PART 1

Higher-dimensional Random Motions and Interactive Particles

1

Random Motions in Higher Dimensions

In this chapter, we discuss many of the previous results in further detail. Section 1.1 focuses on the study of an isotropic random motion at constant absolute velocity, where the direction alternations occur at renewal epochs of the Erlang or a general distribution. In recent years, the study of random walks with non-Markov switching processes is a topic of research. Many researchers have given special attention to the Pearson–Gamma random walk where the length of steps are Gamma distributed. In addition, the number of steps is related to the Dirichlet distribution. This type of motion is called a Pearson–Dirichlet motion. Apparently, the first study in this direction was done by Franceschetti (2007), in which the author studies a random walk in $\mathbb{R}^d$ ($d = 1, 2$) with uniformly distributed directions and the steps are i.i.d. random vectors of exponential length, and the walker travels a fixed total path. This means that the steps of the walk have joint Dirichlet distribution with all parameters equal to $q = 1$. By using the characteristic function technique, the author obtained a closed-form expression for the conditional distribution for the particle position after n steps, and he found a necessary condition for the uniform density of the particle position for a given number of steps in dimension d. It is mentioned in the paper that for dimensions $d \geq 3$ the random walk does not lead to a uniform density. Le Caër (2010, 2011) generalizes most of the results in Franceschetti (2007) to the case of multidimensional Pearson–Dirichlet random walk, whose parameters are all equal to $q > 0$. In Le Caër (2010), the author introduces and studies the so-called "Hyperspherical Uniform" (HU) Pearson–Dirichlet random walk. At renewal epochs (i.e. when the direction is changing), the HU random walk has a distribution in a unit disc identical to the distribution of the projection on the disc of a point uniformly distributed over the surface of a sphere of higher dimension. By using this HU property, Le Caër finds the pdf of the particle position that can be written in a closed-form expression with parameters $q = n - 1, n \geq 2$ and $q = n/2 - 1, n \geq 3$.

Another family of random motions is presented in Le Caër (2011), and it also has a closed-form expression for the distribution under the condition $q = n$. In this work, the author generalizes the approach developed in the paper by Beghin and Orsingher (2010a) for the parameter $q = 2$ on the plane, since he considers solutions for all integer $q = n \geq 2$.

In all of these works, the distribution of the particle position at renewal epochs of the switching directions process is studied. Basically, for the case of the non-Markov switching process, this task is reduced to studying the distribution of the non-Markov process through its embedded Markov chain. Here, we should mention also the paper by Letac and Piccioni (2014), in which the authors study Stiltjes transitions for a Pearson–Dirichlet walk, and by using these transitions they simplify the proof of similar results obtained by Le Caër, Orsingher and Beghin, as well as the limiting properties of the Dirichlet walk in the infinite growth of the number of steps.

We have contributed in high-dimensional random motions and these findings will be presented in this chapter. For instance, in section 1.1.1 we obtain formulas for the recursive calculations of the conditional characteristic functions of a random walk, where we are considering non-Markov switching processes with Erlang distribution for the interarrival times. Namely, we study changes of the conditional characteristic functions at each Poisson event (Pogorui and Rodríguez-Dagnino 2011).

In section 1.1.2, we derive the renewal equation for the characteristic function of the transition density for this multidimensional motion. Furthermore, by using this renewal equation we investigate some properties of the distribution function for the isotropic motion in two- and three-dimensional spaces. Section 1.2 is devoted to random motion with uniformly distributed directions and random velocity with examples for one-, two-, three- and four-dimensional cases. In section 1.2.1, the renewal equation for the characteristic function of isotropic motion with random velocity in a semi-Markov environment is obtained. Section 1.3 deals with multidimensional isotropic random motion at random speed, where the direction alternations occur according to the renewal epochs of a general distribution. Such a model represents a generalization of the model that only considers a constant velocity, which has been the typical assumption in most works related to this topic (Orsingher 1985; Orsingher and De Gregorio 2007; Orsingher and Ratanov 2007; Pinsky 1991; Franceschetti 2007; Le Caër 2010, 2011; Letac and Piccioni 2014; Pogorui and Rodríguez-Dagnino 2011; Pogorui 2011). We find the distribution at a fixed time of the random motions with some velocity distribution for one-, two-, three- and four-dimensional cases.

Section 1.3 deals with the characteristic function for the jump telegraph process in higher dimensions. In section 1.6.1, the explicit formula for this characteristic function is obtained. In section 1.4, we study the correspondence between a telegraph-type equation and random motion in three- and five-dimensional spaces.

1.1. Random motion at finite speed with semi-Markov switching directions process

In this section, we study random motions in $\mathbb{R}^n$ having uniformly distributed changes of direction at finite speed. The direction alternations occur according to a non-Markov switching process. In recent years, there have been many important works related to these processes (Beghin and Orsingher 2010a; Franceschetti 2007; Le Caër 2010, 2011; Letac and Piccioni 2014; Pogorui and Rodríguez-Dagnino 2011). In these studies, most of the authors are interested in non-Markov cases, where they can find closed-form expressions for the pdf and the conditional characteristic function of these semi-Markov transport processes.

In section 5.1 of volume 1 (Volume 1, Chapter 5), an isotropic random motion at finite speed with Erlang-k-distributed direction alternations is studied. Now, we will develop some relations for the recursive calculation of conditional characteristic functions of the random motion in the Erlang media considering that the switching process is not Markov. Namely, we study changes in the conditional characteristic functions at all Poisson epochs. In addition, we obtain the integral Volterra equation for the characteristic function of a Pearson–Gamma random walk with an integer parameter. This formulation aims to achieve a better understanding of the characteristic function of a random walk in the general semi-Markov medium as it satisfies a renewal integral equation.

In section 5.1 of volume 1 (Volume 1, Chapter 5), we also studied the case where the switching process is a general renewal process. In this section, we investigate some properties for the distribution function of the position of a particle in two- and three-dimensional space by using the integral equation for the characteristic functions of this distribution.

Let $\{\xi(t), t \geq 0\}$ be a renewal process such that $\xi(t) = \max\{m \geq 0 : \tau_m \leq t\}$, where $\tau_m = \sum_{k=0}^{m} \theta_k$, $\tau_0 = 0$ and $\theta_k \geq 0$, $k = 1, 2, \ldots$, are independent random variables denoting interarrival times. We assume that these random variables are identically distributed with cumulative distribution function (cdf) $G_\theta(t)$ and that there exists the pdf $g_\theta(t) = \frac{d}{dt}G(t)$. We will study the random motion of a particle that starts from the coordinate origin $\mathbf{0} = (0, 0, \ldots, 0)$ of the space $\mathbb{R}^n$, at time $t = 0$, and continues its motion with a constant velocity v along the direction $\boldsymbol{\eta}_0^{(n)}$, where $\boldsymbol{\eta}_0^{(n)} = (x_1, x_2, \ldots, x_n)$ is a random n-dimensional vector uniformly distributed on the sphere

$$\Omega_1^{n-1} = \left\{(x_1, x_2, \ldots, x_n) : x_1^2 + x_2^2 + \ldots + x_n^2 = 1\right\}.$$

At instant τ_1, the particle changes its direction to $\boldsymbol{\eta}_1^{(n)}$, where $\boldsymbol{\eta}_1^{(n)}$ and $\boldsymbol{\eta}_0^{(n)}$ are i.i.d. random vectors on Ω_1^{n-1}. Then, at instant τ_2 the particle changes its direction to $\boldsymbol{\eta}_2^{(n)}$, where $\boldsymbol{\eta}_2^{(n)}$, $\boldsymbol{\eta}_1^{(n)}$ and $\boldsymbol{\eta}_0^{(n)}$ are i.i.d. random vectors, and so on.

Denote by $\mathbf{x}^{(n)}(t)$, $t \geq 0$, the particle position at time t. We have that

$$\mathbf{x}^{(n)}(t) = v \sum_{j=1}^{\xi(t)} \boldsymbol{\eta}_{j-1}^{(n)} (\tau_j - \tau_{j-1}) + v\, \boldsymbol{\eta}_{\xi(t)}^{(n)} \left(t - \tau_{\xi(t)}\right).$$

Now let us denote as $\nu(t)$ the number of velocity alternations occurred in the interval $(0,t)$, and let us assume first a Poisson process for the epochs of these alternations. Then, the random variables θ_k are exponentially distributed with parameter λ, hence $g_\theta(t) = \lambda e^{-\lambda t}\mathbb{I}_{\{t \geq 0\}}$.

Furthermore, let us denote as $\mathbf{x}_m^{(n)}(t)$ the particle position at time $t > 0$ when $\nu(t) = m \geq 0$, i.e.

$$P\left(\mathbf{x}_m^{(n)}(t) \leq y\right) = P\left(\mathbf{x}^{(n)}(t) \leq y |\, \nu(t) = m\right).$$

It is not difficult to see that $\mathbf{x}_m^{(n)}(t)$ is given by

$$\mathbf{x}_m^{(n)}(t) = v \sum_{j=1}^{m+1} \boldsymbol{\eta}_{j-1}^{(n)} (\tau_j - \tau_{j-1}), \qquad [1.1]$$

where $\tau_{m+1} = t$.

The probabilistic properties of the random vector $\mathbf{x}_m^{(n)}(t)$ are completely determined by those of its projection $x_m^{(n)}(t) = v\sum_{j=1}^{m+1} \eta_{j-1}^{(n)} (\tau_j - \tau_{j-1})$ on a fixed line, where $\eta_j^{(n)}$ is the projection of $\boldsymbol{\eta}_j^{(n)}$ on the line.

Indeed, let us consider the conditional cdf

$$F_x(y |\, \nu(t) = m) = P\left(x_m^{(n)}(t) \leq y |\, \nu(t) = m\right).$$

Then, the conditional characteristic function (Fourier transform) $H_m(t,\alpha) = H_m(t)$ of $\mathbf{x}^{(n)}(t)$, where $\alpha = \|\boldsymbol{\alpha}\| = \sqrt{\alpha_1^2 + \alpha_2^2 + \ldots + \alpha_n^2}$, is given by

$$H_m(t) = \mathbf{E}\left[\exp\left\{i\left(\boldsymbol{\alpha}, \mathbf{x}^{(n)}(t)\right)\right\} |\, \nu(t) = m\right]$$
$$= \mathbf{E}\left[\exp\left\{i\left(\boldsymbol{\alpha}, \mathbf{x}_m^{(n)}(t)\right)\right\}\right] = \mathbf{E}\left[\exp\left\{i\,\|\boldsymbol{\alpha}\|\left(\mathbf{e}, \mathbf{x}_m^{(n)}(t)\right)\right\}\right]$$
$$= \mathbf{E}\left[\exp\left\{i\,\|\boldsymbol{\alpha}\|\, x_m^{(n)}(t)\right\}\right] = \int_0^\infty \exp\{i\,\|\boldsymbol{\alpha}\|\, y\} dF_x(y |\, \nu(t) = m),$$

where $x_m^{(n)}(t)$ is the projection of $\mathbf{x}_m^{(n)}(t)$ onto the unit vector $\mathbf{e} = \frac{\boldsymbol{\alpha}}{\|\boldsymbol{\alpha}\|}$ and it has a conditional cdf $F_x(y |\, \nu(t) = m)$.

Let denote by $f_{\eta^{(n)}}(x)$ the pdf of the projection $\eta_j^{(n)}$ of the vector $\boldsymbol{\eta}_j^{(n)}$ onto a fixed line. It is shown in Pogorui (2009b) that $f_{\eta^{(n)}}(x)$ is of the following form:

$$f_{\eta^{(n)}}(x) = \begin{cases} \frac{\Gamma\left(\frac{n}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{n-1}{2}\right)}\left(1-x^2\right)^{(n-3)/2} & , x \in [-1,1], \\ 0 & , x \notin [-1,1]. \end{cases} \qquad [1.2]$$

We can also consider the case $n = 1$ and it is easily seen that

$$f_{\eta^{(1)}}(x) = \frac{1}{2}\delta\left(|x|-1\right).$$

It is also straightforward to verify that

$$H_m(t) = \mathbf{E}\left[\exp\left\{i\left\|\boldsymbol{\alpha}\right\|v\sum_{j=1}^{m+1}\eta_{j-1}^{(n)}\left(\tau_j-\tau_{j-1}\right)\right\}\right]$$

$$= \frac{m!}{t^m}\int_0^t ds_1\int_{s_1}^t ds_2\ldots\int_{s_{m-1}}^t ds_m \mathbf{E}\left[\exp\left\{i\left\|\boldsymbol{\alpha}\right\|v\sum_{j=1}^{m+1}\eta_{j-1}^{(n)}\left(s_j-s_{j-1}\right)\right\}\right]$$

$$= \frac{m!}{t^m}\int_0^t ds_1\int_{s_1}^t ds_2\ldots\int_{s_{m-1}}^t ds_m\left\{\prod_{j=1}^{m+1}\varphi_{\eta^{(n)}}\left(\left\|\boldsymbol{\alpha}\right\|v\left(s_j-s_{j-1}\right)\right)\right\},$$

where $s_0 = 0$, $s_{m+1} = t$ and $\varphi_{\eta^{(n)}}(\lambda) = \mathbf{E}\left[e^{-i\lambda\eta_j^{(n)}}\right] = \int_{-\infty}^{\infty} e^{-i\lambda x} f_{\eta^{(n)}}(x)dx$ is the characteristic function of $\eta_j^{(n)}$. It is well known that (Pogorui and Rodríguez-Dagnino 2011)

$$\varphi(t) = 2^{\frac{n-2}{2}}\Gamma\left(\frac{n}{2}\right)\frac{J_{\frac{n-2}{2}}\left(\alpha t v\right)}{\left(\alpha t v\right)^{\frac{n-2}{2}}}. \qquad [1.3]$$

1.1.1. *Erlang-K-distributed direction alternations*

Let us assume a fixed integer $K \geq 2$. Now, suppose that the particle changes its direction just at epochs τ_{Kj}, $j = 1, 2, \ldots$, i.e.

$$\mathbf{x}_m^{(n)}(t) = v\sum_{j=1}^{\left\lfloor\frac{m}{K}\right\rfloor}\boldsymbol{\eta}_{K(j-1)}^{(n)}\left(\tau_{Kj}-\tau_{K(j-1)}\right)$$
$$+v\,\boldsymbol{\eta}_{K\left\lfloor\frac{m}{K}\right\rfloor}^{(n)}\left(\tau_{m+1}-\tau_{K\left\lfloor\frac{m}{K}\right\rfloor}\right), \qquad [1.4]$$

where $\tau_{m+1} = t$ and $\sum_{j=1}^{0}\boldsymbol{\eta}_{K(j-1)}^{(n)}\left(\tau_{Kj}-\tau_{K(j-1)}\right) = 0$.

Hence, defining $c = \alpha v$, we have for $m = lK + r$, $l = 0, 1, 2, \ldots$, $0 \leq r < K$

$$H_l^{(r)}(t) = H_m(t) = \mathbf{E}\left[\exp\left\{ic\mathbf{x}_m^{(n)}(t)\right\}\right]$$
$$= \frac{(lK+r)!}{t^{lK+r}} \int_0^t ds_1 \int_{s_1}^t ds_2 \ldots \int_{s_{lK+r-1}}^t ds_{lK+r}$$
$$\times \left\{\prod_{j=1}^{l} \varphi\left((s_{Kj} - s_{Kj-K})\right)\varphi\left((t - s_{Kl})\right)\right\},$$

where $\prod_{j=1}^{0} \varphi\left((s_{Kj} - s_{K(j-1)})\right) = 1$.

We should note that $H_0^{(r)}(t) = \varphi(t)$.

The random variables $\tau_{Kj} - \tau_{K(j-1)}$, $j \geq 1$, are Erlang-K distributed, and we may define the renewal process $\xi_K(t) = \max\left\{j \geq 0 : \tau_j^{(K)} \leq t\right\}$, $t \geq 0$, where $\tau_j^{(K)} = \sum_{l=0}^{j} \theta_l^{(K)}$, $\tau_0^{(K)} = 0$ and $\theta_l^{(K)}$, $l = 1, 2, \ldots$, are i.i.d. interarrival times having an Erlang-K pdf $g_\theta(t) = \frac{\lambda^K t^{K-1}}{(K-1)!} e^{-\lambda t} I_{\{t \geq 0\}}$.

Thus, the stochastic process $\mathbf{x}^{(n)}(t)$ can be considered as the random evolution of the particle in the non-Markov Erlang-K medium $\xi_K(t)$.

The functions $H_l^{(r)}(t)$, $l = 0, 1, 2, \ldots$ are the conditional characteristic functions of $\mathbf{x}^{(n)}(t)$. We should remember that $\nu(t) = m$ represents m Poisson events occurred in $[0, t)$, and the number of direction alternations $\nu_K(t)$ of $\mathbf{x}^{(n)}(t)$ is given by $\nu_K(t) = \left\lfloor \frac{\nu(t)}{K} \right\rfloor$.

For $l = 0, 1, 2, \ldots; 0 \leq r < K$, let us consider the following multiple integral:

$$I_l^{(r)}(t) = \int_0^t ds_1 \int_{s_1}^t ds_2 \ldots \int_{s_{lK+r-1}}^t ds_{lK+r}$$
$$\times \left\{\prod_{j=1}^{l} \varphi\left((s_{Kj} - s_{Kj-K})\right)\varphi\left((t - s_{Kl})\right)\right\}, \qquad [1.5]$$

where $s_0 = 0$. Let us state the notation $\psi_K(t) = \dfrac{t^{K-1}}{(K-1)!}\varphi(t)$.

THEOREM 1.1.– For any $l \geq 0$, we have the following recursive relation:

$$I_{l+1}^{(r)}(t) = \int_0^t \frac{u^{K-1}}{(K-1)!} \varphi\left((u)\right) I_l^{(r)}\left(t-u\right) du = \psi_K * I_l^{(r)}(t), \qquad [1.6]$$

where we have by assumption that $I_0^{(r)}(t) = \frac{t^r}{r!}\varphi(t)$, $r = 0, 1, \ldots, K-1$.

PROOF.– Let us prove [1.6] by induction arguments. First, suppose that $l = r = 0$, then

$$I_1^{(0)}(t) = \int_0^t ds_1 \int_{s_1}^t ds_2 \ldots \int_{s_{K-1}}^t ds_K \left\{\varphi\left((s_K)\right) I_0^{(0)}\left(c\left(t-s_K\right)\right)\right\}$$

$$= \left[s_1 \int_{s_1}^t ds_2 \ldots \int_{s_{K-1}}^t ds_K \left\{\varphi\left((s_K)\right) I_0^{(0)}\left(c\left(t-s_K\right)\right)\right\}\right]_0^t$$

$$+ \int_0^t s_1 \int_{s_1}^t ds_3 \ldots \int_{s_{K-1}}^t ds_K \left\{\varphi\left((s_K)\right) I_0^{(0)}\left(c\left(t-s_K\right)\right)\right\} ds_1$$

$$= \int_0^t \frac{s_1^2}{2} \int_{s_1}^t ds_4 \ldots \int_{s_{K-1}}^t ds_K \left\{\varphi\left((s_K)\right) I_0^{(0)}\left(c\left(t-s_K\right)\right)\right\} ds_1$$

$$\vdots$$

$$= \int_0^t \frac{s_1^{K-1}}{(K-1)!} \varphi\left((s_1)\right) I_0^{(0)}\left(c\left(t-s_1\right)\right) ds_1 = \psi_K * I_0^{(0)}(t). \qquad [1.7]$$

Now, suppose that [1.6] is valid for any $l \leq N-2$, for a fixed $N > 2$ and $r = 0$.

The integral $I_{l+1}^{(0)}(t)$ for $l = N-1$ can be represented in the following form:

$$I_N = \int_0^t ds_1 \ldots \int_{s_{K-1}}^t ds_K \left\{\varphi\left(s_K\right) \int_{s_K}^t ds_{K+1}\right.$$

$$\ldots \int_{s_{2K-1}}^t ds_{2K} \left\{\varphi\left((s_{2K}-s_K)\right) \int_{s_{2K}}^t ds_{2K+1} \ldots \left\{\int_{s_{K(N-1)-1}}^t ds_{K(N-1)}\right.\right.$$

$$\left.\left.\left.\ldots \int_{s_{KN-1}}^t ds_{KN} \varphi\left(\left(s_{KN}-s_{K(N-1)}\right)\right) \varphi\left((t-s_{kN})\right)\right\} \ldots\right\}. \qquad [1.8]$$

Let us consider the following interior integral in $I_N^0(t)$ (with respect to $s_{K(N-1)+1}$) in [1.8]. Let us make the change of variables

$\xi_j = s_{K(N-1)+j} - s_{K(N-1)}$, $j = 1, 2, \ldots, K$, and by taking into account [1.7], we obtain

$$\int_{s_{K(N-1)}}^{t} ds_{K(N-1)+1} \cdots \int_{s_{KN-1}}^{t} ds_{KN} \varphi\left(\left(s_{KN} - s_{K(N-1)}\right)\right) \varphi\left((t - s_{KN})\right)$$

$$= \int_{0}^{t-s_{K(N-1)}} d\xi_1 \int_{\xi_1}^{t-s_{K(N-1)}} \cdots \int_{\xi_{K-1}}^{t-s_{K(N-1)}} d\xi_K \varphi\left(\xi_K\right) \varphi\left(\left(t - s_{K(N-1)} - \xi_K\right)\right)$$

$$= \psi_K * I_1^{(0)}\left(t - s_{K(N-1)}\right).$$

Now, let us deal with the next interior integral in [1.8] (with respect to $s_{K(N-2)+1}$). By defining the change of variables $\xi_j = s_{K(N-2)+j} - s_{K(N-2)}$, $j = 1, 2, \ldots, K$, we obtain

$$\int_{s_{K(N-2)}}^{t} ds_{K(N-2)+1} \cdots \int_{s_{K(N-1)-1}}^{t} ds_{K(N-1)} \varphi\left(\left(s_{K(N-1)} - s_{K(N-2)}\right)\right)$$

$$\times I_1^{(0)}\left(t - s_{K(N-1)}\right)$$

$$= \int_{0}^{t-s_{K(N-1)}} d\xi_1 \int_{\xi_1}^{t-s_{K(N-1)}} \cdots \int_{\xi_{K-1}}^{t-s_{K(N-1)}} d\xi_K \varphi\left(\xi_K\right)$$

$$\times I_1^{(0)}\left(t - s_{K(N-2)} - \xi_K\right) = \psi_K * I_1^{(0)}\left(\left(t - s_{K(N-2)}\right)\right).$$

This is in accordance with [1.7] and the induction assumption.

By continuing this procedure, we can obtain, after the $(N-1)$th step,

$$I_N^{(0)}(t) = \psi_K * I_{N-1}^{(0)}(t).$$

Next, it can be easily verified that for any $0 \leq r \leq K-1$

$$\int_{s_K}^{t} ds_{K+1} \cdots \int_{s_{K+r-1}}^{t} ds_{K+r} = \frac{\left(t - s_K\right)^r}{r!}.$$

Hence, for all $0 < r \leq K-1$, we have

$$I_1^{(r)}(t) = \int_0^t ds_1 \int_{s_1}^t ds_2 \ldots \int_{s_{K-1}}^t ds_K \left\{\varphi\left((s_K)\right) \varphi\left((t - s_K)\right)\right\} \int_{s_K}^t ds_{K+1}$$

$$\cdots \int_{s_{K+r-1}}^{t} ds_{K+r}$$

$$= \int_0^t ds_1 \int_{s_1}^t ds_3 \cdots \int_{s_{K-1}}^t ds_K \left\{ \varphi\left((s_K)\right) \frac{(t-s_K)^r}{r!} \varphi\left((t-s_K)\right) \right\}$$

$$= \int_0^t \frac{s_1^{K-1}}{(K-1)!} \varphi\left((s_1)\right) I_0^{(r)}\left(c\left(t-s_K\right)\right) ds_1 = \psi_K * I_0^{(r)}(t)$$

and by using the recursive relation [1.6] it can be proved for $0 < r \le K-1$, in a similar manner as we did for the case $r = 0$.

COROLLARY 1.1.– It follows from [1.7] and [1.8] that

$$I_l^{(r)}(t) = \psi_K^{l*} * I_0^{(r)}(t).$$

where $l*$ is the l-fold convolution, $l > 0$, $I_0^{(r)}(t) = \frac{t^r}{r!}\varphi(t)$, $r = 0, 1, \ldots, K-1$.

COROLLARY 1.2.– For any $l \ge 0$, the conditional characteristic functions satisfy the following equations:

$$H_{l+1}^{(r)}(t) = \frac{(K(l+1)+r)!}{t^{K(l+1)+r}(Kl+r)!} \int_0^t u^{Kl+r} \psi_K(u) H_l^{(r)}(t-u)\, du.$$

PROOF.– From [1.6], it follows that

$$H_{l+1}^{(r)}(t) = \frac{(K(l+1)+r)!}{t^{K(l+1)+r}} I_{l+1}^{(r)}(t)$$

$$= \frac{(K(l+1)+r)!}{t^{K(l+1)+r}(Kl+r)!} \int_0^t u^{Kl+r} \psi_K(u) \left[\frac{(Kl+r)!}{u^{Kl+r}} I_l^{(r)}(t-u)\right] du$$

$$= \frac{(K(l+1)+r)!}{t^{K(l+1)+r}(Kl+r)!} \int_0^t u^{Kl+r} \psi_K(u) H_l^{(r)}(t-u)\, du.$$

From equation [1.11] and using $H_0^{(r)}(t) = \varphi(t)$, we can calculate $H_l^{(r)}(t)$ for all $l \ge 1$, $r = 0, 1, \ldots, K-1$.

Many authors (such as Le Caër 2010, 2011; Beghin and Orsingher 2010a; De Gregorio and Orsingher 2012; and De Gregorio 2014), have studied the conditional distributions of the particle position at renewal epochs with velocities switched by a renewal process with Dirichlet-distributed interarrival times. These authors studied the characteristic functions $H_l^{(0)}(t)$, $l \ge 1$ in terms of the conditional characteristic function $H_l^{(r)}(t)$.

THEOREM 1.2.– The characteristic function $H(t) = \mathbf{E}\left[\exp\left\{ic\mathbf{x}^{(n)}(t)\right\}\right]$, $t \geq 0$, of the random motion $\mathbf{x}^{(n)}(t)$ is a solution of the following renewal equation:

$$H(t) = e^{-\lambda t}\sum_{r=0}^{K-1}\frac{t^r}{r!}\varphi(t) + \lambda^K\int_0^t \psi_K\left(t-u\right)e^{-\lambda(t-u)}H\left(u\right)du, \qquad [1.9]$$

or in the convolution form

$$H(t) = \varphi(t)\left(1 - F_K(t)\right) + \left(\varphi f_K\right) * H\left(u\right),$$

where $F_K(t)$ is Erlang-K distributed and $f_K(t) = \frac{d}{dt}F_K(t)$.

PROOF.– It is easily seen that

$$H(t) = \exp\left\{i\left(\boldsymbol{\alpha}, \mathbf{x}^{(n)}(t)\right)\right\} = \sum_{l=0}^{+\infty}\sum_{r=0}^{K-1} H_l^{(r)}(t)P\left(\nu(t) = Kl + r\right)$$

$$= e^{-\lambda t}\sum_{l=0}^{\infty}\sum_{r=0}^{K-1}\lambda^{Kl+r}I_l^{(r)}(t).$$

Now, we need to show that for any $t > 0$ the series $\sum_{l=0}^{\infty}\sum_{r=0}^{K-1}\lambda^{Kl+r}I_l^{(r)}(t)$ uniformly converges with respect to $c = \alpha v$.

Indeed, taking into account that $|\varphi\left(\lambda\right)| = \left|\mathbf{E}\left[e^{-i\lambda\eta_j^{(n)}}\right]\right| \leq 1$, we have

$$\left|I_l^{(r)}(t)\right| \leq \int_0^t ds_1\int_{s_1}^t ds_2 \ldots \int_{s_{lK+r-1}}^t ds_{lK+r} = \frac{t^{lK+r}}{(lK+r)!}.$$

By using this bound, we state

$$\left|\sum_{l=0}^{\infty}\sum_{r=0}^{K-1}\lambda^{Kl+r}I_l^{(r)}(t)\right| \leq \sum_{l=0}^{\infty}\sum_{r=0}^{K-1}\frac{t^{lK+r}}{(lK+r)!} = e^{\lambda t}.$$

Therefore, $H(t)$ can be represented by the uniformly convergent series as follows:

$$H(t) = e^{-\lambda t}\sum_{l=0}^{\infty}\sum_{r=0}^{K-1}\lambda^{Kl+r}I_l^{(r)}(t) = e^{-\lambda t}\sum_{l=0}^{\infty}\sum_{r=0}^{K-1}\psi_K^{l*} * I_0^{(r)}(t). \qquad [1.10]$$

Hence, we have

$$H(t) = \exp\left\{i\left(\boldsymbol{\alpha}, \mathbf{x}^{(n)}(t)\right)\right\} = \sum_{l=0}^{+\infty}\sum_{r=0}^{K-1} H_l^{(r)}(t)P\left(\nu(t) = Kl + r\right)$$

$$= e^{-\lambda t}\left\{\sum_{r=0}^{K-1}\frac{t^r}{r!}\varphi(t)+\int_0^t \psi_K\left(t-u\right)\left(\sum_{l=1}^{\infty}\sum_{r=0}^{K-1}\lambda^{Kl+r}I_{l-1}^{(r)}\left(u\right)\right)du\right\}$$

$$= e^{-\lambda t}\left\{\sum_{r=0}^{K-1}\frac{t^r}{r!}\varphi(t)+\lambda^K\int_0^t \psi_K\left(t-u\right)\left(\sum_{l=0}^{\infty}\sum_{r=0}^{K-1}\lambda^{Kl+r}I_{l}^{(r)}\left(u\right)\right)du\right\}$$

$$= e^{-\lambda t}\left\{\sum_{r=0}^{K-1}\frac{t^r}{r!}\varphi(t)+\lambda^K\int_0^t \psi_K\left(t-u\right)e^{\lambda u}H\left(u\right)du\right\}.$$

In the following sections, equation [1.9] is generalized for an arbitrary distribution of time between consecutive changes of the particle direction and for random absolute velocity of the particle.

EXAMPLE 1.1.– Let us consider a three-dimensional random motion with Erlang-2 distribution for times between consecutive changes of the particle direction. In this case, $f_{\eta^{(3)}}\left(x\right) = \frac{1}{2}I\left\{-1\le x\le 1\right\}$. Therefore, $\varphi(t) = \mathbf{E}\left[e^{ict\eta^{(3)}}\right] = \frac{\sin ct}{ct}$, and $H_0^{(0)}(t)=\varphi\left(\alpha vt\right)=\frac{\sin\alpha vt}{\alpha vt}=H_0^{(1)}(t)$, where $\alpha=\sqrt{\alpha_1^2+\alpha_2^2+\alpha_3^2}$.

It is well known that the function $4\pi R\dfrac{\sin\alpha R}{\alpha}$ is the Fourier transform for the function of simple layer $\delta_{S(R)}$ (Vladimirov 1996).

Denote by $\overline{H}_l(t) = H_l^{(0)}(t)P\left[\nu(t)=2l\right] + H_l^{(1)}(t)P\left[\nu(t)=2l+1\right]$, the characteristic function of the particle position $\mathbf{x}_l^{(3)}(t)$ at time t provided that l changes of velocity are occurred. Hence,

$$P\left(\mathbf{x}^{(3)}(t)\in d\mathbf{x}/\nu_2(t)=0\right)=P\left(\mathbf{x}_0^{(3)}(t)\in d\mathbf{x}\right)$$

$$=\mathcal{F}^{-1}\left(\overline{H}_0(t)\right)dx_1dx_2dx_3=\frac{\delta_{S(vt)}}{4\pi\left(vt\right)^2}\left(e^{-\lambda t}+\lambda te^{-\lambda t}\right)dx_1dx_2dx_3,$$

where $\mathbf{x}=\left(x_1,x_2,x_3\right)$.

Let us calculate $H_1^{(0)}(t)$ and $H_1^{(1)}(t)$. It is easily verified that for the three-dimensional space, we have

$$H_1^{(0)}(t)=\frac{2}{t^2}\int_0^t\psi\left(t-u\right)H_0^{(0)}\left(u\right)du$$

$$=\frac{2}{\alpha^2t^2v^2}\int_0^t\frac{\left(t-u\right)\sin\left(\alpha v\left(t-u\right)\right)}{\left(t-u\right)}\frac{\sin\left(\alpha vu\right)}{u}du$$

$$=\frac{2}{\alpha^2t^2v^2}\int_0^t\frac{\sin\left(\alpha v\left(t-u\right)\right)\sin\left(\alpha vu\right)}{u}du$$

$$= \frac{1}{\alpha^2 t^2 v^2} \left(\mathrm{Si}\left(2\alpha vt\right) \sin\left(\alpha vt\right) + \mathrm{Ci}\left(2\alpha vt\right) \cos\left(\alpha vt\right) \right.$$
$$\left. - \left(\ln\left(2\alpha vt\right) + \gamma\right) \cos\left(\alpha vt\right)\right)$$
$$= \frac{1}{\alpha^2 t^2 v^2} \left(\mathrm{Si}\left(2\alpha vt\right) \sin\left(\alpha vt\right) + \int_0^{2\alpha vt} \frac{\cos\xi - 1}{\xi} d\xi \cos\left(\alpha vt\right)\right),$$

where γ is Euler's constant.

Denote by $f\left(\mathbf{x} \,|\, \nu(t) = 2\right) d\mathbf{x} = P\left(\mathbf{x}^{(3)}(t) \in d\mathbf{x} \,|\, \nu(t) = 2\right)$ the conditional pdf of a particle position assuming that there was only one change of speed occurred and after this no more Poisson event occurred.

Then we have $f\left(\mathbf{x} \,|\, \nu(t) = 2\right) = \mathcal{F}^{-1}\left(H_1^{(0)}(t)\right)$, where $\mathcal{F}^{-1}$ is the inverse Fourier transform.

Hence,

$$f\left(\mathbf{x} \,|\, \nu(t) = 2\right) = \frac{1}{4\pi \|\mathbf{x}\| \left(vt\right)^2} \ln\left(\frac{vt + \|\mathbf{x}\|}{vt - \|\mathbf{x}\|}\right) \to \infty, \quad \|\mathbf{x}\| \uparrow vt.$$

Consequently, for the continuous part $f_c\left(\mathbf{x}\right)$ of the distribution of the particle position, we obtain

$$f\left(\mathbf{x} \,|\, \nu(t) = 2\right) \leq f_c\left(\mathbf{x}\right) \to \infty, \quad \|\mathbf{x}\| \uparrow vt.$$

Thus, in this case, the explosion effect holds in a similar manner as the case of the exponential switching process.

For the conditional pdf $f\left(\mathbf{x} \,|\, \nu(t) = 3\right)$ of the particle position assuming that only one change of speed occurred and after that exactly one Poisson event occurred, we have $f\left(\mathbf{x} \,|\, \nu(t) = 3\right) = \mathcal{F}^{-1}\left(H_1^{(1)}(t)\right)$. For $H_1^{(1)}(t)$, we obtain

$$H_1^{(1)}(t) = \frac{6}{\alpha^2 t^3 v^2} \int_0^t u \frac{(t-u) \sin\left(\alpha v\left(t-u\right)\right)}{(t-u)} \frac{\sin\left(\alpha vu\right)}{u} du$$
$$= \frac{6}{t^3 v^2 \alpha^2} \int_0^t \sin\left(\alpha v\left(t-u\right)\right) \sin\left(\alpha vu\right) du = 3 \frac{\sin\left(\alpha vt\right) - \alpha vt \cos\left(\alpha vt\right)}{t^3 v^3 \alpha^3}$$
$$= \chi\left(\alpha\right).$$

Since the function $\chi\left(\alpha\right)$ depends only on $\alpha = \sqrt{\alpha_1^2 + \alpha_2^2 + \alpha_3^2}$, the function $f\left(\mathbf{x} \,|\, \nu(t) = 3\right)$ depends only on $\rho = \|\mathbf{x}\| = \sqrt{x_1^2 + x_2^2 + x_3^2}$, that is $f\left(\mathbf{x} \,|\, \nu(t) = 3\right) = \phi_1^{(1)}\left(\rho\right)$, $\rho \leq vt$.

Thus, we have

$$\chi(\alpha) = \frac{(2\pi)^{3/2}}{\sqrt{\alpha}} \int_0^{vt} \phi_1^{(1)}(\rho)\,\rho^{3/2} J_{1/2}(\rho\alpha) d\rho$$
$$= \frac{4\pi}{\alpha} \int_0^{vt} \phi_1^{(1)}(\rho)\,\rho \sin(\rho\alpha) d\rho.$$

It is easily verified that if $\phi_1^{(1)}(\rho) = \dfrac{3}{4\pi t^3 v^3} I_{\{\rho \leq vt\}}$, then

$$\frac{4\pi}{\alpha} \int_0^{vt} \frac{3}{4\pi t^3 v^3} \rho \sin \alpha\rho d\rho = \chi(\alpha).$$

This means that $f(\mathbf{x} \mid \nu(t) = 3) = \frac{3}{4\pi t^3 v^3} \mathbb{I}_{\{\|\mathbf{x}\| \leq vt\}}$ is the inverse three-dimensional Fourier transform of the function $\chi(\alpha)$.

Therefore, after exactly three Poisson events the particle position is uniformly distributed over the unit ball $B(vt) = \{\|\mathbf{x}\| < vt\}$ as follows:

$$f(\mathbf{x} \mid \nu(t) = 3) = \frac{3}{4\pi t^2 v^3} \mathbb{I}_{\{\|\mathbf{x}\| \leq vt\}}.$$

1.1.2. *Some properties of the random walk in a semi-Markov environment and its characteristic function*

Now, we consider the general cdf $G(t)$, which has the pdf $g(t)$. By using renewal theory one can easily prove that the characteristic function $H(t)$ is a solution of the following Volterra renewal-type integral equation:

$$H(t) = (1 - G(t))\,\varphi(t) + \int_0^t g(u)\,\varphi(u)\,H(t-u)\,du. \qquad [1.11]$$

We will generalize equation [1.11] in the following lemma to the case where the absolute velocity of the particle is a random variable. We should recall that equation [1.11] has a unique solution in the case of a continuous kernel $g\,\varphi$, since it is a Volterra equation.

LEMMA 1.1.– Suppose that $g(t) > 0$ for all $t \geq 0$. Then for $n = 2, 3$, we have

$$f_n(t, \mathbf{x}) \uparrow \infty \text{ as } \|\mathbf{x}\| \uparrow vt.$$

PROOF.– Since $f_n(t, \mathbf{x}) = \mathcal{F}^{-1}(H(t))$, where $\mathcal{F}^{-1}$ is the n-dimensional inverse Fourier transform of $H(t) = (H(t, \|\alpha\|))$ with respect to $\boldsymbol{\alpha}$, then it follows from [1.11] that

$$f_n(t, \mathbf{x}) = \mathcal{F}^{-1}(H(t)) = (1 - G(t)) \frac{\Gamma\left(\frac{n}{2}\right) \delta\left(v^2 t^2 - \|\mathbf{x}\|^2\right)}{2\pi^{\frac{n}{2}} (vt)^{n-1}}$$

$$+\frac{\left(\Gamma\left(\frac{n}{2}\right)\right)^2}{4\pi^n}\int_{\|\mathbf{u}\|\le vt}\int_0^t \frac{(1-G(t-s))\,\delta\left(v^2(t-s)^2-\|\mathbf{x}-\mathbf{u}\|^2\right)}{(v(t-s))^{n-1}}$$
$$\times\frac{g(s)\,\delta\left(v^2s^2-\|\mathbf{u}\|^2\right)}{(vs)^{n-1}}dsd\mathbf{u}+\dots$$

For $n=3$, we have $\varphi(t)=\dfrac{\sin(vt\,\|\boldsymbol{\alpha}\|)}{vt\,\|\boldsymbol{\alpha}\|}$.

It is well known that

$$\mathcal{L}\left(\frac{\sin(vt\,\|\boldsymbol{\alpha}\|)}{vt\,\|\boldsymbol{\alpha}\|}\right)=\frac{1}{v\,\|\boldsymbol{\alpha}\|}\operatorname{arctg}\left(\frac{v\,\|\boldsymbol{\alpha}\|}{s}\right).$$

Thus, we have

$$\frac{\left(\Gamma\left(\frac{3}{2}\right)\right)^2}{4\pi^3}\int_{\|\mathbf{u}\|\le vt}\int_0^t \frac{\delta\left(v^2(t-s)^2-\|\mathbf{x}-\mathbf{u}\|^2\right)}{(v(t-s))^2}\frac{\delta\left(v^2s^2-\|\mathbf{u}\|^2\right)}{(vs)^2}dsd\mathbf{u}$$
$$=\mathcal{F}^{-1}\left(\frac{1}{v^2\|\boldsymbol{\alpha}\|^2}\mathcal{L}^{-1}\left[\left(\operatorname{arctg}\left(\frac{v\,\|\boldsymbol{\alpha}\|}{s}\right)\right)^2\right]\right)=\frac{1}{4\pi v^2t^2\,\|\mathbf{x}\|}\ln\left(\frac{vt+\|\mathbf{x}\|}{vt-\|\mathbf{x}\|}\right).$$

Since for any $s\ge 0$, the function $g(s)>0$, we have

$$C_t=\inf_{0\le s\le t}(1-G(t-s))\,g(s)>0$$

and

$$\frac{C_t}{4\pi v^2t^2\,\|\mathbf{x}\|}\ln\left(\frac{vt+\|\mathbf{x}\|}{vt-\|\mathbf{x}\|}\right)\le f_3(t,\mathbf{x}).$$

Hence, $f_3(t,\mathbf{x})\uparrow\infty$ as $\|\mathbf{x}\|\uparrow vt$, i.e. we have the explosive effect.

For the case $n=2$ (i.e. $\mathbf{u}$, $\mathbf{x}\in\mathbb{R}^2$), we have

$$\frac{2}{4\pi^2}\int_0^t\int_{\|\mathbf{u}\|\le vt}\frac{\delta\left(v^2(t-s)^2-\|\mathbf{x}-\mathbf{u}\|^2\right)}{(v(t-s))^2}\frac{\delta\left(v^2s^2-\|\mathbf{u}\|^2\right)}{(vs)^2}dsd\mathbf{u}$$
$$=\frac{\left(v^2s^2-\|\mathbf{u}\|^2\right)^{-\frac{1}{2}}}{4\pi vt}.$$

By following the same approach, we can show that $f_2(t,\mathbf{x})\uparrow\infty$ as $\|\mathbf{x}\|\uparrow vt$.

In the case of an Erlang pdf $g(t)$, the explosive effect also holds because in these spaces of dimension $n=2,3$ there are not enough directions to leave the "layer" around the sphere of singularity.

1.2. Random motion with uniformly distributed directions and random velocity

Now, we deal with uniformly distributed direction of motion or isotropic motion at random speed or velocity where the direction alternations occur according to the renewal epochs of a general distribution. We derive the renewal equation for the characteristic function of the transition density of the multidimensional motion. Then, by using the renewal equation, we study the behavior of the transition density near the sphere of its singularity for one-, two-, three- and four-dimensional cases. To illustrate our solution methodology, we present detailed calculations of many solvable examples.

1.2.1. *Renewal equation for the characteristic function of isotropic motion with random velocity in a semi-Markov media*

Most papers concerning isotropic random motion on the multidimensional space are devoted to the analysis of models in which motions are driven by the homogeneous Poisson process. Another typical assumption is to consider constant speed or constant velocity; their models are basically Markov processes (e.g. (Orsingher and De Gregorio 2007; Orsingher and Beghin 2006; Pinsky 1991; Pogorui and Rodríguez-Dagnino 2011; Pogorui 2011) and references therein). In this section, we consider multidimensional random motions with uniformly distributed directions having random velocity. We will find a renewal equation for the characteristic function of the distribution of the particle position of this motion.

For the case of four-dimensional space, we give an example having an explosive effect, i.e. the pdf of the particle position goes to infinity when approaching the sphere of singularity if the absolute speed of the particle is some random variable.

As mentioned before, we consider the renewal process $\nu(t) = \max\{m \geq 0 : \ \tau_m \leq t\}$, $t \geq 0$, where $\tau_m = \sum_{k=0}^{m} \theta_k$, $\tau_0 = 0$ and $\theta_k \geq 0$, $k = 1, 2, \ldots$, are i.i.d. random variables denoting the interarrival times. Let $G(t)$ be the cdf of θ_1 and we assume that there exists the pdf $g(t) = \frac{d}{dt}G(t)$.

We will study the random motion of a particle that starts from the coordinate origin $\mathbf{0} = (0, 0, \ldots, 0)$ of the space $\mathbb{R}^n$ at time $t = 0$ and continues its motion with a velocity $\gamma_0 > 0$ along the direction $\boldsymbol{\eta}_0^{(n)}$, where γ_0 is a random variable with the cdf $Z(v)$ and there exists the pdf $z(v) = \frac{d}{dv}Z(v)$, $v \geq 0$, and the direction $\boldsymbol{\eta}_0^{(n)} = (x_1, x_2, \cdots, x_n)$ is a random n-dimensional vector uniformly distributed on the sphere

$$\Omega_1^{n-1} = \left\{(x_1, x_2, \cdots, x_n) : \ x_1^2 + x_2^2 + \ldots + x_n^2 = 1\right\}.$$

The random variables γ_0 and $\boldsymbol{\eta}_0^{(n)}$ are independent.

Then, the particle moves in straight lines and it changes its direction at renewal epochs $\tau_m,\ m = 1,\ 2, \ldots,$ much in the same manner, which was considered in Volume 1, Chapter 5, with the only difference that at epoch τ_k the particle changes not only the direction of its movement from $\boldsymbol{\eta}_{k-1}^{(n)}$ to $\boldsymbol{\eta}_k^{(n)}$ but with the direction it also changes the absolute velocity from γ_{k-1} to γ_k, where γ_{k-1} and γ_k are i.i.d. with the distribution $Z(v)$.

Denote by $\mathbf{x}^{(n)}(t)$, $t \geq 0$, the particle position at time t. As in the previous section, we have

$$\mathbf{x}^{(n)}(t) = \sum_{j=1}^{\nu(t)} \boldsymbol{\eta}_{j-1}^{(n)} \gamma_{j-1} \left(\tau_j - \tau_{j-1}\right) + \boldsymbol{\eta}_{\nu(t)}^{(n)} \gamma_{\nu(t)} \left(t - \tau_{\nu(t)}\right). \qquad [1.12]$$

Let us consider the projection $x^{(n)}(t)$ of $\mathbf{x}^{(n)}(t)$ onto a fixed line. Then we have

$$x^{(n)}(t) = \sum_{j=1}^{m+1} \eta_{j-1}^{(n)} \gamma_{j-1} \left(\tau_j - \tau_{j-1}\right) + \eta_{\nu(t)}^{(n)} \gamma_{\nu(t)} \left(t - \tau_{\xi(t)}\right)$$

where $\eta_j^{(n)}$ is the projection of $\boldsymbol{\eta}_j^{(n)}$ onto the line.

Denote as $F_x\left(y\right) = P\left(x_m^{(n)}(t) \leq y\right)$. For the characteristic function of $\mathbf{x}^{(n)}(t)$, we have

$$H(t) = \mathbf{E}\left[\exp\left\{i\left(\boldsymbol{\alpha}, \mathbf{x}^{(n)}(t)\right)\right\}\right] = \mathbf{E}\left[\exp\left\{i\left\|\boldsymbol{\alpha}\right\|\left(\mathbf{e}, \mathbf{x}^{(n)}(t)\right)\right\}\right]$$

$$= \mathbf{E}\left[\exp\left\{i\left\|\boldsymbol{\alpha}\right\| x^{(n)}(t)\right\}\right] = \int_0^\infty \exp\left\{i\left\|\boldsymbol{\alpha}\right\| y\right\} dF_x\left(y\right),$$

where $\|\boldsymbol{\alpha}\| = \sqrt{\alpha_1^2 + \alpha_2^2 + \cdots + \alpha_n^2}$.

Thus, the probabilistic properties of $\mathbf{x}^{(n)}(t)$ are completely defined by the properties of $x^{(n)}(t)$.

Denote by $f_{\eta^{(n)}}\left(x\right)$ the pdf of $\eta_j^{(n)}$. From [1.2], it follows that

$$f_{\eta^{(n)}}\left(x\right) = \begin{cases} \frac{\Gamma\left(\frac{n}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{n-1}{2}\right)}\left(1 - x^2\right)^{(n-3)/2} & , \ x \in \left[-1, 1\right], \\ 0 & , \ x \notin \left[-1, 1\right]. \end{cases} \qquad [1.13]$$

Hence, it is easily verified that $F_{\eta^{(n)}v}(x) = P\left(\eta_j^{(n)}\gamma_j \le x\right)$ is of the following form:

$$F_{\eta^{(n)}\gamma}(x) = \begin{cases} \frac{1}{2} + \frac{\Gamma\left(\frac{n}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{n-1}{2}\right)} \int_0^1 Z\left(\frac{x}{t}\right)\left(1-t^2\right)^{(n-3)/2} dt, & x \ge 0, \\ \frac{1}{2} - \frac{\Gamma\left(\frac{n}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{n-1}{2}\right)} \int_0^1 Z\left(-\frac{x}{t}\right)\left(1-t^2\right)^{(n-3)/2} dt, & x < 0. \end{cases} \quad [1.14]$$

Let us denote by $\varphi(t) = \mathbf{E}\left[e^{it\alpha\gamma_j\eta_j^{(n)}}\right] = \int_{-\infty}^{\infty} e^{it\alpha x} dF_{\eta^{(n)}\gamma}(x)$ the characteristic function of the random variable $\eta_j^{(n)}\gamma_j$. As it is mentioned in section 1.5.1., if $\gamma_j = v = const > 0$, then the function $\varphi(t)$ is as follows:

$$\varphi(t) = 2^{\frac{n-2}{2}}\Gamma\left(\frac{n}{2}\right)\frac{J_{\frac{n-2}{2}}(\alpha t v)}{(\alpha t v)^{\frac{n-2}{2}}},$$

where $\alpha = \|\boldsymbol{\alpha}\|$.

In the case where γ_j is a random variable with pdf $z(v)$, $v \ge 0$, it is easy to verify that $\varphi(t)$ is of the following form:

$$\varphi(t) = \mathbf{E}\left[e^{it\alpha\gamma_j\eta_j^{(n)}}\right] = 2^{\frac{n-2}{2}}\Gamma\left(\frac{n}{2}\right)\int_0^{\infty}\frac{J_{\frac{n-2}{2}}(\alpha t v)}{(\alpha t v)^{\frac{n-2}{2}}} z(v)\, dv. \quad [1.15]$$

THEOREM 1.3.– The characteristic function $H(t)$, $t \ge 0$, is a solution of the following Volterra renewal-type integral equation:

$$H(t) = (1 - G(t))\,\varphi(t) + \int_0^t g(u)\,\varphi(u)\,H(t-u)\,du. \quad [1.16]$$

PROOF.– From equation [1.12], it follows that

$$\begin{aligned} H(t) &= \mathbf{E}\left[\exp\left\{i\left(\boldsymbol{\alpha}, \mathbf{x}^{(n)}(t)\right)\right\}\right] \\ &= \mathbf{E}\left[\exp\left\{i\left(\boldsymbol{\alpha}, \sum_{j=1}^{\xi(t)} \boldsymbol{\eta}_{j-1}^{(n)}\gamma_{j-1}\theta_j + \gamma_{\xi(t)}\boldsymbol{\eta}_{\xi(t)}^{(n)}\left(t - \tau_{\xi(t)}\right)\right)\right\}\right] \\ &= (1 - G(t))\,Ee^{it\gamma_0\left(\boldsymbol{\alpha}, \boldsymbol{\eta}_0^{(n)}\right)} + \int_0^t g(u)\,Ee^{iu\gamma_0\left(\boldsymbol{\alpha}, \boldsymbol{\eta}_0^{(n)}\right)} H(t-u)\,du. \end{aligned}$$

To complete the proof, we mention that $\varphi(t) = \mathbf{E}\left[e^{it\gamma_0\left(\boldsymbol{\alpha}, \boldsymbol{\eta}_0^{(n)}\right)}\right]$.

Now let us denote by $f_n(t, \mathbf{x})$ the pdf of the particle position at time t. Therefore, $f_n(t, \mathbf{x}) = \mathcal{F}^{-1}(H(t))$, where $\mathcal{F}^{-1}$ is the n-dimensional inverse Fourier transform with respect to $\boldsymbol{\alpha}$.

Let us study functions $f_n(t, \mathbf{x})$ for $n = 1, 2, 3, 4$.

1.2.2. *One-dimensional case*

Let us consider the one-dmensional case, i.e. $n = 1$, with constant velocity $v > 0$, $\varphi(t) = \cos(\alpha t v)$. We have, for the exponential pdf $g(t) = \lambda e^{-\lambda t}\mathbb{I}_{\{t\geq 0\}}$, the renewal equation

$$H(t) = e^{-\lambda t}\cos(\alpha t v) + \lambda \int_0^t e^{-\lambda u}\cos(\alpha u v) H(t-u)\, du. \qquad [1.17]$$

We should note that the process $\mathbf{x}^{(1)}(t)$ is not the telegraph process because at renewal epochs the particle may not change its direction.

Denote by $h(t) = H(t)e^{\lambda t}$, then from equation [1.17] it follows that

$$h(t) = \cos(\alpha t v) + \lambda \int_0^t \cos(\alpha u v)\, h(t-u)\, du.$$

Since $\mathcal{L}(\cos(\alpha t v)) = \int_0^\infty e^{-su}\cos(\alpha u v) du = \frac{s}{s^2+v^2\alpha^2}$, then after applying the Laplace transform to [1.17] (or its equivalent in $h(t)$) we obtain

$$\mathcal{L}(h(t)) = \int_0^\infty e^{-st} h(t) dt = \frac{s}{s^2 - \lambda s + v^2\alpha^2}.$$

Thus, the inverse Laplace transform gives

$$H(t) = e^{-\lambda t}h(t) = e^{-\frac{\lambda t}{2}}\left(\cosh\left(\frac{t}{2}\sqrt{\lambda^2 - 4v^2\alpha^2}\right)\right.$$

$$\left. + \frac{\lambda \sinh\left(\frac{t}{2}\sqrt{\lambda^2 - 4v^2\alpha^2}\right)}{\sqrt{\lambda^2 - 4v^2\alpha^2}}\right). \qquad [1.18]$$

We should recall the formula from Bateman (1954, p. 57, no. 47).

$$\frac{\sinh\left(t\sqrt{1-\alpha^2}\right)}{\sqrt{1-\alpha^2}} = \frac{1}{2}\int_{-t}^t I_0\left(\sqrt{t^2 - x^2}\right) e^{ix\alpha} dx.$$

This can be tailored to our case as

$$\frac{\lambda \sinh\left(\frac{t}{2}\sqrt{\lambda^2 - 4v^2\alpha^2}\right)}{\sqrt{\lambda^2 - 4v^2\alpha^2}} = \frac{\lambda}{2v}\int_{-tv}^{tv} I_0\left(\frac{\lambda}{2v}\sqrt{(tv)^2 - x^2}\right) e^{ix\alpha} dx.$$

Therefore, by applying the inverse Fourier transform (with respect to $\boldsymbol{\alpha}$) to equation [1.18], we obtain

$$f_1(t,x) = \frac{1}{2}\delta\left((vt)^2 - x^2\right)e^{-\frac{\lambda t}{2}}$$

$$+\frac{e^{-\frac{\lambda t}{2}}}{2v}\left[\frac{\lambda}{2}I_0\left(\frac{\lambda}{2v}\sqrt{(tv)^2 - x^2}\right) + \frac{\partial}{\partial t}I_0\left(\frac{\lambda}{2v}\sqrt{(tv)^2 - x^2}\right)\right]$$

$$= \frac{1}{2}\delta\left((vt)^2 - x^2\right)e^{-\frac{\lambda t}{2}} + \frac{\lambda}{4v}I_0\left(\frac{\lambda}{2v}\sqrt{(tv)^2 - x^2}\right)e^{-\frac{\lambda t}{2}}$$

$$+\frac{\lambda t}{4}\frac{I_1\left(\sqrt{(tv)^2 - x^2}\right)}{\sqrt{(tv)^2 - x^2}}e^{-\frac{\lambda t}{2}}.$$

Thus, $f_1(t,x)$ is a solution of the following telegraph equation:

$$\frac{\partial^2 f}{\partial t^2} + \lambda\frac{\partial f}{\partial t} - v^2\frac{\partial^2 f}{\partial x^2} = 0, \;\; f(x,0) = \delta(x),$$

which defines the process with the exponential distribution with parameter $\lambda/2$ for interarrival times of the switching directions process. This is because the switching process for this random walk is a Markov process with the phase space $\{0,1\}$ and the following infinitesimal matrix:

$$Q = \begin{pmatrix} -\lambda/2 & \lambda/2 \\ \lambda/2 & -\lambda/2 \end{pmatrix}.$$

We should recall that for the Goldstein–Kac process the infinitesimal matrix is as follows:

$$2Q = \begin{pmatrix} -\lambda & \lambda \\ \lambda & -\lambda \end{pmatrix}.$$

Therefore, if we consider only real change of direction, the previous process is a Goldstein–Kac process with the parameter $\lambda/2$.

Suppose that γ_1 is a random variable with pdf

$$z(v) = \frac{2}{\pi\sqrt{1-v^2}}, \; 0 \le v \le 1.$$

This pdf is the density of distribution of the projection of a vector uniformly distributed on a unit semicircle onto a line that passes through the center of the circle (see section 1.4.2). Then, we obtain

$$\varphi(t) = \int_0^1 \cos(\alpha t v)\,\frac{2}{\pi\sqrt{1-v^2}}dv = J_0(\alpha t). \qquad [1.19]$$

For $g(t) = \lambda e^{-\lambda t} I_{\{t \geq 0\}}$, we have

$$H(t) = e^{-\lambda t} J_0(\alpha t) + \lambda \int_0^t e^{-\lambda(t-u)} J_0(\alpha(t-u)) H(u)\, du. \qquad [1.20]$$

Let us state as $h(t) = e^{\lambda t} H(t)$. Then equation [1.20] is of the form:

$$h(t) = J_0(\alpha t) + \lambda \int_0^t J_0(\alpha(t-u)) h(u)\, du. \qquad [1.21]$$

It is well known (Bateman 1954) that the Laplace transform of the Bessel function $J_0(\alpha t)$ with respect to t is as follows:

$$\mathcal{L}(J_0(\alpha t)) = \frac{1}{\sqrt{s^2+\alpha^2}}.$$

By obtaining the Laplace transform in equation [1.21], we have

$$\mathcal{L}(h(t)) = \frac{1}{\sqrt{s^2+\alpha^2}} + \frac{\lambda}{\sqrt{s^2+\alpha^2}} \mathcal{L}(h(t)).$$

Hence,

$$\mathcal{L}(h(t)) = \frac{1}{\sqrt{s^2+\alpha^2} - \lambda} = \frac{1}{\lambda} \sum_{n=1}^{\infty} \left(\frac{\lambda}{\sqrt{s^2+\alpha^2}} \right)^n.$$

The inverse Laplace transform of the function $\frac{1}{\lambda}\left(\frac{\lambda}{\sqrt{s^2+\alpha^2}}\right)^n$ with respect to s is of the following form:

$$\mathcal{L}^{-1}\left(\frac{1}{\lambda} \left(\frac{\lambda}{\sqrt{s^2+\alpha^2}} \right)^n \right) = \frac{\sqrt{\pi}}{\Gamma\left(\frac{n}{2}\right)} \left(\frac{\lambda^2 t}{2} \right)^{\frac{n-1}{2}} \frac{J_{\frac{n-1}{2}}(\alpha t)}{\alpha^{\frac{n-1}{2}}}.$$

Therefore,

$$H(t) = e^{-\lambda t} \sum_{n=1}^{\infty} \frac{\sqrt{\pi}}{\Gamma\left(\frac{n}{2}\right)} \left(\frac{\lambda^2 t}{2} \right)^{\frac{n-1}{2}} \frac{J_{\frac{n-1}{2}}(\alpha t)}{\alpha^{\frac{n-1}{2}}}.$$

Then, by reducing the inverse Fourier transform to the Hankel transform, we obtain

$$f_1(t,x) = \mathcal{F}^{-1}(H(t)) = e^{-\lambda t} \sum_{n=1}^{\infty} \frac{\lambda^{n-1}}{\left(\Gamma\left(\frac{n}{2}\right)\right)^2 2^{n-1}} \frac{\left(t^2 - x^2\right)^{\frac{n-2}{2}}}{(2t)^{\frac{n-1}{2}}} \mathbb{I}_{\{t > |x|\}}.$$

For the first term, $\mathcal{F}^{-1}\left(e^{-\lambda t} J_0(\alpha t)\right) = \frac{e^{-\lambda t}}{\pi\sqrt{t^2-x^2}} \mathbb{I}_{\{t>|x|\}}$. Hence, it is easily seen that the pdf $f_1(t,x)$ satisfies $f_1(t,x) = \mathcal{F}^{-1}(H(t)) \to +\infty, \ \ |x| \uparrow t$.

Therefore, we have an explosive effect at lines $|x| = t$ while for the constant velocity v the explosive effect does not appear.

1.2.3. *Two-dimensional case*

We should note that in case of constant velocity $\gamma_j = v > 0$, we have $\varphi(t) = J_0(\alpha t v)$.

Now, let us suppose that the velocity γ_j is a random variable with the following heavy-tail pdf:

$$z(v) = \frac{v}{(1+v^2)^{\frac{3}{2}}}, \quad v \geq 0.$$

Then,

$$\varphi(t) = \int_0^\infty J_0(\alpha t v)\, z(v) dv = \int_0^\infty \frac{J_0(\alpha t v)\, v}{(1+v^2)^{\frac{3}{2}}} dv = e^{-t\alpha}.$$

Now, let us assume an exponential pdf $g(t) = \lambda e^{-\lambda t}\mathbb{I}_{\{t\geq 0\}}$; hence, we have the renewal equation

$$H(t) = e^{-\lambda t} e^{-\alpha t} + \lambda \int_0^t e^{-\lambda(t-u)} e^{-\alpha(t-u)} H(u)\, du. \qquad [1.22]$$

By solving equation [1.22], we have

$$H(t) = e^{-\alpha t}.$$

Now, we can obtain the two-dimensional inverse Fourier transform of $H(t) = H(t,\alpha)$ with respect to α, and we can apply the Hankel transform (Bochner and Chandrasekhar 1949) to obtain the pdf of $\mathbf{x}^{(2)}(t)$

$$f_2(t,x) = \frac{1}{2\pi} \int_0^\infty e^{-\alpha t} \alpha J_0(\alpha x)\, d\alpha = \frac{t}{2\pi (t^2+x^2)^{\frac{3}{2}}},$$

$$x = |\mathbf{x}|, \quad \mathbf{x} = (x_1, x_2).$$

1.2.4. *Three-dimensional case*

1.2.4.1. *Isotropic transport process with random velocity in* $\mathbb{R}^3$

The isotropic random motion of a particle at constant velocity in $\mathbb{R}^n$ driven by a homogeneous Poisson process has been intensively studied by many researchers. For planar ($n = 2$) random motion, the explicit form of the pdf for the particle position at a fixed time was obtained by several methods (Stadje 2007; Masoliver *et al.* 1993b). However, for the most important three-dimensional case, the pdf for the continuous part of the distribution of the process was given in the form a complicated integral

(Stadje 2007), which can hardly be evaluated precisely. We remove the condition of constant velocity of the particle, and consider the case of a random speed, but distributed in the way such that it is close to a constant speed. We obtain the closed form for the pdf of the particle position in $\mathbb{R}^3$.

Let $\{\nu(t)\}$ be a Poisson process with rate $\lambda > 0$ starting at $\nu(0) = 0$. We define the time of the kth jump by $\tau_k = \inf\{t : \nu(t) = k\}$, $k \geq 0$. The random variable τ_k is also called the Poisson event and $\theta_k = \tau_{k+1} - \tau_k$ is called the kth waiting time (or interarrival time).

We will study the random motion of a particle that starts its motion from the origin $\mathbf{0} = (0, 0, 0)$ of the space $\mathbb{R}^{\mathbf{3}}$, at time $t = 0$, and it continues its motion with a velocity $v_0 > 0$ along a direction $\boldsymbol{\eta}_0$ in $\mathbb{R}^{\mathbf{3}}$. At instant τ_1, the particle changes its direction to $\boldsymbol{\eta}_1$ and continues its motion with a velocity $v_1 > 0$ along $\boldsymbol{\eta}_1$. At instant τ_2, the particle changes its direction to $\boldsymbol{\eta}_2$ and continues its motion with a velocity $v_2 > 0$ along $\boldsymbol{\eta}_2$, and so on. We assume that $\{v_i,\ i = 0, 1, 2, \ldots\}$ is a sequence of i.i.d. positive random variables with cdf $Z(s)$ and pdf $z(s) = dZ(s)/ds$, $s \geq 0$, and $\boldsymbol{\eta}_i$, $i = 0, 1, 2, \ldots$ are i.i.d. random three-dimensional vectors uniformly distributed on the unit sphere $\Omega_1^2 = \left\{(x_1, x_2, x_3) : x_1^2 + x_2^2 + x_3^2 = 1\right\}$. The random variables v_i are independent of the random variables $\boldsymbol{\eta}_j$, $i, j \in \{0, 1, 2, \ldots\}$.

Denote by $\mathbf{x}(t)$, $t \geq 0$ the particle position at time t. We have that

$$\mathbf{x}(t) = \sum_{j=1}^{\nu(t)} \boldsymbol{\eta}_{j-1} v_{j-1} \theta_{j-1} + \boldsymbol{\eta}_{\nu(t)} v_{\nu(t)} \left(t - \tau_{\nu(t)}\right). \qquad [1.23]$$

As we know (Pogorui and Rodríguez-Dagnino 2012), the characteristic function $H(t) = \mathbf{E}\left[\exp\left\{i\left(\boldsymbol{\alpha}, \mathbf{x}(t)\right)\right\}\right]$ satisfies the following renewal equation:

$$H(t) = e^{-\lambda t}\varphi(t) + \lambda \int_0^t e^{-\lambda u}\varphi(u) H(t-u)\, du, \qquad [1.24]$$

where

$$\varphi(t) = \int_0^\infty z(v) \frac{\sin(\alpha t v)}{\alpha t v}\, dv, \qquad \alpha = |\boldsymbol{\alpha}|.$$

As far as we know, in the three-dimensional case for a constant velocity of a particle, the distribution of the particle position has not been obtained before, and this case is relevant for applications. Now, we consider the case of random velocity where the particle velocity has the following pdf:

$$z(s) = \frac{4s^2}{\pi\sqrt{1-s^2}}, \quad 0 \leq s < 1. \qquad [1.25]$$

This velocity is "almost equal" to $v = 1$. By making this assumption, we are able to obtain the distribution of particle position in the closed form.

$$\varphi(t) = \int_0^\infty \frac{4v^2}{\pi\sqrt{1-v^2}} \frac{\sin(\alpha t v)}{\alpha t v} dv = \frac{2J_1(at)}{at}.$$

Hence, we have the following equation:

$$H(t) = \varphi(t)e^{-\lambda t} + \lambda \int_0^t \varphi(u)\, e^{-\lambda u} H(t-u)\, du.$$

This renewal equation [1.24] has a unique solution of the following form Kovalenko *et al*. (1983):

$$H(t) = e^{-\lambda t} \sum_{n=0}^{\infty} \lambda^n [\varphi(t)]^{*(n+1)}$$

The relation for the Bessel function (Prudnikov *et al*. 1986, p.190, formula 2.12.33.8), is well known.

$$\int_0^t \frac{J_m(a(t-u))}{t-u} \frac{J_n(au)}{u} du = \frac{m+n}{mn} \frac{J_{n+m}(at)}{t}. \qquad [1.26]$$

By using equation [1.26], we obtain

$$[\varphi(t)]^{*(k)} = \frac{(k-1)\, 2^{k-1}}{a^{k-1}} \int_0^t \frac{J_{k-1}(a(t-u))}{(t-u)} \frac{2J_1(au)}{au} du = k\, 2^k \frac{J_k(at)}{a^k t},$$

$$k = 2, 3, \ldots$$

where $[\varphi(t)]^{*(1)} = \varphi(t)$.

It is easily verified that

$$[\varphi(t)]^{*(k)} = k\, 2^k \frac{J_k(at)}{a^k t}.$$

The inverse Fourier (Hankel) transform of $[\varphi(t)]^{*(k)}$ is as follows:

$$I_k = \mathcal{F}^{-1}\left([\varphi(t)]^{*(k)}, \alpha\right) = \begin{cases} \frac{k2^k\sqrt{2/\pi}}{(k+1)!t^{k+1}} \sqrt{\left(t^2-x^2\right)^{2k-3}}\ , & k \text{ is even} \\ \frac{k2^k\sqrt{2/\pi}}{k!t^{k+1}} \sqrt{\left(t^2-x^2\right)^{2k-3}}\ , & k \text{ is odd,} \end{cases}$$

where $x = \|\mathbf{x}\|$.

Therefore, the pdf $f\left(t,\mathbf{x}\right)$ of the particle position at time t is as follows:

$$\begin{aligned} f\left(t,x\right) &= \mathcal{F}^{-1}\left(H(t)\right) \\ &= e^{-\lambda t}\sqrt{2/\pi}\left(e^{-\lambda t}\sqrt{2/\pi}\left(\sum_{m=1}^{\infty}\frac{(2m-1)\ 2^{(2m-1)}\lambda^{(2m-1)}}{(2m-1)!\,t^{2m}}\right.\right. \\ &\left.\left.\times\sqrt{\left(t^2-x^2\right)^{4m-5}}+\sum_{l=1}^{\infty}\frac{2l\ 4^l\lambda^{2l}}{(2l+1)!\,t^{2l+1}}\sqrt{\left(t^2-x^2\right)^{4l-3}}\right)\right). \end{aligned} \qquad [1.27]$$

The following formula is well known (Arfken *et al*. 2012, pp. 544–545 and 547–548):

$$\Gamma\left(n+\frac{1}{2}\right)=\frac{(2n-1)!}{2^n}\sqrt{\pi}. \qquad [1.28]$$

By using [1.28] and calculating the sum of series, we get for $x^2 < t^2$

$$\begin{aligned} f\left(t,x\right) &= \mathcal{F}^{-1}\left(H(t)\right)=e^{-\lambda t}\sqrt{2}\left(\sum_{l=1}^{\infty}\frac{l\ 2^l\lambda^{2l}}{\Gamma\left(l+3/2\right)t^{2l+1}}\sqrt{\left(t^2-x^2\right)^{4l-3}}\right. \\ &\left.+\sum_{m=1}^{\infty}\frac{(2m-1)\,2^{(m-1)}\lambda^{(2m-1)}}{\Gamma\left(m+1/2\right)t^{2m}}\sqrt{\left(t^2-x^2\right)^{4m-5}}\right) \\ &= \frac{e^{-\lambda t}\sqrt{2}}{4\sqrt{\pi}\lambda t^2\Lambda^3}\left[4\lambda t\Lambda^{3/2}+8\lambda^2\,\Lambda^{5/2}+\left(\lambda^2\sqrt{128\Lambda^5}-t^2\sqrt{8\Lambda}\right.\right. \\ &\left.\left.+\frac{16\lambda^3\sqrt{2\Lambda^7}}{t}\right)e^{\frac{2\lambda^2\Lambda^2}{t^2}}\int_0^{\sqrt{2}\lambda\Lambda/t}e^{-y^2}dy\right], \end{aligned}$$

where $\Lambda = t^2 - x^2$.

REMARK 1.1.– *It is well known that random walks are used to model polymers (Banchoff 1976). One important characteristic of a polymer is the distribution of the distance between starting and ending points of its molecules and the result obtained may be interesting from this point of view.*

1.2.4.2. *Isotropic transport process with random velocity in $\mathbb{R}^3$: small parameter approximation*

Now, we will study a random motion in $\mathbb{R}^3$ with random velocity, distributed according to Maxwell's law of distribution of velocities. We believe that this model is a natural alternative to the Wiener model of Brownian motion. In fact, this is a more realistic model because of the two following facts: first, the velocity of the particles in this model has the Maxwell distribution, unlike the Wiener process, where the absolute particle velocity is infinite. Second, unlike the Wiener particle path, the particle in this model has a non-zero free path.

Let us consider a random motion $x(t)$, which is determined by equation [1.23], where the random velocity v_i is distributed according to the Maxwell distribution with pdf

$$z\left(v\right)=\sqrt{\frac{2}{\pi}}v^{2}\,e^{-\frac{1}{2}v^{2}},\quad v>0.$$

Then, it follows that

$$\varphi(t)=\sqrt{2/\pi}\int_{0}^{\infty}v^{2}\,e^{-\frac{1}{2}v^{2}}\frac{\sin\left(\alpha tv\right)}{\alpha tv}dv=e^{-\frac{t^{2}\alpha^{2}}{2}}.$$

Hence, the equation for the characteristic function of the distribution of particle position is as follows:

$$H(t)=e^{-\lambda t}e^{-\frac{t^{2}\alpha^{2}}{2}}+\lambda\int_{0}^{t}e^{-\lambda u}e^{-\frac{u^{2}\alpha^{2}}{2}}H\left(t-u\right)du.$$

Denote as $h(t)=e^{\lambda t}H(t)$, then we have

$$h(t)=e^{-\frac{1}{2}t^{2}\alpha^{2}}+\lambda\int_{0}^{t}e^{-\frac{1}{2}u^{2}\alpha^{2}}h\left(t-u\right)du.$$

Since the parameter $\varepsilon=1/\lambda$ is very small (e.g. in most gas models), we will study the perturbed integral equation

$$\varepsilon h(t)=\varepsilon e^{-\frac{1}{2}t^{2}\alpha^{2}}+\int_{0}^{t}e^{-\frac{1}{2}u^{2}\alpha^{2}}h\left(t-u\right)du.$$

A solution of such equations can be found by using the following expansion (Angell and Olmstead 1987; Shubin 2006):

$$h(t)=\sum_{k=0}^{\infty}\varepsilon^{k}\left(u_{k}(t)+w_{k}\left(\tau\right)\right),\qquad [1.29]$$

where $\tau=t/\varepsilon$, the terms $u_k(t)$ are called regular, and $w_k\left(\tau\right)$ are called singular terms of [1.29].

For $u_0(t)$, we have

$$\varepsilon u_{0}(t)=\varepsilon e^{-\frac{1}{2}t^{2}\alpha^{2}}+\int_{0}^{t}e^{-\frac{1}{2}u^{2}\alpha^{2}}u_{0}\left(t-u\right)du.\qquad [1.30]$$

From [1.30], it follows

$$\int_0^t e^{-\frac{1}{2}u^2\alpha^2} u_0\left(t-u\right)du = 0. \qquad [1.31]$$

It is well known that equation [1.31] has only the trivial solution

$$u_0(t) = 0.$$

For $w_0\left(\tau\right)$, we have

$$\varepsilon w_0\left(\tau\right) = \varepsilon e^{-\frac{1}{2}t^2\alpha^2} + \int_0^t e^{-\frac{1}{2}(t-u)^2\alpha^2} w_0\left(\frac{u}{\varepsilon}\right)du.$$

By letting $s = \frac{u}{\varepsilon}$, we have

$$w_0\left(\tau\right) = e^{-\frac{1}{2}\tau^2\alpha^2\varepsilon^2} + \int_0^\tau e^{-\frac{1}{2}(\tau-s)^2\alpha^2\varepsilon^2} w_0\left(s\right)ds. \qquad [1.32]$$

In equation [1.32], we can only use the first term of the expansion for the exponent, so

$$e^{-\frac{1}{2}\tau^2\alpha^2\varepsilon^2} = 1 - \frac{1}{2}\tau^2\alpha^2\varepsilon^2 + \frac{1}{4}\tau^4\alpha^4\varepsilon^4 + \cdots, \qquad [1.33]$$

$$e^{-\frac{1}{2}(\tau-s)^2\alpha^2\varepsilon^2} = 1 - \frac{1}{2}(\tau-s)^2\alpha^2\varepsilon^2 + \frac{1}{4}(\tau-s)^4\alpha^4\varepsilon^4 + \cdots.$$

Hence, we obtain

$$w_0\left(\tau\right) = 1 + \int_0^\tau w_0\left(s\right)ds.$$

It is easily verified that

$$w_0\left(\tau\right) = e^\tau.$$

Next, for $u_1(t)$ we have

$$\varepsilon^2 u_1(t) = \varepsilon e^{-\frac{1}{2}t^2\alpha^2} + \varepsilon\int_0^t e^{-\frac{1}{2}(t-u)^2\alpha^2} u_1\left(u\right)du$$

or

$$e^{-\frac{1}{2}t^2\alpha^2} = \int_0^t e^{-\frac{1}{2}(t-u)^2\alpha^2} u_1\left(u\right)du.$$

For $w_1(\tau)$, we have

$$\varepsilon^2 w_1(\tau) = \varepsilon\left(e^{-\frac{1}{2}t^2\alpha^2} - 1\right) + \varepsilon \int_0^t e^{-\frac{1}{2}(t-u)^2\alpha^2} w_1\left(\frac{u}{\varepsilon}\right) du.$$

Hence, by changing variable $s = \frac{u}{\varepsilon}$, we have

$$w_1(\tau) = -\varepsilon \frac{1}{2}\tau^2\alpha^2 + \int_0^\tau e^{-\frac{1}{2}(\tau-s)^2\alpha^2\varepsilon^2} w_1(s)\, du$$

or

$$w_1(\tau) = \int_0^\tau e^{-\frac{1}{2}(\tau-s)^2\alpha^2\varepsilon^2} w_1(s)\, du$$

and as a consequence $w_1(\tau) = 0$.

For $w_2(\tau)$

$$\varepsilon^3 w_2(\tau) = \varepsilon\left(e^{-\frac{1}{2}t^2\alpha^2} - 1\right) + \varepsilon^2 \int_0^t e^{-\frac{1}{2}(t-u)^2\alpha^2} w_1\left(\frac{u}{\varepsilon}\right) du$$
$$+ \int_0^t \left(e^{-\frac{1}{2}(t-u)^2\alpha^2} - 1\right) w_0\left(\frac{u}{\varepsilon}\right) du.$$

Introducing the change of variable $s = \frac{u}{\varepsilon}$, we have

$$\varepsilon^3 w_2(\tau) = \varepsilon\left(e^{-\frac{1}{2}t^2\alpha^2} - 1\right) + \varepsilon^3 \int_0^\tau e^{-\frac{1}{2}(t-u)^2\alpha^2} w_2(s)\, ds$$
$$+ \int_0^\tau \left(e^{-\frac{1}{2}(\tau-s)^2\alpha^2\varepsilon^2} - 1\right) w_0(s)\, ds.$$

Whence, we have

$$w_2(\tau) = -\frac{1}{2}\tau^2\alpha^2 + \int_0^\tau w_2(s)\, ds - \frac{1}{2}\alpha^2 \int_0^\tau (\tau - s)^2 w_0(s)\, ds.$$

Taking into account that $w_0(\tau) = e^\tau$, we have

$$w_2(\tau) = -\frac{1}{2}\tau^2\alpha^2 + \int_0^\tau w_2(s)\, ds - \frac{1}{2}\alpha^2\left(2e^\tau - \tau^2 - 2\tau - 1\right)$$

$$w_2(\tau) = \int_0^\tau w_2(s)\, ds - \alpha^2 e^\tau + \alpha^2\tau + \frac{1}{2}\alpha^2$$

$$w_2(0) = \frac{1}{2}\alpha^2$$

$$\frac{d}{d\tau}w_2(\tau) = w_2(\tau) - \alpha^2 e^{\tau} + \alpha^2$$

$$w_2(\tau) = -\alpha^2 - \alpha^2 \tau e^{\tau} + Ce^{\tau}$$

$$C = \frac{1}{2}\alpha^2$$

$$w_2(\tau) = \frac{1}{2}\alpha^2 e^{\tau} - \alpha^2 - \alpha^2 \tau e^{\tau}.$$

$$\varepsilon\varepsilon^4 w_4(\tau) = \varepsilon\varepsilon^4 \frac{1}{4}\tau^4\alpha^4 + \varepsilon^4 \int_0^t e^{-\frac{1}{2}(t-u)^2\alpha^2} w_4\left(\frac{u}{\varepsilon}\right) du$$

$$+\varepsilon^2 \int_0^t e^{-\frac{1}{2}(t-u)^2\alpha^2} w_2\left(\frac{u}{\varepsilon}\right) du + \int_0^t e^{-\frac{1}{2}(t-u)^2\alpha^2} w_0\left(\frac{u}{\varepsilon}\right) du$$

$$w_4(\tau) = \frac{1}{4}\tau^4\alpha^4 + \int_0^{\tau} w_4(s)\,ds - \int_0^{\tau} \frac{(\tau-s)^2\alpha^2}{2} w_2(s)\,ds$$

$$+ \int_0^t \frac{(\tau-s)^4\alpha^4}{4} w_0(s)\,ds.$$

Hence,

$$\frac{d}{d\tau}w_4(\tau) = w_4(\tau) + \frac{\alpha^4}{12}\left(12\tau e^{\tau} + 42e^{\tau} - 30\tau^2 - 54\tau - 42\right).$$

Since $w_4(0) = 0$, we obtain

$$w_4(\tau) = 13\alpha^4 + \frac{19}{2}\alpha^4\tau + \frac{5}{2}\alpha^4\tau^2 + \frac{7}{2}\alpha^4\tau e^{\tau} + \frac{1}{2}\alpha^4\tau^2 e^{\tau} - 13\alpha^4 e^{\tau}.$$

The following recursive formula holds:

$$w_{2k}(\tau) = \frac{(-1)^k}{2^k k!}\alpha^{2k}\tau^{2k} + \int_0^{\tau} w_{2k}(s)ds - \int_0^{\tau} \frac{\alpha^2(\tau-s)^2}{2} w_{2k-2}(s)\,ds$$

$$+ \int_0^{\tau} \frac{\alpha^4(\tau-s)^4}{2\cdot 2!} w_{2k-4}(s)\,ds - \int_0^{\tau} \frac{\alpha^6(\tau-s)^6}{2\cdot 3!} w_{2k-6}(s)\,ds + \cdots.$$

Taking into account equation [1.29], we have

$$h(t) = e^{-\frac{1}{2}\tau^2\alpha^2\varepsilon^2} + \varepsilon^2 \int_0^{\tau} e^{-\frac{1}{2}(\tau-s)^2\alpha^2\varepsilon^2} e^s ds$$

$$+\varepsilon^4 \int_0^{\tau} e^{-\frac{1}{2}(\tau-s)^2\alpha^2\varepsilon^2} \left(\frac{1}{2}\alpha^2 e^s - \alpha^2 - \alpha^2 s e^s\right) ds + \cdots.$$

Consider the following approximation:

$$h(t) = e^{-\frac{1}{2}\tau^2\alpha^2\varepsilon^2} + \varepsilon^2 \int_0^\tau e^{-\frac{1}{2}(\tau-s)^2\alpha^2\varepsilon^2} e^s ds.$$

After obtaining the inverse three-dimensional Fourier transform with respect to α (by using the Hankel transform), we get

$$g(\tau, x) = \mathcal{F}^{-1}(h(\tau)) = \frac{1}{2\pi^2\tau^3} e^{-\frac{x^2}{2\tau^2}} + \frac{\varepsilon^2}{2\pi^2} \int_0^\tau \frac{1}{(\tau-s)^3} e^{-\frac{x^2}{2(\tau-s)^2}} e^s ds$$

$$= \frac{1}{\sqrt{8\pi^3}\tau^3} e^{-\frac{x^2}{2\tau^2}} + \frac{\varepsilon^2}{\sqrt{8\pi^3}} e^\tau \int_0^\tau \frac{1}{u^3} e^{-\frac{x^2}{2u^2}} e^{-u} du + o\left(\varepsilon^2\right).$$

Then, for the pdf $f(t, x)$ of particle position at time t we have

$$f(t, x) = e^{-\lambda t} g(\lambda t, x) = \frac{e^{-\lambda t}}{\sqrt{8\pi^3}(\lambda t)^3} e^{-\frac{x^2}{2(\lambda t)^2}}$$

$$+ \frac{1}{\sqrt{8\pi^3}\lambda^2} \int_0^\infty \frac{1}{u^3} e^{-\frac{x^2}{2u^2}} e^{-u} du + o\left(\frac{1}{\lambda^2}\right).$$

$$\mathcal{F}^{-1}\left(e^{\frac{1}{2}t^2\alpha^2}\right) = \frac{1}{\sqrt{8\pi^3}} \int_0^\infty e^{-\frac{1}{2}t^2\alpha^2} \alpha^2 \frac{J_{\frac{1}{2}}(\alpha x)}{\sqrt{\alpha x}} d\alpha = \frac{1}{\sqrt{8\pi^3}} \frac{1}{t^3} e^{-\frac{x^2}{2t^2}}.$$

1.2.5. *Four-dimensional case*

We have that $\varphi(t) = \frac{J_1(\alpha t v)}{\alpha t v}$ when $v = const > 0$ in the four-dimensional case.

Now, let us assume that the velocity v_j is a random variable with the pdf

$$z(v) = \frac{v}{\sqrt{1-v^2}}, \quad 0 \le v < 1.$$

From [1.14], it follows that

$$\varphi(t) = \int_0^1 \frac{J_1(\alpha t v)}{\alpha t v} \frac{v}{\sqrt{1-v^2}} dv = 2\left(\frac{\sin(\alpha t/2)}{\alpha t}\right)^2,$$

with the corresponding renewal equation

$$H(t) = e^{-\lambda t}\left(\frac{\sin(\alpha t/2)}{\alpha t}\right)^2$$

$$+2\lambda \int_0^t e^{-\lambda(t-u)} \left(\frac{\sin\left(\frac{\alpha}{2}(t-u)\right)}{\alpha(t-u)}\right)^2 H(u)\, du, \qquad [1.34]$$

for the exponential pdf $g(t) = \lambda e^{-\lambda t}\mathbb{I}_{\{t \ge 0\}}$.

We cannot solve equation [1.34] but it is easily seen that

$$f(t,\mathbf{x}) = \mathcal{F}^{-1}(H(t)) \geq e^{-\lambda t}\mathcal{F}^{-1}\left(\left(\frac{\sin(\alpha t/2)}{\alpha t}\right)^2\right).$$

We have, by using the Hankel transform (Bochner and Chandrasekhar 1949), that

$$\mathcal{F}^{-1}\left(\left(\frac{\sin(\alpha t/2)}{\alpha t}\right)^2\right) = \frac{1}{4\pi^2}\int_0^\infty \frac{J_1(\alpha t x)}{\alpha t x}\alpha^3\left(\frac{\sin(\alpha t/2)}{\alpha t}\right)^2 d\alpha$$

$$= \frac{1}{4\pi^2 t x^2}\frac{1}{\sqrt{t^2-x^2}}, \quad x<t, \quad x=|\mathbf{x}|, \quad \mathbf{x}=(x_1,x_2,x_3,x_4).$$

Hence,

$$f(t,\mathbf{x}) \to \infty, \quad x \to t.$$

This explosive effect phenomenon was not observed in the four-dimensional case for constant velocity, for the absolute continuous part of the distribution of the particle position.

1.3. The distribution of random motion at non-constant velocity in semi-Markov media

This section deals with random motion at a non-constant speed, with uniformly distributed directions where the direction alternations occur according to renewal epochs having a general distribution. We derive the renewal equation for the characteristic function of the transition density of the multidimensional motion. By using the renewal equation, we study the behavior of the transition density near the sphere of its singularity, for two- and four-dimensional cases and variable velocity and the three-dimensional case for constant velocity. As an example, we have derived the distribution for one-, two- and three-dimensional random motion.

As in the case of one-dimensional random motion, many of the papers on random motion with uniformly distributed directions on the multidimensional space are devoted to the analysis of models where the motions are driven by a homogeneous Poisson process, so their processes are Markov. In various studies (Di Crescenzo 2001; Pogorui 2011), a non-Markov generalization of one-dimensional random evolutions of the telegrapher's random process, where the motion is driven by an alternating semi-Markov process with Erlang-distributed interrenewal times, is considered. Random flights in $\mathbb{R}^n$, with K-Erlang-distributed displacements and uniformly distributed directions, have been studied in Pogorui and Rodríguez-Dagnino (2011). A planar random motion performed by a particle that changes direction at even-valued Poisson events is studied in Beghin and Orsingher (2010a). In previous studies (Le Caër 2010, 2011), a random walk with steps of a uniform orientation and Dirichlet-distributed lengths is considered. The transition density, which has simple analytical forms for two- and four-dimensional Markov random motion, has been derived by Orsingher and De Gregorio (2007). In this work, we consider multidimensional random motions with uniformly distributed directions, with general distributed steps and non-constant velocity, and we have extended some results of Pogorui and Rodríguez-Dagnino (2005), Pogorui (2011) and Pogorui and Rodríguez-Dagnino (2011). We show some interesting solvable cases that have not been reported by other studies.

Let us consider the renewal process $\nu(t) = \max\{m \geq 0 : \ \tau_m \leq t\}, \ t \geq 0$, where $\tau_m = \sum_{k=1}^{m} \theta_k$, $\tau_0 = 0$ and $\theta_k \geq 0$, $k = 1, 2, \ldots$, are i.i.d. with a distribution function $G(t)$ and the probability density function $g(t) = \frac{d}{dt}G(t)$.

We assume that a particle starting from the coordinate origin $(0, 0, \ldots, 0)$ of the space R^n, at time $t = 0$, continues its motion with a velocity $v(s) > 0$, $0 \leq s < \theta_1$ along the direction of $\boldsymbol{\eta}_0^{(n)}$, where $\boldsymbol{\eta}_0^{(n)} = (x_1, x_2, \cdots, x_n)$ is a random n-dimensional vector uniformly distributed on the unit sphere

$$\Omega_1^{n-1} = \left\{(x_1, x_2, \cdots, x_n) : \ x_1^2 + x_2^2 + \cdots + x_n^2 = 1\right\}.$$

At instant τ_1, the particle changes its direction to $\boldsymbol{\eta}_1^{(n)}$, where $\boldsymbol{\eta}_1^{(n)}$ and $\boldsymbol{\eta}_0^{(n)}$ are independent and identically distributed on Ω_1^{n-1}, and continues its motion with the velocity $v(s)$, $0 \leq s < \theta_2$ along the direction of $\boldsymbol{\eta}_1^{(n)}$. Then at instant τ_2, the particle changes its direction to $\boldsymbol{\eta}_2^{(n)}$, where $\boldsymbol{\eta}_2^{(n)}$ is also uniformly distributed on Ω_1^{n-1} and independent of $\boldsymbol{\eta}_0^{(n)}$, $\boldsymbol{\eta}_1^{(n)}$, and continues its motion with the velocity $v(s)$, $0 \leq s < \theta_3$ along the direction of $\boldsymbol{\eta}_2^{(n)}$, and so on.

Denote by $\mathbf{x}^{(n)}(t)$, $t \geq 0$ the particle position at time t. We have that

$$\mathbf{x}^{(n)}(t) = \sum_{j=1}^{\nu(t)} \boldsymbol{\eta}_{j-1}^{(n)} \int_0^{\theta_j} v(s)\, ds + \boldsymbol{\eta}_{\nu(t)}^{(n)} \int_0^{t-\tau_{\nu(t)}} v(s)\, ds, \qquad [1.35]$$

where we assume that $\sum_{j=1}^{0} \boldsymbol{\eta}_{j-1}^{(n)} \int_0^{\theta_j} v(s)\, ds = 0$.

Basically, equation [1.35] determines the random evolution in the semi-Markov medium $\nu(t)$.

The counting process $\{\nu(t)\}$ gives the number of velocity alternations that have occurred in the interval $(0, t)$.

The probabilistic properties of a random vector $\mathbf{x}^{(n)}(t)$ are completely determined by its projection $x^{(n)}(t) = \sum_{j=1}^{\nu(t)} \eta_{j-1}^{(n)} \int_0^{\theta_j} v(s)\,ds + \eta_{\nu(t)}^{(n)} \int_0^{t-\tau_{\nu(t)}} v(s)\,ds$ on a fixed line, where $\eta_j^{(n)}$ is the projection of $\boldsymbol{\eta}_j^{(n)}$ on the line.

Indeed, let us consider the distribution function $F_x(y) = P\left(x^{(n)}(t) \le y\right)$. Then, the characteristic function $H(t)$ of $\mathbf{x}^{(n)}(t)$ is given by:

$$H(t) = E\exp\left\{i\left(\boldsymbol{\alpha}, \mathbf{x}^{(n)}(t)\right)\right\} = E\exp\left\{i\,\|\boldsymbol{\alpha}\|\left(\mathbf{e}, \mathbf{x}^{(n)}(t)\right)\right\}$$
$$= E\exp\left\{i\,\|\boldsymbol{\alpha}\|\,x^{(n)}(t)\right\} = \int_0^{\infty} \exp\{i\,\|\boldsymbol{\alpha}\|\,y\}\,dF_x(y),$$

where $\|\boldsymbol{\alpha}\| = \sqrt{\alpha_1^2 + \alpha_2^2 + \cdots + \alpha_n^2}$.

Let us denote by $f_{\eta^{(n)}}(x)$, the pdf of the projection $\eta_j^{(n)}$ of vector $\boldsymbol{\eta}_j^{(n)}$ onto a fixed line. In Pogorui (2010b), we proved that:

$$f_{\eta^{(n)}}(x) = \begin{cases} \frac{\Gamma\left(\frac{n}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{n-1}{2}\right)}\left(1-x^2\right)^{(n-3)/2}, & x \in [-1,1], \\ 0, & x \notin [-1,1], \end{cases} \qquad [1.36]$$

and

$$\varphi(t,\alpha) = \varphi_{\eta^{(n)}}\left(\alpha\int_0^t v(s)\,ds\right) = \mathrm{E}\left[e^{i\int_0^t v(s)ds\left(\boldsymbol{\alpha},\boldsymbol{\eta}_j^{(n)}\right)}\right]$$
$$= \int_{-1}^{1} e^{i\alpha\int_0^t v(s)dsx}\, f_{\eta^{(n)}}(x)dx,$$

where $\alpha = \|\boldsymbol{\alpha}\|$.

It is not hard to verify that

$$\varphi(t,\alpha) = 2^{\frac{n-2}{2}}\Gamma\left(\frac{n}{2}\right)\frac{J_{\frac{n-2}{2}}\left(\alpha\int_0^t v(s)\,ds\right)}{\left(\alpha\int_0^t v(s)\,ds\right)^{\frac{n-2}{2}}}.$$

1.3.1. *Renewal equation for the characteristic function*

THEOREM 1.4.– The characteristic function $H(t) = \mathbf{E}\left[\exp\left\{i\left(\boldsymbol{\alpha}, \mathbf{x}^{(n)}(t)\right)\right\}\right]$, $t \ge 0$, of the random motion $\mathbf{x}^{(n)}(t)$ is a solution of the following Volterra integral equation:

$$H(t) = (1 - G(t))\,\varphi(t,\alpha) + \int_0^t g(u)\,\varphi(u,\alpha)\,H(t-u)\,du. \qquad [1.37]$$

PROOF.– It follows from equation [1.35] that

$$H(t) = \mathbf{E}\left[\exp\left\{i\left(\boldsymbol{\alpha}, \mathbf{x}^{(n)}(t)\right)\right\}\right]$$

$$= \mathbf{E}\left[\exp\left\{i\left(\boldsymbol{\alpha}, \sum_{j=1}^{\nu(t)} \boldsymbol{\eta}_{j-1}^{(n)} \int_0^{\theta_j} v\left(s\right) ds + \boldsymbol{\eta}_{\nu(t)}^{(n)} \int_0^{t-\tau_{\nu(t)}} v\left(s\right) ds\right)\right\}\right]$$

$$= \left(1 - G(t)\right) \mathbf{E}\left[e^{i\int_0^t v(s)ds\left(\boldsymbol{\alpha}, \boldsymbol{\eta}_0^{(n)}\right)}\right]$$

$$+ \int_0^t g\left(u\right) \mathbf{E}\left[e^{i\int_0^u v(s)ds\left(\boldsymbol{\alpha}, \boldsymbol{\eta}_0^{(n)}\right)}\right] H\left(t-u\right) du.$$

To complete the proof, observe that $\varphi\left(t, \alpha\right) = \mathbf{E}\left[e^{i\int_0^t v(s)ds\left(\boldsymbol{\alpha}, \boldsymbol{\eta}_0^{(n)}\right)}\right]$.

Denote by $f_n\left(t, \mathbf{x}\right)$, the pdf of the particle position at time t. It is easily seen that $f_n\left(t, \mathbf{x}\right) = \mathcal{F}^{-1}\left(H(t)\right)$.

Our purpose is to study $f_n\left(t, \mathbf{x}\right)$. It is not hard to verify that $f_n\left(t, \mathbf{x}\right)$ is a radial function, i.e. $f_n\left(t, \mathbf{x}\right) = f_n\left(t, \mathbf{y}\right)$, for $\|\mathbf{x}\| = \|\mathbf{y}\|$ and we will also use the notation $f_n\left(t, x\right)$, where $x = \|\mathbf{x}\|$.

1.3.2. *Two-dimensional case*

In the two-dimensional case, $\varphi\left(t, \alpha\right) = J_0\left(\alpha \int_0^t v\left(s\right) ds\right)$. Let us consider the case where $g(t) = \lambda e^{-\lambda t} \mathbb{I}_{\{t \geq 0\}}$, $v\left(s\right) = \frac{c}{2\sqrt{s}}$, $c > 0$. Therefore, $\varphi\left(t, \alpha\right) = J_0\left(\alpha c \sqrt{t}\right)$ and we have

$$H(t) = e^{-\lambda t} J_0\left(\alpha c \sqrt{t}\right) + \int_0^t \lambda e^{-\lambda u} J_0\left(\alpha c \sqrt{u}\right) H\left(t-u\right) du.$$

We first define $h(t) = e^{\lambda t} H(t)$, and then we obtain

$$h(t) = J_0\left(\alpha c \sqrt{t}\right) + \lambda \int_0^t J_0\left(\alpha c \sqrt{u}\right) h\left(t-u\right) du. \qquad [1.38]$$

Denote by $\hat{h}\left(p\right) = \mathcal{L}\left(h(t), p\right) = \int_0^\infty e^{-tp} h(t) dt$, the Laplace transform of $h(t)$.

By applying the Laplace transform in equation [1.38], we have

$$\hat{h}\left(p\right) = \frac{1}{p} e^{-\frac{\alpha^2 c^2}{4p}} + \frac{\lambda}{p} e^{-\frac{\alpha^2 c^2}{4p}} \hat{h}\left(p\right).$$

Hence,

$$\hat{h}(p) = \frac{1}{\lambda}\sum_{n=1}^{\infty}\left(\frac{\lambda}{p}e^{-\frac{\alpha^2c^2}{4p}}\right)^n.$$

Since this series converges uniformly for $Re(p) \geq \lambda + \sigma$, where σ is a fixed positive number, we can obtain the inverse Laplace, which is given by

$$h(t) = \sum_{n=1}^{\infty}\left(\frac{4t\lambda^2}{n\alpha^2c^2}\right)^{\frac{n-1}{2}} J_{n-1}(\alpha c\sqrt{tn})$$

or equivalently

$$H(t) = e^{-\lambda t}\sum_{n=1}^{\infty}\left(\frac{4t\lambda^2}{n\alpha^2c^2}\right)^{\frac{n-1}{2}} J_{n-1}(\alpha c\sqrt{tn}).$$

It is well known that

$$\mathcal{F}^{-1}\left(J_0\left(\alpha c\sqrt{t}\right)\right) = \frac{1}{2\pi}\delta\left(c^2t - x^2\right),$$

where $\mathcal{F}^{-1}$ is the two-dimensional inverse Fourier transform, and $\delta(\cdot)$ is the Dirac delta function.

To calculate the two-dimensional inverse Fourier transform of $H(t)$, we use the Hankel transform as follows:

$$\frac{1}{2\pi}\int_0^{\infty}\left(\frac{4t\lambda^2}{n\alpha^2c^2}\right)^{\frac{n-1}{2}} \alpha J_{n-1}\left(\alpha c\sqrt{tn}\right) J_0(\alpha x)\,d\alpha$$

$$= \frac{1}{\pi}\frac{\lambda^{n-1}\left(nc^2t - x^2\right)^{n-2}}{\Gamma(n-1)\,n^{n-1}c^{2n-1}}, \quad n \geq 2.$$

It is well known (Di Crescenzo 2001; Orsingher and De Gregorio 2007) that

$$f_2(t, \mathbf{x}) = s_2(t, \mathbf{x}) + c_2(t, \mathbf{x}),$$

where $s_2(t, \mathbf{x})$ is the singular part and $c_2(t, \mathbf{x})$ is the continuous part of the pdf $f_2(t, \mathbf{x})$.

It is not hard to verify that

$$s_2(t, \mathbf{x}) = \frac{1}{2\pi}\delta\left(c^2t - x^2\right)$$

and

$$c_2(t,\mathbf{x}) = \frac{e^{-\lambda t}}{\pi}\sum_{n=2}^{\infty}\frac{\lambda^{n-1}\left(nc^2t - x^2\right)^{n-2}}{\Gamma(n-1)\,n^{n-1}c^{2n-1}}.$$

Hence, for $c^2t \geq x^2$, we have

$$f_2(t,\mathbf{x}) = \mathcal{F}^{-1}(H(t),\alpha) = \frac{e^{-\lambda t}}{2\pi}\delta\left(c^2t - x^2\right) + \frac{e^{-\lambda t}}{\pi}\sum_{n=2}^{\infty}\frac{\lambda^{n-1}\left(nc^2t - x^2\right)^{n-2}}{\Gamma(n-1)\,n^{n-1}c^{2n-1}}.$$

It is easily seen that the series $c_2(t,\mathbf{x})$ converges uniformly for $|\mathbf{x}| \leq c\sqrt{t}$ and there is no explosion effect as $|\mathbf{x}| \to c\sqrt{t}$, i.e.

$$\begin{aligned}\lim_{|\mathbf{x}|\to c\sqrt{t}} c_2(t,\mathbf{x}) &= \lim_{|\mathbf{x}|\to c\sqrt{t}} \frac{e^{-\lambda t}}{\pi}\sum_{n=2}^{\infty}\frac{\lambda^{n-1}\left(nc^2t - x^2\right)^{n-2}}{\Gamma(n-1)\,n^{n-1}c^{2n-1}} \\ &= \frac{e^{-\lambda t}}{\pi}\sum_{n=2}^{\infty}\frac{\lambda^{n-1}\left((n-1)\,c^2t\right)^{n-2}}{\Gamma(n-1)\,n^{n-1}c^{2n-1}} \\ &= \frac{e^{-\lambda t}}{\pi c^3}\sum_{n=2}^{\infty}\frac{\lambda^{n-1}\left(\left(1-\frac{1}{n}\right)t\right)^{n-2}}{\Gamma(n-1)\,n} < +\infty.\end{aligned}$$

Therefore, $c_2(t,\mathbf{x})$ does not go to infinity, as $|\mathbf{x}|$ goes to the circle of singularity $C\left(0, c\sqrt{t}\right) = \left\{\mathbf{x} = (x_1,x_2)\,|\, x_1^2 + x_2^2 = c^2t\right\}$ of $f_2(t,\mathbf{x})$, which takes place for the movement of a particle with a constant velocity on a plane (Orsingher and De Gregorio 2007). The absence of the explosion effect can be explained by the consideration that the particle velocity $v(s) = \frac{c}{2\sqrt{s}}$, which is very large just after renewal epoch, makes the particle leaves very quickly from the layer closest to the singularity area.

1.3.3. ***Three-dimensional case***

It was proved in Kolesnik and Pinsky (2011), that in the case of $g(t) = \lambda e^{-\lambda t}\mathbb{I}\{t \geq 0\}$ and $v(s) = v = constant$, the n-dimensional random evolutions ($n \geq 3$) are driven by the hyperparabolic operators composed of the telegraph operators and their integer powers. However, it is not easy to find closed-form solutions in this case.

We consider the case where $g(t)$ is Erlang-2 pdf and $v(s) = v = const$. We will show that these three-dimensional random evolutions are driven by the three-dimensional telegraph equation.

Let us consider the case where $g(t) = \lambda^2 t e^{-\lambda t} \mathbb{I}\{t \geq 0\}$ and $v(s) = v = const.$ Thus, we have

$$\varphi(t,\alpha) = \frac{\sin(\alpha v t)}{\alpha v t}, \qquad \alpha = \sqrt{\alpha_1^2 + \alpha_2^2 + \alpha_3^2}$$

Therefore,

$$\begin{aligned} H(t) &= \left(e^{-\lambda t} + \lambda t e^{-\lambda t}\right) \frac{\sin(\alpha v t)}{\alpha v t} \\ &+ \int_0^t \lambda^2 (t-u) e^{-\lambda(t-u)} \frac{\sin(\alpha v (t-u))}{\alpha v (t-u)} H(u)\, du \\ &= \left(e^{-\lambda t} + \lambda t e^{-\lambda t}\right) \frac{\sin(\alpha v t)}{\alpha v t} \\ &+ \frac{\lambda^2}{\alpha v} \int_0^t e^{-\lambda(t-u)} \sin(\alpha v (t-u)) H(u)\, du. \end{aligned} \qquad [1.39]$$

The pdf $f_3(t,\mathbf{x}) = \mathcal{F}^{-1}(H(t),\alpha)$, where $\mathcal{F}^{-1}$ is the three-dimensional inverse Fourier transform with respect to α, is a radial function (i.e. it depends only on $x = \sqrt{x_1^2 + x_2^2 + x_3^2}$), and it has the following form:

$$f_3(t,x) = s_3(t,x) + c_3(t,x),$$

where $s_3(t,x)$ is the singular part and $c_3(t,x)$ is the continuous part of the pdf $f_3(t,x)$.

It is well known that the function $4\pi R \frac{\sin \alpha R}{\alpha}$ is the Fourier transform of the simple layer $\delta_{S(R)}$, (Vladimirov 1996, p. 154). Hence,

$$s_3(t,x) = \mathcal{F}^{-1}\left(\left(e^{-\lambda t} + \lambda t e^{-\lambda t}\right) \frac{\sin(\alpha v t)}{\alpha v t}\right) = \left(e^{-\lambda t} + \lambda t e^{-\lambda t}\right) \frac{\delta_{S(vt)}}{4\pi (vt)^2}.$$

Denote by $H_s(t) = \int_{\mathbb{R}^3} e^{i(\boldsymbol{\alpha},\mathbf{x})} s_3(t,\mathbf{x}) d\mathbf{x}$ and by $H_c(t) = \int_{\mathbb{R}^3} e^{i(\boldsymbol{\alpha},\mathbf{x})} c_3(t,\mathbf{x}) d\mathbf{x}$. It is easily seen that $H(t) = H_s(t) + H_c(t)$.

Since $H_s(t) = \left(e^{-\lambda t} + \lambda t e^{-\lambda t}\right) \frac{\sin(\alpha v t)}{\alpha v t}$ from equation [1.39], then it follows that

$$H_c(t) = \frac{\lambda^2}{\alpha v} \int_0^t e^{-\lambda(t-u)} \sin(\alpha v (t-u)) \left(H_s(u) + H_c(u)\right) du$$

or equivalently,

$$H_c(t) = \frac{\lambda^2}{\alpha v} e^{-\lambda t} \int_0^t \sin\left(\alpha v \left(t-u\right)\right)\ \left(1+\lambda u\right) \frac{\sin\left(\alpha v u\right)}{\alpha v u} du$$
$$+ \frac{\lambda^2}{\alpha v} \int_0^t e^{-\lambda(t-u)} \sin\left(\alpha v \left(t-u\right)\right) H_c\left(u\right) du.$$

Let us consider

$$h(t) = \lambda^2 \int_0^t \frac{\sin\left(\alpha v \left(t-u\right)\right)}{\alpha v}\ \left(1+\lambda u\right) \frac{\sin\left(\alpha v u\right)}{\alpha v u} du$$
$$+ \frac{\lambda^2}{\alpha v} \int_0^t \sin\left(\alpha v \left(t-u\right)\right) h\left(u\right) du, \qquad [1.40]$$

where $h(t) = e^{\lambda t} H_c(t)$.

Differentiating equation [1.40] with respect to t two times, we obtain

$$\frac{dh(t)}{dt} - \lambda^2 \int_0^t \cos\left(\alpha v \left(t-u\right)\right) h\left(u\right) du$$
$$= \lambda^2 \int_0^t \cos\left(\alpha v \left(t-u\right)\right)\ \left(1+\lambda u\right) \frac{\sin\left(\alpha v u\right)}{\alpha v u} du,$$

and

$$\frac{d^2 h(t)}{dt} - \lambda^2 h(t) + \lambda^2 \alpha v \int_0^t \sin\left(\alpha v \left(t-u\right)\right) h\left(u\right) du$$
$$= \left(\lambda^2 + \lambda^3 t\right) \frac{\sin\left(\alpha v t\right)}{\alpha v t} - \lambda^2 \int_0^t \sin\left(\alpha v \left(t-u\right)\right)\ \left(1+\lambda u\right) \frac{\sin\left(\alpha v u\right)}{u} du. \qquad [1.41]$$

Combining the equations [1.40] and [1.41], we obtain

$$\frac{d^2 h(t)}{dt} - \lambda^2 h(t) + \alpha^2 v^2 h(t) = \left(\lambda^2 + \lambda^3 t\right) \frac{\sin\left(\alpha v t\right)}{\alpha v t}. \qquad [1.42]$$

Denote by $g\left(t, \mathbf{x}\right) = \mathcal{F}^{-1}\left(h(t), \alpha\right)$ (recall that $h(t)$ depends on α), the three-dimensional inverse Fourier transform. Here, we use the notation $g\left(t, \mathbf{x}\right)$, and we should remember that g is a radial function and $g\left(t, \mathbf{x}\right) = g\left(t, \mathbf{y}\right)$ for all $\mathbf{x},\ \mathbf{y} \in \mathbb{R}^3$, such that $|\mathbf{x}| = |\mathbf{y}|$.

After obtaining the three-dimensional inverse Fourier transform of equation [1.42], for $0 \leq x \leq vt$ we have

$$\frac{\partial^2 g\left(t, \mathbf{x}\right)}{\partial t^2} - \lambda^2 g\left(t, \mathbf{x}\right) - v^2 \sum_{i,j=1}^{3} \frac{\partial^2 g\left(t, \mathbf{x}\right)}{\partial x_i \partial x_j} = \left(\lambda^2 + \lambda^3 t\right) \frac{\delta_{S(vt)}}{4\pi (vt)^2}. \qquad [1.43]$$

Since $e^{\lambda t}c_3(t,\mathbf{x}) = \mathcal{F}^{-1}(H_c(t),\alpha) = g(t,\mathbf{x})$ from equation [1.43], for $0 \le x \le vt$ we obtain

$$\frac{\partial^2 c_3(t,\mathbf{x})}{\partial t^2} + 2\lambda \frac{\partial}{\partial t} c_3(t,\mathbf{x}) - v^2 \sum_{i,j=1}^{3} \frac{\partial^2 c_3(t,\mathbf{x})}{\partial x_i \partial x_j}$$
$$= \left(\lambda^2 + \lambda^3 t\right) e^{-\lambda t} \frac{\delta_{S(vt)}}{4\pi (vt)^2}, \qquad [1.44]$$

with the initial conditions $c_3(t,\mathbf{x}) = 0$ and $\left.\frac{\partial c_3(t,\mathbf{x})}{\partial t}\right|_{t=0} = 0$, which can be obtained directly from equation [1.40]. We have $s_3(t,\mathbf{x}) = \left(e^{-\lambda t} + \lambda t e^{-\lambda t}\right) \frac{\delta_{S(vt)}}{4\pi(vt)^2}$ by solving equation [1.44]. Therefore, we obtain the pdf $f_3(t,\mathbf{x})$.

REMARK 1.2.– *It was the subject of discussion among researchers on whether the multidimensional random flights could be described by the telegraph equations, similarly to the one-dimensional case (Kolesnik and Pinsky 2011). We think that for the three-dimensional case, equation [1.44] answers this question. Then, it seems natural to call equation [1.44] the generalized Goldstein–Kac equation in three dimensions.*

1.3.4. *Four-dimensional case*

In this case

$$\varphi(t,\alpha) = \frac{J_1\left(\alpha \int_0^t v(s)\,ds\right)}{\alpha \int_0^t v(s)\,ds},$$

where J_1 is the Bessel function of order 1.

Let us consider the case for a 2-Erlang pdf, i.e. $g(t) = \lambda^2 t e^{-\lambda t}\mathbb{I}\{t \ge 0\}$ and velocity $v(s) = \frac{1}{2\sqrt{s}}$.

Therefore,

$$H(t) = \left(e^{-\lambda t} + \lambda t e^{-\lambda t}\right) \frac{J_1\left(\alpha\sqrt{t}\right)}{\alpha\sqrt{t}} + \int_0^t \lambda^2 u e^{-\lambda u} \frac{J_1\left(\alpha\sqrt{u}\right)}{\alpha\sqrt{u}} H(t-u)\,du,$$

or

$$h(t) = (1+\lambda t) \frac{J_1\left(\alpha\sqrt{t}\right)}{\alpha\sqrt{t}} + \frac{\lambda^2}{\alpha} \int_0^t J_1\left(\alpha\sqrt{u}\right)\sqrt{u}\, h(t-u)\,du, \qquad [1.45]$$

where $h(t) = H(t)e^{\lambda t}$.

After obtaining the Laplace transform in equation [1.45], we have

$$\hat{h}\left(p\right)=\frac{2-2e^{-\frac{\alpha^2}{4p}}}{\alpha^2}+\frac{\lambda e^{-\frac{\alpha^2}{4p}}}{2p^2}+\frac{\lambda^2 e^{-\frac{\alpha^2}{4p}}}{2p^2}\hat{h}\left(p\right).$$

Hence,

$$\hat{h}\left(p\right)=\frac{2-2e^{-\frac{\alpha^2}{4p}}}{\alpha^2}+\frac{\lambda e^{-\frac{\alpha^2}{4p}}}{2p^2}+\frac{2-2e^{-\frac{\alpha^2}{4p}}}{\alpha^2}\sum_{n=1}^{\infty}\left(\frac{\lambda^2 e^{-\frac{\alpha^2}{4p}}}{2p^2}\right)^n$$
$$+\frac{1}{\lambda}\sum_{n=2}^{\infty}\left(\frac{\lambda^2 e^{-\frac{\alpha^2}{4p}}}{2p^2}\right)^n. \qquad [1.46]$$

By obtaining the Laplace transform in equation [1.46], we have

$$h(t)=\left(1+\lambda t\right)\frac{J_1\left(\alpha\sqrt{t}\right)}{\alpha\sqrt{t}}+\sum_{n=1}^{\infty}\lambda^{2n}t^{\frac{2n-1}{2}}2^n$$
$$\times\left(\frac{\left(n+1\right)^{n-\frac{1}{2}}}{n^{n-\frac{1}{2}}}\frac{J_{2n-1}\left(\sqrt{n}\alpha\sqrt{t}\right)}{\alpha^{2n+1}}-\frac{n^{n-\frac{1}{2}}}{\left(n+1\right)^{n-\frac{1}{2}}}\frac{J_{2n-1}\left(\sqrt{n+1}\alpha\sqrt{t}\right)}{\alpha^{2n+1}}\right)$$
$$+\sum_{n=2}^{\infty}\frac{2^{n-1}\lambda^{2n-1}\sqrt{n}}{n^n}\frac{t^{\frac{2n-1}{2}}J_{2n-1}\left(\sqrt{n}\alpha\sqrt{t}\right)}{\alpha^{2n-1}}. \qquad [1.47]$$

We will use the following Hankel transforms to calculate the four-dimensional inverse Fourier transforms (Vladimirov 1996, p. 321):

$$\int_0^{\infty}\frac{J_{2n-1}\left(\sqrt{n}\alpha\sqrt{t}\right)}{\alpha^{2n+1}}\alpha^2 J_1\left(\alpha x\right)d\alpha=\frac{2\left(nt\right)^{n-1/2}\left(1-\left(\frac{nt-x^2}{nt}\right)^{2n-1}\right)}{2^{2n}x\Gamma\left(2n\right)},$$

$$\int_0^{\infty}\frac{J_{2n-1}\left(\sqrt{n+1}\alpha\sqrt{t}\right)}{\alpha^{2n+1}}\alpha^2 J_1\left(\alpha x\right)d\alpha$$
$$=\frac{2\left(nt+t\right)^{n-1/2}\left(1-\left(\frac{(n+1)t-x^2}{(n+1)t}\right)^{2n-1}\right)}{2^{2n}x\Gamma\left(2n\right)},$$

$$\int_0^{\infty}\frac{J_{2n-1}\left(\sqrt{n}\alpha\sqrt{t}\right)}{\alpha^{2n-1}}\alpha^2 J_1\left(\alpha x\right)d\alpha=\frac{8x\left(nt-x^2\right)^{2n-3}}{4^n\left(tn\right)^{n-1/2}\Gamma\left(2n-2\right)}. \qquad [1.48]$$

Then, by considering equation [1.48] we can obtain the four-dimensional inverse Fourier transform in equation [1.47]. For instance, for $t \geq x^2$ we have

$$f_4(t, \mathbf{x}) = \mathcal{F}^{-1}(H(t), \alpha) = \frac{\left(e^{-\lambda t} + \lambda t e^{-\lambda t}\right)}{2\pi} \delta\left(t - x^2\right)$$

$$+e^{-\lambda t} \sum_{n=1}^{\infty} \lambda^{2n} t^{\frac{2n-1}{2}} \left(\frac{2(nt+t)^{n-1/2}\left(1 - \left(\frac{nt-x^2}{nt}\right)^{2n-1}\right)}{2^n x \Gamma(2n)} \right.$$

$$\left. - \frac{2(nt)^{n-1/2}\left(1 - \left(\frac{(n+1)t-x^2}{(n+1)t}\right)^{2n-1}\right)}{2^n x\, \Gamma(2n)} \right)$$

$$+e^{-\lambda t} \sum_{n=2}^{\infty} \frac{\lambda^{2n-1} t^{\frac{2n-1}{2}} \sqrt{n}}{n^n} \frac{4x\left(nt - x^2\right)^{2n-3}}{2^n (tn)^{n-1/2} \Gamma(2n-2)}.$$

Therefore, the singular part of the pdf $f_4(t, \mathbf{x}) = s_4(t, \mathbf{x}) + c_4(t, \mathbf{x})$, is given by

$$s_4(t, \mathbf{x}) = \frac{\left(e^{-\lambda t} + \lambda t e^{-\lambda t}\right)}{2\pi} \delta\left(t - x^2\right)$$

and the continuous part of $f_4(t, \mathbf{x})$ is

$$c_4(t, \mathbf{x}) = e^{-\lambda t} \sum_{n=1}^{\infty} \lambda^{2n} t^{\frac{2n-1}{2}}$$

$$\times \left(\frac{2(nt+t)^{n-1/2}\left(1 - \left(\frac{nt-x^2}{nt}\right)^{2n-1}\right)}{2^n x \Gamma(2n)} - \frac{2(nt)^{n-1/2}\left(1 - \left(\frac{(n+1)t-x^2}{(n+1)t}\right)^{2n-1}\right)}{2^n x \Gamma(2n)} \right)$$

$$+e^{-\lambda t} \sum_{n=2}^{\infty} \frac{\lambda^{2n-1} t^{\frac{2n-1}{2}} \sqrt{n}}{n^n} \frac{4x\left(nt - x^2\right)^{2n-3}}{2^n (tn)^{n-1/2} \Gamma(2n-2)}.$$

It is easily verified that the series $c_4(t, \mathbf{x})$ converges uniformly for $|\mathbf{x}| \leq \sqrt{t}$ and that there is no explosion effect, as $|\mathbf{x}| \to \sqrt{t}$, i.e.

$$\lim_{|\mathbf{x}| \to \sqrt{t}} c_4(t, \mathbf{x}) = \lim_{|x| \to \sqrt{t}} e^{-\lambda t} \sum_{n=1}^{\infty} \lambda^{2n} t^{\frac{2n-1}{2}}$$

$$\times \left(\frac{2(nt+t)^{n-1/2}\left(1 - \left(\frac{nt-x^2}{nt}\right)^{2n-1}\right)}{2^n x \Gamma(2n)} \right.$$

$$-\frac{2(nt)^{n-1/2}\left(1-\left(\frac{(n+1)t-x^2}{(n+1)t}\right)^{2n-1}\right)}{2^n x\,\Gamma(2n)}\Bigg)$$

$$+e^{-\lambda t}\sum_{n=2}^{\infty}\frac{\lambda^{2n-1}t^{\frac{2n-1}{2}}\sqrt{n}}{n^n}\frac{4x\left(nt-x^2\right)^{2n-3}}{2^n(tn)^{n-1/2}\Gamma(2n-2)}$$

$$=e^{-\lambda t}\sum_{n=1}^{\infty}\lambda^{2n}t^{n-1}\left(\frac{2(nt+t)^{n-1/2}\left(1-\left(\frac{nt-t}{nt}\right)^{2n-1}\right)}{2^n\Gamma(2n)}\right.$$

$$\left.-\frac{2(nt)^{n-1}\left(1-\left(\frac{nt}{(n+1)t}\right)^{2n-1}\right)}{2^n\sqrt{n}\,\Gamma(2n)}\right)$$

$$+e^{-\lambda t}\sum_{n=2}^{\infty}\frac{\lambda^{2n-1}t^n\sqrt{n}}{n^n}\frac{4(nt-t)^{2n-3}}{2^n(tn)^{n-1/2}\Gamma(2n-2)}<+\infty.$$

Therefore, $c_4(t,\mathbf{x})$ does not go to infinity as $|x|$ goes to the circus of singularity $C\left(0,c\sqrt{t}\right)=\left\{\mathbf{x}=(x_1,x_2,x_3,x_4)|\,x_1^2+x_2^2+x_3^2+x_4^2=c^2t\right\}$ of $f_4(t,\mathbf{x})$.

1.4. Goldstein–Kac telegraph equations and random flights in higher dimensions

In this section, we deal with random motions in dimensions two, three, and five, where the governing equations are telegraph-type equations. Our methodology is first applied to the second-order telegraph equation, then we refine the well-known results found by other methods. Next, we show that the (2,λ)-Erlang distribution for sojourn times defines the underlying stochastic process for the three-dimensional Goldstein–Kac type telegraph equation. By finding the corresponding fundamental solution of this equation, we have obtained the approximated expression for the transition density of the three-dimensional movement. Our results are more complete than previous ones, and may have important consequences in applications. We also obtain the five-dimensional telegraph-type equation by assuming a random motion with a (4,λ)-Erlang distribution for sojourn times. Such an equation can be factorized as the product of two telegraph-type equations, where one of them is the Goldstein–Kac five-dimensional telegraph equation. In our analysis, the dimension n is related to the ($n-1$,λ)-Erlang distribution for sojourn times of the random walks.

We should remember that the problem of a one-dimensional random motion, defined by the movement of a particle at a constant velocity v, traveling in a direction for a random distance drawn from an exponential distribution, and the particle change to the opposite direction under the same stochastic conditions, was studied and solved by Goldstein (1951) and Kac (1974) in 1951 and 1956, respectively. This problem is specified as a random motion governed by a switching Poisson process with alternating directions, but having exponentially distributed sojourn times, i.e. it is essentially a Markov process. The solution of this problem satisfies the one-dimensional telegraph-type equation (this equation has a similar form as the Heaviside telegraph equation for wave propagation in transmission lines)

$$\frac{\partial^2 f(t,x)}{\partial t^2} + 2\lambda \frac{\partial}{\partial t} f(t,x) = v^2 \frac{\partial^2 f(t,x)}{\partial x^2}, \qquad [1.49]$$

and it is called the Goldstein–Kac telegraph equation. Kac (1974) states the possibility of extending the same analysis for modeling multidimensional random motions, satisfying, Goldstein–Kac telegraph equations in higher dimensions. In the last paragraph of his paper, Kac states:

> This same proof goes also for a higher number of dimensions. Again it is simply a matter of writing the Laplace transform and verifying the same formula.

We should mention that the problem of assessing the validity of this statement has been an elusive problem for researchers working in this area since then, i.e. to find underlying stochastic processes describing multidimensional random evolutions and that satisfy telegraph-type equations. Hitherto there is a definitive answer to this problem, so we cannot ensure that multidimensional random flights can be described by multidimensional telegraph equations, similarly to the one-dimensional case, i.e. equations of the following type:

$$\frac{\partial^2 f(t,\mathbf{x})}{\partial t^2} + 2\lambda \frac{\partial}{\partial t} f(t,\mathbf{x}) = v^2 \sum_{i=1}^{n} \frac{\partial^2 f(t,\mathbf{x})}{\partial x_i^2}, \qquad [1.50]$$

where $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, $n \geq 2$.

Some remarkable efforts in this direction are presented in De Gregorio and Orsingher (2012) and Garra and Orsingher (2014), where they show that for random motions in three dimensions, with particles that only change direction at even-valued Poisson events, the governing partial differential equation is a Goldstein–Kac telegraph equation. However, the solution presented by these authors seems to be in complete and it is not a true pdf.

In this work, we provide the governing partial differential equations for random evolutions in two, three and five dimensions. We focus on these two odd dimension cases since the first one is very important in applications. We are also including the two-dimensional random flight, whose transition, density is the fundamental solution to the two-dimensional Goldstein-Kac telegraph equation. This problem has been studied by several researchers in the past and our main purpose is to show that it is also consistent with our assumption of having n-dimensional random flights with $(n-1,\lambda)$-Erlang-distributed sojourn times. Furthermore, we are adding some new insights to the solution of this two-dimensional random flight.

1.4.1. *Preliminaries about our modeling approach*

Let $\nu(t)$ be the renewal process that starts with $\nu(0)=0$. We define the time of the kth jump by $\tau_k=\inf\{t:\ \nu(t)=k\}$, $k\geq 0$. The random variables τ_k are called jump times, $\theta_k=\tau_{k+1}-\tau_k$ is called the kth renewal interval (or interarrival time) and $\theta_k,\ k\in$ are non-negative, independent, identically distributed random variables with cdf $G(t)$.

We will study the random motion of a particle that starts its motion from the origin $\mathbf{0}=(0,0,\ldots,0)$ of the space $\mathbb{R}^n$, at time $t=0$, and continues its motion with a constant velocity $v>0$ along a direction $\boldsymbol{\eta}_0$ in $\mathbb{R}^n$. At instant τ_1, the particle changes its direction to $\boldsymbol{\eta}_1$ and continues its motion with a velocity $v>0$ along $\boldsymbol{\eta}_1$. At instant τ_2, the particle changes its direction to $\boldsymbol{\eta}_2$ and continues its motion with a velocity $v>0$ along $\boldsymbol{\eta}_2$, and so on. We assumed that $\boldsymbol{\eta}_i$, $i=0,1,2,\ldots$ are independent and identically distributed random n-dimensional vectors uniformly distributed on the unit sphere $\Omega_1^{(n-1)}=\left\{(x_1,x_2,\ldots,x_n):\ x_1^2+x_2^2+\ldots+x_n^2=1\right\}$.

Denote by $\mathbf{x}(t)$, $t\geq 0$, the particle position at time t. We have:

$$\mathbf{x}(t)=v\sum_{j=1}^{\nu(t)}\boldsymbol{\eta}_{j-1}\theta_{j-1}+v\,\boldsymbol{\eta}_{\nu(t)}\left(t-\tau_{\nu(t)}\right).$$

As we know, the characteristic function $H(t,\alpha)=\mathrm{E}\left[\exp\left\{i\left(\boldsymbol{\alpha},\mathbf{x}(t)\right)\right\}\right]$, $\boldsymbol{\alpha}=(\alpha_1,\alpha_2,\ldots,\alpha_n)$, $\alpha=|\boldsymbol{\alpha}|=\sqrt{\alpha_1^2+\alpha_2^2+\ldots+\alpha_n^2}$ satisfies the following renewal equation (Pogorui and Rodríguez-Dagnino 2011, 2013):

$$H(t,\alpha)=(1-G(t))\,\varphi(t)+\int_0^t g(u)\,\varphi(u)H(t-u,\alpha)\,du,$$

where $\varphi(t,\alpha)=2^{\frac{n-2}{2}}\Gamma\left(\frac{n}{2}\right)\frac{J_{\frac{n-2}{2}}(\alpha tv)}{(\alpha tv)^{\frac{n-2}{2}}}$, $G(t)$ is the (m,λ)-Erlang distribution function, and $g(t)=\frac{dG(t)}{dt}=\frac{\lambda^m}{(m-1)!}t^{m-1}e^{-\lambda t}I_{\{t\geq 0\}}$ is the corresponding pdf.

In the following, we will omit α in $H(t,\alpha)$ and $\varphi(t,\alpha)$ to avoid cumbersome formulas, i.e. $H(t)=H(t,\alpha)$, $\varphi(t)=\varphi(t,\alpha)$.

Denote by $f_n(t,\mathbf{x})$ the generalized pdf of the particle position at point $\mathbf{x}$ and time t. It is well known that the function $f_n(t,\mathbf{x})$ is radial, i.e. it only depends on $x=|\mathbf{x}|=$

$\sqrt{x_1^2 + x_2^2 + \ldots + x_n^2}$ Pogorui and Rodríguez-Dagnino (2011, 2013); Orsingher and De Gregorio (2007). Thus, $f_n(t,x) = f_n(t,\mathbf{x}) = \mathcal{F}^{-1}(H(t),\alpha)$, where $\mathcal{F}^{-1}$ is the n-dimensional inverse Fourier transform with respect to α. We will show that for an n-dimensional space $\mathbb{R}^n$ ($n \geq 2$), in the case where $G(t)$ is an $(n-1,\lambda)$-Erlang distribution, the generalized pdf $f_n(t,x)$ is the solution of a telegraph-type equation of order n. We will do this for important cases when n is equal to two, three and five dimensions. The case $n = 4$ will be published elsewhere.

Hence, for $g(t) = \frac{\lambda^{n-1}}{\Gamma(n-1)} t^{n-2} e^{-\lambda t}$, we have

$$H(t) = \left(\sum_{k=0}^{n-2} \frac{(\lambda t)^k e^{-\lambda t}}{k!}\right) 2^{\frac{n-2}{2}} \Gamma\left(\frac{n}{2}\right) \frac{J_{\frac{n-2}{2}}(\alpha t v)}{(\alpha t v)^{\frac{n-2}{2}}} + \frac{\lambda^{n-1} 2^{\frac{n-2}{2}} \Gamma\left(\frac{n}{2}\right)}{\Gamma(n-1)(\alpha v)^{\frac{n-2}{2}}}$$

$$\times \int_0^t (t-u)^{\frac{n-2}{2}} e^{-\lambda(t-u)} J_{\frac{n-2}{2}}(\alpha(t-u)v) H(u)\, du. \qquad [1.51]$$

It is well known that the function $f_n(t,x) = f_n(t,\mathbf{x}) = \mathcal{F}^{-1}(H(t),\alpha)$, where $\mathcal{F}^{-1}$ is the n-dimensional inverse Fourier transform with respect to α, has the following form (Pogorui and Rodríguez-Dagnino 2011, 2013; Orsingher and De Gregorio 2007):

$$f_n(t,\mathbf{x}) = c_n(t,\mathbf{x}) + s_n(t,\mathbf{x}).$$

In this expression, $s_n(t,\mathbf{x})$ represents the singular part and $c_n(t,\mathbf{x})$ is the continuous part of the generalized pdf $f_n(t,\mathbf{x})$. We should note that the generalized pdf $f_n(t,\mathbf{x})$ and functions $s_n(t,\mathbf{x})$, $c_n(t,\mathbf{x})$ are radial, i.e. $f_n(t,\mathbf{x}) = f_n(t,x)$, $s_n(t,\mathbf{x}) = s_n(t,x)$, $c_n(t,\mathbf{x}) = c_n(t,x)$, where $x = |\mathbf{x}|$. The singular term $s_n(t,\mathbf{x})$ corresponds to event $\tau_1 > t$, which has probability $\left(\sum_{k=0}^{n-2} \frac{(\lambda t)^k e^{-\lambda t}}{k!}\right)$, and reflects the fact that the particle has not yet become "smeared out" by the random velocity alternation.

We define $H_c(t) = H_c(t,\alpha) = \int_{\mathbb{R}^n} e^{i(\boldsymbol{\alpha},\mathbf{x})} c_n(t,\mathbf{x})\, d\mathbf{x}$, $H_s(t) = H_s(t,\alpha) = \int_{\mathbb{R}^n} e^{i(\boldsymbol{\alpha},\mathbf{x})} s_n(t,\mathbf{x})\, d\mathbf{x}$, hence $H(t) = H_c(t) + H_s(t)$. The characteristic function for the singular part is

$$H_s(t) = \left(\sum_{k=0}^{n-2} \frac{(\lambda t)^k e^{-\lambda t}}{k!}\right) 2^{\frac{n-2}{2}} \Gamma\left(\frac{n}{2}\right) \frac{J_{\frac{n-2}{2}}(\alpha t v)}{(\alpha t v)^{\frac{n-2}{2}}}.$$

Thus, from equation [1.51] we have

$$H_c(t) + H_s(t) = \left(\sum_{k=0}^{n-2} \frac{(\lambda t)^k e^{-\lambda t}}{k!}\right) 2^{\frac{n-2}{2}} \Gamma\left(\frac{n}{2}\right) \frac{J_{\frac{n-2}{2}}(\alpha t v)}{(\alpha t v)^{\frac{n-2}{2}}}$$

$$+\frac{\lambda^{n-1}2^{\frac{n-2}{2}}\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(n-1\right)\left(\alpha v\right)^{\frac{n-2}{2}}}$$
$$\times\int_0^t\left(t-u\right)^{\frac{n-2}{2}}e^{-\lambda(t-u)}J_{\frac{n-2}{2}}\left(\alpha v\left(t-u\right)\right)\left(H_c\left(u\right)+H_s\left(u\right)\right)du.$$

Furthermore,

$$H_c(t)=\frac{\lambda^{n-1}2^{n-2}\left(\Gamma\left(\frac{n}{2}\right)\right)^2}{\Gamma\left(n-1\right)\left(\alpha v\right)^{n-2}}\int_0^t\left(t-u\right)^{\frac{n-2}{2}}e^{-\lambda(t-u)}J_{\frac{n-2}{2}}\left(\alpha v\left(t-u\right)\right)$$
$$\times\left(\sum_{k=0}^{n-2}\frac{\left(\lambda u\right)^k e^{-\lambda u}}{k!}\right)\frac{J_{\frac{n-2}{2}}\left(\alpha uv\right)}{u^{\frac{n-2}{2}}}du$$
$$+\frac{\lambda^{n-1}2^{\frac{n-2}{2}}\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(n-1\right)\left(\alpha v\right)^{\frac{n-2}{2}}}\int_0^t\left(t-u\right)^{\frac{n-2}{2}}e^{-\lambda(t-u)}J_{\frac{n-2}{2}}\left(\alpha v\left(t-u\right)\right)H_c\left(u\right)du.$$

Now, by setting $H_c(t)e^{\lambda t}=h_c(t)$, we have

$$h_c(t)=\frac{\lambda^{n-1}2^{n-2}\left(\Gamma\left(\frac{n}{2}\right)\right)^2}{\Gamma\left(n-1\right)\left(\alpha v\right)^{n-2}}\int_0^t\left(t-u\right)^{\frac{n-2}{2}}J_{\frac{n-2}{2}}\left(\alpha v\left(t-u\right)\right)\frac{J_{\frac{n-2}{2}}\left(\alpha uv\right)}{u^{\frac{n-2}{2}}}$$
$$\times\left(\sum_{k=0}^{n-2}\frac{\left(\lambda u\right)^k}{k!}\right)du$$
$$+\frac{\lambda^{n-1}2^{\frac{n-2}{2}}\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(n-1\right)\left(\alpha v\right)^{\frac{n-2}{2}}}\int_0^t\left(t-u\right)^{\frac{n-2}{2}}J_{\frac{n-2}{2}}\left(\alpha v\left(t-u\right)\right)h_c\left(u\right)du. \qquad [1.52]$$

Suppose that $S_n\left(R\right)$ is a sphere with a radius R in $\mathbb{R}^n$ and f is a continuous function defined over $S_n\left(R\right)$. Let us consider the generalized function $f\delta_{S_n(R)}$, acting according to the following rule:

$$\left(f\delta_{S_n(R)},\varphi\right)=\int_{S_n(R)}f\left(\mathbf{x}\right)\varphi\left(\mathbf{x}\right)\mu\left(d\mathbf{x}\right),$$

where φ is an infinitely differentiable function with compact support and μ is the Lebesgue measure in $\mathbb{R}^n$. The function is called a simple layer on the spherical surface $S_n\left(R\right)$ (Vladimirov 1996).

1.4.2. *Two-dimensional case*

For $n = 2$, equation [1.51] can be written in the following form:

$$H(t) = e^{-\lambda t} J_0(\alpha t v) + \lambda \int_0^t e^{-\lambda(t-u)} J_0(\alpha(t-u)v) H(u)\, du$$

and equation [1.52] has the form

$$h_c(t) = \lambda \int_0^t J_0(\alpha v(t-u)) J_0(\alpha v u)\, du$$

$$+\lambda \int_0^t J_0(\alpha v(t-u)) h_c(u)\, du. \qquad [1.53]$$

THEOREM 1.5.– The continuous part $c_2(t, \mathbf{x})$ of the generalized pdf $f_2(t, \mathbf{x})$ satisfies the following equation for $0 \leq x \leq vt$:

$$\frac{\partial^2 c_2(t, \mathbf{x})}{\partial t^2} + 2\lambda \frac{\partial}{\partial t} c_2(t, \mathbf{x}) - v^2 \left(\frac{\partial^2 c_2(t, \mathbf{x})}{\partial x_1^2} + \frac{\partial^2 c_2(t, \mathbf{x})}{\partial x_2^2} \right)$$

$$= \lambda^2 e^{-\lambda t} \frac{\delta_{S_2(vt)}}{2\pi v t}, \qquad [1.54]$$

with the initial conditions $c_2(0) = 0$ and $\left.\frac{\partial c_2(t)}{\partial t}\right|_{t=0} = \lambda \delta(x)$.

DEMONSTRATION.– Let us consider the first two derivatives of $h_c(t)$

$$\frac{\partial}{\partial t} h_c(t) = \lambda J_0(\alpha v t) - \lambda \alpha v \int_0^t J_1(\alpha v(t-u)) J_0(\alpha v u)\, du$$

$$+\lambda h_c(t) - \lambda \alpha v \int_0^t J_1(\alpha v(t-u)) h_c(u)\, du \qquad [1.55]$$

and

$$\frac{\partial^2}{\partial t^2} h_c(t) = \lambda \frac{\partial}{\partial t}(J_0(\alpha v t)) - \lambda (\alpha v)^2 \int_0^t J_0(\alpha v(t-u)) J_0(\alpha v u)\, du$$

$$+\lambda (\alpha v)^2 \int_0^t \frac{J_1(\alpha v(t-u))}{\alpha v(t-u)} J_0(\alpha v u)\, du + \lambda \frac{d}{dt} h_c(t)$$

$$-\lambda (\alpha v)^2 \int_0^t J_0(\alpha v(t-u)) h_c(u)\, du$$

$$+\lambda (\alpha v)^2 \int_0^t \frac{J_1(\alpha v(t-u))}{\alpha v(t-u)} h_c(u)\, du. \qquad [1.56]$$

The last integral in equation [1.56] can be written in a more convenient form by doing some equivalent transformations

$$\int_0^t \frac{J_1\left(\alpha v\left(t-u\right)\right)}{\alpha v\left(t-u\right)} h_c\left(u\right) du$$

$$= \lambda \int_0^t \frac{J_1\left(\alpha v\left(t-u\right)\right)}{\alpha v\left(t-u\right)} \int_0^u J_0\left(\alpha v\left(u-s\right)\right) J_0\left(\alpha v s\right)\, ds\, du$$

$$+\lambda \int_0^t \frac{J_1\left(\alpha v\left(t-u\right)\right)}{\alpha v\left(t-u\right)} \int_0^u J_0\left(\alpha v\left(u-s\right)\right) h_c\left(s\right)\, ds\, du$$

$$= \lambda \int_0^t \int_0^{t-s} \frac{J_1\left(\alpha v\left(t-s-u\right)\right)}{\alpha v\left(t-s-u\right)} J_0\left(\alpha v u\right)\, du\, J_0\left(\alpha v s\right)\, ds$$

$$+\lambda \int_0^t \int_0^{t-s} \frac{J_1\left(\alpha v\left(t-s-u\right)\right)}{\alpha v\left(t-s-u\right)} J_0\left(\alpha v u\right)\, du\, h_c\left(s\right)\, ds$$

$$= \frac{\lambda}{\alpha v} \int_0^t J_1\left(\alpha v\left(t-s\right)\right) J_0\left(\alpha v s\right)\, ds + \frac{\lambda}{\alpha v} \int_0^t J_1\left(\alpha v\left(t-s\right)\right) h_c\left(s\right)\, ds,$$

where we used the formula 6.533 in Gradshteyn and Ryzhik (1980); we repeat that formula here for easy reference,

$$\int_0^x \frac{J_1\left(x-u\right)}{x-u} J_0\left(u\right) du = J_1\left(x\right).$$

Thus, combining equations [1.53]–[1.56], and taking into account that $\lambda \frac{\partial}{\partial t} J_0(\alpha v t) + \lambda \alpha v J_1(\alpha v t) = 0$, we obtain

$$\frac{\partial^2}{\partial t^2} h_c(t) = \left\{ \lambda \frac{\partial}{\partial t}\left(J_0\left(\alpha v t\right)\right) + \lambda \alpha v J_1(\alpha v t) \right\}$$

$$-\lambda (\alpha v)^2 \int_0^t J_0\left(\alpha v\left(t-u\right)\right) J_0\left(\alpha v u\right)\, du$$

$$-\lambda (\alpha v)^2 \int_0^t J_0\left(\alpha v\left(t-u\right)\right) h_c\left(u\right) du + \lambda \frac{\partial}{\partial t} h_c(t)$$

$$+\lambda^2 \alpha v \int_0^t J_1\left(\alpha v\left(t-u\right)\right) J_0\left(\alpha v u\right)\, du + \lambda^2 \alpha v \int_0^t J_1\left(\alpha v\left(t-u\right)\right) h_c\left(u\right) du$$

$$= -(\alpha v)^2 h_c(t) + \lambda \frac{\partial}{\partial t} h_c(t) - \lambda \left(\frac{\partial}{\partial t} h_c(t) - \lambda J_0\left(\alpha v t\right) - \lambda h_c(t) \right).$$

Therefore, we have the equivalent differential equation

$$\frac{\partial^2}{\partial t^2} h_c(t) + (\alpha v)^2 h_c(t) - \lambda^2 h_c(t) = \lambda^2 J_0\left(\alpha v t\right). \qquad [1.57]$$

with the initial conditions $h_c(0) = 0$ and $\left.\frac{\partial h_c(t)}{\partial t}\right|_{t=0} = \lambda$, which can be obtained directly from equation [1.53].

It is well known (it can be easily obtained from formula 4.644 of Gradshteyn and Ryzhik 1980) that the function $J_0(\alpha vt)$ is the two-dimensional Fourier transform with respect to $\mathbf{x} = (x_1, x_2)$ of the function $\frac{\delta_{S_2(vt)}}{2\pi vt}$, where $\delta_{S_2(vt)}$ is a simple layer for a circle with radius vt. Thus, $\mathcal{F}^{-1}(J_0(\alpha vt)) = \frac{\delta_{S_2(vt)}}{2\pi vt}$. Denote by $g(t, \mathbf{x}) = \mathcal{F}^{-1}(h_c(t))$ the two-dimensional inverse Fourier transform of $h_c(t)$ with respect to α. Here, we use the notation $g(t, \mathbf{x})$, where g is a radial function, and $g(t, \mathbf{x}) = g(t, \mathbf{y})$ for all $\mathbf{x}$, $\mathbf{y} \in \mathbb{R}^2$ such that $|\mathbf{x}| = |\mathbf{y}|$.

We have the following two-dimensional inverse Fourier transform:

$$\mathcal{F}^{-1}\left(\alpha^2 h_c(t), \alpha\right) = -\sum_{i=1}^{2} \frac{\partial^2 g(t, \mathbf{x})}{\partial x_i^2}.$$

Here, we used the fact that $\mathcal{F}^{-1}\left(\alpha_i^2 h_c(t), \alpha\right) = -\frac{\partial^2 g(t,\mathbf{x})}{\partial x_i^2}$. Passing to the two-dimensional inverse Fourier transform in equation [1.57], for $0 \le x \le vt$ we can obtain

$$\frac{\partial^2}{\partial t^2} g(t, \mathbf{x}) + (\alpha v)^2 g(t, \mathbf{x}) - \lambda^2 g(t, \mathbf{x}) = \lambda^2 \frac{\delta_{S_2(vt)}}{2\pi vt}. \qquad [1.58]$$

Let us denote by $c_2(t, \mathbf{x}) = \mathcal{F}^{-1}(H_c(t), \alpha)$, the absolute continuous part of the generalized pdf of the particle's position at time t.

Since $e^{\lambda t} c_2(t, \mathbf{x}) = g(t, \mathbf{x})$, from equation [1.58] we obtain equation [1.54], with the initial conditions $c_2(0) = 0$ and $\left.\frac{\partial c_2(t)}{\partial t}\right|_{t=0} = \lambda \delta(x)$, which can be obtained directly from the initial conditions of equation [1.57].

The fact that $c_2(t, \mathbf{x})$ is a fundamental solution of the two-dimensional Goldstein–Kac equation, has been found in literature by using other methods. However, in our case, equation [1.54] differs from the corresponding similar published equation, and it is a consequence of specifying the expression on the right side of the equation and the initial conditions.

At first glance, it may seem strange that on the right side of equation [1.54] we have a singular function for the continuous part $c_2(t, \mathbf{x})$. But there is no contradiction if we consider that the continuous part $c_2(t, \mathbf{x})$ goes to infinity as $|x| \to vt$, and it has the so-called explosion effect for just the continuous part of the generalized density (Pogorui and Rodríguez-Dagnino 2011, and the references therein).

Thus, the generalized pdf of the particle position is given by $f(t, \mathbf{x}) = c_2(t, \mathbf{x}) + s_2(t, \mathbf{x})$, where $c_2(t, \mathbf{x})$ is the solution of the two-dimensional Goldstein–Kac telegraph equation [1.54] and $s_2(t, \mathbf{x}) = e^{-\lambda t}\delta(vt - x)$.

REMARK 1.3.– *We will apply the basic method developed for $n = 2$, to the cases $n=3$ and $n = 5$ in the following sections. We should remark that most of the published works in this area have solved the cases with even dimensions, and the cases of odd dimensions remain as elusive cases. We will show, by using our method, that it is possible to find the governing telegraph-type partial differential equations for these odd dimensions, and we recall the corresponding solution in the three-dimensional case.*

1.4.3. *Three-dimensional case*

In this case for a $(2, \lambda)$-Erlang pdf (i.e. $g(t) = \lambda^2 te^{-\lambda t},\ t \geq 0$), the characteristic function $H(t)$ satisfies the following integral equation:

$$H(t) = (1 + \lambda t)\, e^{-\lambda t}\, \frac{\sin(\alpha vt)}{\alpha vt}$$
$$+\frac{\lambda^2}{\alpha v}\int_0^t e^{-\lambda(t-u)}\sin(\alpha v\,(t-u))H(u)du. \qquad [1.59]$$

The generalized pdf $f_3(t,x) = \mathcal{F}^{-1}(H(t),\alpha)$, where $x = \sqrt{x_1^2 + x_2^2 + x_3^2} = |\mathbf{x}|$, $\mathcal{F}^{-1}$ is the three-dimensional Fourier transform with respect to α, can be written as follows:

$$f_3(t,x) = s_3(t,x) + c_3(t,x)\,,$$

where $s_3(t,x)$ is the singular part and $c_3(t,x)$ is the continuous part of the generalized pdf $f_3(t,x)$.

Let us define $\Lambda_* = \sqrt{(tv)^2 - x^2}$, then our main result in this section is established in the following basic theorem.

THEOREM 1.6.– For $0 \leq x \leq vt$, the generalized pdf $f_3(t,x)$ is given by

$$f_3(t,x) = (1+\lambda t)\,e^{-\lambda t}\,\frac{\delta_{S_3(vt)}}{4\pi(vt)^2} + \frac{\lambda^2 e^{-\lambda t}}{4\pi v^2}\frac{I_1\left(\frac{\lambda}{v}\Lambda_*\right)}{\Lambda_*}\,\Theta(tv - x)$$
$$+\frac{t\lambda e^{-\lambda t}\left(\lambda\Lambda_* I_0\left(\frac{\lambda}{v}\Lambda_*\right) - 2vI_1\left(\frac{\lambda}{v}\Lambda_*\right)\right)}{4\pi v\Lambda_*^3}\,\Theta(tv - x) + r(t,\mathbf{x})\,, \qquad [1.60]$$

where $\Theta(tv - x)$ is the Heaviside function and the reminder $r(t,\mathbf{x})$ is non-negative, $r(t,\mathbf{x}) = 0$ for $tv < x$ and satisfies (see Gradshteyn and Ryzhik 1980, formula 4.642)

$$\int_{|\mathbf{x}|\leq vt} r(t,\mathbf{x})\,d\mathbf{x} = 4\pi\int_0^{vt} x^2 r(t,x)dx \leq \frac{\lambda^2 t^2}{2}e^{-\lambda t}.$$

DEMONSTRATION.– As shown in Vladimirov (1996), the function $4\pi R\frac{\sin\alpha R}{\alpha}$ is the three-dimensional Fourier transform of the simple layer $\delta_{S_3(R)}$. Hence,

$$s_3(t,x) = \mathcal{F}^{-1}\left((1+\lambda t)\,e^{-\lambda t}\,\frac{\sin(\alpha vt)}{\alpha vt}\right) = (1+\lambda t)\,e^{-\lambda t}\,\frac{\delta_{S_3(vt)}}{4\pi(vt)^2}.$$

It follows from equation [1.59] that

$$h_c(t) = \lambda^2\int_0^t \frac{\sin(\alpha v(t-u))}{\alpha v}\,(1+\lambda u)\,\frac{\sin(\alpha vu)}{\alpha vu}du$$
$$+\frac{\lambda^2}{\alpha v}\int_0^t \sin(\alpha v(t-u))\,h_c(u)\,du, \qquad [1.61]$$

The second derivative of equation [1.61] with respect to t is given by

$$\frac{\partial^2 h_c(t)}{\partial t^2} - \lambda^2 h_c(t) + \lambda^2\alpha v\int_0^t \sin(\alpha v(t-u))\,h_c(u)\,du$$
$$= \left(\lambda^2+\lambda^3 t\right)\frac{\sin(\alpha vt)}{\alpha vt}$$
$$-\lambda^2\alpha v\int_0^t \sin(\alpha v(t-u))\,(1+\lambda u)\,\frac{\sin(\alpha vu)}{u}\,du. \qquad [1.62]$$

Combining equations [1.61] and [1.62], we obtain a differential equation for $h_c(t)$

$$\frac{\partial^2 h_c(t)}{\partial t^2} - \lambda^2 h_c(t) + \alpha^2 v^2 h_c(t) = \left(\lambda^2+\lambda^3 t\right)\frac{\sin(\alpha vt)}{\alpha vt} \qquad [1.63]$$

with the initial conditions $h_c(0) = 0$, $\left.\frac{\partial h_c(t)}{\partial t}\right|_{t=0} = 0$ that can be calculated directly from equation [1.61].

Since a solution of equation [1.63] with the given initial conditions is unique, we can split equation [1.63] into a set of two equations. Namely,

$$\frac{\partial^2 h_c^{(1)}(t)}{\partial t^2} - \lambda^2 h_c^{(1)}(t) + \alpha^2 v^2 h_c^{(1)}(t) = \lambda^2\,\frac{\sin(\alpha vt)}{\alpha vt} \qquad [1.64]$$

with the initial conditions $h_c^{(1)}(0) = 0$, $\left.\frac{\partial h_c^{(1)}(t)}{\partial t}\right|_{t=0} = 0$, and

$$\frac{\partial^2 h_c^{(2)}(t)}{\partial t^2} - \lambda^2 h_c^{(2)}(t) + \alpha^2 v^2 h_c^{(2)}(t) = \lambda^3\,\frac{\sin(\alpha vt)}{\alpha v} \qquad [1.65]$$

with similar initial conditions $h_c^{(2)}(0) = 0$, $\left.\frac{\partial h_c^{(2)}(t)}{\partial t}\right|_{t=0} = 0$.

The solutions of equations [1.64] and [1.65] with the given initial conditions are also unique. Hence, the solution $h_c(t)$ of equation [1.63] is

$$h_c(t) = h_c^{(1)}(t) + h_c^{(2)}(t).$$

It is easily verified that by solving equation [1.64], we can obtain

$$h_c^{(1)}(t) = \frac{\lambda^2}{2\alpha v \Lambda_o}\Big(\cos(t\Lambda_o)\,\mathrm{Ci}\,(t(\alpha v - \Lambda_o)) - \cos(t\Lambda_o)\,\mathrm{Ci}\,(t(\Lambda_o + \alpha v)) + \sin(t\Lambda_o)\,\mathrm{Si}\,(t(\Lambda_o - \alpha v)) - \sin(t\Lambda_o)\,\mathrm{Si}\,(t(\Lambda_o - \alpha v)) + \cos(t\Lambda_o)\,(\ln(\alpha v + \Lambda_o) - \ln(\alpha v - \Lambda_o))\Big),$$

where $\Lambda_o = \sqrt{\alpha^2 v^2 - \lambda^2}$, $\mathrm{Ci}(\cdot)$ is the cosine integral, and $\mathrm{Si}(\cdot)$ is the sine integral.

Then, by solving equation [1.65], we obtain

$$h_c^{(2)}(t) = \frac{\lambda \sin(t\Lambda_o)}{\Lambda_o} - \frac{\lambda \sin(\alpha v t)}{\alpha v}.$$

By using formula 2.15.10 (4) of Prudnikov *et al.* (1986) and the Hankel transform of order $\frac{1}{2}$, after some elementary mathematical simplifications, we obtain the three-dimensional inverse Fourier transform of $h_c^{(2)}(t)$ for $x \leq vt$

$$g^{(2)}(t,\mathbf{x}) = \mathcal{F}^{-1}\left(h_c^{(2)}(t),\alpha\right) = \lambda\,\mathcal{F}^{-1}\left(\frac{\sin(t\Lambda_o)}{\Lambda_o} - \frac{\sin(\alpha v t)}{\alpha v},\alpha\right) = \frac{\lambda^2}{4\pi v^2}\frac{I_1\left(\frac{\lambda}{v}\Lambda_*\right)}{\Lambda_*}\,\Theta(tv - x).$$

We should note that this result coincides with the result obtained in Garra and Orsingher (2014, equation 4.1). However, it represents only a partial result in our analysis and to obtain $c_3(t,x)$, we should calculate $g^{(1)}(t,\mathbf{x}) = \mathcal{F}^{-1}\left(h_c^{(1)}(t),\alpha\right)$. Unfortunately, we cannot obtain a closed-form solution for this expression. A good approximated solution can be obtained by taking into account the inverse three-dimensional Fourier transform $\mathcal{F}^{-1}\left(\alpha^2 h_c^{(1)},\alpha\right) = -\sum_{i=1}^{3}\frac{\partial^2 g^{(1)}(t,\mathbf{x})}{\partial x_i^2}$, and passing to the inverse three-dimensional Fourier transform of equation [1.63], it is easily seen that for $x < vt$

$$\frac{\partial^2 g^{(1)}(t,\mathbf{x})}{\partial t^2} - \lambda^2 g^{(1)}(t,\mathbf{x}) - v^2\sum_{i=1}^{3}\frac{\partial^2 g^{(1)}(t,\mathbf{x})}{\partial x_i^2} = 0. \qquad [1.66]$$

Here we used the fact that $\mathcal{F}^{-1}\left(\frac{\sin(\alpha v t)}{\alpha v t},\alpha\right) = \frac{\delta_{S(vt)}}{4\pi(vt)^2} = 0$ for $x < vt$, where $\delta_{S(vt)}$ is a three-dimensional simple layer.

It is straightforward to verify that

$$\tilde{g}^{(1)}(t,\mathbf{x}) = \frac{\lambda}{4\pi v^2}\frac{\partial}{\partial t}\frac{I_1\left(\frac{\lambda}{v}\Lambda_*\right)}{\Lambda_*} = \frac{t\lambda\left(\lambda\Lambda_* I_0\left(\frac{\lambda}{v}\Lambda_*\right) - 2vI_1\left(\frac{\lambda}{v}\Lambda_*\right)\right)}{4\pi v\Lambda_*^3}$$

is a solution of equation [1.66], for $x < vt$.

Then, since

$$\lim_{\alpha\to 0} h_c^{(2)}(t) = \lim_{\alpha\to 0}\left(\frac{\lambda\sin(t\Lambda_o)}{\Lambda_o} - \frac{\lambda\sin(\alpha vt)}{\alpha v}\right) = \sinh(\lambda t) - \lambda t,$$

and by using (Gradshteyn and Ryzhik 1980, formula 4.642) we have

$$\int_{|\mathbf{x}|\le vt} g^{(2)}(t,\mathbf{x})\, d\mathbf{x} = 4\pi\int_0^{vt} x^2 g^{(2)}(t,x)\,dx$$

$$= \frac{\lambda^2}{v^2}\int_0^{tv} x^2\frac{I_1\left(\frac{\lambda}{v}\Lambda_*\right)}{\Lambda_*}\,dx = \sinh(\lambda t) - \lambda t.$$

Hence,

$$\frac{\lambda}{v^2}\int_0^{tv} x^2\frac{\partial}{\partial t}\frac{I_1\left(\frac{\lambda}{v}\Lambda_*\right)}{\Lambda_*}\,dx = \cosh(\lambda t) - 1 - \frac{\lambda^2 t^2}{2}.$$

In a similar manner,

$$\int_{|\mathbf{x}|\le vt}\tilde{g}^{(1)}(t,\mathbf{x})\,d\mathbf{x} = \frac{\lambda}{v}\int_0^{tv} x^2\frac{t\left(\lambda\Lambda_* I_0\left(\frac{\lambda}{v}\Lambda_*\right) - 2vI_1\left(\frac{\lambda}{v}\Lambda_*\right)\right)}{\Lambda_*^3}\,dx$$

$$= \cosh(\lambda t) - 1 - \frac{\lambda^2 t^2}{2}.$$

On the other hand, it can be easily verified that

$$\lim_{\alpha\to 0} h_c^{(1)}(t) = \lim_{\alpha\to 0}\frac{\lambda^2}{2\alpha v\Lambda_o}\Big(\cos(t\Lambda_o)\,\mathrm{Ci}\,(t(\alpha v - \Lambda_o))$$

$$-\cos(t\Lambda_o)\,\mathrm{Ci}\,(t(\Lambda_o + \alpha v)) + \sin(t\Lambda_o)\,\mathrm{Si}\,(t(\Lambda_o - \alpha v))$$

$$-\sin(t\Lambda_o)\,\mathrm{Si}\,(t(\Lambda_o - \alpha v)) + \cos(t\Lambda_o)\left(\ln(\alpha v + \Lambda_o) - \ln(\alpha v - \Lambda_o)\right)\Big)$$

$$= \cosh(\lambda t) - 1.$$

Hence, taking into account (Gradshteyn and Ryzhik 1980, formula 4.642), we have

$$\int_{|\mathbf{x}|<vt} g^{(1)}(t,\mathbf{x})\,d\mathbf{x} = 4\pi \int_0^{vt} x^2 g^{(1)}(t,x)\,dx = \cosh(\lambda t) - 1.$$

By defining $r(t,x) = g^{(1)}(t,x) - \tilde{g}^{(1)}(t,x)$ and considering that

$$c_3(t,\mathbf{x}) = \mathcal{F}^{-1}(H_c(t),\alpha) = \left(g^{(1)}(t,\mathbf{x}) + g^{(2)}(t,\mathbf{x})\right)e^{-\lambda t}$$

the proof is concluded. □

REMARK 1.4.– *We should mention that* $c_3(t,\mathbf{x})$ *satisfies*

$$\frac{\partial^2 c_3(t,\mathbf{x})}{\partial t^2} + 2\lambda \frac{\partial}{\partial t} c_3(t,\mathbf{x}) - v^2 \sum_{i=1}^{3} \frac{\partial^2 c_3(t,\mathbf{x})}{\partial x_i^2}$$

$$= \left(\lambda^2 + \lambda^3 t\right) e^{-\lambda t} \frac{\delta_{S_3(vt)}}{4\pi (vt)^2}. \qquad [1.67]$$

$$c_3(0,\mathbf{x}) = 0, \left.\frac{\partial c_3(t,\mathbf{x})}{\partial t}\right|_{t=0} = 0.$$

We have a singular function on the right side of equation [1.67] *for the continuous part* $c_3(t,\mathbf{x})$ *since it goes to infinity as* $|x| \to vt$*, and it has the explosion effect just as well as in the two-dimensional case for* $c_2(t,\mathbf{x})$.

Let us assume $g(t,\mathbf{x}) = \mathcal{F}^{-1}(h_c(t),\alpha)$, where $h_c(t)$ is a solution of equation [1.63]. Since $e^{\lambda t} c_3(t,\mathbf{x}) = g(t,\mathbf{x})$, and in passing equation [1.63] to the three-dimensional inverse Fourier transform, we obtain equation [1.67].

The generalized pdf $f_3(t,x)$ found in equation [1.60] depends on λ and v, and we illustrate its behavior in Figure 1.1 for $\lambda = 2$ and $v = 3$.

Similarly, we plot equation [4.1] of Garra and Orsingher (2014) in Figure 1.2.

In order to do more comparisons, we will include the singular part to the generalized pdf $f(t,\mathbf{x}) = f(t,x)$ found in Garra and Orsingher (2014), i.e.

$$f(t,x) = (1+\lambda t)\,e^{-\lambda t} \frac{\delta_{S(vt)}}{4\pi (vt)^2} + \frac{\lambda^2 e^{-\lambda t}}{4\pi v^2} \frac{I_1\left(\frac{\lambda}{v}\Lambda_*\right)}{\Lambda_*}\,\Theta(tv - x). \qquad [1.68]$$

We were not able to find a closed-form expression for $r(t,x)$, however, we are now including the singular part of $f_3(t,x) - r(t,x)$. The comparison of the integrals on the whole range of equations [1.60] and [1.68] are shown in Figures 1.3–1.6 for different combinations of λ and v.

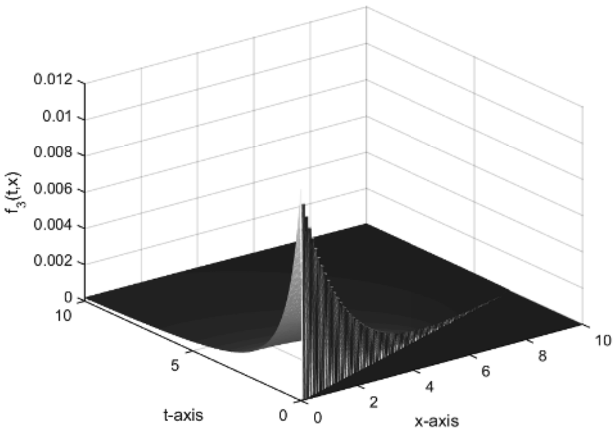

Figure 1.1. *Approximated probability density function $f_3(t,x) - r(t,x)$ for $\lambda = 2$ and $v = 3$. The singular term is not included in this plot. For a color version of this figure, see www.iste.co.uk/pogorui/random2.zip*

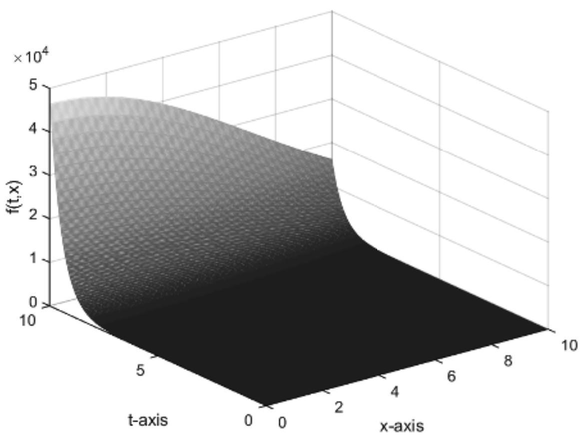

Figure 1.2. *$f(t,x)$ for $\lambda = 2$ and $v = 3$, according to Garra and Orsingher (2014). For a color version of this figure, see www.iste.co.uk/pogorui/random2.zip*

The results in these plots show that $f_3(t,x)$ is a pdf and the contribution of $r(t,x)$ is not significant for t large, since the integral approaches 1 very quickly. We should mention that the result in Garra and Orsingher (2014) seems to just concern the probability of finding the particle after an odd number of direction changes. So, it is not a true pdf since "half" of the time particle moves after odd and "half" after even direction changes. It is not surprising that $f(t, x)$ goes to 1/2 as t is large.

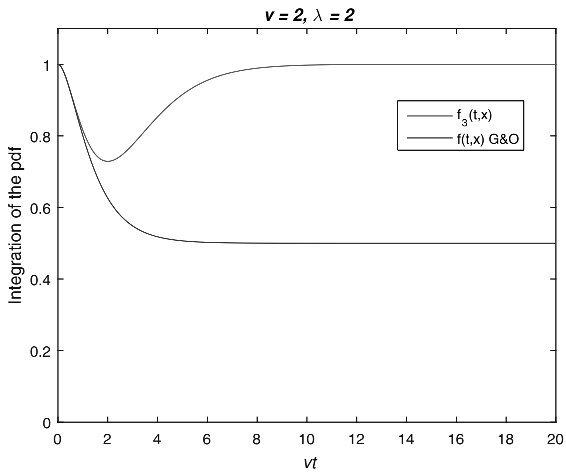

Figure 1.3. *Integration of $f_3(t,x) - r(t,x)$ and $f(t,x)$ for $\lambda = 2$ and $v = 2$. The singular part is included in both cases. For a color version of this figure, see www.iste.co.uk/pogorui/random2.zip*

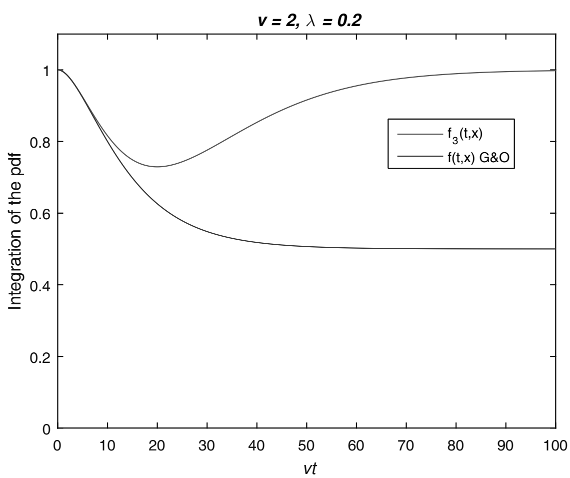

Figure 1.4. *Integration of $f_3(t,x) - r(t,x)$ and $f(t,x)$ for $\lambda = 0.2$ and $v = 2$. The singular part is included in both cases. For a color version of this figure, see www.iste.co.uk/pogorui/random2.zip*

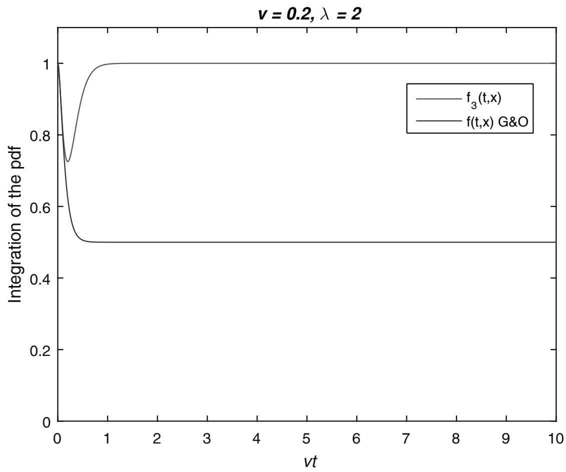

Figure 1.5. *Integration of $f_3(t, x) - r(t, x)$ and $f(t, x)$ for $\lambda = 2$ and $v = 0.2$. The singular part is included in both cases. For a color version of this figure, see www.iste.co.uk/pogorui/random2.zip*

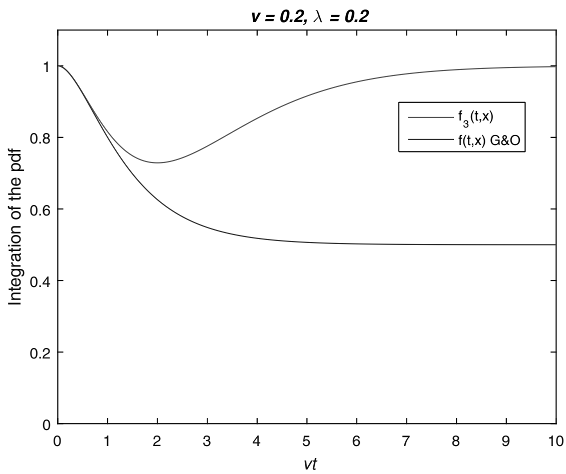

Figure 1.6. *Integration of $f_3(t, x)$ and $f(t, x)$ for $\lambda = 0.2$ and $v = 0.2$. The singular part is included in both cases. For a color version of this figure, see www.iste.co.uk/pogorui/random2.zip*

We should mention that equation 7.4.18 in Morse and Feshbach (1953) was proposed as a model for the transmission of heat in a gas, where the velocity of the propagation of the heat is included, or the velocity of the propagation of a disturbance in a gas. A similar reasoning was considered in Tautz and Lerche (2016) for modeling the transport of solar energetic particles, where the authors emphasize the fact that the telegraph-type equation has the potential to make a difference between early ballistic motion and later diffusive transport (Kac's limit). We should emphasize that the underlying stochastic process for this random flight is governed by 2-Erlang-distributed sojourn times. So, in this work we found that the non-homogeneous three-dimensional Goldstein–Kac telegraph equation is the partial differential equation for modeling this random flight.

1.4.4. *Five-dimensional case*

In this case, we have

$$\varphi(t,\alpha) = 2^{\frac{3}{2}}\Gamma\left(\frac{5}{2}\right)\frac{J_{\frac{3}{2}}(\alpha v t)}{(\alpha v t)^{\frac{3}{2}}} = 3\,\frac{\sin(\alpha v t) - \alpha v t\cos(\alpha v t)}{(\alpha v t)^3},$$

where

$$\alpha = \sqrt{\alpha_1^2 + \alpha_2^2 + \alpha_3^2 + \alpha_4^2 + \alpha_5^2}.$$

Considering the case of a $(4,\lambda)$-Erlang pdf, i.e. $g(t) = \frac{1}{6}\lambda^4 t^3 e^{-\lambda t}\mathbb{I}_{\{t\geq 0\}}$, we obtain

$$H(t) = \left(\sum_{k=0}^{3}\frac{(\lambda t)^k}{k!}\right)e^{-\lambda t}\,3\,\frac{\sin(\alpha v t) - \alpha v t\cos(\alpha v t)}{(\alpha v t)^3}$$
$$+\frac{\lambda^4}{2(\alpha v)^3}\int_0^t e^{-\lambda(t-u)}\left\{\sin(\alpha v(t-u)) - \alpha v(t-u)\cos(\alpha v(t-u))\right\}$$
$$\times H(u)\,du. \qquad [1.69]$$

THEOREM 1.7.– The continuous part $c_5(t,\mathbf{x})$ of the generalized pdf $f_5(t,\mathbf{x})$ satisfies the following equation for $0 \leq x \leq vt$:

$$\left(\frac{\partial^2}{\partial t^2} + 2\lambda\frac{\partial}{\partial t} - v^2\sum_{i=1}^{5}\frac{\partial^2}{\partial x_i^2} + 2\lambda^2\right)\left(\frac{\partial^2}{\partial t^2} + 2\lambda\frac{\partial}{\partial t} - v^2\sum_{i=1}^{5}\frac{\partial^2}{\partial x_i^2}\right)c_5(t,\mathbf{x})$$
$$= \lambda^4 e^{-\lambda t}\left(\sum_{k=0}^{3}\frac{(\lambda t)^k}{k!}\right)\delta_{S_5(vt)}. \qquad [1.70]$$

DEMONSTRATION.– It is well known that the function $3\,\frac{\sin(\alpha v t) - \alpha v t\cos(\alpha v t)}{(\alpha v t)^3}$ is the five-dimensional Fourier transform with respect to x of the function $\frac{8\delta_{S_5(vt)}}{3\pi^2(vt)^4}$, where

$\mathbf{x} = (x_1, x_2, x_3, x_4, x_5)$, $x = |\mathbf{x}|$, and $S_5(R)$ is the simple layer on the spherical surface, with radius R in $\mathbb{R}^5$. So,

$$s_5(t,x) = \mathcal{F}^{-1}\left(\left(\sum_{k=0}^{3}\frac{(\lambda t)^k}{k!}\right)e^{-\lambda t}\,3\,\frac{\sin(\alpha vt) - \alpha vt\cos(\alpha vt)}{(\alpha vt)^3}, \alpha\right)$$

$$= \left(\sum_{k=0}^{3}\frac{(\lambda t)^k}{k!}\right)e^{-\lambda t}\,\delta_{S(x,vt)}.$$

From equation [1.69], it follows that

$$h_c(t) = \frac{3\lambda^4}{2(\alpha v)^3}\int_0^t \{\sin(\alpha v(t-u)) - \alpha v(t-u)\,\cos(\alpha v(t-u))\}$$

$$\times\left(\sum_{k=0}^{3}\frac{(\lambda u)^k}{k!}\right)\frac{\sin(\alpha vu) - \alpha vu\cos(\alpha vu)}{(\alpha vu)^3}\,du$$

$$+\frac{\lambda^4}{2(\alpha v)^3}\int_0^t \{\sin(\alpha v(t-u)) - \alpha v(t-u)\,\cos(\alpha v(t-u))\}$$

$$\times h_c(u)\,du. \qquad [1.71]$$

Consider the following derivatives:

$$\frac{\partial^2}{\partial t^2}\int_0^t \{\sin(\alpha v(t-u)) - \alpha v(t-u)\,\cos(\alpha v(t-u))\}\,h_c(u)\,du$$

$$= \int_0^t \left\{(\alpha v)^2\sin(\alpha v(t-u)) + (\alpha v)^3(t-u)\cos(\alpha v(t-u))\right\}$$

$$\times h_c(u)\,du, \qquad [1.72]$$

and

$$\frac{\partial^4}{\partial t^4}\int_0^t \{\sin(\alpha v(t-u)) - \alpha v(t-u)\,\cos(\alpha v(t-u))\}\,h_c(u)\,du$$

$$= \int_0^t \left\{-3(\alpha v)^4\sin(\alpha v(t-u)) - (\alpha v)^5(t-u)\,\cos(\alpha v(t-u))\right\}$$

$$\times h_c(u)\,du + 2(\alpha v)^3 h_c(t). \qquad [1.73]$$

By combining equations [1.71]–[1.73], the following equation is obtained:

$$\left(\frac{\partial^4}{\partial t^4} + 2(\alpha v)^2\frac{\partial^2}{\partial t^2} + (\alpha v)^4 - \lambda^4\right)h_c(t)$$

$$= 3\lambda^4 \left(\sum_{k=0}^{3} \frac{(\lambda t)^k}{k!} \right) \frac{\sin(\alpha v t) - \alpha v t \cos(\alpha v t)}{(\alpha v t)^3}.$$

The initial conditions for $h_c(t)$ are given by $h_c(0) = 0$, $\left.\frac{\partial h_c(t)}{\partial t}\right|_{t=0} = 0$, $\left.\frac{\partial^2 h_c(t)}{\partial t^2}\right|_{t=0} = 0$ and $\left.\frac{\partial^3 h_c(t)}{\partial t^3}\right|_{t=0} = 0$, which can be obtained directly from equation [1.71].

Passing to the inverse Fourier transform of equation [1.74], for $0 \leq x \leq vt$ we have

$$\left(\frac{\partial^4}{\partial t^4} - 2v^2 \sum_{i=1}^{5} \frac{\partial^4}{\partial x_i^2 \partial t^2} + v^4 \sum_{1 \leq i < j \leq 5} \frac{\partial^4}{\partial x_i^2 \partial x_j^2} - \lambda^4 \right) g(t, \mathbf{x})$$

$$= \lambda^4 \left(\sum_{k=0}^{3} \frac{(\lambda t)^k}{k!} \right) \delta_{S_5(vt)}, \qquad [1.74]$$

where $g(t, \mathbf{x}) = \mathcal{F}^{-1}(h_c(t), \alpha)$.

After factorizing equation [1.74], we obtain:

$$\left(\frac{\partial^2}{\partial t^2} - v^2 \sum_{i=1}^{5} \frac{\partial^2}{\partial x_i^2} + \lambda^2 \right) \left(\frac{\partial^2}{\partial t^2} - v^2 \sum_{i=1}^{5} \frac{\partial^2}{\partial x_i^2} - \lambda^2 \right) g(t, \mathbf{x})$$

$$= \lambda^4 \left(\sum_{k=0}^{3} \frac{(\lambda t)^k}{k!} \right) \delta_{S_5(vt)}. \qquad [1.75]$$

Since $e^{\lambda t} c_5(t, \mathbf{x}) = g(t, \mathbf{x})$, then equation [1.70] follows from equation [1.75].

The left-hand factor of equation [1.70] is the five-dimensional version of the Heaviside telegraph equation and the other factor is the five-dimensional version of the Goldstein–Kac telegraph equation.

Under the Kac condition, i.e. when $\frac{v^2}{\lambda} \to \sigma^2$, $\lambda \to +\infty$, and dividing the equation [1.70] by λ^3, we obtain the five-dimensional diffusion equation:

$$\frac{\partial}{\partial t} w(t, \mathbf{x}) - \frac{\sigma^2}{2} \sum_{i=1}^{5} \frac{\partial^2}{\partial x_i^2} w(t, \mathbf{x}) = 0.$$

1.5. The jump telegraph process in $\mathbb{R}^n$

Let us consider a similar mathematical framework as before, i.e. let the following renewal process $\xi(t) = \max\{m \geq 0 : \tau_m \leq t\}$, $t \geq 0$, where $\tau_m = \sum_{k=0}^{m} \theta_k$, $\tau_0 = 0$ and $\theta_k \geq 0$, $k = 1, 2, \ldots$, are i.i.d. random variables with cdf $G(t)$ and corresponding pdf $g(t) = \frac{d}{dt}G(t)$. Suppose that a particle starting from $(0, 0, \ldots, 0) \in \mathbb{R}^n$, at $t = 0$, continues its motion with an absolute velocity v along a direction $\boldsymbol{\eta}_0^{(n)}$, where $\boldsymbol{\eta}_0^{(n)} = (x_1, x_2, \cdots, x_n)$ is a random vector in $\mathbb{R}^n$, uniformly distributed on the unit sphere

$$\Omega_1^{n-1} = \left\{(x_1, x_2, \cdots, x_n) : x_1^2 + x_2^2 + \ldots + x_n^2 = 1\right\}.$$

At instant τ_1, the particle changes its direction to $\boldsymbol{\eta}_1^{(n)}$ and jumps by a random vector $\boldsymbol{\beta}_1$, where $\boldsymbol{\eta}_1^{(n)}$, $\boldsymbol{\eta}_0^{(n)}$ and $\boldsymbol{\beta}_1 \in \mathbb{R}^n$ are independent. Then, at time τ_2 the particle changes its direction to $\boldsymbol{\eta}_2^{(n)}$ and jumps by $\boldsymbol{\beta}_2$, where $\boldsymbol{\eta}_2^{(n)}$, $\boldsymbol{\eta}_1^{(n)}$, $\boldsymbol{\eta}_0^{(n)}$, $\boldsymbol{\beta}_1$ and $\boldsymbol{\beta}_2$ are independent, and so on. We assume that all $\boldsymbol{\eta}_i^{(n)}$, $i = 0, 1, 2, \ldots$ are identically distributed and all $\boldsymbol{\beta}_i$, $i = 1, 2, \ldots$ are identically distributed and radial (isotropic).

Denoting by $\mathbf{x}^{(n)}(t)$, $t \geq 0$, the particle position at time t, we then have

$$\mathbf{x}^{(n)}(t) = \sum_{j=1}^{\xi(t)} \left\{v\boldsymbol{\eta}_{j-1}^{(n)} (\tau_j - \tau_{j-1}) + \boldsymbol{\beta}_j\right\} + v\boldsymbol{\eta}_{\xi(t)}^{(n)} \left(t - \tau_{\xi(t)}\right). \qquad [1.76]$$

Here, we have assumed $\sum_{j=1}^{0} = 0$.

Consider $\varphi(t, \alpha) = \mathbf{E}\left[e^{itv\left(\boldsymbol{\alpha}, \boldsymbol{\eta}_0^{(n)}\right)}\right]$, where $\alpha = \|\boldsymbol{\alpha}\|$. We should note that the function $\varphi(t, \alpha)$ is well known (Pogorui and Rodríguez-Dagnino 2012) and is of the following form:

$$\varphi(t, \alpha) = 2^{\frac{n-2}{2}} \Gamma\left(\frac{n}{2}\right) \frac{J_{\frac{n-2}{2}}(\alpha t v)}{(\alpha t v)^{\frac{n-2}{2}}}. \qquad [1.77]$$

THEOREM 1.8.– The characteristic function $H(t, \alpha) = \mathbf{E}\left[e^{itv\left(\boldsymbol{\alpha}, \mathbf{x}^{(n)}(t)\right)}\right]$, $t \geq 0$, is a solution of the following integral Volterra equation of a convolution type:

$$H(t, \alpha) = (1 - G(t))\, \varphi(t, \alpha)$$

$$+\varphi_\beta(\alpha) \int_0^t g(u)\, \varphi(u, \alpha)\, H(t - u, \alpha)\, du. \qquad [1.78]$$

PROOF.– It follows from equation [1.76] that:

$$H(t,\alpha) = \mathbf{E}\Big[\exp\Big\{i\Big(\boldsymbol{\alpha}, \mathbf{x}^{(n)}(t)\Big)\Big\}\Big] = \mathbf{E}\Big[\exp\Big\{i\Big(\boldsymbol{\alpha}, \mathbf{x}^{(n)}(t)\Big)\Big\}\mathbb{I}\{\theta_1 > t\}\Big]$$

$$+\mathbf{E}\Big[\exp\Big\{i\Big(\boldsymbol{\alpha}, \mathbf{x}^{(n)}(t)\Big)\Big\}\mathbb{I}\{\theta_1 \le t\}\Big] = \mathbf{E}\Big[\exp\Big\{i\Big(\boldsymbol{\alpha}, v\boldsymbol{\eta}_0^{(n)}t\Big)\Big\}\mathbb{I}\{\theta_1 > t\}\Big]$$

$$+\mathbf{E}\Big[\exp\Big\{i\Big(\boldsymbol{\alpha}, v\boldsymbol{\eta}_0^{(n)}\theta_1 + \boldsymbol{\beta}_1 + S_\xi + v\boldsymbol{\eta}_{\xi(t)}^{(n)}\left(t - \tau_{\xi(t)}\right)\Big)\Big\}\mathbb{I}\{\theta_1 \le t\}\Big]$$

$$= (1 - G(t))\,\mathbf{E}\Big[e^{itv\left(\boldsymbol{\alpha}, \boldsymbol{\eta}_0^{(n)}\right)}\Big] + \int_0^t g(u)\,\mathbf{E}\Big[e^{iuv\left(\boldsymbol{\alpha}, \boldsymbol{\eta}_0^{(n)}\right)}\Big]$$

$$\times\varphi_\beta(\alpha)H(t-u,\alpha)\,du,$$

where $S_\xi = \sum_{j=2}^{\xi(t)}\left(v\boldsymbol{\eta}_j^{(n)}(\tau_{j+1} - \tau_j) + \boldsymbol{\beta}_j\right)$, and $\varphi_\beta(\alpha) = \mathbf{E}\left[e^{i(\boldsymbol{\alpha}, \boldsymbol{\beta}_1)}\right]$.

Denote by $f_n(t,\mathbf{x})$ the pdf of the particle position at time t. By using the n-dimension inverse Fourier transform $\mathcal{F}^{-1}$ with respect to α, we get $f_n(t,\mathbf{x}) = \mathcal{F}^{-1}(H(t,\alpha))$.

1.5.1. *The jump telegraph process in* $\mathbb{R}^3$

Let us consider the three-dimensional case, that is $n = 3$ and $\varphi(t,\alpha) = \frac{\sin(\alpha tv)}{\alpha tv}$. For the Erlang-2 distribution $g(t) = \lambda^2 te^{-\lambda t}\mathbb{I}_{\{t\ge 0\}}$, we have

$$H(t,\alpha) = \left(e^{-\lambda t} + \lambda te^{-\lambda t}\right)\frac{\sin(\alpha tv)}{\alpha tv}$$

$$+\frac{\lambda^2\varphi_\beta(\alpha)}{\alpha v}\int_0^t e^{-\lambda u}\sin(\alpha uv)H(t-u,\alpha)\,du.$$

By denoting $h(t,\alpha) = H(t,\alpha)\,e^{\lambda t}$, we obtain

$$h(t,\alpha) = (1 + \lambda t)\frac{\sin(\alpha tv)}{\alpha tv}$$

$$+\frac{\lambda^2\varphi_\beta(\alpha)}{\alpha v}\int_0^t \sin(\alpha(t-u)v)h(u,\alpha)\,du. \qquad [1.79]$$

By differentiating equation [1.79] twice, we obtain

$$\frac{\partial^2}{\partial t^2}h(t,\alpha) + \left(\alpha^2v^2 - \lambda^2\varphi_\beta(\alpha)\right)h(t,\alpha)$$

$$= 2\frac{\sin(\alpha tv) - \alpha tv\cos(\alpha tv)}{\alpha t^3 v}, \qquad [1.80]$$

with initial conditions $h(0,\alpha) = 1$, $\frac{\partial}{\partial t} h(0,\alpha) = \lambda$, which can be obtained directly from equation [1.79].

After solving equation [1.80] with the initial conditions, we get

$$H(t,\alpha) = e^{-\lambda t} h(t,\alpha) = \frac{\sin(\alpha t v)}{\alpha t v} e^{-\lambda t} + \frac{\sin(At)}{A} \lambda e^{-\lambda t}$$

$$+ \left[\ln \frac{\alpha v - A}{\alpha v + A} + (\mathrm{Si}(t(A - \alpha v)) - \mathrm{Si}(t(A + \alpha v))) \sin(At) \right.$$

$$\left. + (\mathrm{Ci}(t(A - \alpha v)) - \mathrm{Ci}(t(A + \alpha v))) \cos(At) \right] \frac{\lambda^2 \varphi_\beta(\alpha)}{2 A \alpha v} e^{-\lambda t},$$

where $A = \sqrt{\alpha^2 v^2 - \lambda^2 \varphi_\beta(\alpha)}$, $\mathrm{Si}(x) = \int_0^x \frac{\sin(t)}{t} dt$, $\mathrm{Ci}(x) = \gamma + \ln(x) + \int_0^x \frac{\cos(t)-1}{t} dt$, and γ is the Euler–Mascheroni constant.

1.5.2. *Conclusions and final remarks*

The core of our analysis is based on integral equations for the characteristic function of the movement of a particle. For the Erlang case used in this work, the proof of the integral equation was carried out in two different ways. First, in Pogorui and Rodríguez-Dagnino (2011), the proof was carried out in a detailed manner and it reduces to an exponential or Markov case, which is well accepted in other published papers. Second, in Pogorui and Rodríguez-Dagnino (2013), the proof was carried out on the basis of renewal theory and corresponds to the semi-Markov case.

Moreover, our results obtained for 2D do not contradict known results stating that a continuous part of the density is a solution of a two-dimensional telegraph equation with a zero right-hand side. However, we must remark that this statement is not the complete view of this problem. For instance, for $|x| < vt$, the right-hand side is also zero. At $|x| = vt$, the continuous part of the generalized pdf has singularity (it goes to infinity as $|x| \to vt$), and it has the so-called explosion effect for just the continuous part of the density. In addition, in our approach we used the integral equation for the characteristic function of motion on the plane for the Markov case, and we formulated the characteristic function for the motion density as the sum of the characteristic function of the singular part and the continuous part, which is barely considered in other works. Some recent results in this area are associated with the Dirichlet distribution (see De Gregorio and Orsingher 2012). However, it is not clear that by considering the Dirichlet distribution, the three-dimensional telegraph equation will appear as natural as it does for the Erlang case, and as a consequence it is not clear that our explanation of the physical process related to three-dimensional telegraph equation can be obtained from the Dirichlet distribution, as it has been done in our work. The physical relevance of our analysis can be appreciated after looking at the physical models in Tautz and Lerche (2016) and Morse and Feshbach (1953). We have to mention that the incomplete distribution density for the position of such a process described by the three-dimensional telegraph equation, was recently obtained in Tautz and Lerche (2016).

Furthermore, we are including plots of the main results to clarify some of the theoretical issues stated in this work. We are also including the five-dimensional case in our analysis, which may have some theoretical interest.