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Bolotin's Dynamic Edge Effect Method Revisited (Review)

A comprehensive review of Bolotin's edge effect method is presented. This chapter begins with a toy problem and is concluded by nonlinear considerations that have not been developed by Bolotin himself. Various generalizations and modifications of the method are described, along with a variety of solved problems for which a wide list of references is provided. Attempts are also made to frame the method among the known methods for finding rapidly oscillating solutions.

1.1. Introduction

Professor Isaac E. Elishakoff was a doctoral student of the world-renowned scientist V.V. Bolotin (March 29, 1926 to May 28, 2008) (Bolotin 2006). The first research works of I. Elishakoff and his PhD thesis were devoted to the application and development of the dynamic edge effect (EE) method proposed by V.V. Bolotin. After moving from the Soviet Union to the Western world, Prof. Elishakoff made great efforts to popularize the dynamic EE method in the Western scientific community (Elishakoff 1974, 1976; Elishakoff and Wiener 1976).

Therefore, the appearance of a review of papers related to Bolotin's method in the volume devoted to Prof. Elishakoff's 75th birthday seems quite reasonable. Moreover, the previous comprehensive reviews of the subject were published in 1976 (Elishakoff 1976) and 1984 (Bolotin 1984).

In the early 1960s, V.V. Bolotin put forward an asymptotic method for studying natural oscillations of plates and shells, which used the inverse of the dimensionless

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vibration frequency as a small parameter (Bolotin 1960a, 1960b). In a more general formulation, it is a method for solving self-adjoint eigenvalue problems defined in a rectangular domain, called the boundary value problems with quasi-separable variables, according to Bolotin's terminology. For this reason, the method is referred to as Bolotin's method or the dynamic edge effect method (DEEM). And despite the fact that 60 years have passed since the method creation, it is still relevant. The purpose of this review is describing various generalizations and modifications of DEEM, the problems solved with the use of this method and also trying to determine the place of DEEM among the known methods for finding rapidly oscillating solutions. Thus, we demonstrate that DEEM can be broadly applied for solving modern problems.

1.2. Toy problem: natural beam oscillations

Demonstrate the main idea of DEEM on a spatially 1D problem, which can be reduced to a transcendental equation and solved numerically with any degree of accuracy (Weaver *et al*. 1990). Consider the natural oscillations of a beam of length L, described by the following PDE:

$$\frac{\partial^4 w}{\partial x^4} + a^2 \frac{\partial^2 w}{\partial t^2} = 0, \quad a^2 = \frac{\rho F}{EI}. \qquad [1.1]$$

Here, w is the normal displacement, E is the Young modulus, F is the cross-sectional area of the beam, I is the axial inertia moment of the beam cross-section, and ρ is the density of the beam material.

Let us compare two versions of boundary conditions:

$$w = 0, \quad \frac{\partial^2 w}{\partial x^2} = 0 \quad \text{at } x = 0, L, \qquad [1.2]$$

$$w = 0, \quad \frac{\partial w}{\partial x} = 0 \quad \text{at } x = 0, L. \qquad [1.3]$$

We use the following ansatz:

$$w(x, t) = W(x)\exp(i\omega t),$$

where ω is the eigenfrequency and $W(x)$ is the eigenfunction.

The equation for eigenfunction $W(x)$ has the form

$$\frac{d^4W}{dx^4} - a^2\omega^2 W = 0\,. \qquad [1.4]$$

The solution of the eigenvalue problem [1.4], [1.2] is given by

$$W = \sin \frac{m\pi}{L}x \quad , \quad m = 1, 2, \ldots ; \qquad [1.5]$$

$$\omega_m = \frac{1}{a}\left(\frac{m\pi}{L}\right)^2 . \qquad [1.6]$$

The eigenvalue problem [1.4], [1.3] does not allow separation of variables. However, if the eigenfunction oscillates rapidly along x (i.e. a rather high form of oscillations is considered), then we can hope that in this case a solution of the form [1.5] is also valid for the inner domain sufficiently distant from the boundaries (Figure 1.1). Such an expression does not satisfy the boundary conditions. However, if the solution that compensates the residuals at the boundary conditions and decays rapidly, then the approximate expressions for the eigenfunctions and eigenfrequencies can be obtained.

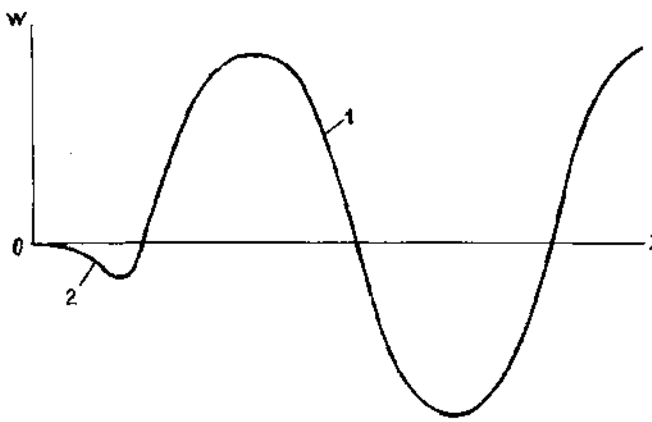

Figure 1.1. *Curve 1 corresponds to the rapidly oscillating solution in the inner domain, and curve 2 corresponds to the sum of DEE and the rapidly oscillating solution*

Let us suppose the solution of equation [1.4] in the form

$$W_0 = \sin\frac{\pi(x - x_0)}{\lambda}. \quad [1.7]$$

The oscillation frequency ω is

$$\omega = \frac{1}{a}\left(\frac{\pi}{\lambda}\right)^2. \quad [1.8]$$

Factorization of ODE [1.4] is (Vakhromeev and Kornev 1972)

$$\left(\frac{d^2}{dx^2} + a\omega\right)\left(\frac{d^2}{dx^2} - a\omega\right)W = 0. \quad [1.9]$$

The general solution of ODE [1.9] is given by

$$W = W_0 + W_1 + W_2,$$

where functions W_0 and $W_{1,2}$ are the general solutions of the following equations:

$$\frac{d^2W_0}{dx^2} + a\omega W_0 = 0, \quad [1.10]$$

$$\frac{d^2W_{1,2}}{dx^2} - a\omega W_{1,2} = 0. \quad [1.11]$$

For large frequencies $(a\omega >> 1)$, the following estimates for the derivatives of functions W_0 and $W_{1,2}$ are obtained:

$$\frac{dW_0}{dx} \sim a\omega W_0, \quad \frac{dW_{1,2}}{dx} \sim a\omega W_{1,2}.$$

The behavior of these solution components is different: W_1 is the rapidly oscillating function, and W_2 is the sum of exponentials rapidly decreasing from the edges of the beam.

Therefore, the situation under consideration is fundamentally different from the case when the characteristic equation has small and large modulo roots, which is typical for boundary layer theory. In our case, we are talking about the separation of solutions, one of which oscillates at the same rate as the EE decays (i.e. the characteristic equation has large real and imaginary roots with moduli of the same order). The self-adjoint eigenvalue problem [1.1], [1.2] can be referred to as the boundary value problem with quasi-separable variables (Bolotin 1960a, 1960b, 1961a, 1961b, 1961c; Bolotin *et al*. 1950, 1961).

We proceed to the construction of the EE described by equation [1.11]. Taking into account the expression for the natural frequency [1.8], we obtain the following relations for EEs localized in the vicinity of the edges $x=0$ and $x=L$, respectively:

$$W_1 = C_1 \exp\left(-\pi\lambda^{-1}x\right), \qquad [1.12]$$

$$W_2 = C_2 \exp\left[-\pi\lambda^{-1}\left(x-L\right)\right]. \qquad [1.13]$$

We assume that the beam is so long such that EEs do not affect each other, i.e. $\exp\left(-\pi\lambda^{-1}L\right) << 1$.

Now, it remains to find the quantities x_0, λ and constants C_1, C_2 from the boundary conditions to determine the eigenmodes and eigenfrequencies:

$$W_0 + W_1 = 0, \quad \frac{d}{dx}\left(W_0 + W_1\right) = 0 \quad \text{at } x = 0, \qquad [1.14]$$

$$W_0 + W_2 = 0, \quad \frac{d}{dx}\left(W_0 + W_2\right) = 0 \quad \text{at } x = L. \qquad [1.15]$$

Note that in original works by Bolotin, a slightly different matching procedure is used. Namely, the matching conditions [1.14], [1.15] are not given at the domain boundaries. They are set at some points, which are then determined from the solution of the system of transcendental equations along with other unknown constants.

Substituting expressions [1.7] and [1.12] into conditions [1.14], and expressions [1.7] and [1.13] into [1.15], we obtain

$$C_1 - \sin\frac{\pi x_0}{\lambda} = 0, \quad C_1 - \cos\frac{\pi x_0}{\lambda} = 0, \qquad [1.16]$$

$$C_2 + \sin\left(\pi \frac{L - x_0}{\lambda}\right) = 0, \quad C_2 + \cos\left(\pi \frac{L - x_0}{\lambda}\right) = 0\,. \tag{1.17}$$

For λ and x_0, we have the following expressions:

$$\lambda = \frac{L}{m + 0.5}, \quad m = 1, 2, \ldots; \quad x_0 = \lambda(0.25 + n), \quad n = 1, 2, \ldots$$

Finally, the formula for the natural oscillation frequencies of the clamped beam is written as follows:

$$\omega_m = \pi^2 \frac{(m + 0.5)^2}{aL^2}, \quad m = 1, 2, \ldots \tag{1.18}$$

The same formula follows from the analysis of the transcendental equation (Weaver *et al*. 1990). Its difference from the numerical solution is less than 1% even for the first natural oscillation frequency.

Using DEEM to solve various problems proved its high accuracy even when calculating the first eigenfrequencies (Bolotin 1960a, 1960b, 1961a, 1961b, 1961c, 1961d; Gavrilov 1961a, 1961b; Bolotin 1963, 1970; Elishakoff and Wiener 1976; Elishakoff and Steinberg 1979; Emmerling 1979; Bolotin 1984; Elishakoff *et al*. 1993, 1994). It confirms “Crighton’s principle” (Crighton 1994):

> All experience suggests that asymptotic solutions are useful numerically far beyond their nominal range of validity, and can often be used directly, at least at a preliminary product design stage, for example, saving the need for accurate computation until the final design stage where many variables have been restricted to narrow ranges.

The problem of natural oscillations was considered above. In the case of forced oscillations, the response is given as a series expansion in terms of normal modes of natural oscillations. It is important that one can use only the rapidly oscillating part of solution W_0, neglecting the dynamic edge effect (DEE) (Bolotin 1961b, 1961c, 1970, 1984).

1.3. Linear problems solved

DEEM in its original form can be effectively used for calculating natural, free and forced oscillations of rectangular plates, shells of revolution and shallow shells with a rectangular base.

In particular, DEEM is widely used for the analysis of rectangular isotropic (Bolotin *et al*. 1958; Bolotin 1961d, 1963; Kudryavtsev 1964; Dickinson and Warburton 1967; Bolotin 1970; King and Lin 1974; Elishakoff 1976; Emmerling 1979; Bolotin 1984; Elishakoff *et al*. 1994) and orthotropic (Dickinson 1971; Elishakoff 1974; Kaza and Ramaiah 1978; Vijaykumar and Ramaiah 1978) plates with various types of boundary conditions, plates on elastic foundations (Gibigaye *et al*. 2016), laminated and stiffened plates and panels (Lin and King 1974; Ueng and Nickels 1978; Meilani 2012, 2015). Oscillations and buckling of stressed plates were studied in Dickinson (1971, 1975), and box structures were studied in Dickinson (1975b). The Timoshenko theory of beams, and Mindlin, Reissner and Ambartsumyan theories of plates were also used in some works (Kudryavtsev 1960; Moskalenko 1961; Nelson 1978).

In the paper by Dickinson and Warburton (1967), DEEM was applied to consider free flexural vibrations of systems built up from rectangular plates.

Many papers are devoted to multi-span plates (Moskalenko and Chen 1965; Moskalenko 1968, 1969; Elishakoff and Steinberg 1979; Elishakoff *et al*. 1993).

Dynamics of cylindrical, conical and shallow spherical shells were analyzed in Bolotin (1960b, 1961c, 1984), Gavrilov (1961a), Zhinzher (1975) and Elishakoff and Wiener (1976).

One of the problems of the DEEM application is the degeneracy of DEE when decaying solutions cannot be constructed for some wave numbers (Bolotin 1984). The resolution of this problem was proposed in Elishakoff (1974) and Elishakoff and Wiener (1976). The solution of the original problem is represented as a sum of solutions of two subproblems. Each of these solutions satisfy the boundary conditions at two opposite boundaries only. The matching conditions described in Bolotin's original papers (Bolotin 1960a, 1960b, 1961a, 1961b, 1961c) are not used, and it gives us the possibility to avoid difficulties caused by the degeneracy of DEEM.

Thereby, we note the following point. Asymptotic methods can be used in two versions (Andrianov *et al*. 2014). From the very beginning, a small parameter can be introduced into the PDEs or ODEs and then asymptotic fractional analysis (Kline 1965) can be used. However, we can use variational approaches (Rayleigh–Ritz, Bubnov–Galerkin, Kantorovich, Trefftz, etc.) to solve the original problem and reduce it to the infinite systems of coupled ODEs or algebraic equations.

Then, a small parameter, caused by the physical nature of the problem or an artificial (homotopy) one, can be introduced into the infinite system to split it into simplified subsystems. The second approach makes it possible to avoid the degeneracy of DEE.

Other generalizations of DEEM are described in the following sections.

1.4. Generalization for the nonlinear case

To describe the generalization of DEEM to the nonlinear case, we use the nonlinear Kirchhoff beam equation (Kauderer 1958):

$$EI\frac{\partial^4 w}{\partial x^4}-\frac{EF}{2L}\left(\int_0^L\left(\frac{\partial w}{\partial x}\right)^2 dx\right)\frac{\partial^2 w}{\partial x^2}+\rho F\frac{\partial^2 w}{\partial t^2}=0. \qquad [1.19]$$

Let the beam be elastically supported:

$$w=0,\quad \frac{\partial^2 w}{\partial x^2}-c^*\frac{\partial w}{\partial x}=0 \quad \text{at } x=0,L, \qquad [1.20]$$

where $c^*=c/EI$, c is the coefficient characterizing elastic support.

We search a generating solution in the form

$$w_0=A\sin\frac{\pi\left(x-x_0\right)}{\lambda}\xi\left(t\right). \qquad [1.21]$$

Substituting ansatz [1.21] into PDE [1.19], we obtain an ODE for determining the time function $\xi\left(t\right)$:

$$\frac{d^2\xi}{dt^2}+\omega^2\left(1+\gamma\xi^2\right)\xi=0, \qquad [1.22]$$

where

$$\omega^2=\frac{EI}{\rho}\left(\frac{\pi}{\lambda}\right)^2,\quad \gamma=0.25\left(1+\lambda_1\right)\left(\frac{A}{r}\right)^2,\quad r=\sqrt{I/F},$$
$$\lambda_1=\frac{\lambda}{2\pi L}\left(\sin\frac{2\pi(L-x_0)}{\lambda}+\sin\frac{2\pi x_0}{\lambda}\right).$$

ODE [1.22] with initial conditions

$$\xi = 1, \quad \frac{d\xi}{dt} = 0 \text{ at } t = 0 \qquad [1.23]$$

has the solution

$$\xi(t) = \mathrm{cn}(\sigma t, k), \quad \sigma = \omega\sqrt{1+\gamma}, \qquad [1.24]$$

where $\mathrm{cn}(\sigma t, k)$ is the Jacobi cosine elliptic functions with period $T = 4K$, $K = \int\limits_0^{\pi/2} \left(1 - k^2 \sin^2\varphi\right)^{-1/2} d\varphi$ is the complete elliptic integral of the first kind with modulus $k = \sqrt{0.5\gamma/(1+\gamma)}$ (Abramowitz and Stegun 1965).

The solution to the problem far from the edges is

$$w_0 = W_0(x)\,\mathrm{cn}(\sigma t, k), \qquad [1.25]$$

where $W_0(x) = A\sin\dfrac{\pi(x - x_0)}{\lambda}$.

Solution [1.25] satisfies the original equation [1.19], but does not satisfy the boundary conditions [1.20]. To construct the states localized near the edges, we represent the solution of the original problem in the form

$$w = w_0 + w_{ee}. \qquad [1.26]$$

Substituting ansatz [1.26] into ODE [1.19], we obtain

$$\begin{aligned} &\frac{\partial^4}{\partial x^4}(w_0 + w_{ee}) - 0.5\left(r^2 L\right)^{-1} \frac{\partial^2}{\partial x^2}(w_0 + w_{ee}) \int\limits_0^L \left(\frac{\partial w_0}{\partial x} + \frac{\partial w_{ee}}{\partial x}\right)^2 dx \\ &+ \frac{\rho}{EI}\frac{\partial^2}{\partial t^2}(w_0 + w_{ee}) = 0. \end{aligned} \qquad [1.27]$$

In contrast to the previously considered linear case, the equations for functions w_0 and w_{ee} are coupled due to the nonlinearity of the problem. At the same time, the solution in the inner domain and EEs differ energetically since the EEs are localized in a small vicinity of the beam ends (Andrianov *et al.* 1979; Awrejcewicz *et al.* 1998;

Andrianov *et al.* 2004, 2014). Let us estimate the orders of the integrand terms in equation [1.27] with respect to $L/\lambda >> 1$:

$$\int_0^L \left(\frac{\partial w_0}{\partial x}\right)^2 dx \sim \left(\frac{L}{\lambda}\right)^2, \quad \int_0^L \frac{\partial w_0}{\partial x}\frac{\partial w_{ee}}{\partial x} dx \sim \left(\frac{L}{\lambda}\right), \quad \int_0^L \left(\frac{\partial w_{ee}}{\partial x}\right)^2 dx \sim 1\,. \qquad [1.28]$$

Restricting ourselves to the term of order $(\pi/\lambda)^2 >> 1$ in equation [1.27], we reduce it to the form:

$$\begin{aligned}&\frac{\partial^4 w_0}{\partial x^4} - 0.5\left(r^2 L\right)^{-1}\frac{\partial^2 w_0}{\partial x^2}\int_0^L \left(\frac{\partial w_0}{\partial x}\right)^2 dx + \frac{\rho}{EI}\frac{\partial^2 w_0}{\partial t^2} + \frac{\partial^4 w_{ee}}{\partial x^4}\\ &- 0.5\left(r^2 L\right)^{-1}\frac{\partial^2 w_{ee}}{\partial x^2}\int_0^L \left(\frac{\partial w_0}{\partial x}\right)^2 dx + \frac{\rho}{EI}\frac{\partial^2 w_{ee}}{\partial t^2} = 0.\end{aligned} \qquad [1.29]$$

Substituting function w_0 into equation [1.29], we obtain a PDE for function w_{ee}:

$$\frac{\partial^4 w_{ee}}{\partial x^4} - B\,\mathrm{cn}^2\left(\sigma t, k\right)\frac{\partial^2 w_{ee}}{\partial x^2} + \frac{\rho}{EI}\frac{\partial^2 w_{ee}}{\partial t^2} = 0\,, \qquad [1.30]$$

where $B = \gamma\left(\frac{\pi}{\lambda}\right)^2$.

It is important that PDE [1.30] is linear. The spatial and time variables are not separated exactly; therefore, we apply the Kantorovich variational method (Kantorovich and Krylov 1958) to solve equation [1.30], presenting w_{ee} in the form

$$w_{ee}\left(x,t\right) \cong W_{ee}\left(x\right)\mathrm{cn}\left(\sigma t, k\right). \qquad [1.31]$$

On substituting ansatz [1.31] into PDE [1.30] and applying the Kantorovich method (Kantorovich and Krylov 1958), the following ODE is obtained:

$$\frac{d^4 W_{ee}}{dx^4} - B_1\frac{d^2 W_{ee}}{dx^2} - \left(\frac{\pi}{\lambda}\right)^2\left[\left(\frac{\pi}{\lambda}\right)^2 + B_1\right]W_{ee} = 0\,, \qquad [1.32]$$

with

$$B_1 = A\left(\frac{2k^2-1}{2k^2}+\frac{\sqrt{1-k^2}}{2k\arcsin k}\right). \qquad [1.33]$$

Hereinafter, we use the principal value of the $\arcsin(...)$ function.

Among the four roots of the characteristic equation for ODE [1.32], two purely imaginary ones correspond to the generating solution W_0. To construct DEE, we should use real roots of the characteristic equation. Then, the DEE solution is

$$W_{ee}(x) = C_1 \exp\left[\left(-\sqrt{\left(\frac{\pi}{\lambda}\right)^2 + B_1}\right)x\right] + C_2 \exp\left[\left(\sqrt{\left(\frac{\pi}{\lambda}\right)^2 + B_1}\right)x\right]. \qquad [1.34]$$

Let us construct DEE near the edge $x = 0$. For a sufficiently long beam, we can suppose

$$C_2 = 0. \qquad [1.35]$$

Then, at $x = 0$, we have from the boundary conditions

$$W_0 + W_{ee} = 0, \quad \frac{d^2W_0}{dx^2}+\frac{d^2W_{ee}}{dx^2} = c^*\left(\frac{dW_0}{dx}+\frac{dW_{ee}}{dx}\right). \qquad [1.36]$$

Using expressions [1.34]–[1.36], we obtain

$$C_1 = A\sin\frac{\pi x_0}{\lambda}, \qquad [1.37]$$

$$x_0 = \frac{\lambda}{\pi}\arctan\frac{\pi}{\lambda\left[\left(2(\pi/\lambda)^2 + B_1\right)/c^* + \sqrt{(\pi/\lambda)^2 + B_1}\right]}. \qquad [1.38]$$

Note that when $c^* \to 0$ and $c^* \to \infty$, formulas [1.34]–[1.38] yield solutions for simply supported and clamped ends of the beam, respectively.

Similarly, we can construct DEE localized at the edge $x = L$.

The modes of natural nonlinear oscillations of the beam can be divided into groups according to the types of symmetry. For the modes that are symmetric relative to the point $x = L/2$, from the condition

$$\frac{dW_0}{dx} = 0 \quad \text{at } x = L/2,$$

we obtain

$$L - 2x_0 = (2m+1)\pi, \quad m = 1,2,... \qquad [1.39]$$

For antisymmetric modes, from the condition

$$W_0 = 0 \quad \text{at } x = L/2,$$

we have

$$L - 2x_0 = 2n\pi, \quad n = 1,2,... \qquad [1.40]$$

Equations [1.39] and [1.40] can be reduced to the following form:

$$L - 2x_0 = m\pi, \quad m = 1,2,..., \qquad [1.41]$$

in which even values of *m* correspond to antisymmetric modes, and odd values of *m* to symmetric modes relative to the point $x = L/2$.

Thus, the system of equations [1.37], [1.38] and [1.41] can be applied to determine the constants λ and x_0.

The described technique was used to study nonlinear oscillations of isotropic (Andrianov *et al*. 1979; Zhinzher and Denisov 1983; Awrejcewicz *et al*. 1998; Andrianov *et al*. 2004) and orthotropic (Zhinzher and Khromatov 1984) plates, circular cylindrical and shallow shells (Zhinzher and Denisov 1983; Andrianov and Kholod 1985; Zhinzher and Khromatov 1990; Andrianov and Kholod 1993a, 1993b, 1995).

1.5. DEEM and variational approaches

DEEM, designed to calculate high eigenfrequencies, also gives enough accurate results for lower vibration modes at kinematic boundary conditions. For static conditions, the accuracy of determining the lowest natural frequencies decreases.

Attempts to apply the method to stability problems have shown that the error of determining the buckling load is quite high.

One of the promising ways to improve the DEEM accuracy is its combination with variational approaches. The first works in this direction were the papers (Vijaykumar and Ramaiah 1978a, 1978b), where the Rayleigh–Ritz method (RRM) was applied and the asymptotic expressions for natural modes were used as basis functions (the Rayleigh–Ritz–Bolotin method, RRBM). According to the comparative estimates, this modification grants a much more accurate determination of natural frequencies (see also Krizhevskii 1988, 1989).

As an example, we use RRBM for natural oscillations of a square plate $\left(0 \le x, y \le a\right)$ with free contour. The governing equation is

$$D\nabla^4 w + \rho h \frac{\partial^2 w}{\partial t^2} = 0 . \qquad [1.42]$$

Here, $D = \dfrac{Eh^3}{12\left(1-\nu^2\right)}$, h is the plate thickness and ν is Poisson's ratio.

Boundary conditions have the form

$$w_{xx} + \nu w_{yy} = 0, \quad w_{xxx} - 2(1-\nu) w_{xyy} = 0 \quad \text{at} \quad x = 0, x = a, \qquad [1.43]$$

$$w_{yy} + \nu w_{xx} = 0, \quad w_{yyy} - 2(1-\nu) w_{yxx} = 0 \quad \text{at} \quad y = 0, y = a. \qquad [1.44]$$

According to the principle of virtual work,

$$U + V = 0, \qquad [1.45]$$

where U and V are, respectively, the potential and kinetic energy, defined as follows:

$$U = \frac{D}{2}\int_0^a\int_0^a \left(w_{xx}^2 + w_{yy}^2 + 2\nu w_{xx}^2 w_{yy}^2 + 2(1-\nu) w_{xy}^2 \right) dxdy , \qquad [1.46]$$

$$V = \frac{\rho h}{2}\int_0^a\int_0^a w_t^2 dxdy . \qquad [1.47]$$

Using the ansatz

$$w(x,y,t)=W(x,y)\exp(i\omega t),$$

we obtain from equations [1.45]–[1.47]

$$\lambda^2=\omega^2 a^4\frac{\rho h}{D}$$
$$=a^4\left[\int_0^a\int_0^a\left(W_{xx}^2+W_{yy}^2+2\nu W_{xx}^2W_{yy}^2+2(1-\nu)W_{xy}^2\right)dxdy\right]\left[\int_0^a\int_0^a W^2dxdy\right]^{-1}. \qquad [1.48]$$

The expression for the eigenfunction $W(x,y)$ obtained using DEEM has the form

$$W(x,y)=W_0(x,y)+W_1(x)\sin(\beta_2 y+l_2)+W_2(y)\sin(\beta_1 x+l_1), \qquad [1.49]$$

where

$$W_0(x,y)=\sin(\beta_1 x+l_1)\sin(\beta_2 y+l_2), \qquad [1.50]$$

$$W_1(x)=C_{11}\exp\left[\alpha_1(x-a)\right]+C_{12}\exp(-\alpha_1 x), \qquad [1.51]$$

$$W_2(y)=C_{21}\exp\left[\alpha_2(y-a)\right]+C_{22}\exp(-\alpha_2 y). \qquad [1.52]$$

On satisfying the boundary conditions [1.43] and [1.44] to determine the wave numbers, we obtain a system of transcendental equations

$$\beta_i a=2l_i+m\pi,\quad i=1,2;\quad m=0,1,2,..., \qquad [1.53]$$

where $l_i=\arctan\left[\frac{\beta_i}{\alpha_i}\left(\frac{\beta_i^2+(2-\nu)\beta_k^2}{\beta_i^2+\nu\beta_k^2}\right)^2\right],\quad \alpha_i=\left(\beta_i^2+2\beta_k^2\right)^{1/2},\quad i,k=1,2;\quad i\neq k.$

For constants C_{ij}, we obtain

$$C_{i1}=\frac{\alpha_i^2\sin l_i}{\alpha_i^2-\nu\beta_k^2},\quad C_{i2}=\frac{\alpha_i^2\sin(\beta_i a+l_i)}{\alpha_i^2-\nu\beta_k^2},\quad i,k=1,2;\quad i\neq k. \qquad [1.54]$$

Using the DEEM solution [1.49]–[1.54], we can determine the desired frequency from expression [1.48].

The square of dimensionless frequencies λ for $\nu = 0.225$ and various *m*, obtained by RRM (Gontkevich 1964), RRBM and DEEM are shown in Table 1.1. Wave forms along a cylindrical surface are not considered since in this case an exact solution can be obtained. The numbers corresponding to the indicated modes of vibration are omitted in Table 1.1.

m	λ, RRM (Gontkevich 1964)	λ, RRBM	Discrepancy with Gontkevich (1964), %	λ, DEEM	Discrepancy with Gontkevich (1964), %
1	14.10	14.48	2.7	12.41	13.6
3	35.96	36.68	2.0	34.60	3.9
5	65.24	66.33	1.7	63.44	2.8
6	74.45	75.28	1.1	73.59	2.5
7	109.30	109.10	0.2	106.30	2.8

Table 1.1. *Comparison of frequencies obtained using various approximation methods*

RRBM gives more accurate results than DEEM for the first natural frequencies. When *m* increases, both solutions asymptotically approach the exact one, namely, from above in the case of applying RRMB and from below in the case of using DEEM.

RRBM can also be used for stability problems of plates and shells with complicated boundary conditions. This method was applied to plates of complicated form (skew, circle, sector (Andrianov and Krizhevskiy 1988, 1989, 1991)) and structures (Andrianov and Krizhevskiy 1987, 1993).

An interesting modification of DEEM for determining natural frequencies and mode shapes of isotropic and orthotropic rectangular plates with various types of boundary conditions was given in Pevzner *et al.* (2000). This approach does not postulate the formula for the eigenfrequency, but rather it is based on the condition that the frequency obtained from the governing differential equations has to be equal to that given by the Rayleigh method. The paper by Pevzner *et al.* (2000) claims that this modification is more straightforward and computationally faster, and the mode shapes derived are valid on a larger part of the plate.

1.6. Quasi-separation of variables and normal modes of nonlinear oscillations of continuous systems

Bolotin did not give an exact definition of the concept of quasi-separation of variables. Intuitively, this means that the difference between solutions of boundary value problems with separated and quasi-separated variables is sufficient only near the boundaries. In other words, the energy accumulated in the EE zone is small compared to that accumulated in the inner zone. This allows us to not take into account DEE when expanding the natural mode of vibration during the calculation of forced oscillations. Bolotin's conception of quasi-separation of variables (Bolotin 1961c, 1984) can be used in the theory of normal modes of nonlinear oscillations for continuous systems.

When studying linear oscillatory systems with a finite number of DOF, normal oscillation modes play a key role. Kauderer (1958) indicated the existence of solutions in a nonlinear system, which were, in a sense, similar to the normal modes of linear systems. He called these solutions the principal ones and showed how to construct their trajectories in the configuration space. Rosenberg (1962) defined normal vibrations of nonlinear systems with a finite number of DOF, formulated the problem in the configuration space and found several classes of nonlinear systems that allowed solutions with straight-line trajectories (for details, see Mikhlin and Avramov 2011; Avramov and Mikhlin 2013). Generalizations of this concept to continuous systems are related to the exact separation of spatial and time variables (Wah 1964; Avramov and Mikhlin 2013), i.e. to the possibility of representing the sought solutions in the form

$$U(\mathbf{x},t) = X(\mathbf{x})T(t).$$

The restriction of this approach is clear since the separation of variables only works for some boundary conditions. Based on Bolotin's conception of the quasi-separation of variables, we can propose the following definition (Andrianov 2008): a function $U(\mathbf{x},t)$ is called the normal mode of nonlinear oscillations of a continuous system if

$$U(\mathbf{x},t) = X(\mathbf{x})T(t) + Y(\mathbf{x},t),$$

where $T(t)$ and $Y(\mathbf{x},t)$ are the periodic and quasi-periodic functions in time, respectively; and function $Y(\mathbf{x},t)$ is small compared to function $X(\mathbf{x})T(t)$ in some energy norm. The last condition can be verified both *a priori* and *a posteriori*.

1.7. Short-wave (high-frequency) asymptotics. Possible generalizations of DEEM

DEEM can be considered a special case of short-wave (high-frequency) asymptotics. The corresponding algorithms are known as the method of geometric optics, the ray method, the semi-classical approximation, the WKBJ (Wentzel–Kramers–Brillouin–Jeffreys) approach, the method of edge waves, the Keller–Rubinow method, etc. (Keller and Rubinow 1960; Maslov and Fedoryuk 1981; Babich *et al.* 1985; Babich and Buldyrev 1991; Chen *et al.* 1991, 1992; Chen and Zhou 1993; Bauer *et al.* 2015). They were independently developed in various fields of mathematics, mechanics and physics.

Note an interesting fact: Ufimtsev proposed the asymptotic method of edge waves (Ufimtsev 1962, 2003, 2014). According to Rich and Janos (1994) and Mitzner (2003), this theory played a critical role in the design of American stealth aircrafts F-117 and B-2. It is a fascinating example of the direct application of asymptotic formulas in engineering practice!

The key to short-wave asymptotics is the ansatz $\varphi(x)\exp\left(i\varepsilon^{-1}S(x)\right)$, in the nonlinear case – $\varphi(x)\Phi\left(i\varepsilon^{-1}S(x)\right)$, where $i=\sqrt{-1},\ 0<\varepsilon<<1.$ In the DEE method, $\varepsilon=1/\lambda$, where λ is the nondimensional frequency. As a result, the construction of the asymptotics can be reduced to solving the eikonal and transport equations (Birger and Panovko 1968). However, short-wave (high-frequency) asymptotics can be treated as a multiscale approach (Maslov and Fedoryuk 1981).

Let us show the generalization of DEEM based on the WKBJ approach using the toy problem – natural oscillation of a beam of variable cross-section (Bauer *et al.* 2015):

$$\frac{d^2}{dx^2}\left[EI\varphi_1(x)\frac{d^2w}{dx^2}\right]-\rho\varphi_2(x)F\omega^2w=0\,, \qquad [1.55]$$

with clamped edges.

In dimensionless variables, we obtain

$$\varepsilon^4\frac{d^2}{d\xi^2}\left[\varphi_1\frac{d^2w}{d\xi^2}\right]-\varphi_2w=0\,, \qquad [1.56]$$

$$w=\frac{dw}{d\xi}=0 \quad \text{at } \xi=0,1\,, \qquad [1.57]$$

where $\varepsilon^4 = \dfrac{EI}{\omega^2 \rho F L^4}, \quad \xi = \dfrac{x}{L}$.

Functions φ_1, φ_2 are supposed to be smooth enough to avoid turning points.

The solution to equation [1.56] is sought in the form:

$$w = \exp\left(\varepsilon^{-1} \int_0^{\xi} \psi(\tau) d\tau \right) \left[u_0(\xi) + \varepsilon u_1(\xi) + \varepsilon^2 u_2(\xi) + \ldots \right]. \qquad [1.58]$$

Substituting ansatz [1.58] into equation [1.56] and applying ε-splitting, we obtain a recurrent system of equations:

$$\left(\varphi_1 \psi^4 - \varphi_2 \right) u_0 = 0 , \qquad [1.59]$$

$$4\varphi_1 \psi^3 u_0' + 6\varphi_1 \psi^2 \psi' u_0 + 2\psi^3 \varphi_1' u_0 = 0, \qquad [1.60]$$

The eikonal equation [1.59] has the following solutions:

$$\psi_{1,2} = \pm \left(\frac{\varphi_2}{\varphi_1} \right)^{1/4}, \quad \psi_{1,2} = \pm i \left(\frac{\varphi_2}{\varphi_1} \right)^{1/4}.$$

From the transport equation [1.60], we obtain

$$u_0(\xi) = \frac{1}{\psi^{3/2} \varphi_1^{1/2}}.$$

In the first approximation, the general solution of equation [1.56] can be written as follows:

$$\begin{aligned} w = {} & C_1 \sin\left(\varepsilon^{-1} \int_0^{\xi} \psi(\tau) d\tau \right) u_0(\xi) + C_2 \cos\left(\varepsilon^{-1} \int_0^{\xi} \psi(\tau) d\tau \right) u_0(\xi) \\ & + C_3 \exp\left(-\varepsilon^{-1} \psi(0) \xi \right) u_0(0) + C_4 \exp\left(-\varepsilon^{-1} \psi(1)(1-\xi) \right) u_0(1). \end{aligned} \qquad [1.61]$$

In expression [1.61], function $u_0(\xi)$ is “frozen” at either end of the interval (i.e. we can change $u_0(\xi)$ to $u_0(0)$ or $u_0(\xi)$ to $u_0(1)$) for rapidly decaying components).

Using solution [1.61] and boundary conditions [1.57], we obtain the frequency of oscillations:

$$\omega = \pi^2 (n+0.5)^2 \sqrt{\frac{EI}{\rho F L^4}} \left[\int_0^1 \left(\frac{\varphi_2(x)}{\varphi_1(x)} \right)^{1/4} dx \right]^{-2}, \quad n = 1,2,\ldots \qquad [1.62]$$

Formula [1.62] at $\varphi_1 = \varphi_2 = 1$ coincides with Bolotin's formula [1.18]. Thus, the WKBJ method generalizes the DEE method to the problems with variable coefficients.

As it is mentioned in Chen *et al.* (1991, 1992), the short-wave (high-frequency) asymptotics gives the same results as the DEE approach for domains of simple geometry. At the same time, DEE does not cover the cases of different geometries (circular, elliptical, etc.) or non-self-adjoint problems. The short-wave asymptotics in the form of the Keller–Rubinow approach (Keller and Rubinow 1960) in Chen *et al.* (1991) allows more ready extension to other geometries and is more aptly generalizable to dissipative boundary conditions. In other words, it gives the possibility to overcome degeneration of the DEE case.

1.8. Conclusion: DEEM, highly recommended

The importance and usefulness of a particular calculation method is determined by its wide application when studying practically important systems and phenomena. From this point of view, the importance of DEEM is not in doubt.

From the very beginning, DEEM was originated by Bolotin for the analysis of overhead power lines (Bolotin *et al.* 1958).

DEEM was used to obtain estimates for the density of natural frequencies of shallow shell vibrations (Gavrilov 1961b; Bolotin 1963; Stearn 1970; Zhinzher and Khromatov 1971, 1972a, 1972b; Moskalenko 1972, 1975). This is very important during the study of random vibrations of elastic structures (Bolotin 1966; Birger and Panovko 1968; Moskalenko 1968; Bolotin 1984; Elishakoff *et al.* 1994).

The influence of the magnetic field on the distribution of plate and shell vibration frequencies was studied in Bagdasaryan (1986), Koreshkova and Khromatov (2009), Golubeva *et al.* (2013) and Khromatov and Golubeva (2013).

A lot of research devoted to the problems of aeroelastic stability and supersonic flutter can be mentioned (Zhinzher 1983; Zhinzher and Kadarmetov 1984, 1986; Dubovskikh *et al*. 1996).

We also mention the optimal control problem for continuous systems (Andrianov and Iskra 1991).

DEEM and its generalizations are important particular cases of high-frequency asymptotics. The effectiveness of this method for analyzing the main types of plates and shells used in engineering practices has been proven through experience. The main advantage of DEEM consists of its simplicity and good compatibility with variational approaches.

Naturally, DEEM is not a panacea. For example, when considering a mixed boundary value problem with many points of change in the boundary conditions, the method based on the homotopy parameter (Andrianov *et al*. 2014) seems more suitable.

Nevertheless, in general, we hope that our review has convinced researchers that DEEM and its generalizations occupied an honorable place in the arsenal of analytical methods for solving the dynamics and stability problems of thin-walled structures.

1.9. Acknowledgments

Several years ago, Professor I. Elishakoff pointed out that it would be useful to prepare a new review of Bolotin's method, since his previous review on this topic was written in 1976. We are grateful to him for this idea.

CONFLICTS OF INTEREST.– The authors declare no conflict of interest.

1.10. Appendix

Professor Elishakoff enjoys historiography of science and his historical research is read with great interest. Bubnov or Galerkin? Timoshenko or Ehrenfest? The chicken or the egg?

We also provided a little historical research. Bolotin wrote about the possibility of extending the range of applicability of DEEM to PDEs with variable coefficients (Bolotin *et al.* 1961, Chapter II): "If the coefficients of the equation change slowly, then it is advisable to combine this method with the Wentzel–Brillouin–Kramers method or its related Blumenthal-Shtaerman approach" (*translated by us*). After reading the relevant papers, we were convinced that Blumenthal used the "WKB method", created to solve problems of quantum mechanics in 1926, already in 1912 (Blumenthal 1912, 1914). He created this asymptotic method for solving problems of the shell theory. H. Reissner used Blumenthal's approach in 1912 (Reissner 1912), as did Shtaerman in 1924 (Shtaerman 1924).

With these remarks, we are certainly not going to interfere with the complex priority history of the WKB approach (Wikipedia 2020). We recall Nayfeh's remark concerning one well-known asymptotic method (Nayfeh 2000, p. 232): "The method of multiple scales is so popular that it is being rediscovered just about every 6 months". A lot of phenomena in completely different fields of science are described using similar or directly identical equations. Researchers, as a rule, do not search for methods of their solution in areas far from them, but simply rediscover them. The corresponding methods are naturally given different names in different fields of science. Surprisingly, this does not lead to the "Tower of Babel effect".

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