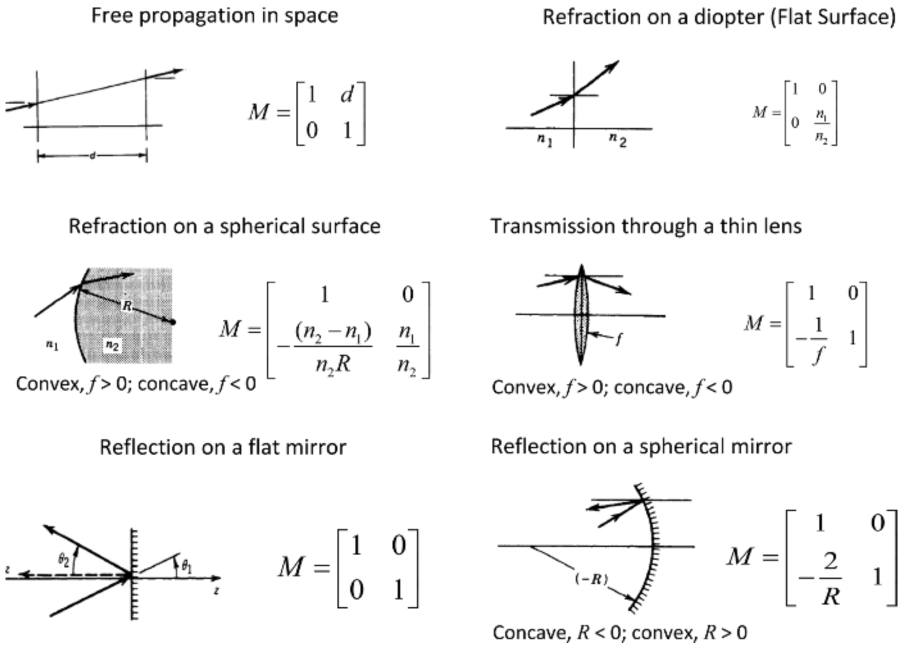
where $(1- n_1/n_2)\,(1/R)$ is the diopter focus.

Thus, to model the propagation of light in an optical system, a transfer matrix M is used, the elements of which denoted A, B, C, D characterize the optical system. This matrix describes the transmission of rays in optical components. For various common optical systems, the transfer matrices, which are given in Figure 1.5, are obtained.

Examples of optical assemblies are given in Figures 1.6 and 1.7.

The diameter of the beams is given in a ratio $D_2/D_1 = f_2/f_1$ in the telescope and in the case of Gaussian beams, the "waist" w is widened from the invariant: $w\theta = \lambda/\pi$.

Mirrors or cube corners can be used to reflect light and reflect back the beam (retro reflection) (Figure 1.8).



Free propagation in space

$$M = \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix}$$

Refraction on a diopter (Flat Surface)

$$M = \begin{bmatrix} 1 & 0 \\ 0 & \frac{n_1}{n_2} \end{bmatrix}$$

Refraction on a spherical surface

$$M = \begin{bmatrix} 1 & 0 \\ -\frac{(n_2 - n_1)}{n_2 R} & \frac{n_1}{n_2} \end{bmatrix}$$

Convex, $f > 0$; concave, $f < 0$

Transmission through a thin lens

$$M = \begin{bmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{bmatrix}$$

Convex, $f > 0$; concave, $f < 0$

Reflection on a flat mirror

$$M = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Reflection on a spherical mirror

$$M = \begin{bmatrix} 1 & 0 \\ -\frac{2}{R} & 1 \end{bmatrix}$$

Concave, $R < 0$; convex, $R > 0$

Figure 1.5. *Optical transfer matrices of different centered systems*

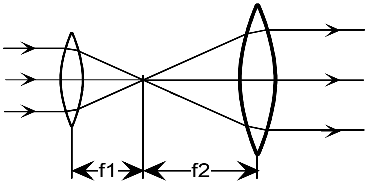
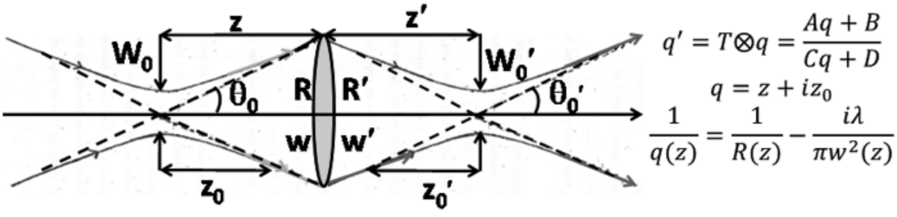


Figure 1.6. *Optical assembly for a telescope in a lidar*



$$q' = T \otimes q = \frac{Aq + B}{Cq + D}$$

$$q = z + iz_0$$

$$\frac{1}{q(z)} = \frac{1}{R(z)} - \frac{i\lambda}{\pi w^2(z)}$$

Figure 1.7. *Transmission of a Gaussian beam by a thin lens*

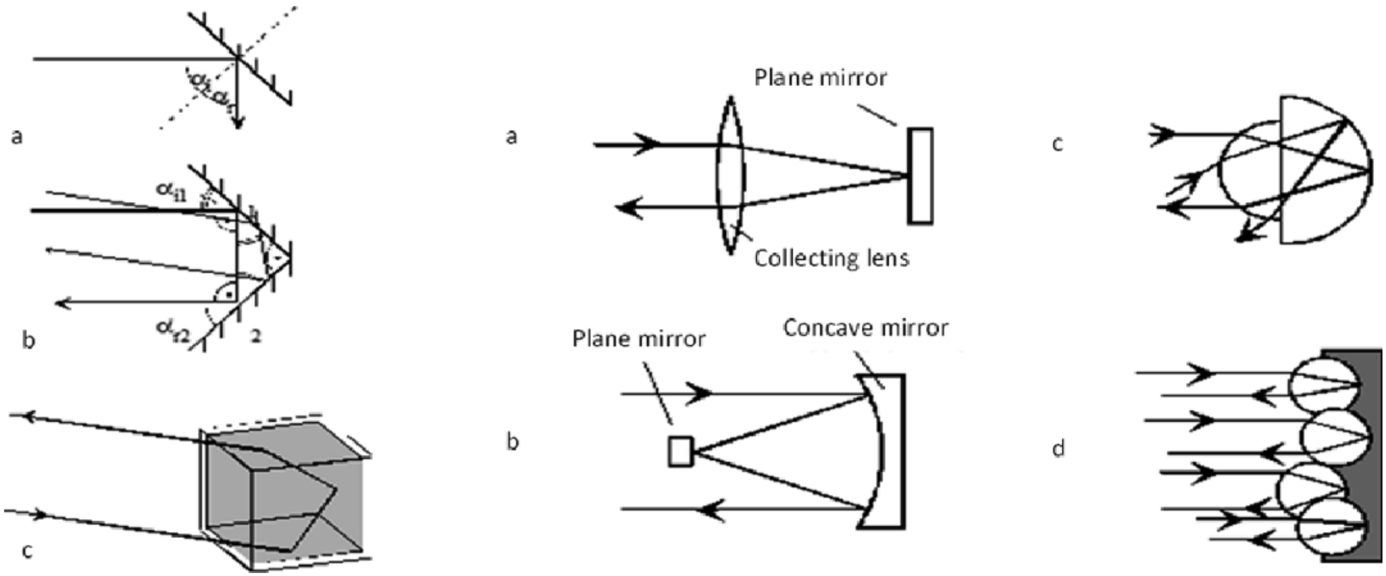


Figure 1.8. *Optical mounting of mirrors and retro-reflectors*

The propagation of light can be studied with 2×2 matrices in wave optics also by considering the two polarization states of light in Jones formalism. The optical axis is taken parallel to the Oz axis which corresponds to the direction of propagation of the wave and the plane of polarization of the wave is the Oxy plane, perpendicular to the direction of propagation since the electromagnetic wave is transverse. Any state of polarization can be described as a linear combination of two vibrations, as shown in Figure 1.9.

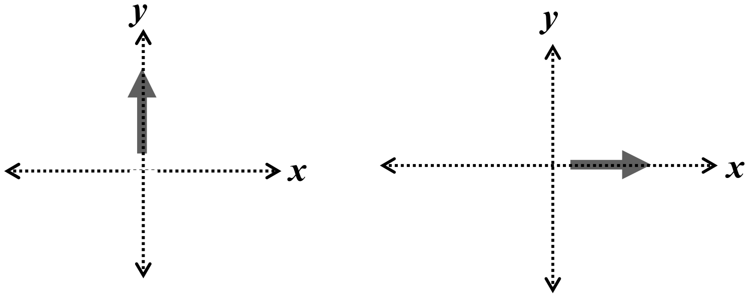


Figure 1.9. *Light polarization states. For a color version of this figure, see www.iste.co.uk/dahoo/metrology2.zip*

The mathematical expression is given by:

$$\vec{E} = E_{0x} e^{i(kz-\omega t+\varphi_x)} \hat{x} + E_{0y} e^{i(kz-\omega t+\varphi_y)} \hat{y} \qquad [1.3]$$

When the phases related to amplitudes and propagation are separated, the electric field of the wave is expressed as the product of a complex amplitude which contains all the information on polarization and a function which represents the propagation of the wave in the positive z direction and which gives the variation of the phase of the wave during its displacement on the trajectory of the light. The electric field of the polarized wave is written by:

$$\vec{E} = \left(E_{0x} e^{i\varphi_x} \hat{x} + E_{0y} e^{i\varphi_y} \hat{y} \right) e^{i(kz-\omega t)} \qquad [1.4]$$

The composition of the electrical vibration of each of the perpendicular components as a function of the phase shift between the x and y components of the wave ($\Delta\varphi = \varphi_y - \varphi_x$) is shown for different situations in Figure 1.10.

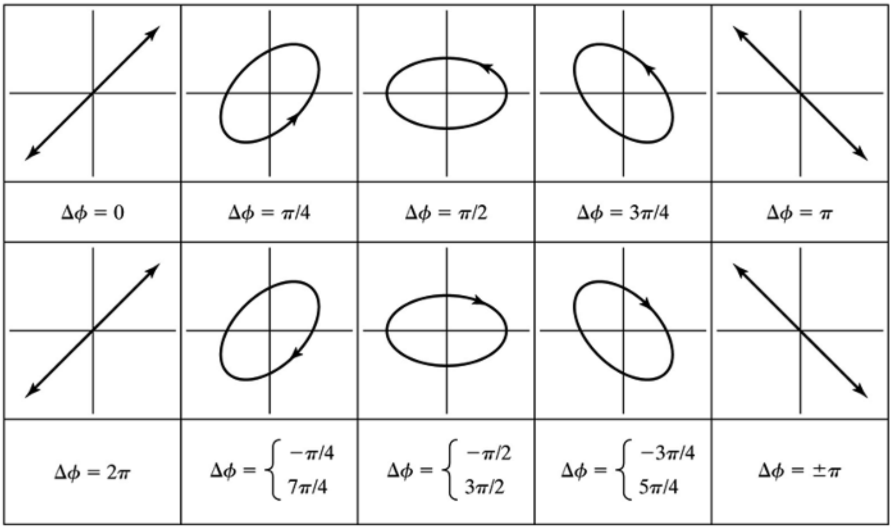


Figure 1.10. *Electric field vibrations in the polarization plane as a function of phase shifts*

In the case of a linear polarizer, all the vibrations of the *E* field are eliminated so that after the polarizer, there remains only one vibration which corresponds to a given direction which is the proper axis of the polarizer. Figure 1.8 shows an example of the effect of a linear polarizer.

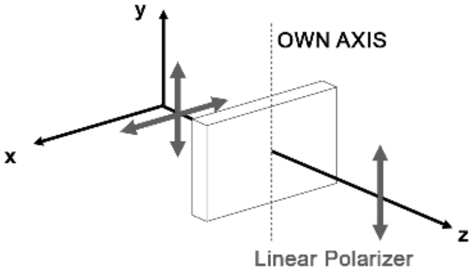


Figure 1.11. *Electric field vibration after a linear polarizer. For a color version of this figure, see www.iste.co.uk/dahoo/metrology2.zip*

In Jones formalism (Chapter 3 of [DAH 16]), the effect of the linear polarizer is described by a 2×2 transfer matrix, whose elements A, B, C and

D correspond to the modifications of the polarization state of the light. In the case of Figure 1.8, $A = B = C = 0$ and $D = 1$. The effect of the polarizer is to let the Ey component of the field pass according to:

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} E_x \\ E_y \end{pmatrix} = \begin{pmatrix} 0 \\ E_y \end{pmatrix} \qquad [1.5]$$

The 2×2 matrices, which correspond to the effects of the various optical elements used in an experimental setup or a device, translate the actions of these optical elements. The wave is represented by the Jones vector with two components. Figure 1.11 gives the example of a phase retarder which introduces a phase difference between the two polarization components parallel to the own axes of the retarder (neutral axes of the birefringent material). In the figure on the right, it can be noticed that the higher the index, the more the wave is "packed" in the material. For $n = 2$, there are 3 and a half periods and for $n = 1.5$, there are 2 and a half periods over the thickness of the plate. The slow axis corresponds to the high index (more time is needed to cross the plate) or the lower phase speed and the fast axis to the lower index or the higher phase speed.

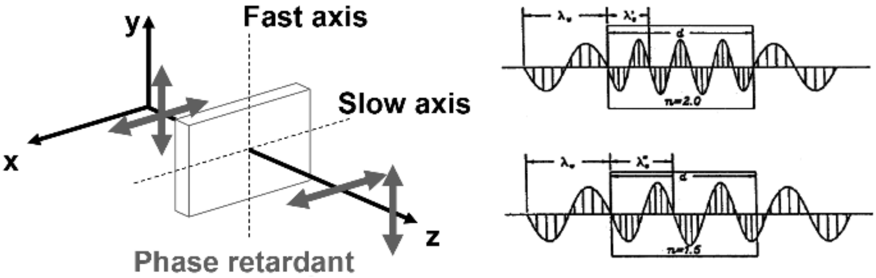


Figure 1.12. *Vibration of the electric field after a linear polarizer and difference in optical path along the axes. For a color version of this figure, see www.iste.co.uk/dahoo/metrology2.zip*

The effect of the self-timer is given by:

$$\begin{pmatrix} e^{i\varphi_x} & 0 \\ 0 & e^{i\varphi_y} \end{pmatrix} \begin{pmatrix} E_x \\ E_y \end{pmatrix} = \begin{pmatrix} E_x e^{i\varphi_x} \\ E_y e^{i\varphi_y} \end{pmatrix} \qquad [1.6]$$

For example, for a thickness d, the phase shifts are: $\Delta\varphi_0 = -\frac{2\pi}{\lambda} n_0 d$ and $\Delta\varphi_e = -\frac{2\pi}{\lambda} n_e d$ on the ordinary and extraordinary axis, respectively. This leads to a phase shift between the paths of the wave along the ordinary axis and along the extraordinary axis of: $\Delta\varphi = \Delta\varphi_0 - \Delta\varphi_e = -\frac{2\pi}{\lambda}(n_e - n_0)d$.

If the phase difference $\Delta\varphi = \varphi_x - \varphi_y$ introduced between the two components x and y is π/2, the plate is said to be a quarter-wave plate. If this difference is π, the plate is said to be a half-wave plate. When the slow axis is vertical, the matrices are given by:

$$e^{-i\frac{\pi}{4}}\begin{pmatrix}1 & 0\\ 0 & i\end{pmatrix} \text{ and } e^{-i\frac{\pi}{2}}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \qquad [1.7]$$

And when the slow axis is horizontal, by:

$$e^{i\frac{\pi}{4}}\begin{pmatrix}1 & 0\\ 0 & i\end{pmatrix} \text{ and } e^{i\frac{\pi}{2}}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \qquad [1.8]$$

Figure 1.13 shows an example of a rotator that rotates the plane of polarization of the light at a certain angle. The rotator matrix is given by:

$$\begin{pmatrix}cos\theta & -sin\theta\\ sin\theta & cos\theta\end{pmatrix} \qquad [1.9]$$

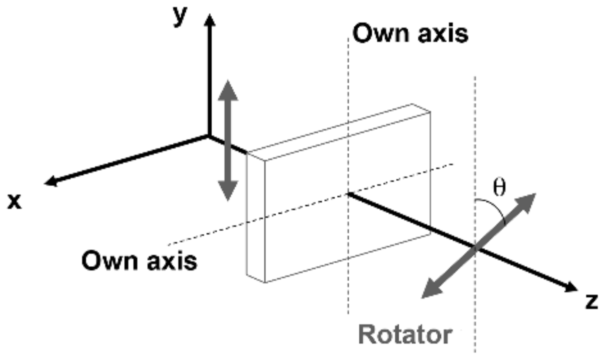


Figure 1.13. *Electric field vibration after a rotator. For a color version of this figure, see www.iste.co.uk/dahoo/metrology2.zip*

To model the effects of several devices on the state of polarization, it suffices to multiply the Jones vector of the input polarization by all the Jones matrices while respecting the order in which the elements act during propagation. The matrix of the first element is placed on the far right, first in the product, then the second element on the left of the first matrix and so on. Thus, a single Jones matrix, which is obtained by the product of the individual Jones matrices, describes the combination of several devices. Figure 1.14 gives a diagram of three devices that leads to $M = M_3M_2M_1$.

Figure 1.14. *Combination of three devices. For a color version of this figure, see www.iste.co.uk/dahoo/metrology2.zip*

In the case of crossed polarizers (Figure 1.15), we have:

$$\begin{pmatrix}E_{0x}\\E_{0y}\end{pmatrix}\begin{pmatrix}0&0\\0&1\end{pmatrix}\begin{pmatrix}1&0\\0&0\end{pmatrix}=\begin{pmatrix}0&0\\0&0\end{pmatrix}\begin{pmatrix}E_{0x}\\E_{0y}\end{pmatrix}=\begin{pmatrix}E_{1x}\\E_{1y}\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix} \qquad [1.10]$$

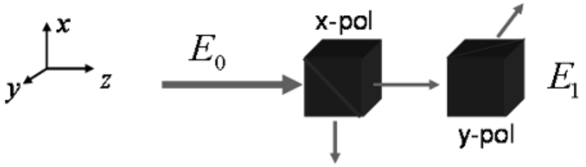


Figure 1.15. *Cross polarizers. For a color version of this figure, see www.iste.co.uk/dahoo/metrology2.zip*

1.3. Photon emission and detection

In interference devices, the light source is either a thermal source or a coherent laser source (Chapters 3 and 6 of [DAH 16]). Considering the states of creation and annihilation (section 3.4, Chapter 3 of [DAH 16]) and the quantum Hamiltonian expressed in terms of vibrators, the statistical operator in the canonical set is obtained in the form $\rho = \frac{\exp(-\beta H)}{Tr[\exp(-\beta H)]}$, where $\beta = 1/kT$ and H is the Hamiltonian operator whose expression is $H = \hbar\omega(a^+a + \frac{1}{2})$, i.e. $H = \left(\hat{n} + \frac{1}{2}\right)\hbar\omega$, where $\hat{n}$ is the operator number of photons. Considering a radiation field made up of a single mode of the photon, the probability of finding the system in the state with n photons $|n\rangle$ is given by the Bose Einstein statistic, i.e.: $P_n = \frac{\exp(-\beta n\hbar\omega)}{Z} = \exp(-\beta n\hbar\omega)$ $(1-\exp(-\beta\hbar\omega))$, where Z is the partition function. The mean value of the distribution is given by $\langle n\rangle = \frac{1}{1-\exp(-\beta\hbar\omega)}$ and the standard deviation is given by: $\Delta n = \sqrt{\langle n\rangle + \langle n\rangle^2}$. The discrete probability distribution (Figure 1.16) can thus be expressed in the form: $P_n = \frac{(\langle n\rangle-1)^n}{(\langle n\rangle)^{n+1}}$.

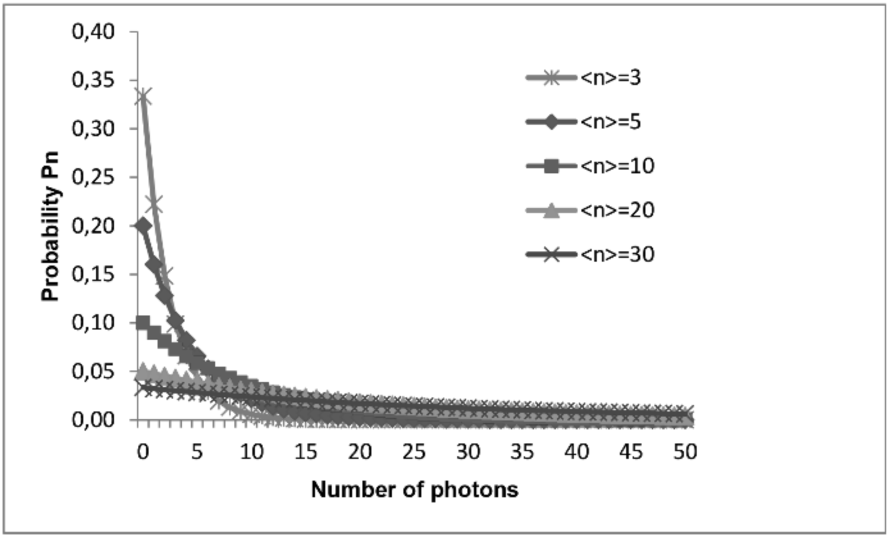


Figure 1.16. *Probability distribution for a thermal source. For a color version of this figure, see www.iste.co.uk/dahoo/metrology2.zip*

The Glauber state or the coherent state of an ideal single-mode laser is given by (equation [3.65], Chapter 3 of [DAH 16]) $|\alpha\rangle$, an eigenstate of the annihilation operator a of the photon, of eigenvalue α. As a is a non-hermitic operator, the phase α is a complex number which corresponds to the complex amplitude of the wave in classical optics of an electromagnetic field [GLA 63, SUD 63, GLA 67, ARE 72, DAV 96]. $|\alpha\rangle$ can be written in the basis of kets $|n\rangle$ of the Fock space as:

$$|\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle \equiv \left|re^{i\theta}\right\rangle.$$

This equation relates the wave nature, to the corpuscular nature of light, and shows that in the coherent state, the number of photons is not defined while the phase is (except for the uncertainty equation). These coherent states represent quasi-classical states insofar as a phase θ and an average amplitude *r* are associated with them.

The probability of having n photons in a coherent state $|\alpha\rangle$ is $P_n = |\langle n|\alpha\rangle|^2$. It can be shown that the distribution of photons follows a Poisson law (Figure 1.17): $P_n = e^{-|\alpha|^2}\frac{\alpha^{2n}}{n!}$, where the term $|\alpha|^2$corresponds to the mean value of the number of photons $\langle n\rangle$ associated with the operator $N(n) = a^{\dagger}a\rangle$. This mean value is given by $\langle n\rangle = \langle\alpha|N(n)|\alpha\rangle = |\alpha|^2$and the variance by:

$$\sigma^2 = \langle\alpha|N^2(n)|\alpha\rangle - |\langle\alpha|N(n)|\alpha\rangle|^2 = |\alpha|^2,$$

so that the standard deviation is given by $|\alpha|$. Thus:

$$P_n = e^{-\langle n\rangle}\frac{\langle n\rangle^n}{n!}.$$

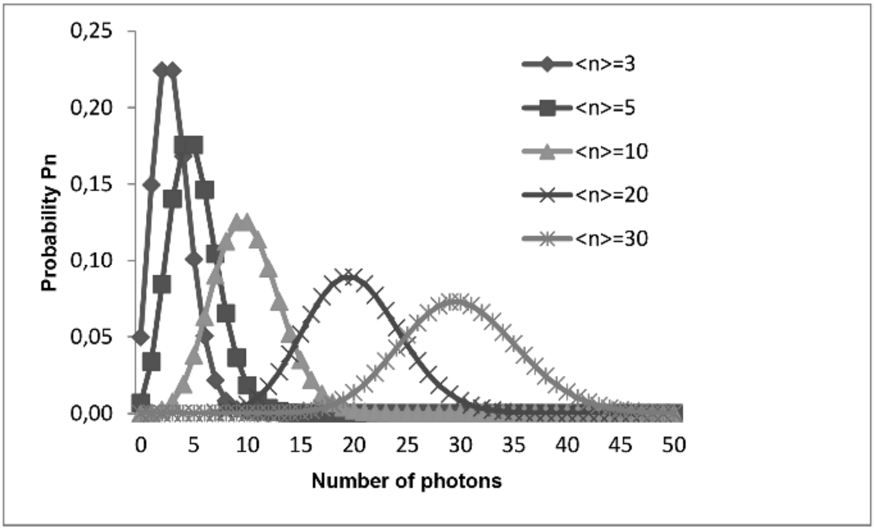


Figure 1.17. *Probability distribution for a coherent source. For a color version of this figure, see www.iste.co.uk/dahoo/metrology2.zip*

Photon detection is the result of a light–matter interaction. The electromagnetic field of a light wave has an electrical component E and a magnetic component B arranged perpendicular to each other while being perpendicular to the direction of propagation of the wave. The Poynting vector $\boldsymbol{S}$ of such a wave represents the power transported by the unit of surface and is expressed by: $\vec{S} = \frac{1}{2}\left(\vec{E} \wedge \vec{H}^*\right)$, where $\left|\vec{H}\right| = \frac{\vec{E}}{Z_0}$, where Z_0 is the characteristic impedance of the medium, which in the vacuum is equal to $\sqrt{(\mu_0/\varepsilon_0)}$. The Poynting vector is parallel to the direction of wave propagation in a homogeneous isotropic medium (Figure 1.17). Detection of the energy carried by a wave is done by placing the surface of a detector perpendicular to the path of the wave (Figure 1.18).

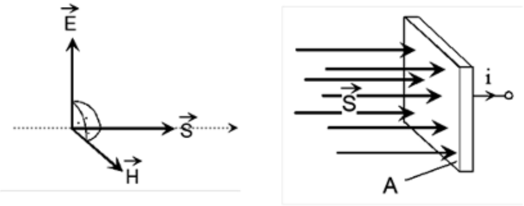


Figure 1.18. *Light detection*

The total power P carried by the wave and detected on a surface A is given by: $P = \oiint \vec{S}\vec{n}dxdy$ which leads to $P = \frac{1}{2Z_0}\oiint\left|\vec{E}\right|^2 dxdy = \frac{A}{2Z_0}\left|\vec{E}\right|^2$.

A detector being quadratic, if R is the response of the detector, the photo-current is given by: $i = RP = R\frac{A}{2Z_0}\left|\vec{E}\right|^2 = K\left|\vec{E}\right|^2$. As the intensity I of the light is given by $I = \left|\vec{E}\right|^2$, the following relation is obtained: for the photo-current: $i = K\left|\vec{E}\right|^2 = KI$. Detection is a nonlinear process.

Let W be the energy incident on the detector. The number n of incident photons on the detector is given by: $n = \frac{W}{W_{ph}} = \frac{W}{h\nu}$. The total charge is equal to: $q = \eta e\frac{W}{h\nu}$, where η is the detection efficiency. The photo-current is given by:

$$i = \frac{dq}{dt} = \frac{\eta e}{h\nu}\frac{dW}{dt} = \frac{\eta e}{h\nu}P = RP.$$

1.4. Application exercises on interferometry

In its operating principle, an interferometer is a two-wave interference optical device, which is based on the separation of a wave (Figure 1.19) by amplitude or wave-front splitting. The Michelson interferometer is used for interference experiments. It was made famous following the experiments of Michelson–Morley [MIC 87] on the verification of the existence of the ether in an approach based on high resolution measurement of the phase to determine a possible Doppler effect on the speed of the light.

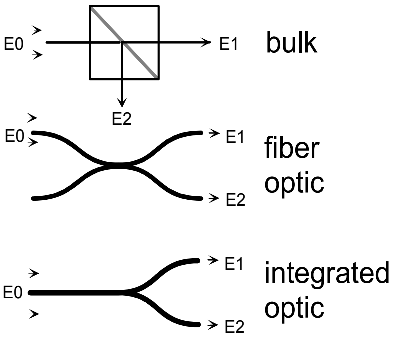


Figure 1.19. *Beam splitter devices*

In operation, an incident ray from a light source is partially reflected on the separator towards the mirror M_1 and partially transmitted towards M_2. After reflection on the two mirrors, the rays on paths 1 and 2 pass again through the separator. The rays coming out of the interferometer after having traveled different optical paths interfere and, since they are coherent, coming from the same wave surface, give rise to light interference.

The device consists of a fixed separator, and an adjustable compensator using adjustment screws, to orient it parallel to the separator. The light beam, which separates into two paths, is reflected on two mirrors, M_1 and M_2. The mirror M_2, facing the source in Figure 1.20, is mobile and can be translated and oriented using a three-screw system (line, point, plane system) by a so-called coarse adjustment. The parallelism between the mirrors M_1 and M_2 is obtained in a fine manner by fine adjustment screws on the fixed mirror.

Depending on the path difference (Δx) between the two rays which depends on the relative position of the two mirrors M_1 and M_2, the interference is constructive or destructive. The fact that the rays coming from M_2 cross twice the separator whereas those coming from M_1 cross it once induces additional phase shifts which are compensated by a compensator, which symmetrizes the optical paths in the two arms of the Michelson.

After reflection on each of the mirrors, the beam, which is recombined again at the level of the intensity separator cube, is sent to a detector which delivers an electrical signal to a counter whose role is to determine the number of fringes which pass as and measurement of the displacement of the movable mirror. A fringe is a whole cycle of varying light intensity, going from light to dark and back to light. Each cycle corresponds to a displacement of $\lambda_0 = 2n$. Knowing λ_0 and n, one deduces displacement from it.

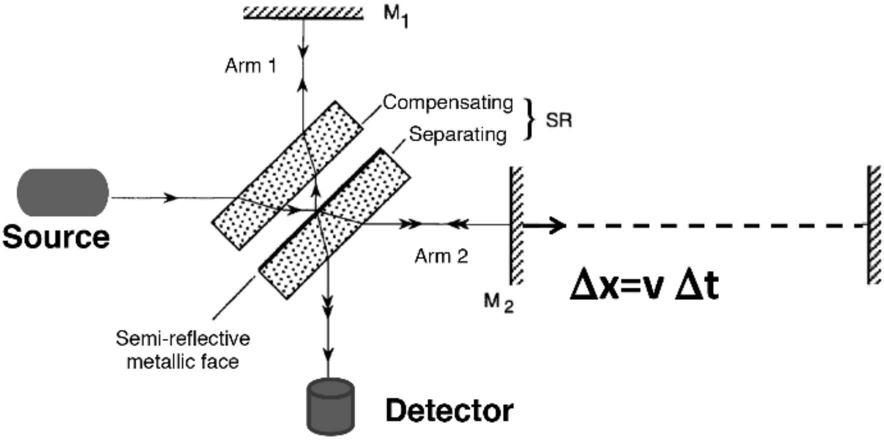


Figure 1.20. *Diagram of an interferometer. For a color version of this figure, see www.iste.co.uk/dahoo/metrology2.zip*

Interferometers are used in spectroscopy to identify molecules that absorb light in the IR domain (Reliability of Multiphysical Systems Set, Volume 1 [CAR 16]).

1.4.1. *Propagation of electromagnetic waves in a Fabry–Pérot cavity*

Two concave mirrors with radii R_1 and R_2 are considered (Figure 1.21). For paraxial rays, for which all the angles are small, the relation between (y_{m+1}, θ_{m+1}) and (y_m, θ_m) is linear and can be written in a matrix form by:

$$\begin{bmatrix} y_{m+1} \\ \theta_{m+1} \end{bmatrix} = \begin{bmatrix} A\,B \\ C\,D \end{bmatrix} \begin{bmatrix} y_m \\ \theta_m \end{bmatrix}$$

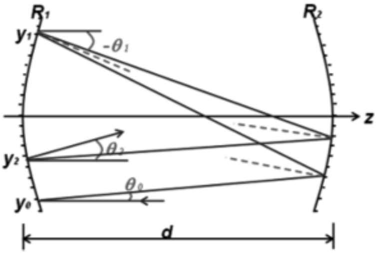


Figure 1.21. *Diagram of a Fabry–Pérot cavity. For a color version of this figure, see www.iste.co.uk/dahoo/metrology2.zip*

A ray which starts from the point m, (y_m, θ_m) of the mirror M_1 (radius of curvature R_1), propagates in the vacuum over a distance d to then arrive at the mirror 2 where it undergoes a reflection on M_2 is considered. It then starts again from M_2, propagates in the vacuum over a distance d and is reflected on M_1 at the point (y_{m+1}, θ_{m+1}).

1.4.1.1. *Question*

Demonstrate from the parameters of the cavity that the elements of the matrix M are expressed by:

$$A = 1 + {}^{2d}\!/_{R_2}$$

$$B = 2d(1 + {}^{d}\!/_{R_2})$$

$$C = {}^{2}\!/_{R_1} + {}^{2}\!/_{R_2} + {}^{4d}\!/_{R_1R_2}$$

$$D = 1 + {}^{4d}\!/_{R_1} + {}^{2d}\!/_{R_2} + {}^{4d^2}\!/_{R_1R_2}$$

1.4.1.2. *Solution*

The matrix M, which makes it possible to pass from the mirror of radius R_1 to the mirror of radius R_2, is obtained by multiplying the matrices corresponding to the path from mirror 1 to mirror 2 (M_1), then the reflection on the mirror of radius R_2 (M_2), then the path from mirror 2 to mirror 1 (M_3) and, finally, the reflection on mirror 2 (M_4).

Thus:

$$\begin{bmatrix} y_{m+1} \\ \theta_{m+1} \end{bmatrix} = M4M3M2M1 = \begin{bmatrix} AB \\ CD \end{bmatrix} \begin{bmatrix} y_m \\ \theta_m \end{bmatrix}$$

with:

$$M1 = \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix}, M2 = \begin{bmatrix} 1 & 0 \\ \dfrac{2}{R_1} & 1 \end{bmatrix}, M3 = \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix} \text{and } M4 = \begin{bmatrix} 1 & 0 \\ \dfrac{2}{R_2} & 1 \end{bmatrix}.$$

We can therefore write that the matrix M has the expression:

$$M = M4M3M2M1 = \begin{bmatrix} 1 & 0 \\ \dfrac{2}{R_2} & 1 \end{bmatrix} \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \dfrac{2}{R_1} & 1 \end{bmatrix} \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 + \dfrac{2d}{R_2} & 2d(1 + \dfrac{d}{R_2}) \\ \dfrac{2d}{R_1} + \dfrac{2d}{R_2} + \dfrac{4d}{R_1R_2} & 1 + {}^{4d}/_{R_1} + {}^{2d}/_{R_2} + {}^{4d^2}/_{R_1R_2} \end{bmatrix}$$

1.4.2. ***Propagation of electromagnetic waves in a material***

A retarder plate of thickness d is considered (Figure 1.11). This plate is characterized by main axes, a fast axis (A_R) of index nr and a slow axis (A_L) of index n_l.

1) What do these axes represent?

Solution:

The slow and fast axes correspond to the own axes of the plate which are determined by the symmetry properties (see Reliability of Multiphysical Systems Set, Volume 9, Chapter 4 [DAH 21a]). Along these axes, the electric field propagates at different phase speeds. The proper axes are characterized by the so-called ordinary and extraordinary indices of the plate and which are determined by the properties of symmetry (see Reliability of Multiphysical Systems Set, Volume 9, Chapter 4 [DAH 21a]). Along these axes, the electric field propagates at different phase speeds. The displacement field D is parallel to E and its amplitude is proportional to that of E, such that $D_i = \varepsilon_i E_i$, where $\varepsilon_l = (n_i)^2$.

2) To which physical parameter do the terms fast and slow axes refer?

Solution:

The terms slow and fast axes correspond to the phase speeds given by $v_r = c/n_r$ and $v_l = c/n_l$, where n_i is the index along the axis. The vacuum index is 1 and the index in matter is greater than 1. The phase speed is therefore slower than that of light in vacuum. The slow term corresponds to the axis for which the phase speed is the lowest and conversely the fast term corresponds to the axis for which the phase speed is the highest.

3) Show that the axes introduce a phase difference $\Delta\varphi$ between the orthogonal components of the wave given by: $\Delta\varphi = \left({}^{2\pi}/_{\lambda}\right)(n_l - n_r)d$.

Solution:

On the slow axis, the phase of the wave is: $\Delta\varphi_l = \left({}^{2\pi}/_{\lambda}\right)n_l d$.

On the fast axis, the phase of the wave is: $\Delta\varphi_r = \left({}^{2\pi}/_{\lambda}\right)n_r d$.

The phase difference is: $\Delta\varphi = \Delta\varphi_l - \Delta\varphi_r = \left({}^{2\pi}/_{\lambda}\right)(n_l - n_r)d$.

4) A quarter-wave plate introduces a phase shift $\Delta\varphi = \pi/2$ and a half-wave plate introduces a phase shift $\Delta\varphi = \pi$ between the components of the electric field. Give the corresponding matrices of these plates.

Solution:

If the wave s is parallel to the axis Ox, Ox,

$$\vec{E}_s(\vec{r},t) = \left|\vec{E}_s\right| \exp\left(-i(\omega t)\right)\overrightarrow{e_x}.$$

If the wave p is parallel at axis Ox,

$$\vec{E}_p(\vec{r},t) = \left|\vec{E}_p\right| \exp\left(-i(\omega t)\right)\overrightarrow{e_y}.$$

After crossing the plate, the polarized wave has the expression $\vec{E}_s(\vec{r},t) = |\vec{E}_s| \exp\left(i\left(-\omega t + \left({}^{2\pi}/_{\lambda}\right)n_r d\right)\right)\overrightarrow{e_x}$ and $\vec{E}_p(\vec{r},t) = |\vec{E}_p| \exp\left(i\left(-\omega t + \left({}^{2\pi}/_{\lambda}\right)n_l d\right)\right)\overrightarrow{e_y}$. When the wave propagates along the z axis in the positive direction, the sign between the temporal and spatial phases is negative.

If the elements of the matrix *M* are denoted by *A*, *B*, *C* and *D*, such that $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$, *B* and *C* are zero, so that the non-zero terms are $A = \exp\left(i\left(\left({}^{2\pi}/_{\lambda}\right)n_l d\right)\right) = \exp(i\Delta\varphi_l)$ and $D = \exp\left(i\left(\left({}^{2\pi}/_{\lambda}\right)n_r d\right)\right) = \exp(i\Delta\varphi_r)$. Then, the matrix can be expressed as:

$$\begin{pmatrix} e^{i\Delta\varphi_l} & 0 \\ 0 & e^{i\Delta\varphi_r} \end{pmatrix} = e^{\frac{i(\Delta\varphi_l+\Delta\varphi_r)}{2}} \begin{pmatrix} e^{\frac{i(\Delta\varphi_l-\Delta\varphi_r)}{2}} & 0 \\ 0 & e^{-\frac{i(\Delta\varphi_l-\Delta\varphi_r)}{2}} \end{pmatrix}$$

In the case of a quarter-wave plate, Δ*nd* = λ/4 and a matrix *M* is obtained:

$$M = e^{\frac{i(\Delta\varphi_l+\Delta\varphi_r)}{2}} \begin{pmatrix} e^{\frac{i\pi}{4}} & 0 \\ 0 & e^{-\frac{i\pi}{4}} \end{pmatrix} = e^{i\psi} e^{\frac{i\pi}{4}} \begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix}$$

In the case of a half-wave plate, Δ*nd* = λ/2, and a matrix *M* is obtained:

$$M = e^{\frac{i(\Delta\varphi_l+\Delta\varphi_r)}{2}} \begin{pmatrix} e^{\frac{i\pi}{2}} & 0 \\ 0 & e^{-\frac{i\pi}{2}} \end{pmatrix} = e^{i\psi} e^{\frac{i\pi}{2}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

As we generally calculate the intensity, we have $e^{i\psi} * e^{-i\psi} = 1$ and the matrices are:

$$M(\tfrac{\lambda}{4}) = e^{\frac{i\pi}{4}} \begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix} \text{ and } M(\tfrac{\lambda}{2}) = e^{\frac{i\pi}{2}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

1.4.3. *Interferometry and optical lambda meter*

In photonic systems, it is often necessary to precisely know the wavelength of the laser sources used by the systems. This is the case when the atmosphere is studied by lidars which use various lasers at various wavelengths or in telemetry for the measurement of distances. The Earth–Moon distance, for example, is measured from a pulsed laser (Côte d'Azur Observatory, Calern Plateau) which sends 10 pulses per second towards the Moon. The reflected photons are collected by a 1.5 m diameter telescope. The number of reflected photons is very low, of the order of one photon per 100 shots. The time interval between the emission of the light pulses and the reception of the return signal is between 2.3 and 2.8 seconds. This time interval is measured with an accuracy of 7–10 ps, which provides a distance between the transmitter and the receiver to within 3 mm on average.

Wavelength measuring systems are called lambda meters and are based on the use of a Michelson interferometer. The diagram of the device of a lambda meter is given in Figure 1.22. The main optical elements are a quarter-wave plate oriented at 45° (Q_{45}) with respect to the s axis, a polarizer *s* oriented at 45° with respect to the *s* axis (P_{45}) and a cube polarization splitter (CSP). A major difference compared to the conventional interferometer is that: in the case of a lambda meter, the two arms of the interferometer are mobile. This is done using a double-sided mirror, a mirror whose two sides are reflective (see Figure 1.22).

If $\vec{E}_S$ is the field of the light wave propagating on one of the arms of the interferometer and $\vec{E}_P$ the field of the wave propagating on the other arm, their expressions are given by:

$$\vec{E}_S(\vec{r},t) = \left|\vec{E}_S\right| \mathrm{e}^{i(-\omega t+\vec{k}\vec{r}_{0S})}\,\overrightarrow{e_S} = \left|\vec{E}_S\right| \mathrm{e}^{i(-\omega t+\varphi_{0S})}\,\overrightarrow{e_S} \qquad [1.11]$$

and

$$\vec{E}_P(\vec{r},t) = \left|\vec{E}_P\right| \mathrm{e}^{i(-\omega t+\vec{k}\vec{r}_{0P})}\,\overrightarrow{e_P} = \left|\vec{E}_P\right| \mathrm{e}^{i(-\omega t+\varphi_{0P})}\,\overrightarrow{e_P} \qquad [1.12]$$

where $\omega = 2\pi\nu_0$ is the pulse of the wave, ν_0 is the frequency of the wave equal to $\nu_0 = c_0/\lambda_0$ in a vacuum and $k = 2\pi/\lambda_0$ is the norm of the wave vector. $c_0 = 299\ 792\ 458\ \mathrm{ms}^{-1}$.

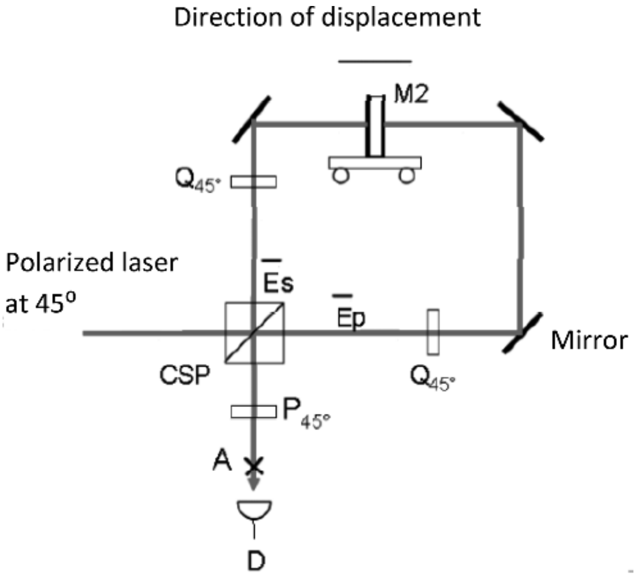


Figure 1.22. *Schematic diagram of the device of a lambda meter. For a color version of this figure, see www.iste.co.uk/dahoo/metrology2.zip*

In the Jones basis:

$$\vec{E}_L(\vec{r},t) = e^{-i\omega t}\begin{pmatrix} |\vec{E}_S|e^{i\varphi_{0S}} \\ |\vec{E}_P|e^{i\varphi_{0p}} \end{pmatrix} \qquad [1.13]$$

If the laser wave is polarized at 45°, then $e^{i\varphi_{0S}} = e^{i\varphi_{0P}} = e^{i\varphi_0}$ and $|\vec{E}_S| = |\vec{E}_P|$.

To determine the expressions of the components of the resulting field at point A (Figure 1.20) after passing through the photonic device, the Jones matrix approach is used. The matrices of the different components are given in the appendix. r_s and r_p represent, respectively, the total distances traveled (source-round trip-detector) traveled by the *s* wave and the *p* wave between the polarization splitter cube and the mirror M_2.

1) On the outward path, the plate is Q_{45}, and on the return path, it is Q_{-45}. Why?

2) Which of these four statements corresponds to the mounting of the lambda meter (circle the correct answer)?

A: At the level of the cube CSP, if the electric field on the outward path is E_S (respectively, E_P), on the return path it becomes E_p ((respectively, E_S) after crossing the cube.

B: At the level of the CSP cube, if the electric field on the outward path is E_S (respectively, E_P), on the return path it becomes E_S ((respectively, E_P) after crossing the cube.

C: At the level of the CSP cube, if the electric field on the outward path is E_S (respectively, E_P), on the return path it becomes E_p ((respectively, E_P) after crossing the cube.

D: At the level of the CSP cube, if the electric field on the outward path is E_S (respectively, E_P), on the return path it becomes E_S ((respectively, E_S) after crossing the cube.

3) On the matrices associated with Q_{45} and Q_{-45} which are given by:

$$Q_{45} = \frac{1}{2}\begin{pmatrix} a & a^* \\ a^* & a \end{pmatrix} \text{and } Q_{-45} = \frac{1}{2}\begin{pmatrix} a & -a^* \\ -a^* & a \end{pmatrix}$$

where $a = 1\text{-}i$ with $i^2 = -1$ and a^* is the complex conjugate of a: $a^* = 1+i$.

Check that $aa^*= 2$ and that $a^2+a^{*2} = 0$.

4) It will be assumed that P_{45} is defined on the return path. Show that:

$$P_{45} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

5) Check that $M^2 = I$.

The initial laser field is given by: $\vec{E}_L = \vec{E}_S + \vec{E}_P = \begin{pmatrix} |E_S|e^{i\varphi_{0s}} \\ |E_P|e^{i\varphi_{0p}} \end{pmatrix} e^{-i\omega t}$ with:

$$\vec{E}_s(\vec{r},t) = \left|\vec{E}_s\right| \exp\left(i\left(-\omega t + \vec{k}\vec{r}_{0s}\right)\right)\overrightarrow{e_s} = \left|\vec{E}_s\right| \exp\left(i\left(-\omega t + \varphi_{0s}\right)\right)\overrightarrow{e_s}$$

and:

$$\vec{E}_p(\vec{r},t) = \left|\vec{E}_p\right| \exp\left(i\left(-\omega t + \vec{k}\vec{r}_{0p}\right)\right)\overrightarrow{e_p} = \left|\vec{E}_p\right| \exp\left(i\left(-\omega t + \varphi_{0p}\right)\right)\overrightarrow{e_p}$$

By using the formalism of the matrix equation for each path, we can calculate the electric field $E_S(A)$ arriving on the detector at A through the path S and the corresponding electric field $E_P(PD)$ on the path P. Like the laser is polarized at 45°, we have

6) From the initial laser field, we get on the path of the arm initially S, the field in A from the multiplication of the following matrices: $\vec{E}_S(A) = D(\varphi_S) * P_{45}^{S} * CSP_P * Q_{-45} * M_2 * Q_{45} * CSP_S * \vec{E}_L$. We do not have to consider the effect of the mirror between Q_{45} and M_2 which allows us to modify the path, because $M^2=I$, the identity matrix. Justify the matrices present in this product using the diagram of the device in Figure 1.22.

7) From the interpretation of the various Jones matrices: CSP_S, Q_{45}, M_2, Q_{-45}, CSP_P, $D(\varphi_s)$ and P^S_{45} give the expanded matrix expressions.

8) By multiplying the matrices, determine the field: $\vec{E}_S(A) = -K_S\frac{|E_S|e^{i\varphi_{rs}}}{2}\begin{pmatrix}1\\1\end{pmatrix}$, where K_S, which includes the term $e^{i\omega t}$, is a complex constant and φ_{rS} is a phase shift. Give the expression of φ_{rS} as a function of n, λ_0 and r_s, the path traveled in the interferometer (use the expression of $\phi = 2\pi nd/\lambda_0$ given in the attached form, specifying d).

9) Calculate $E_P(A)$, the wave propagating on the arm P. By proceeding in the same way as for the field $E_S(A)$ give its expression in terms of *ABCD* matrices.

10) Explain the matrices allowing us to calculate $EP(A)$ $E_P(A)$.

11) Show that: $\vec{E}_P(A) = K_P\frac{|E_P|e^{i\varphi_{rP}}}{2}\begin{pmatrix}1\\1\end{pmatrix}$, where K_P is a complex constant and φ_{rP} a phase shift. Express φ_{rp} as a function of n, λ_0 and r_p, the path traveled in the interferometer (use the expression of $\phi = 2\pi nd/\lambda_0$ given in the form in the appendix, specifying d).

12) Deduce the expression for the intensity of the light wave at the output of the interferometer when $r_s = r_p$, i.e. the two paths are identical in the two arms. Set $I_0 = |E_S|^2 + |E_P|^2 = 1$ and $I = \varepsilon_0 c <E>^2$.

13) The mobile mirror M_2 is moved by an amount Δz in the direction shown in Figure 1.22 to the right. Rewrite the previous expression as a function of Δz.

14) To which displacement (or variation of optical path) does a phase variation of 2π correspond?

15) This displacement corresponds to the scrolling of an interference fringe. Denote by N the number of interference fringes obtained for a displacement d. Give the expression of N as a function of n, d and λ_0. Calculate N for a displacement of 10 cm knowing that $\lambda_0 = 632.991458$ nm and $n = 1.000247$ (round to the nearest integer).

16) The previous laser is replaced by a laser whose wavelength is unknown. Let λ_i be this wavelength. For the same displacement, a number of fringes N_i is obtained. Give the expression of λ_i as a function only of λ_0, N and N_i. Deduce the value of λ_i knowing that $N_i = 512\ 643$. Provide the result with six significant digits after the decimal point.

17) It can be shown that the relative uncertainty on this measurement varies as $1/N$. What must then be the minimum displacement d_{min} corresponding to a relative uncertainty on λ_i of 10^{-6} (i.e. 0.001 nm). Calculate d_{min} within 1 mm.

1.4.3.1. *Answer to question 1*

On the outward path, the plate is Q_{45} and on the return path it is Q_{-45}. Why?

The axes of the path are locally defined so as to position the Oz axis on the light path and the Ox axis horizontally along the s polarization and the Oy axis vertically along the Oy axis parallel to the p polarization. On the figure given in appendix and in Figure 1.23, the go wave is in the direction Oz, which corresponds to a phase $\Psi = -\omega t + kr_s$ for the component s and a phase $\Psi = \omega t + kr_p$ for the *p* component for a phase velocity in the direction of increasing *z*.

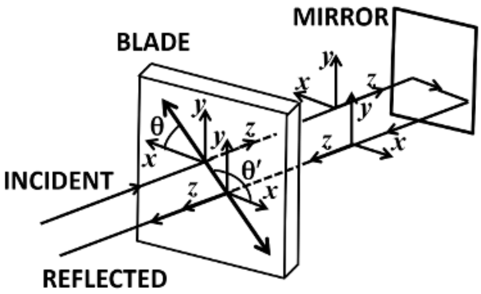


Figure 1.23. *Diagram of the light path and reference axes*

On the outward angle between the optical axis is 45° in the counterclockwise direction and on the return, the local axes are positioned differently on the plate and the angle made by the optical axis is 135° or -45° counterclockwise.

1.4.3.2. *Answer to question 2*

Which of these four statements corresponds to the mounting of the lambda meter (circle the correct answer)?

After reflection, the wave is in an s-type state and after transmission in a p-type state. The correct answer is given by Proposition A.

1.4.3.3. *Answer to question 3*

According to the form given in the appendix, aa*=$(1-i)(1+i)=1-(-1)=2$.

Thus: $a^2+a^{*2}=(1-i)^2+(1+i)^2=-2i+2i=0$.

1.4.3.4. *Answer to question 4*

Using the formulas:

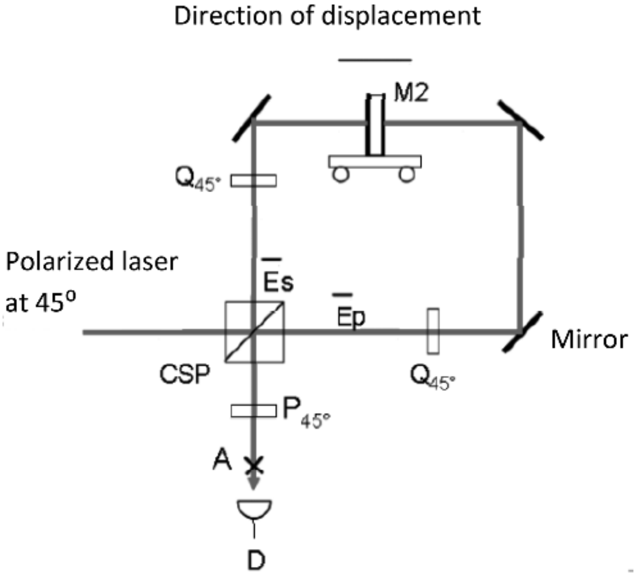
$M(\theta)=R(\theta)M(0)R(-\theta)$ and $R(\theta)=\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ with θ = 45°, we calculate:

$$P(45°)=\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}=\frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}=\frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

1.4.3.5. *Answer to question 5*

$$M * M = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

1.4.3.6. *Answer to question 6*



Reminder: Figure 1.22. *Schematic diagram of the device of a lambda meter*

In Figure 1.22, the matrices are in order: *CSPS*, which reflects the component *s* of the incident field, then the quarter-wave plate Q^{45}, then a first mirror, then the mirror M2, the mirror again (2 times reflection, M2 = 1), the quarter-wave plate on the return path, i.e. Q-^{45}, transmission in the cube, therefore transformation into a *p* wave, hence *CSPp*, and the polarizer *s* at 45° before the detector.

To these transformations by the optical elements should be added the path of the beam retaining the *s* part of the wave, hence the multiplication by the matrix $D(\varphi s)$ which switches with all the other matrices.

1.4.3.7. *Answer to question 7*

$$CSP_S = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad Q_{45} = \frac{1}{2}\begin{pmatrix} 1+i & 1-i \\ 1-i & 1+i \end{pmatrix} \quad M_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$Q_{-45} = \frac{1}{2}\begin{pmatrix} 1+i & -(1-i) \\ -(1-i) & 1+i \end{pmatrix} \quad CSP_p = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad P_{S45} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

$$D(\varphi) = \begin{pmatrix} \exp(i\varphi) & 0 \\ 0 & \exp(i\varphi) \end{pmatrix}$$

$\varphi = nk_0\delta = 2\pi n\delta/\lambda_0$ with $\delta = r_{is}$ the round trip between the source and the detector, traversed by the light on the arm where the cube returns by reflection the polarization *s*.

Under MATLAB (appendix): >> M_S= $P^S{}_{45}$*CSP_P *Q_{-45}*$M2$*$Q45$*$CSPS$; leads to the matrix $M = -\frac{1}{2}\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$.

Multiplication by $D(\varphi)$ leads to $-\frac{1}{2}\begin{pmatrix} \exp(i\varphi) & 0 \\ \exp(i\varphi) & 0 \end{pmatrix}$

1.4.3.8. *Answer to question 8*

$$\vec{E}_S(A) = -\frac{1}{2}\begin{pmatrix} \exp(i\varphi) & 0 \\ \exp(i\varphi) & 0 \end{pmatrix} * \vec{E}_L = -\frac{1}{2}\begin{pmatrix} \exp(i\varphi) & 0 \\ \exp(i\varphi) & 0 \end{pmatrix}\begin{pmatrix} |E_S|e^{i\varphi_{0s}} \\ |E_P|e^{i\varphi_{0p}} \end{pmatrix} e^{-i\omega t}$$

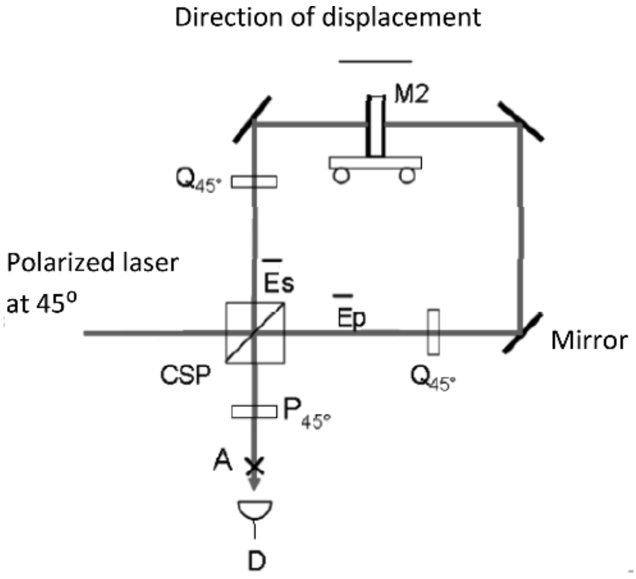
$$\vec{E}_S(A) = -\frac{1}{2}e^{i\varphi}\begin{pmatrix} |E_S|e^{i\varphi_{0s}} \\ |E_S|e^{i\varphi_{0s}} \end{pmatrix} e^{-i\omega t} = -K_S\frac{|E_S|e^{i\varphi_{rs}}}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

By identification: a $e^{i\varphi} = e^{i\varphi_{rs}}$ and $K_S = e^{i\varphi_{0s}}e^{-i\omega t}$

The expression of $\varphi_{rS} = 2\pi n(r_s)/\lambda_0$.

1.4.3.9. *Answer to question 9*

On the other arm, we have:



Reminder: Figure 1.22. *Schematic diagram of the device of a lambda meter*

A path by transmission in the cube, either *CSPP*, then the crossing of the quarter-wave plate, or *Q*45, the reflection on the mirror once on the outward path and once on the return, which amounts to multiplying by *I*, of even for the following mirror, the reflection on M2, or M2, the crossing of the quarter-wave plate on the return path or Q^{-45}, the reflection on the cube corner, or *CSPS*, then the crossing of the polarizer at 45° and the propagation matrix $D(\varphi_p)$ on the arm *p* by transmission.

Thus: $\vec{E}_P(A) = D(\varphi_P) * P_{45}^{S} * CSP_S * Q_{-45} * M_2 * Q_{45} * CSP_P * \vec{E}_L$.

1.4.3.10. *Answer to question 10*

From the matrices given in question 7 only $D(\varphi)$ differs:

$$D(\varphi) = \begin{pmatrix} \exp(i\varphi) & 0 \\ 0 & \exp(i\varphi) \end{pmatrix}$$

where $\varphi = nk_0\delta = 2\pi n\delta/\lambda_0$ with $\delta = r_{ip}$ the round trip between the source and the detector, traveled by light on the arm where the cube operates in transmission and returns the polarization *p*.

1.4.3.11. *Answer to question 11*

The multiplication of matrices leads to:

$$\vec{E}_P(A) = \tfrac{1}{2}\begin{pmatrix} 0 & \exp(i\varphi) \\ 0 & \exp(i\varphi) \end{pmatrix} * \vec{E}_L = \tfrac{1}{2}\begin{pmatrix} 0 & \exp(i\varphi) \\ 0 & \exp(i\varphi) \end{pmatrix}\begin{pmatrix} |E_S|e^{i\varphi_{0s}} \\ |E_P|e^{i\varphi_{0p}} \end{pmatrix} e^{-i\omega t}$$

$$\vec{E}_P(A) = \frac{1}{2}e^{i\varphi}\begin{pmatrix} |E_P|e^{i\varphi_{0P}} \\ |E_P|e^{i\varphi_{0P}} \end{pmatrix} e^{-i\omega t} = K_P\frac{|E_P|e^{i\varphi_{rP}}}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

It can be obtained by identification: a $e^{i\varphi} = e^{i\varphi_{rP}}$ and $K_P = e^{i\varphi_{0P}}e^{-i\omega t}$

and $\varphi_{rP}=2\pi n(r_P)/\lambda_0$.

1.4.3.12. *Answer to question 12*

$$\frac{I}{\varepsilon_0 c} = \left|\vec{E}_S(A) + \vec{E}_P(A)\right|^2 = \left(\vec{E}_S(A) + \vec{E}_p(A)\right)^* \left(\vec{E}_S(A) + \vec{E}_p(A)\right)$$

where:

$$\frac{I}{\varepsilon_0 c} = \vec{E}_S(A)\left(\vec{E}_S(A)\right)^* + \vec{E}_p(A)\left(\vec{E}_p(A)\right)^* + \vec{E}_S(A)\left(\vec{E}_p(A)\right)^* + \vec{E}_p(A)\left(\vec{E}_S(A)\right)^*$$

Since: $\vec{E}_S(A) = -K_S\frac{|E_S|e^{i\varphi_{rs}}}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\vec{E}_P(A) = K_P\frac{|E_P|e^{i\varphi_{rP}}}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, then:

$$\vec{E}_S(A)\left(\vec{E}_S(A)\right)^* = (K_S)^*\frac{|E_S|e^{-i\varphi_{rs}}}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} K_S\frac{|E_S|e^{i\varphi_{rs}}}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$
$$= |K_S|^2\frac{|E_S|^2}{4}\begin{pmatrix} 1 \\ 1 \end{pmatrix}.\begin{pmatrix} 1 \\ 1 \end{pmatrix}\left|\vec{E}_S(A)\right|^2 = |K_S|^2\frac{|E_S|^2}{4}2 = |K_S|^2\frac{|E_S|^2}{2} \text{ and}$$
$$\left|\vec{E}_P(A)\right|^2 = |K_P|^2\frac{|E_P|^2}{4}2 = |K_P|^2\frac{|E_P|^2}{2}$$

For the cross term:

$$\vec{E}_S(A)\left(\vec{E}_p(A)\right)^* = -K_S\frac{|E_S|e^{i\varphi_{rs}}}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}.(K_P)^*\frac{|E_P|e^{-i\varphi_{rP}}}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\vec{E}_S(A)\left(\vec{E}_p(A)\right)^* = -K_S(K_P)^*\frac{|E_S||E_P|e^{i(\varphi_{rs}-\varphi_{rP})}}{2}$$

$$\vec{E}_P(A)\left(\vec{E}_S(A)\right)^* = -K_P(K_S)^* \frac{|E_S||E_P|e^{-i(\varphi_{rs}-\varphi_{rP})}}{2}$$

Given the expression of the initial laser field:

$$\vec{E}_L = \vec{E}_S + \vec{E}_P = \begin{pmatrix} |E_S|e^{i\varphi_{0s}} \\ |E_P|e^{i\varphi_{0P}} \end{pmatrix} e^{-i\omega t}$$

then: $\frac{I_L}{\varepsilon_0 c} = \vec{E}_L\left(\vec{E}_L\right)^* = |E_S|^2 + |E_P|^2 = I_0$ and in this case:

$$cos\alpha = \frac{|E_S|}{\sqrt{(|E_S|^2+|E_P|^2)}} \text{ and } sin\alpha = \frac{|E_P|}{\sqrt{(|E_S|^2+|E_P|^2)}}.$$

As $K_S = e^{i\varphi_{0S}}e^{-i\omega t}$ and $K_P = e^{i\varphi_{0P}}e^{-i\omega t}$; using the fact that at the output of the laser, the wave is polarized at 45°, then $e^{i\varphi_{0S}} = e^{i\varphi_{0P}}$, i.e. $K_S = K_P = K_0$, and $|E_S|^2 = |E_P|^2 = 1/2$, thus:

$$\frac{I}{\varepsilon_0 c} = |K_0|^2\frac{|E_S|^2}{2} + |K_0|^2\frac{|E_P|^2}{2} - |K_0|^2|E_S||E_P|\frac{e^{i(\varphi_{rs}-\varphi_{rP})} + e^{-i(\varphi_{rs}-\varphi_{rP})}}{2}$$

$$\frac{I}{\varepsilon_0 c} = \frac{|K_0|^2}{2}(|E_S|^2 + |E_P|^2)(1 - \frac{2|E_S||E_P|}{(|E_S|^2 + |E_P|^2)}\cos(\varphi_{rs} - \varphi_{rP}))$$

and since $\varphi_{rs} - \varphi_{rP} = \frac{2\pi n}{\lambda}(r_s - r_P)$:

$$\frac{I}{\varepsilon_0 c} = \frac{|K_0|^2}{2}(1 - \cos(\frac{2\pi n}{\lambda}(r_s - r_P))$$

Since $r_s - r_P = 0$:

$$\frac{I}{\varepsilon_0 c} = \frac{|K_0|^2}{2}(1 - 1) = 0$$

1.4.3.13. *Answer to question 13*

If the mirror is moved to the right, on path r_s, the path is $2\Delta z$ longer, and on path r_P, the path is $2\Delta z$ shorter. Then:

$$\frac{I}{\varepsilon_0 c} = \frac{|K_0|^2}{2}(1 - \cos(\frac{2\pi n}{\lambda}(r_s + 2\Delta z - (r_P - 2\Delta z))))$$

$$\frac{I}{\varepsilon_0 c} = \frac{|K_0|^2}{2}(1 - \cos(\frac{2\pi n}{\lambda}(r_s - r_P + 4\Delta z)))$$

$$\frac{I}{\varepsilon_0 c} = \frac{|K_0|^2}{2}(1 - \cos(\frac{2\pi n}{\lambda}4\Delta z))$$

1.4.3.14. *Answer to question 14*

$$\frac{2\pi n}{\lambda}4\Delta z = 2\pi \Rightarrow \Delta z = \frac{\lambda}{4n}$$

1.4.3.15. *Answer to question 15*

$$N = \frac{d}{\Delta z} \Rightarrow N = 4n\frac{d}{\lambda}$$

Digital application: $N = 4 \text{x} 1.000247 \text{x} 10\ 10^{-2}/(632.991458\ 10^{-9}) = 632\ 076$

1.4.3.16. *Answer to question 16*

$\Delta z = \frac{\lambda}{4n}$ and $N = \frac{d}{\Delta z} \Rightarrow N = 4n\frac{d}{\lambda}$

The ratio is thus: $\frac{N_i}{N} = \frac{4n\frac{d}{\lambda_i}}{4n\frac{d}{\lambda}} \Rightarrow \lambda_i = \lambda\frac{N}{N_i}$

Digital application:

$\lambda_I = 632.991458 10^{-9} \times (632\ 076/512\ 643) = 780.462639\ 10^{-9}$ m

1.4.3.17. *Answer to question 17*

$$N_i = 4n\frac{d}{\lambda_i}$$

If the relative uncertainty is equal to 10^{-6}, then N is equal to 10^6. In this case, d is given by:

$d = \frac{N_i\lambda_i}{4n} = 10^6 \times 780.462639\ 10^{-9}/(4 \times 1.000247) = 19.5$ mm.

1.4.4. *The homodyne interferometer and refractometer*

The electric field is usually composed of a component E_p which is parallel to the plane of incidence and of a component E_s perpendicular to the plane of incidence. The coordinate axes for the *p* wave and *s* waves are given in Figure 1.24.

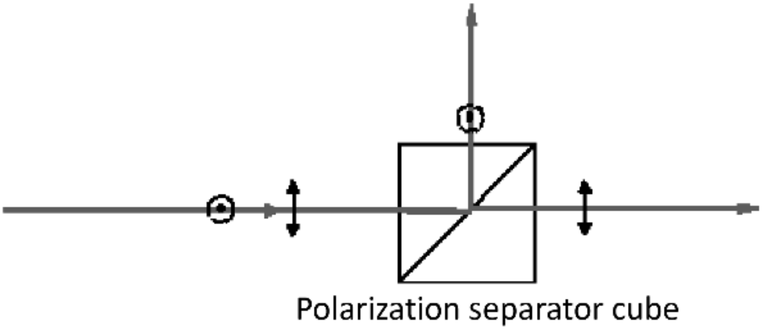


Figure 1.24. *Coordinate axes for p wave and s wave. For a color version of this figure, see www.iste.co.uk/dahoo/metrology2.zip*

The refractive index of a liquid medium can be measured using a laser interferometer. The sample to be analyzed is placed in one of the arms of the interferometer, and the refractive index of the medium is obtained by comparing the phase difference between the two arms of the measuring instrument.

Homodyne interferometers are commonly used in quantum optics [GRO 01, WEN 05, FUW 15].

A homodyne interferometer is represented schematically in Figure 1.25. The laser used is a helium–neon laser emitting an unpolarized light beam at a wavelength of 632.991 nm in vacuum. The instrument is equipped with an S polarizer (P^{45}) oriented at 45°, a quarter-wave plate oriented at 45° (- Q^{45}) and a polarization splitter cube (CSP) which is called CSPS (reflection mode) or CSPS (transmission mode).

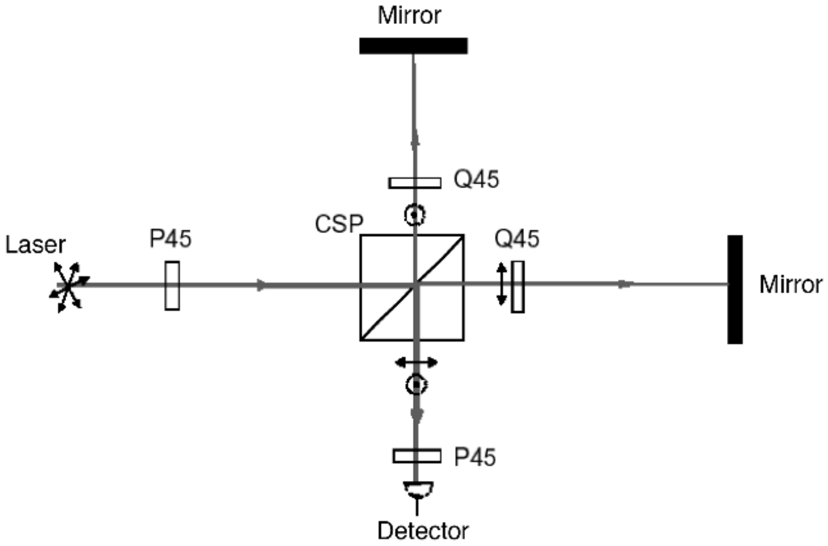


Figure 1.25. *Diagram of a homodyne interferometer and of the paths s and p. For a color version of this figure, see www.iste.co.uk/dahoo/metrology2.zip*

Let E_s be the field of the light wave propagating on one of the interferometer arms, and E_p, the field of the wave propagating on the other arm such that:

$$\vec{E}_s(\vec{r},t) = \left|\vec{E}_s\right| \exp\left(i\vec{k}\vec{r_0}\right) \exp\left(i\left(-\omega t + \vec{k}\vec{r_s}\right)\right) \vec{e_s}$$

and

$$\vec{E}_p(\vec{r},t) = \left|\vec{E}_p\right| \exp\left(i\vec{k}\vec{r_0}\right) \exp\left(i\left(-\omega t + \vec{k}\vec{r_p}\right)\right) \vec{e_p}$$

where $\omega = 2\pi\nu_0$ is the pulse of the wave, and ν_0 the frequency of the wave being equal to $\nu_0 = c/\lambda_0$ in a vacuum. We give $c = 299\ 792\ 458\ \text{ms}^{-1}$.

To obtain the resulting electric field at the photodetector apply Jones formalism, and use the vectors and matrices associated with the optical elements and the rotation matrices, the reference defined by the polarization s and p of the incident wave (see the Appendix).

1) Give the expression of the components of the resulting field at the level of the photodetector. This result will be obtained applying the Jones matrix calculation. The matrices of the different components are given in the appendix. r_s and r_p represent, respectively, the total distances traveled by the s wave and by the p wave between the laser source after the polarizer and the photodetector on the corresponding arms.

2) Obtain the expression of the intensity of the light wave at the output of the interferometer. Set $I_0 = |E_S|^2 + |E_P|^2 = 1$ and $I = \varepsilon_0 c < E>^2$.

3) In order to measure the refractive index n_L of the liquid medium, a sample containing this liquid is placed in the fixed arm of the interferometer (Figure 1.26).

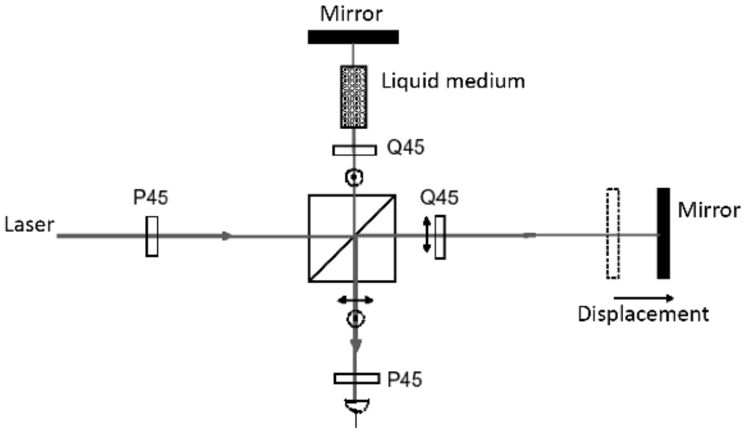


Figure 1.26. *Diagram of the interferometric measurement device. For a color version of this figure, see www.iste.co.uk/dahoo/metrology2.zip*

Since the length of the cell containing the liquid medium is L and its index is n_L, obtain from question 2 the expression of the light intensity. The mobile mirror is fixed and it is taken that $r_s = r_p$.

4) The moving mirror is displaced by a quantity Δz in the direction indicated in Figure 1.26 in order to compensate for the phase shift

induced by the introduction of the liquid medium and measure n_L. Express the light intensity as a function of Δz.

5) A displacement of 13.35 mm is measured. Since $n_{air} = 1.000274$ and L = 10.32 mm, calculate the refractive index of the liquid medium.

1.4.4.1. *Answer to question 1*

From the paths made by the two components *s* and *p*, the component *s* of the electric field on the photodetector is expressed by:

$$\vec{E}_s(PD) = D(\varphi_s) * P_{45}^{s} * CSPP * Q_{-45} * M * Q_{45} * CSPS * P_{45} * \vec{E}_L$$

The component *p* on the photodetector is expressed by:

$$\vec{E}_P(PD) = D(\varphi_P) * P_{45}^{s} * CSPS * Q_{-45} * M * Q_{45} * CSPP * P_{45} * \vec{E}_L$$

with: $\vec{E}_L = \begin{pmatrix} |\vec{E}_0| e^{ikr_0} e^{-i\omega t} \\ |\vec{E}_0| e^{ikr_0} e^{-i\omega t} \end{pmatrix}$

P_{45} (*S*), Q_{45}, Q_{-45} and *CSP* are then calculated:

$$P_{45}^{S} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},\ Q_{45} = \frac{1}{2}\begin{pmatrix} 1+i & 1-i \\ 1-i & 1+i \end{pmatrix},\ Q_{-45} = \frac{1}{2}\begin{pmatrix} 1+i & -1+i \\ -1+i & 1+i \end{pmatrix}$$

$$CSP_S = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad CSP_P = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad M = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$D(\varphi_S) = \begin{pmatrix} \exp(i\varphi_S) & 0 \\ 0 & \exp(i\varphi_S) \end{pmatrix} and\ D(\varphi_P) = \begin{pmatrix} \exp(i\varphi_P) & 0 \\ 0 & \exp(i\varphi_P) \end{pmatrix}$$

With $\varphi_S = nkr_S = 2\pi n r_S/\lambda$ and $\varphi_P = nkr_P = 2\pi n r_P/\lambda$, where r_S and r_P correspond to the round trips between the source and the detector on each arm.

Multiplying the matrices leads to:

$$\vec{E}_s(PD) = -\frac{1}{2}\begin{pmatrix} e^{inkr_s} & 0 \\ e^{inkr_s} & 0 \end{pmatrix} * \vec{E}_L = -\frac{1}{2}|\vec{E}_0| e^{ikr_0} e^{-i\omega t} e^{inkr_s} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

and

$$\vec{E}_p(PD) = \frac{1}{2}\begin{pmatrix}0 & e^{inkr_P}\\ 0 & e^{inkr_P}\end{pmatrix} * \vec{E}_L = \frac{1}{2}\left|\vec{E}_0\right| e^{ikr_0} e^{-i\omega t} e^{inkr_P}\begin{pmatrix}1\\ 1\end{pmatrix}$$

where: $\vec{E}_L(PD) = \vec{E}_S(PD) + \vec{E}_P(PD) = \frac{1}{2}\begin{pmatrix}-e^{inkr_s} & e^{inkr_P}\\ -e^{inkr_s} & e^{inkr_P}\end{pmatrix} * \vec{E}_L$

$$\vec{E}_L(PD) = -\frac{1}{2}\left|\vec{E}_0\right| e^{ikr_0} e^{-i\omega t} e^{inkr_s}\begin{pmatrix}1\\ 1\end{pmatrix} + \frac{1}{2}\left|\vec{E}_0\right| e^{ikr_0} e^{-i\omega t} e^{inkr_P}\begin{pmatrix}1\\ 1\end{pmatrix}$$

$$\vec{E}_L(PD) = -\frac{1}{2}\left|\vec{E}_0\right| e^{ikr_0} e^{-i\omega t} e^{inkr_s}\begin{pmatrix}1\\ 1\end{pmatrix} + \frac{1}{2}\left|\vec{E}_0\right| e^{ikr_0} e^{-i\omega t} e^{inkr_P}\begin{pmatrix}1\\ 1\end{pmatrix}$$

$$\vec{E}_L(PD) = \frac{1}{2}\left|\vec{E}_0\right| e^{ikr_0} e^{-i\omega t}\begin{pmatrix}-e^{inkr_s} + e^{inkr_P}\\ -e^{inkr_s} + e^{inkr_P}\end{pmatrix}$$

1.4.4.2. *Answer to question 2*

As $I = \varepsilon_0 c_0 \langle \vec{E}_L(PD).\vec{E}_L(PD)^* \rangle$

$$\frac{I}{\frac{\varepsilon_0 c_0}{2}} = \left|\vec{E}_S(PD)\right|^2 + \left|\vec{E}_P(PD)\right|^2 + 2(\vec{E}_S(PD)\vec{E}_P{}^*(PD) + \vec{E}_P(PD)\vec{E}_S{}^*(PD)$$

and: $\left|\vec{E}_S(PD)\right|^2 = \frac{1}{2}\left|\vec{E}_0\right|^2$

and $\left|\vec{E}_P(PD)\right|^2 = \frac{1}{2}\left|\vec{E}_0\right|^2$

and $\left|\vec{E}_S(PD)\right|^2 + \left|\vec{E}_P(PD)\right|^2 = \left|\vec{E}_0\right|^2$

The crossed terms lead to:

$$\vec{E}_S(PD)\vec{E}_P{}^*(PD) + \vec{E}_P(PD)\vec{E}_S{}^*(PD) = -\frac{1}{2}\left|\vec{E}_0\right|^2 (e^{in(kr_S - kr_P)} + e^{-in(kr_S - kr_P)})$$

$$\frac{I}{\frac{\varepsilon_0 c_0}{2}} = \left|\vec{E}_0\right|^2 - \left|\vec{E}_0\right|^2 \cos(\, kr_s - kr_P) = \left|\vec{E}_0\right|^2 \left(1 - \cos(\frac{2\pi}{\lambda} n(r_s - r_P)\right)$$

Since $I_0 = \left|\vec{E}_0\right|^2 = 1$

$$\frac{I}{\frac{\varepsilon_0 c_0}{2}} = \left(1 - \cos(\frac{2\pi}{\lambda} n(r_s - r_P)\right)$$

This can be written using the matrix form by:

$$\vec{E}_L(PD) = \vec{E}_S(PD) + \vec{E}_P(PD) = \frac{1}{2}\begin{pmatrix} -e^{ikr_s} & e^{ikr_P} \\ -e^{ikr_s} & e^{ikr_P} \end{pmatrix} * \vec{E}_L$$

$$\frac{I}{\frac{\varepsilon_0 c_0}{2}} = \vec{E}_L(PD).\vec{E}_L(PD)^*$$

$$\frac{I}{\frac{\varepsilon_0 c_0}{2}} = \begin{pmatrix} -e^{ikr_s} & e^{ikr_P} \\ -e^{ikr_s} & e^{ikr_P} \end{pmatrix} \begin{pmatrix} \left|\vec{E}_0\right| e^{ikr_0} e^{-i\omega t} \\ \left|\vec{E}_0\right| e^{ikr_0} e^{-i\omega t} \end{pmatrix} . \begin{pmatrix} -e^{-ikr_s} & e^{-ikr_P} \\ -e^{-ikr_s} & e^{-ikr_P} \end{pmatrix} \begin{pmatrix} \left|\vec{E}_0\right| e^{-ikr_0} e^{i\omega t} \\ \left|\vec{E}_0\right| e^{-ikr_0} e^{i\omega t} \end{pmatrix}$$

Finally: $I_F(PD) = \frac{\varepsilon_0 c_0}{2}\left(1 - \cos(\frac{2\pi n}{\lambda}(r_p - r_s)\right)$

1.4.4.3. *Answer to question 3*

Since $r_p = r_s$, the only path difference between the two paths in each arm comes from the passage of the light beam in the liquid of index n_L, in the arm connected to the fixed mirror where the polarization is *s*. Consequently:

$$I_F(PD) = \frac{\varepsilon_0 c_0}{2}\left(1 - \cos(\frac{2\pi}{\lambda}((r_p - (r_s - 2L + 2n_L L)\right)\right)$$

$$I_F(PD) = \frac{\varepsilon_0 c_0}{2}\left(1 - \cos(\frac{2\pi}{\lambda}(2(n_L - 1)L)\right)$$

1.4.4.4. *Answer to question 4*

If the mobile mirror moves to Δz, there is a differential path length on the arm connected to the mobile mirror, hence:

$$I_F(PD) = \frac{\varepsilon_0 c_0}{2}\left(1 - \cos(\frac{2\pi}{\lambda}(2n_{air}\Delta z - 2(n_L - 1)L)\right)$$

1.4.4.5. *Answer to question 5*

The intensity is zero at the detector when the cosine is equal to 1, hence the following equation is obtained:

$$2n_{air}\Delta z - 2(n_L - 1)L = 0 \Rightarrow n_L - 1 = \frac{n_{air}\Delta z}{L}.$$

Consequently: $n_L = 1 + \frac{n_{air}\Delta z}{L} = 1 + \frac{1{,}000274 \times 10{,}05}{10{,}32} = 1{,}974\ 104$

1.4.5. *The heterodyne interferometer*

In the diagram of the Michelson interferometer in Figure 1.27, a laser beam is split into two beams by a non-polarizing 50/50 beam splitter plate. The arm where the reflection takes place on the moving mirror is the measurement beam and the other arm where the beam is reflected on the fixed mirror is the reference beam. On the return optical path, the two beams cross the splitter plate. A part of each beam returns to the laser source, while the rest of the beams are superimposed on the photodiode. The intensity on the photodiode is directly related to the displacement δ of the mobile mirror. In this configuration, the displacement information is a direct current (d.c.). However, d.c. signals are difficult to deal with in the low frequency domain because of multiple disturbances such as, for example, measurement time drifts caused by amplifier gains. It is much easier to obtain and analyze the displacement information in the form of an alternative current (a.c.) signal. To generate this a.c. signal, a heterodyne interferometer design based on two lasers of frequencies ω_1 and ω_2 is used.

The principle of the heterodyne interferometer is based on the Doppler effect. A heterodyne interferometer has two beams of frequencies ω_1 and ω_2 close to a few megahertz, which are linearly and orthogonally polarized with respect to each other and are spatially separated by a polarization splitter cube.

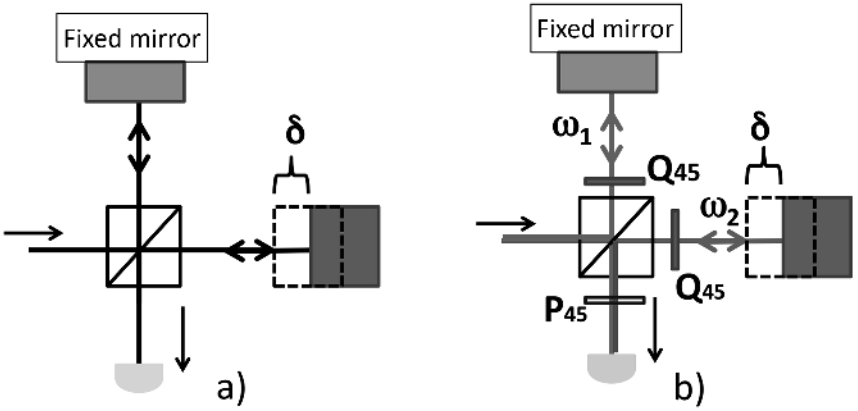


Figure 1.27. *Laser interferometer: a) homodyne and b) heterodyne. For a color version of this figure, see www.iste.co.uk/dahoo/metrology2.zip*

The behavior of the reference beam is similar to that of the homodyne interferometer. Initially, it is s polarized. Then, it is reflected by the beam splitter plate in the direction of the reference mirror. After a double passage through a quarter-wave plate, it passes through the separator plate again to be mixed with the measurement beam. The measurement beam is reflected by the moving mirror moving at a constant speed v, which has the effect of shifting its frequency by Δv_D by the Doppler effect. At the output of the interferometer, the two beams, measure and reference, are mixed using a polarizer generating an interference pattern. After passing through a polarizer (45°), the two beams are superimposed on a photodiode. The information on the displacement of the witness mirror is in phase with the sinusoidal signal at the frequency $\omega = |\omega_1 - \omega_2|$. The beat frequency between the two beams, observed at the photodetector, thus constitutes the measurement signal S_3.

The fields of application are dimensional metrology, microelectronics and production chains for high-tech industries. Integrated circuits are made by photolithography. Photolithography is a process that reproduces a pattern, specific to the desired chip, by projection of light onto a silicon substrate (or wafer) of a mechanical mask. A semiconductor component is made from several layers layered on top of each other. At each layer, the wafer is removed from the device for treatment and then put back into place for the projection of the next layer. In order to ensure the interconnection of integrated circuits on the previous layer, the wafer must be repositioned to the nanometer scale (10^{-9} m). For example, in 256 MB Flash memories, the distance between two lines is 80 nm. Decreasing this distance amounts to increasing the capacity of the component or its speed. The economic stakes involved in wanting to have positioning systems that are as exact as possible are important. These systems are monitored in manufacturing plants by Michelson interferometers (Figure 1.28).

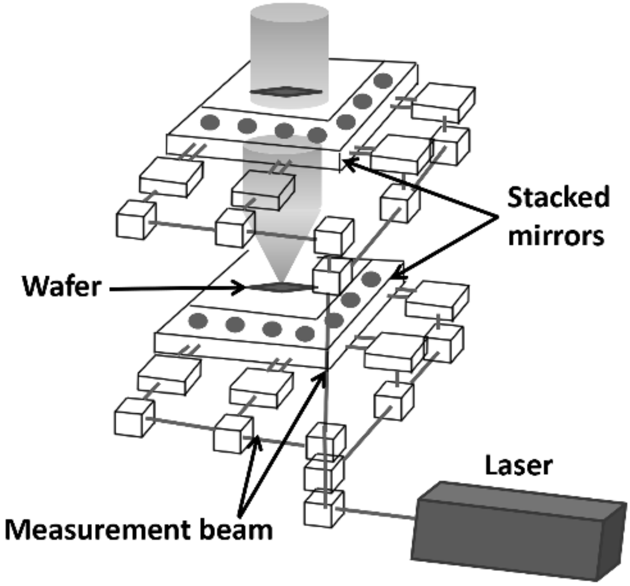


Figure 1.28. *Dimensional metrology. For a color version of this figure, see www.iste.co.uk/dahoo/metrology2.zip*

1.4.5.1. *Exercise: heterodyne interferometer*

The heterodyne interferometer [SUT 87, OLD 93, DEM 98, TOP 03, SCH 06] uses a laser source emitting two monochromatic plane waves of the same amplitude and the same initial phase but with crossed polarizations. The angular frequencies (or pulsations) of the two beams are different. ω_1 is the pulsation of the *s* wave and ω_2 that of the *p* wave. The principle of operation is as follows: when the moving mirror moves, the frequency of the light wave is shifted due to the Doppler effect. By measuring the Doppler frequency, it is possible to deduce the speed of the mirror (RADAR principle but at radiometric frequencies). Knowing the speed, the displacement by means of a time measurement is obtained.

The following Figure 1.29 displays the optical device design and the elements of a heterodyne interferometer with cube corners as reflectors.

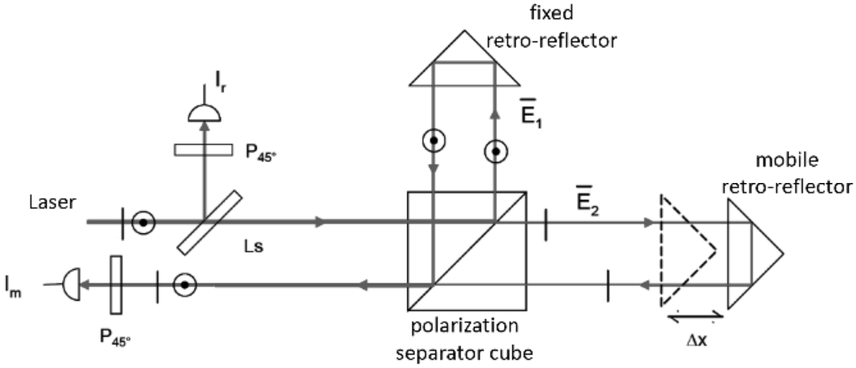


Figure 1.29. *Heterodyne laser interferometer with cube corners. For a color version of this figure, see www.iste.co.uk/dahoo/metrology2.zip*

Several modern technologies use the Doppler effect. Radar, for example (RAdio Detection And Ranging), was developed just before World War II to detect and locate enemy airplanes at long distances. Radar pointers are based on the Doppler effect. Such direct read pointers are used on sports fields to measure ball speed. The laser source emits a monochromatic beam stabilized in frequency which passes through an acousto-optical modulator (AOM) which generates two beams polarized at 90°, of wavelengths λ_1 = 632.991528 nm and λ_2 = 632.991501 nm at different frequencies [DEM 98, TOP 03]. The frequency difference is 20 MHz.

Using the Lorentz transformation formulas, it can be deduced in spectroscopy [DAH 19] that a molecule moving with a speed $\vec{v}$, in the laboratory frame of reference, absorbs radiation of wave number $\omega = 2\pi c/\lambda$ given by:

$$\omega = \omega_0 \left(1 + \frac{\vec{v}.\vec{k}}{c}\right) \qquad [1.14]$$

where ω_0 is the laser probe wave number corresponding to the transition between the two energy levels of the molecule, $\vec{k}$ is a unit vector in the direction of the beam and c is the speed of light.

Likewise, if ω_0 is the frequency of the light wave arriving on a mobile mirror moving at a speed V, after reflection on the mirror, the reflected beam frequency is $\omega_d = \omega_0(1 \pm \frac{2nV}{c})$, where $c = 299\,792\,458$ ms^{-1} is the speed of light in vacuum and $n = 1.000247$ is the refractive index of air. The sign ± depends on the direction of movement. If the mirror gets closer to (away from) the source, then the + (-). sign will be used. The speed of the mirror V can then be deduced.

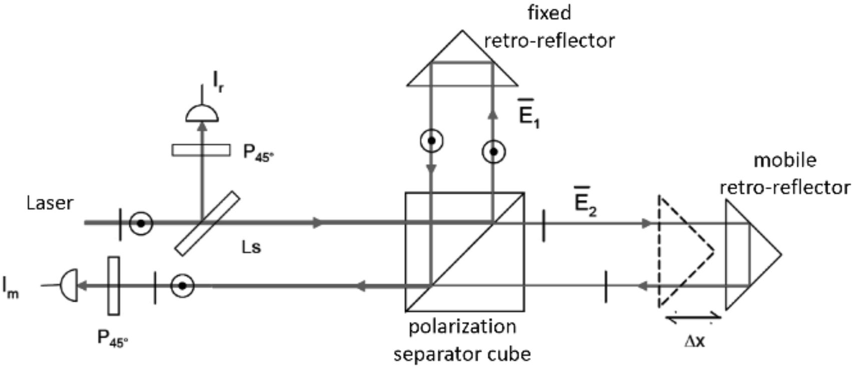
1) Calculate using matrix calculus, the expressions for the intensities I_{ref} and I_{mes}. Set $|\vec{E}_s| = |\vec{E}_p| = 1$, $\Delta\omega = \omega_2 - \omega_1$ and neglect the initial phase terms.

2) The associated electronics circuit measures the phase difference and uses this difference to provide the position and speed as measurement data. Calculate the Doppler frequency associated with the displacement of the mobile mirror at a speed of 2 mm.s^{-1}. Since the detector bandwidth is 12 GHz, calculate the limit speed measurable by this method if the associated electronics circuit has a 26.6 MHz bandwidth.

3) The travel time is measured with an ultra-stable Quartz clock. Deduce from (1) the displacement Δx.

4) What should be the time measurement uncertainty to obtain an uncertainty of 1 nm on the displacement.

1.4.5.1.1. Solution to question 1



Reminder: Figure 1.29. *Heterodyne laser interferometer with cube corners*

The reference beam

The reference beam is detected before it passes through the interferometer by a detector while crossing a 45° polarizer (P_{45}).

Consequently:

$$\vec{E}_L = \begin{pmatrix} |\vec{E}_0| e^{ikr_0} e^{-i\omega_1 t} \\ |\vec{E}_0| e^{ikr_0} e^{-i\omega_2 t} \end{pmatrix} = \begin{pmatrix} |\vec{E}_0| e^{ikr_0} e^{-i\omega_1 t} \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ |\vec{E}_0| e^{ikr_0} e^{-i\omega_2 t} \end{pmatrix}$$

From the paths of the two components *s* and *p*, the component *s* of the electric field on the photodetector is:

$$\vec{E}_s(PD) = D(\varphi_s) * P_{45}^{s} * \vec{E}_S$$

and that the component *p* of the electric field on the photodetector is:

$$\vec{E}_P(PD) = D(\varphi_P) * P_{45}^{s} * \vec{E}_P$$

Since $P_{45}^{S} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ and $D(\varphi) = \begin{pmatrix} e^{i\varphi} & 0 \\ 0 & e^{i\varphi} \end{pmatrix}$

$$\vec{E}_s(PD) = \begin{pmatrix} e^{i\varphi_S} & 0 \\ 0 & e^{i\varphi_S} \end{pmatrix} \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} |\vec{E}_0| e^{ikr_0} e^{-i\omega_1 t} \\ 0 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} |\vec{E}_0| e^{i\varphi_S} e^{ikr_0} e^{-i\omega_1 t} \\ |\vec{E}_0| e^{i\varphi_S} e^{ikr_0} e^{-i\omega_1 t} \end{pmatrix}$$

and

$$\vec{E}_P(PD) = \begin{pmatrix} e^{i\varphi_P} & 0 \\ 0 & e^{i\varphi_P} \end{pmatrix} \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ |\vec{E}_0| e^{ikr_0} e^{-i\omega_1 t} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} |\vec{E}_0| e^{i\varphi_P} e^{ikr_0} e^{-i\omega_2 t} \\ |\vec{E}_0| e^{i\varphi_P} e^{ikr_0} e^{-i\omega_2 t} \end{pmatrix}$$

The intensity calculation is obtained using the relation:

$$I = \varepsilon_0 c_0 \vec{E}_L(PD). \vec{E}_L(PD)^*$$

$$\frac{I}{\varepsilon_0 c_0} = \left|\vec{E}_S(PD)\right|^2 + \left|\vec{E}_P(PD)\right|^2 + 2(\vec{E}_S(PD)\vec{E}_P{}^*(PD) + \vec{E}_P(PD)\vec{E}_S{}^*(PD)$$

Since: $\left|\vec{E}_S(PD)\right|^2 = \frac{1}{2}\left|\vec{E}_0\right|^2$ and $\left|\vec{E}_P(PD)\right|^2 = \frac{1}{2}\left|\vec{E}_0\right|^2$

and: $\left|\vec{E}_S(PD)\right|^2 + \left|\vec{E}_P(PD)\right|^2 = \left|\vec{E}_0\right|^2$

Considering equal initial phases, the crossed terms lead to:

$$\vec{E}_S(PD)\vec{E}_P{}^*(PD) + \vec{E}_P(PD)\vec{E}_S{}^*(PD) = \frac{1}{2}\left|\vec{E}_0\right|^2 (e^{-i(\omega_1-\omega_2)t} + e^{i(\omega_1-\omega_2)t})$$

$$\vec{E}_S(PD)\vec{E}_P{}^*(PD) + \vec{E}_P(PD)\vec{E}_S{}^*(PD) = \frac{1}{2}\left|\vec{E}_0\right|^2 (e^{i(\Delta\omega)t} + e^{-i\Delta\omega)t})$$

$$\frac{I}{\varepsilon_0 c_0} = \left|\vec{E}_0\right|^2 - \left|\vec{E}_0\right|^2 \cos(\,\Delta\omega t) = \left|\vec{E}_0\right|^2 (1 - \cos(\Delta\omega t))$$

Since $\left|\vec{E}_0\right|^2 = 1$, the calculation of the intensity leads to: $\frac{I}{\varepsilon_0 c_0} = (1 - \cos(\Delta\omega t))$.

The measurement beam

The measurement beam travels through the interferometer.

The S-type measurement beam is reflected by the polarization splitter cube (*CSPS*), then on two cube corners ($M^2 = 1$), before being reflected again on the polarization splitter cube (*CSPS*) and crosses a 45° polarizer (P_{45}) before reaching the detector (P_{45} * *CSPS* * *CSPS*).

The P-type measurement beam passes through the polarization splitter cube (*CSPP*), then it is reflected on two cube corners ($M^2 = 1$) moving *D*(*d*), before crossing the splitter cube again polarization (CSPP) and finally passes through a 45° polarizer (P_{45}) before reaching the detector (P_{45} * *CSPP* * *D*(*d*) * *CSPP*).

Analyzing the paths made by the two components *s* and *p*, the component s of the electric field on the photodetector is obtained:

$$\vec{E}_s(PD) = D(\varphi_s) * P^s_{45} * CSPS ** CSPS * \vec{E}_S$$

and the component *p* of the electric field on the photodetector is:

$$\vec{E}_P(PD) = D(\varphi_P) * P^s_{45} * CSPP * D(d) * CSPP * \vec{E}_P$$

Since:

$$\vec{E}_L = \begin{pmatrix} |\vec{E}_0| e^{ikr_0} e^{-i\omega_1 t} \\ |\vec{E}_0| e^{ikr_0} e^{-i\omega_2 t} \end{pmatrix} = \begin{pmatrix} |\vec{E}_0| e^{ikr_0} e^{-i\omega_1 t} \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ |\vec{E}_0| e^{ikr_0} e^{-i\omega_2 t} \end{pmatrix}$$

Using the P_{45} (*S*), *CSPS* and *CSPP* matrices, the phase shift matrix *D*(φ) and the Doppler effect matrix *D*(*d*), it is obtained that:

$$P^S_{45} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, CSP_S = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad CSP_P = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad M = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$D(\varphi_S) = \begin{pmatrix} \exp(i\varphi_S) & 0 \\ 0 & \exp(i\varphi_S) \end{pmatrix} and\ D(\varphi_P) = \begin{pmatrix} \exp(i\varphi_P) & 0 \\ 0 & \exp(i\varphi_P) \end{pmatrix}$$

$$D(d) = \begin{pmatrix} 0 & 0 \\ 0 & \exp(-i\omega_D t) \end{pmatrix}$$

with $\varphi_S = nkr_S = 2\pi nr_S/\lambda$ and $\varphi_P = nkr_P = 2\pi nr_P/\lambda 1$ where r_S and r_P correspond to the paths between the source and the detector on each arm, the multiplication of matrices leads to:

$$\vec{E}_S(PD) = \frac{1}{2}\begin{pmatrix} e^{inkr_s} & 0 \\ e^{inkr_s} & 0 \end{pmatrix} * \vec{E}_L = \frac{1}{2}|\vec{E}_0| e^{ikr_0} e^{-i\omega_1 t} e^{inkr_s} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

and

$$\vec{E}_P(PD) = \frac{1}{2}\begin{pmatrix}0 & e^{inkr_P}\\ 0 & e^{inkr_P}\end{pmatrix} * \begin{pmatrix}0 & 0\\ 0 & e^{\pm i\omega_D t}\end{pmatrix} * \vec{E}_L = \frac{1}{2}\left|\vec{E}_0\right| e^{ikr_0} e^{-i(\omega_2 \pm \omega_D)t} e^{inkr_P}\begin{pmatrix}1\\1\end{pmatrix}$$

From where:

$$\vec{E}_L(PD) = \vec{E}_S(PD) + \vec{E}_P(PD) = \frac{1}{2}\begin{pmatrix}e^{inkr_S} & e^{\pm i\omega_D t} e^{inkr_P}\\ e^{inkr_S} & e^{\pm i\omega_D t} e^{inkr_P}\end{pmatrix} * \vec{E}_L$$

$$\vec{E}_L(PD) = \frac{1}{2}\left|\vec{E}_0\right| e^{ikr_0} e^{-i\omega_1 t} e^{inkr_S}\begin{pmatrix}1\\1\end{pmatrix} + \frac{1}{2}\left|\vec{E}_0\right| e^{ikr_0} e^{-i\omega_2 t} e^{\pm i\omega_D t} e^{inkr_P}\begin{pmatrix}1\\1\end{pmatrix}$$

The intensity calculation is made from: $I = \varepsilon_0 c_0 \vec{E}_L(PD).\vec{E}_L(PD)^*$

$$\frac{I}{\varepsilon_0 c_0} = \left|\vec{E}_S(PD)\right|^2 + \left|\vec{E}_P(PD)\right|^2 + 2(\vec{E}_S(PD)\vec{E}_P{}^*(PD) + \vec{E}_P(PD)\vec{E}_S{}^*(PD)$$

Since:

$$\left|\vec{E}_S(PD)\right|^2 = \frac{1}{2}\left|\vec{E}_0\right|^2 \text{ and } \left|\vec{E}_P(PD)\right|^2 = \frac{1}{2}\left|\vec{E}_0\right|^2$$

and by hypothesis $\left|\vec{E}_S(PD)\right|^2 + \left|\vec{E}_P(PD)\right|^2 = \left|\vec{E}_0\right|^2$, the crossed terms lead to:

$$\vec{E}_S(PD)\vec{E}_P{}^*(PD) + \vec{E}_P(PD)\vec{E}_S{}^*(PD) = \frac{1}{2}\left|\vec{E}_0\right|^2 (e^{-i(\omega_1 - \omega_2 \pm \omega_D)t} + e^{i(\omega_1 - \omega_2 \pm \omega_D)t})$$

$$\vec{E}_S(PD)\vec{E}_P{}^*(PD) + \vec{E}_P(PD)\vec{E}_S{}^*(PD) = \frac{1}{2}\left|\vec{E}_0\right|^2 (e^{i(\Delta\omega \mp \omega_D)t} + e^{-i\Delta\omega \mp \omega_D)t})$$

$$\frac{I}{\varepsilon_0 c_0} = \left|\vec{E}_0\right|^2 - \left|\vec{E}_0\right|^2 \cos(\,(\Delta\omega \mp \omega_D)t) = \left|\vec{E}_0\right|^2 (1 - \cos((\Delta\omega \mp \omega_D)t))$$

Since $\left|\vec{E}_0\right|^2 = 1$,

$$\frac{I}{\varepsilon_0 c_0} = (1 - \cos((\Delta\omega \mp \omega_D)t))$$

Note that the photodetectors, which detect the intensities, carry out the multiplication of the signals and deliver electrical signals at frequencies $\Delta\omega$ and $\Delta\omega \mp \omega_D$, where $\omega_D = \omega_2 \frac{2nV}{c}$.

1.4.5.1.2. Answer to question 2

The laser source emits a monochromatic beam of wavelength λ = 632.991501 nm in vacuum. After reflection on the mirror, the pulse of the wave becomes $\omega_d = \omega_2(1 \pm \frac{2nV}{c})$, where c = 299 792 458 ms^{-1} is the speed of light in vacuum and n = 1.000247 is the air refractive index.

In this case: $\omega_D = \omega_2 \frac{2nV}{c} = \frac{2\pi c}{\lambda}\frac{2nV}{c}$

and: $\omega_2 = \frac{2\pi c}{\lambda} = \frac{2\pi \times 299\ 792\ 458}{632.991501\ 10^{-9}} = 2.975792825\ 10^{15} rad/s$

$$\frac{2nV}{c} = \frac{2 \times 1.000247 \times 210^{-3}}{299\ 792\ 458} = 1.33459\ 10^{-11}$$

$$\omega_D = \omega_2 \frac{2nV}{c} = 2.975792825\ 10^{15} \times 1.33459\ 10^{-11} = 3.9714\ 10^{4} rad/s$$

Expressed as a frequency, $f_D = 6320.76 Hz = 6.32 kHz$

In the pulse space, the reference signal is observed at the frequency Δω and the measuring signal at the pulse $\Delta\omega + \omega_D$.

Since λ_1 = 632.991528 nm and λ_2 = 632.991501 nm,

$\Delta\omega = \frac{2\pi c}{\lambda_2} - \frac{2\pi c}{\lambda_1} = 2\pi\ \times 299\ 792\ 458 \left(\frac{1}{632.991501\ 10^{-9}} - \frac{1}{632.991528\ 10^{-9}}\right) = 1.26931251\ 10^{8} \frac{rad}{s}$.

In frequency, this corresponds to: Δf = 20.201736 MHz. The spectrum of the signal in the frequency space is shown in Figure 1.30.

Since f the bandwidth is 26.6 MHz, the maximum signal is 13.3 MHz to the right or left of the frequency Δf.

In frequency, this corresponds to: $f_D = 13.3\ MHz$; in maximum displacement speed: $V = \frac{c f_D}{2n f_2} = \frac{\lambda_2 f_D}{2n} = \frac{632.991501\ 10^{-9} \times 13{,}310^{6}}{2 \times 1.000247} = 4.2 m/s$.

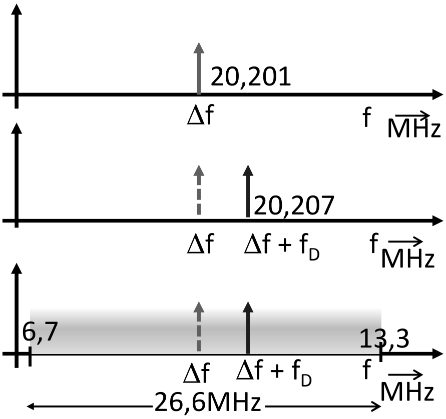


Figure 1.30. *Electrical signals measured in the detection bandwidth. For a color version of this figure, see www.iste.co.uk/dahoo/metrology2.zip*

Using a double-pass interferometer on the measuring path, a speed of 2.1 m/s can be measured. Note that a speed measurement of 1.9 km/s could be performed if the bandwidth was 12 GHz.

1.4.5.1.3. Answer to question 3

The speed is given by the displacement per unit of time. Consequently:

$$\Delta x = V\Delta t$$

The displacement at instant t is calculated by: $x(t) = \int_0^t \frac{\omega_D(t')}{2n\omega_2} c dt'$

If the measurement data are digital to improve resolution and precision, the displacement at time t is obtained by:

$$x(t) = \frac{c}{4\pi n\omega_2} \sum_{i=1}^{N} \Delta\Phi_i, \text{ where } N = t/\Delta T.$$

1.4.5.1.4. Answer to question 4

Resoluton is limited by the system noise and the phase measurement method [OLD 93]. The conventional digital phase meter detection techniques are mainly classified into three types: phase lock, fringe counting and zero crossing. A phase meter usually measures the rising time duration of the measurement signal relative to the counting clock, which is often set at 40 MHz. The phase resolution determines the position resolution. The output values of the phase meter are encoded on 8 bits. The incremental value, which is a fraction of the period of the clock at 40 MHz, for example, corresponds to the measurement of the rise time duration of the measurement signal. The phase resolution, which is 1 over 512 (9 bits because the state is 0 or 1), h, gives the position resolution of $\lambda/512\ M$, where M is the number of passes of the beam through the interferometer. For $M = 1$, we have $\lambda/512=1$ nm.

Regarding the measurement time, the equation, $\Delta t = \Delta x / V$, makes it possible to relate the uncertainty on t to the uncertainty on x.

1.4.6. *Application exercises on ellipsometry*

Ellipsometry is a non-destructive photonic technique, sensitive to surfaces and interfaces and which characterizes from a structural and optical point of view thin, mono or multilayer layers and massive materials (Chapter 8 of [DAH 16]). It is widely used in the microelectronics industry to control the thickness of “wafers”. It is also used to monitor in real time the growth of a thin film or a multilayer stack (Reliability of Multiphysical Systems Set, Volume 9, Chapter 1 [DAH 21a]).

This optical analysis technique is based on the study of the polarization state of polarized light after it is reflected from a surface. In most cases, the state of polarization is elliptical (Figure 1.31) (Chapter 3 of [DAH 16]). By determining the characteristics of this elliptical state, the thicknesses and optical indices of the materials under study can be determined, hence the name ellipsometry (Ψ, Δ).

An ellipsometer consists of optical devices on two arms and a sample holder (Chapter 8, [DAH 16]). One of the arms has a light source and a set of optical devices for obtaining from the source a wave in a known state of polarization. The second arm includes an analyzer to determine the state of polarization of the wave after its reflection on the sample is characterized. A quarter-wave plate or a more elaborate optical device is generally used as a compensator (C), either on the polarizer side (P) (PCSA mount) or on the analyzer side (A) (PSCA mount) to cancel the effect of the sample (S) on the state of polarization of the reflected light. Figure 1.32 displays the diagram of a phase modulation ellipsometer which includes a birefringent modulator whose optical axes can be modulated by an alternating electric voltage, thus modulating the state of polarization of the light passing through it. The polarizer or the analyzer can be rotated to vary the polarization state of light over time. A detection synchronous with the modulation frequency of the polarized light is performed to extract the signal corresponding to the light which probes the material under study.

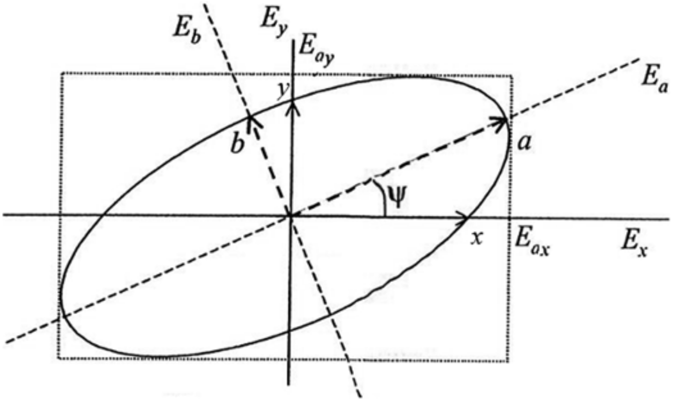


Figure 1.31. *Elliptical polarization state and ellipsometric parameters [DAH 16]*

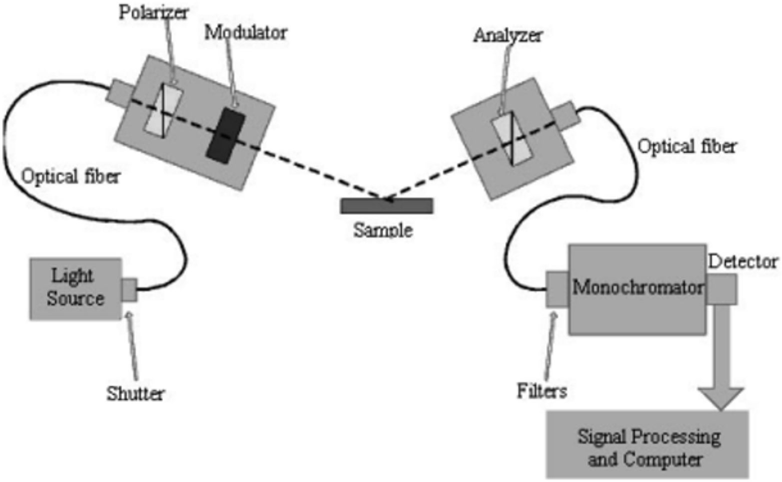


Figure 1.32. *Diagram of a phase modulation ellipsometer. For a color version of this figure, see www.iste.co.uk/dahoo/metrology2.zip*

1.4.6.1. *Exercise: ellisometry*

The photonic system presented in Figure 1.33 is in RAE mode (Rotating Analyzer Ellipsometer). The source is a helium–neon laser emitting a light beam with a wavelength in vacuum of 632, 791 nm. S designates the sample that interacts with light, and P^s (45°) is an S polarizer whose optical axis is oriented at 45° with respect to the axes of the frame of reference (*S*, *P*). P^s (*A*) is an *S* polarizer whose optical axis rotates at a variable angle *A* with respect to the axes of the frame of reference (*S*, *P*).

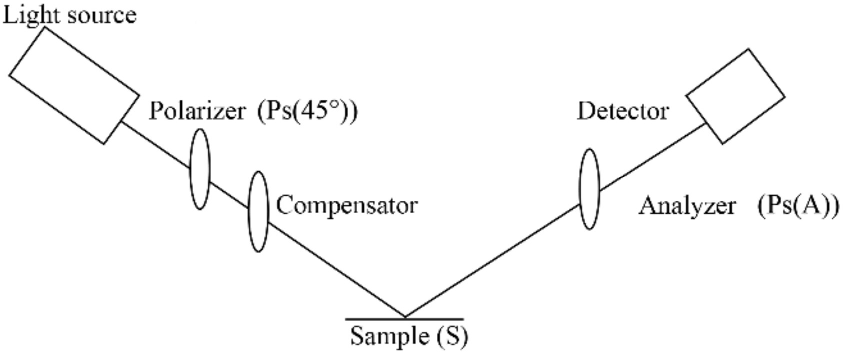


Figure 1.33. *Diagram of an ellipsometer in PCSA mode and reflected light. For a color version of this figure, see www.iste.co.uk/dahoo/metrology2.zip*

1) Formally, write the Jones equation that governs this photonic system.

Solution:

Since:

$$\vec{E}_L = \begin{pmatrix} |\vec{E}_0| e^{ikr_0} e^{-i\omega_1 t} \\ |\vec{E}_0| e^{ikr_0} e^{-i\omega_2 t} \end{pmatrix} = \begin{pmatrix} |\vec{E}_0| e^{ikr_0} e^{-i\omega_1 t} \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ |\vec{E}_0| e^{ikr_0} e^{-i\omega_2 t} \end{pmatrix}$$

P_{45} (*S*), *C*, the effect of the sample *SE*, P^S (*A*), the phase shift matrix *D*(φ) are used to calculate the electric field at the detector:

$$\vec{E}_L(PD) = D(\varphi) * P_{45}^A * SE * C * P_{45}^S * \vec{E}_L$$

and: $P_{45}^{S} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, $C = \begin{pmatrix} e^{i\varphi_1} & 0 \\ 0 & e^{i\varphi_2} \end{pmatrix}$, $SE = \begin{pmatrix} \sin(\Psi)\exp(i\nabla) & 0 \\ 0 & \cos(\Psi) \end{pmatrix}$,

$$P_{45}^{A} = \begin{pmatrix} \cos^2 A & sinAcosA \\ sinAcosA & \sin^2 A \end{pmatrix}, D(\varphi) = \begin{pmatrix} \exp(i\varphi) & 0 \\ 0 & \exp(i\varphi) \end{pmatrix}$$

$\varphi_I = nkr_i = 2\pi nr_i/\lambda$, where r_i corresponds to the path between the source and the detector in the area concerned.

2) Using Jones' matrix calculation and assuming that an S-type polarizer is placed in front of the detector show that the E_A laser field at the output of the device is of the following form:

$$\vec{E}_A(PD) = \left|\vec{E}_0\right| e^{-i\omega t} \begin{pmatrix} sinAcosA\cos(\Psi) + \cos^2 A \sin(\Psi)\exp(i\nabla)e^{i\varphi} \\ \sin^2 A \cos(\Psi)\, e^{i\varphi_2} + sinAcosA \sin(\Psi)\exp(i\nabla)e^{i\varphi} \end{pmatrix}$$

Solution:

After crossing the polarizer oriented at 45°, the electric field is:

$$\vec{E}_L(P_{45}) = \begin{pmatrix} e^{i\varphi} & 0 \\ 0 & e^{i\varphi} \end{pmatrix} \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \left|\vec{E}_0\right| e^{ikr_0} e^{-i\omega t} \\ \left|\vec{E}_0\right| e^{ikr_0} e^{-i\omega t} \end{pmatrix} = \begin{pmatrix} \left|\vec{E}_0\right| e^{i\varphi} e^{ikr_0} e^{-i\omega t} \\ \left|\vec{E}_0\right| e^{i\varphi} e^{ikr_0} e^{-i\omega t} \end{pmatrix}$$

It is a circularly polarized wave.

After the compensator:

$$\vec{E}_C(PD) = \begin{pmatrix} e^{i\varphi_1} & 0 \\ 0 & e^{i\varphi_2} \end{pmatrix} \begin{pmatrix} \left|\vec{E}_0\right| e^{-i\omega t} \\ \left|\vec{E}_0\right| e^{-i\omega t} \end{pmatrix} = \begin{pmatrix} \left|\vec{E}_0\right| e^{i\varphi_1} e^{-i\omega t} \\ \left|\vec{E}_0\right| e^{i\varphi_2} e^{-i\omega t} \end{pmatrix}$$

where we reset the phase corresponding to the displacement.

After the sample:

$$\vec{E}_{SE}(PD) = \begin{pmatrix} \sin(\Psi)\exp(i\nabla) & 0 \\ 0 & \cos(\Psi) \end{pmatrix} \begin{pmatrix} \left|\vec{E}_0\right| e^{i\varphi_1} e^{-i\omega t} \\ \left|\vec{E}_0\right| e^{i\varphi_2} e^{-i\omega t} \end{pmatrix}$$
$$= \begin{pmatrix} \left|\vec{E}_0\right| \sin(\Psi)\exp(i\nabla) e^{i\varphi_1} e^{-i\omega t} \\ \left|\vec{E}_0\right| \cos(\Psi)\, e^{i\varphi_2} e^{-i\omega t} \end{pmatrix}$$

At the level of the analyzer, the matrix corresponding to the 45° polarizer rotating through an angle A must be calculated:

$$P_{45}^{A} = \begin{pmatrix} \cos^2 A & sinAcosA \\ sinAcosA & \sin^2 A \end{pmatrix}$$

$$\vec{E}_P(PD) = \begin{pmatrix} \cos^2 A & sinAcosA \\ sinAcosA & \sin^2 A \end{pmatrix} \begin{pmatrix} \left|\vec{E}_0\right| \sin(\Psi) \exp(i\nabla) e^{i\varphi_1} e^{-i\omega t} \\ \left|\vec{E}_0\right| \cos(\Psi)\, e^{i\varphi_2} e^{-i\omega t} \end{pmatrix}$$

$$\vec{E}_L(PD) = \left|\vec{E}_0\right| e^{-i\omega t} \begin{pmatrix} sinAcosA\cos(\Psi)\, e^{i\varphi_2} + \cos^2 A \sin(\Psi) \exp(i\nabla) e^{i\varphi_1} \\ \sin^2 A \cos(\Psi)\, e^{i\varphi_2} + sinAcosA \sin(\Psi) \exp(i\nabla) e^{i\varphi_1} \end{pmatrix}$$

thus:

$$\vec{E}_L(PD) = \left|\vec{E}_0\right| e^{-i\omega t} e^{i\varphi_2} \begin{pmatrix} sinAcosA\cos(\Psi) + \cos^2 A \sin(\Psi) \exp(i\nabla) e^{i\varphi} \\ \sin^2 A \cos(\Psi) + sinAcosA \sin(\Psi) \exp(i\nabla) e^{i\varphi} \end{pmatrix}$$

3) Deduce that the expression of the intensity of the light wave IA at the level of the detector is of the form:

$$\frac{I}{\varepsilon_0 c_0} = \left|\vec{E}_0\right|^2 (\alpha + \beta cos2A + \gamma sin2A + \eta cos4A + \cdots)$$

Solution:

The intensity calculation is made from: $I = \varepsilon_0 c_0 \vec{E}_A(PD).\vec{E}_A(PD)^*$

$$\frac{I}{\varepsilon_0 c_0} = \left|\vec{E}_A(PD)\right|^2$$

$$\begin{aligned} \left|\vec{E}_S(PD)\right|^2 = \left|\vec{E}_0\right|^2 &\times \left(sinAcosA\cos(\Psi) + \cos^2 A \sin(\Psi) \exp(i\nabla) e^{i\varphi}\right) \\ &\times \left(sinAcosA\cos(\Psi) + \cos^2 A \sin(\Psi) \exp(-i\nabla)\, e^{-i\varphi}\right) \\ &+ \ (\sin^2 A \cos(\Psi) + sinAcosA \sin(\Psi) \exp(i\nabla)\, e^{i\varphi} \\ &\times (\sin^2 A \cos(\Psi) + sinAcosA \sin(\Psi) \exp(-i\nabla) e^{-i\varphi}) \end{aligned}$$

where:

$$\begin{aligned}\left|\vec{E}_S(PD)\right|^2 = \left|\vec{E}_0\right|^2 \Big(& (sin^2Acos^2A\cos^2(\Psi)) + (\cos^4A\sin^2(\Psi)) \\ & + 2\,sinAcosA\cos(\Psi)\cos^2A\sin(\Psi)\cos(\nabla+\varphi)\Big) \\ & + \Big((\sin^4A\cos^2(\Psi)) + (sin^2Acos^2A\sin^2(\Psi))\Big) \\ & + 2\,sinAcosA\sin(\Psi)\sin^2A\cos(\Psi)\cos(\nabla+\varphi)\end{aligned}$$

Using the trigonometric formulas given in the appendix, we can group the terms into a trigonometric series in terms of *cos*(*pA*) and *sin*(*pA*) so that we have:

$$\frac{I}{\varepsilon_0 c_0} = \left|\vec{E}_0\right|^2 (\alpha + \beta cos2A + \gamma sin2A + \eta cos4A + \cdots)$$

with:

$$\alpha = \frac{3+4cos2\Psi}{8};\ \beta = \frac{cos2\Psi}{2};\ \gamma = \frac{3sin2\Psi\cos(\Delta+\varphi)}{4};\ \eta = -\frac{3}{8}cos^2\Psi + sin^2\Psi$$

4) Explain how an FFT (Fast Fourier Transform) analysis of the I_A signal makes it possible to go back to the values of the ellipsometric parameters (Ψ, Δ).

Solution:

By locating the terms at 2 A, i.e. β and γ in the frequency space, we can determine the ellipsometric parameters Δ and Ψ. Beforehand, the device must be calibrated with a substrate in order to correctly adjust the parameters β and γ.

1.5. Appendices

1.5.1. *Conventions used for Jones vectors and Jones ABCD matrices*

The electric field, in the most general case, consists of an E_p component parallel to the plane of incidence and an E_s component perpendicular to the plane of incidence. The convention given in Figure 1.34 will be adopted such that ((Ox, Oy, Oz) ≡ (Os, Op, Oz)).

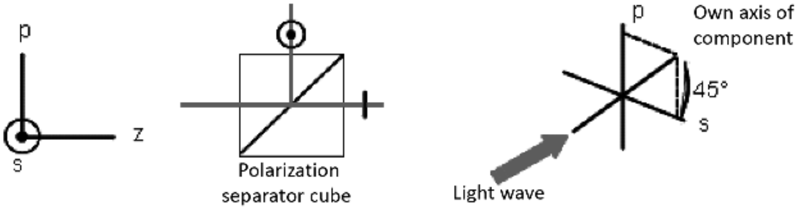


Figure 1.34. *Reference mark and sign convention to be adopted. For a color version of this figure, see www.iste.co.uk/dahoo/metrology2.zip*

Polarization *S* and *P*: $\vec{e}_S = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\vec{e}_P = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

An electromagnetic wave in the base defined by $\boldsymbol{e_s}$ and $\boldsymbol{e_p}$:

$$\vec{E}_L = \begin{pmatrix} |E_S| \\ |E_P| \end{pmatrix} e^{-i\omega t}$$

A plane electromagnetic wave of zero phase polarized at 45° $\vec{E}_L = \vec{E}_S + \vec{E}_P$.

A base change matrix by angle rotation θ in a plane

$$R(\theta) = \begin{pmatrix} cos\,\theta & -sin\theta \\ sin\theta & cos\theta \end{pmatrix}$$

A reference change by rotation of θ $M(\theta) = R(\theta)M(0)R(-\theta)$.

A mirror at normal incidence $M = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

A separator 50/50 $LS = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$.

A polarizer cube oriented along *S* $CSP_S = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.

A polarizer cube oriented according to *P* $CSP_P = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$.

Polarizers according to *P* and *S*: $P_0^P = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ $P_0^S = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.

A polarizer *S* oriented at -45: $P_{-45}^S = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$.

A quarter-wave plate along the proper axes:

$$Q_0 = e^{i\frac{\pi}{4}}\begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix} = e^{i\frac{\pi}{4}}\begin{pmatrix} 1 & 0 \\ 0 & e^{-i\frac{\pi}{2}} \end{pmatrix}$$

A half-wave plate along the proper axes: $L_0 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = e^{i\frac{\pi}{2}}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

A quarter wave plate oriented at 45° and -45°:

$$Q_{45} = \frac{1}{2}\begin{pmatrix} e^{-i\frac{\pi}{4}} & e^{i\frac{\pi}{4}} \\ e^{i\frac{\pi}{4}} & e^{-i\frac{\pi}{4}} \end{pmatrix} \quad Q_{-45} = \frac{1}{2}\begin{pmatrix} e^{-i\frac{\pi}{4}} & e^{-i\frac{3\pi}{4}} \\ e^{i\frac{3\pi}{4}} & e^{-i\frac{\pi}{4}} \end{pmatrix}$$

A phase shift matrix due to displacement δ for propagation in space:

$$D = \begin{pmatrix} \exp(i\phi) & 0 \\ 0 & \exp(i\phi) \end{pmatrix} \text{ with } \phi = nk_0\delta = 2\pi n\delta/\lambda_0$$

A phase shift matrix due to the Doppler effect along the p axis:

$$D_d = \begin{pmatrix} 0 & 0 \\ 0 & exp(-i\omega_d t) \end{pmatrix}$$

with $\omega_d = \omega_0 (1 \pm 2nv/c)$, v being the speed of movement of the object.

A matrix representing the sample that changes the polarization of light in an ellipsometric setup: $SE = \begin{pmatrix} sin(\Psi)\, exp(i\nabla) & 0 \\ 0 & cos(\Psi) \end{pmatrix}$.

1.5.2. *2×2 transfer dies*

Propagation in homogeneous matter over a distance d:

$$M = \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix}$$

Propagation through a diopter separating two media with indices n_1 and n_2:n_1 and n_2:

$$M = \begin{bmatrix} 1 & 0 \\ 0 & \dfrac{n_1}{n_2} \end{bmatrix}$$

Transmission through a thin lens of focal length f ($f > 0$ convex, $f < 0$ concave):

$$M = \begin{bmatrix} 1 & 0 \\ -\dfrac{1}{f} & 1 \end{bmatrix}$$

Reflection on a concave mirror with a radius of curvature R:

$$M = \begin{bmatrix} 1 & 0 \\ \dfrac{2}{R} & 1 \end{bmatrix}$$

1.5.3. *2×2 matrix multiplication*

It is customary to use our matrix elements with the letters A, B, C and D:

$$M_2 = \begin{bmatrix} A_2 & C_2 \\ B_2 & D_2 \end{bmatrix} M_1 = \begin{bmatrix} A_1 & C_1 \\ B_1 & D_1 \end{bmatrix}$$

Identity matrix

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Multiplication of M_1 and M_2

$$M_1 M_2 = \begin{bmatrix} A_1 & C_1 \\ B_1 & D_1 \end{bmatrix} \begin{bmatrix} A_2 & C_2 \\ B_2 & D_2 \end{bmatrix} = \begin{bmatrix} A_1A_2 + C_1B_2 & A_1C_2 + C_1D_2 \\ B_1A_2 + D_1B_2 & B_1C_2 + D_1D_2 \end{bmatrix}$$

Determinant and inverse of the matrix M

$$\det(M) = \begin{vmatrix} A & C \\ B & D \end{vmatrix} = AD - BC \quad M^{-1} = \frac{1}{\det(M)} \begin{bmatrix} D & -C \\ -B & A \end{bmatrix}$$

1.5.4. *Trigonometric forms*

$$\sin^2(x) = (1 - cos2x)/2$$

$$\cos^2(x) = (1 + cos2x)/2$$

$$\sin^3(x) = \frac{sinx - sinxcos2x}{2} = (cosx - \frac{1}{2}sinx + \frac{sin3x}{2})/2$$

$$\cos^3(x) = \frac{cosx + cosxcos2x}{2} = (cosx + \frac{1}{2}cosx + \frac{cos3x}{2})/2$$

$$\sin^4(x) = (3 - 4cos2x + cos4x)/8$$

$$\cos^4(x) = (3 + 4cos2x + cos4x)/8$$

$$\cos(x)\cos(y) = \frac{\cos(x-y) - \cos(x+y)}{2} \quad \sin(x)\sin(y) = \frac{\cos(x-y) - \cos(x+y)}{2}$$

$$\sin(x)\cos(y) = \frac{\sin(x+y) + \sin(x-y)}{2} \quad \cos(x)\sin(y) = \frac{\sin(x+y) - \sin(x-y)}{2}$$

1.5.5. *Solution by MATLAB (exercises 1.4.3, 1.4.4 and 1.4.5)*

MATLAB

```
clear all
% Exercice 543
% Matrice de Jones
CSPS=[1 0; 0 0]
Q45=0.5*[1+1i 1-1i;1-1i 1+1i]
M2=[1 0; 0 -1]
Qm45=0.5*[1+1i -1+1i;-1+1i 1+1i]
CSPP=[0 0; 0 1]
PS45=0.5*[1 1;1 1]
%
%543
%
MS=PS45*CSPP*Qm45*M2*Q45*CSPS
MP=PS45*CSPS*Qm45*M2*Q45*CSPP
%
% expressions symbolic
% syms
% help syms
%
syms phirp  phirs
Dphirs=[exp(1i*phirs) 0;0 exp(1i*phirs)]
Dphirp=[exp(1i*phirp) 0;0 exp(1i*phirp)]
M843SPD=MS*Dphirs
M843PPD=MP*Dphirp
%
%
syms phis0 phip0 Es Ep
EL=[Es*exp(i*phis0) Ep*exp(i*phip0)]
ELt=transpose(EL)
%
%543
%
ESPD=M843SPD*ELt
EPPD=M843PPD*ELt
%
ER543=ESPD+EPPD
%
```

Solution

```
CSPS =
       1    0
       0    0
```

For a color version of this solution, see www.iste.co.uk/dahoo/metrology2.zip.

```
Q45 =
          0.5000 + 0.5000i   0.5000 - 0.5000i
          0.5000 - 0.5000i   0.5000 + 0.5000i
M2 =
          1    0
          0   -1
Qm45 =
           0.5000 + 0.5000i  -0.5000 + 0.5000i
          -0.5000 + 0.5000i   0.5000 + 0.5000i
CSPP =
          0    0
          0    1
PS45 =
           0.5000   0.5000
           0.5000   0.5000
MS =
          -0.5000       0
          -0.5000       0
MP =
          0   0.5000
          0   0.5000
Dphirs =
          [ exp(i*phirs),          0]
          [          0, exp(i*phirs)]

Dphirp =
          [ exp(i*phirp),          0]
          [          0, exp(i*phirp)]
M543SPD =
          [ -1/2*exp(i*phirs),              0]
          [ -1/2*exp(i*phirs),              0]
M543PPD =
          [             0, 1/2*exp(i*phirp)]
          [             0, 1/2*exp(i*phirp)]
EL =
          [ Es*exp(i*phis0), Ep*exp(i*phip0)]
ELt =
          [Es*exp(i*phis0) ]
          [Ep*exp(i*phip0) ]
```

```
ESPD =
          [-1/2*exp(i*phirs)*Es*exp(i*phis0) ]
         [-1/2*exp(i*phirs)*Es*exp(i*phis0) ]
EPPD =
          [1/2*exp(i*phirp)*Ep*exp(i*phip0) ]
          [1/2*exp(i*phirp)*Ep*exp(i*phip0) ]
ER843 =
[-
1/2*exp(i*phirs)*Es*exp(i*phis0)+1/2*exp(i*phirp)*Ep*exp(i*phi
p0) ]
[-
1/2*exp(i*phirs)*Es*exp(i*phis0)+1/2*exp(i*phirp)*Ep*exp(i*phi
p0) ]
```

MATLAB

```
%EXERCICE 544
clear all
% Matrice de Jones
CSPS=[1 0; 0 0]
Q45=0.5*[1+1i 1-1i;1-1i 1+1i]
M2=[1 0; 0 -1]
Qm45=0.5*[1+1i -1+1i;-1+1i 1+1i]
CSPP=[0 0; 0 1]
PS45=0.5*[1 1;1 1]
MS=PS45*CSPP*Qm45*M2*Q45*CSPS
MP=PS45*CSPS*Qm45*M2*Q45*CSPP
% expressions symbolic
syms phirp  phirs
Dphirs=[exp(1i*phirs) 0;0 exp(1i*phirs)]
Dphirp=[exp(1i*phirp) 0;0 exp(1i*phirp)]
MS544PD=MS*Dphirs
MP544PD=MP*Dphirp
%
%
syms phis phip Es Ep phi0  E0
%EL=[Es*exp(i*phis) Ep*exp(i*phip)]
EL=[E0*exp(i*phi0) E0*exp(i*phi0)]
ELt=transpose(EL)
ESPD=MS544PD*ELt
EPPD=MP544PD*ELt
%E0L=PS45*ELt
```

Solution

```
MS =
  -0.5000       0
  -0.5000       0
MP =
       0   0.5000
       0   0.5000
Dphirs =
          [ exp(i*phirs),          0]
          [          0, exp(i*phirs)]
Dphirp =
          [ exp(i*phirp),          0]
          [          0, exp(i*phirp)]
 MS544PD =
          [ -1/2*exp(i*phirs),               0]
          [ -1/2*exp(i*phirs),               0]
MP544PD =
          [              0, 1/2*exp(i*phirp)]
          [              0, 1/2*exp(i*phirp)]
EL =
[ E0*exp(i*phi0), E0*exp(i*phi0)]
ELt =
          [ E0*exp(i*phi0) ]
          [ E0*exp(i*phi0) ]
ESPD =
          [ -1/2*exp(i*phirs)*E0*exp(i*phi0) ]
          [-1/2*exp(i*phirs)*E0*exp(i*phi0) ]
EPPD =
           [1/2*exp(i*phirp)*E0*exp(i*phi0) ]
           [1/2*exp(i*phirp)*E0*exp(i*phi0) ]
```

MATLAB

```
%
%EXO 545 Heterodyne
clear all
% Matrice de Jones
CSPS=[1 0; 0 0]
Q45=0.5*[1+1i 1-1i;1-1i 1+1i]
M2=[1 0; 0 -1]
Qm45=0.5*[1+1i -1+1i;-1+1i 1+1i]
CSPP=[0 0; 0 1]
PS45=0.5*[1 1;1 1]
MS=PS45*CSPS*CSPS
MP=PS45*CSPP*CSPP
```

```
% expressions symbolic
syms phirp  phirs
Dphirs=[exp(1i*phirs) 0;0 exp(1i*phirs)]
Dphirp=[exp(1i*phirp) 0;0 exp(1i*phirp)]
MS545=Dphirs*MS
MP545=Dphirp*MP
%
syms omegaD
DphiD=[0 0;0 exp(1i*omegaD)]
MP545=MP545*DphiD
%
syms omegas omegap Es Ep
EL=[Es*exp(i*omegas) Ep*exp(i*omegap)]
ELt=transpose(EL)
%
ESPD=MS545*ELt
EPPD=MP545*ELt
```

SOLUTION

```
MS545 =
       [ 1/2*exp(i*phirs),              0]
       [ 1/2*exp(i*phirs),              0]

MP545 =
       [              0, 1/2*exp(i*phirp)]
       [              0, 1/2*exp(i*phirp)]
DphiD =

       [           0,           0]
       [           0, exp(i*omegaD)]
MP545 =

       [                          0, 1/2*exp(i*phirp)*exp(i*omegaD)]
       [                          0, 1/2*exp(i*phirp)*exp(i*omegaD)]
EL =

       [ Es*exp(i*omegas), Ep*exp(i*omegap)]

ELt =
       [Es*exp(i*omegas) ]
       [Ep*exp(i*omegap) ]
```

```
ESPD =
          [1/2*exp(i*phirs)*Es*exp(i*omegas) ]
          [1/2*exp(i*phirs)*Es*exp(i*omegas) ]
EPPD =
          [1/2*exp(i*phirp)*exp(i*omegaD)*Ep*exp(i*omegap) ]
          [1/2*exp(i*phirp)*exp(i*omegaD)*Ep*exp(i*omegap) ]
```

1.6. Conclusion

In this chapter, a few examples have illustrated the application of Jones matrices to model the operation of interference devices in polarized light, in particular the homodyne, heterodyne interferometer or the ellipsometer. Some programs for MATLAB are given in the Appendix to simulate the operation of these devices.