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Single-Server Queues Embedded at Departure Epochs

Wherever we go and whatever we do, we invariably have to wait for our turn to get service or attention or execute our job. When we look at the total time spent in executing the job, it includes the actual time for executing the job plus the waiting time (the time spent before our job gets into service). The study of such waiting line problems both from the customers' and the service facilities' points of view, so as to optimize the time in the system and to best utilize the resources, falls under the general topic of queueing models.

Queueing theory plays an important role in many areas, notably telecommunications, production and manufacturing engineering, computer science and other service offering facilities. Queueing theory was developed originally in the context of telephone traffic congestion by Danish Mathematician A.K. Erlang in 1909. He noticed that the telephone systems were generally characterized by either Poisson input and exponential holding times and multiple channels or by Poisson input and constant holding times and a single channel. Erlang was responsible for the concept of stationary equilibrium. E.C. Molina, in 1927, published *Applications of the Theory of Probability to Telephone Trunking Problems*. In the early 1930s, F. Pollaczek did some fundamental work on Poisson input, arbitrary output and single/multiple channel problems. At the same time, A. Kolmogorov and A. Khintchine (from Russia), C.D. Crommelin (France) and C. Palm (Sweden) did some additional work. Other additional contributions include the works by D.R. Lindely, L. Takacs, D.G. Kendall, S. Karlin, J.L. McGregor, J.D.C. Little, M.F. Neuts, N.U. Prabhu, U.N. Bhat, T.L. Saaty and R. Syski. Modern queueing theory incorporating the computational aspects is attributed to Neuts through his pioneering work on matrix-analytic methods (MAM) that started in the 1970s.

Thus, it is safe and fair to say that queueing theory was developed in three stages since the identification and the solution of simple practical queueing problems in the early 20th century. In the first stage, queueing system analysis developed along with the growth of the area of stochastic processes into topics such as birth and death processes and their analysis using generating functions, Laplace transforms and integral equation techniques. The second stage started with the introduction of the imbedded Markov chain technique by David Kendall in order to handle non-Markovian arrival and service processes. However, all of the analysis used a complex variable approach (using Laplace transforms and/or generating functions) and explicit results, if there were any, gave formulas for perhaps the steady-state mean queue length or waiting time. Others involved rather complicated formulas for Laplace-Stieltjes or other transforms. The third stage of development of queueing theory research spearheaded by Neuts in the form of MAM is central to the theme of this two-volume book.

It is customary, as can be seen in many books on queueing theory (see Alfa (2016); Bhat (2015); Dudin et al. (2020); Gross and Harris (1998); Kleinrock (1975)), to present queueing models starting with the most simple ones, such as $M/M/1$, $M/M/c$ and $M/M/\infty$, and then move on to more complex ones. It should be pointed out that the book by Alfa (2016) focuses on discrete-time and hence the discrete analog of $M/M/1$ is $Geo/Geo/1$ ("Geo" stands for geometric distribution). However, here we take a different approach. This is mainly to show how one can unify many queueing models and also establish different techniques to solve the same queueing model.

In this second volume, we focus on the study of queues using MAM techniques, which were presented in Volume 1. As mentioned earlier, we freely refer to the materials, where appropriate, from Volume 1. Thus, it is highly encouraged that the readers go through Volume 1 (or at least refer to the appropriate sections when needed) to fully understand the materials presented here. Most of the chapters in this volume depend heavily on the materials seen in Volume 1. However, readers familiar with the basic concepts in MAM may find this volume to be self-contained. Otherwise, readers should ensure that they fully understand the materials and notations from Volume 1 before continuing.

In this chapter, after briefly describing the basics of queueing theory, we focus on the study of queueing models by looking at the departure epochs. It is worth mentioning that Chapter 9 in Volume 1 discusses briefly queueing theory applications. Queueing models play an important role in many industries, notably the service sectors. Some of these services are offered online. Thus, the arrivals to the service sectors (as well as to other industries) are from different sources and hence the assumption that the inter-arrival times are independent is a very novice idea. How do we build correlation into the inter-arrival times but at the same time make the queueing models amenable to practical implementation? This was one of the main concerns that motivated Neuts to lay the foundation for MAM through introducing phase type (PH) distributions (see Chapters 3 and 4 of Volume 1) and batch Markovian arrival processes (see Chapters 5 and 6 of Volume 1).

This chapter relies heavily on the materials presented in Chapters 7 and 8 of Volume 1. Astute readers might want to know the difference between those materials and the ones in this chapter. They are as follows:

1) In those chapters in Volume 1, the (matrix) elements appearing in the transition probability matrix (TPM) or the generator (depending on the context) are assumed to be known. In this chapter, as well as the rest of the book, these (matrix) elements are obtained based on the model under study.

2) Also in Volume 1, there is no prior assumption about the connection, if any, between the matrices $\{A_n\}$ and $\{B_n\}$ (see sections 7.2 and 7.4). However, in this book they are connected and hence result in additional insights as well as simplifications.

A queueing model is characterized by the following key quantities (see Bhat (2015); Gross and Harris (1998)):

1) *Input process (arrival process)*: Here the arrival pattern of customers (or jobs) is described. One specifies the following: single or multiple (group) arrivals and the distribution of the inter-arrival times between successive arrivals.

2) *Service process*: This describes the number of servers (or machines) in the system as well as the distribution of the service times. In the case of more than one server, the servers may be identical or may have different service time distributions. Servers may offer services singly or in groups. In the case of multiple servers, they may be in parallel or in series or each customer has to be attended by more than one server at a time.

3) *Service discipline*: This plays a major role. The mechanism of serving the customers is described here. There are many such schemes and some of these are first come, first served; last come, first served; service in random order; and round robin services. In some applications, customers are grouped into various priorities and within each priority the customers will be served according to some order. There are many priority schemes. Some of which are dynamic or static; preemptive or non-preemptive; or a combination of some of these. In this book, we focus only on a first come, first served (FCFS) basis.

4) *Buffer size (waiting room)*: Arriving customers are queued up before being offered services. The waiting room size could be 0, any positive finite number or infinite.

5) *Performance measures*: Any system, especially a queueing system, should be evaluated by one or more performance measures. These include (1) mean, (2) median, (3) mode, (4) standard deviation and (5) percentiles of various stationary

queue length densities as well as stationary waiting time distributions. Furthermore, the measures like the system idle probability, the server utilization, the mean number of busy servers, the mean busy period and the mean number of customers served during a busy period play an important role in managerial decisions.

6) *Busy Period (BP)*: The BP can be defined in a number of ways. In this book, we define a *partial busy period* to be the duration of the time interval that begins with an arrival of a customer to an empty system and ends with the system becoming empty again at their departure. In queueing literature (see Artalejo and Lopez-Herrero (2001); Daley and Servi (1998)), many authors use full and partial BPs when dealing with multiple-server systems. A full BP is one in which a BP starts from all servers becoming busy and ends when at least one server becomes free. Note that in a single-server queueing system the partial and full BPs are one and the same. Note that a partial BP comprises the full period and the duration where at least one server is free and at least one server is idle. The type of BP used in this volume should be clear from the context in which it is used.

There are many variations to the type of queueing systems just described. It is impossible to describe all of them here. Every year hundreds of papers appear in the literature on queueing and related fields. Some key journals that focus on the practical use of queueing models are (not in any specific order) *Operations Research*, *Management Science*, *Stochastic Models*, *Queueing Systems*, *Performance Evaluation*, *IEEE Transactions on Communications*, *European Journal of Operational Research*, *Computers and Industrial Engineering*, *Computers and Operations Research*, *Naval Research Logistics*, *Queueing Models and Service Management*, *Applied Mathematical Modeling*, *Applied Mathematics and Computation*, *Opsearch* and *IIE Transactions*.

The following is the standard notation, due to David Kendall, in queueing literature. $GI/G/s/K$ refers to an s-server queueing model in which the inter-arrival times are i.i.d random variables; the service times are i.i.d random variables; and there is a finite waiting room of size *K*. In the case when $K = \infty$, the notation will be $GI/G/s$. In the case when the inter-arrival times are exponential, GI will be replaced by M. When the service times are exponential, G will be replaced by M. For example, $M/M/1$ is a single-server queue with Poisson arrivals, exponential service times and infinite buffer size. When we look at an $M/PH/1$ queueing model, we are looking at a single-server infinite capacity queue with Poisson arrivals and phase type services. The queueing models can be studied either in discrete-time or in continuous-time. Here, our focus is on continuous-time and we refer the reader to Alfa (2016) for queues in discrete-time.

Before we discuss specific queueing models, we register some general observations. Toward this end, let $\{a_n : n \geq 1\}$ denote the inter-arrival times of the customers, assumed to be i.i.d; $X(t)$ denote the number of customers in the system at time t; $N(t)$ denote the number of customers arriving by time t (i.e. number arriving during $[0, t)$); and T denote the total time a customer spends in the system before leaving. We assume that the limit, $\lim_{n\to\infty} P(X(t) = k)$, exists and is given by $\pi_k, k \geq 0$. Suppose, for $k \geq 0$, we denote by u_k and v_k, respectively, the steady-state probability that an arriving customer and a departing customer see exactly k customers in the system. For any system in which arrivals occur singly and services are offered to one at a time, then we have $u_k = v_k, k \geq 0$. However, note that π_k may differ from u_k (or v_k). The average arrival rate is given by:

$$\lambda_a = \lim_{t\to\infty} \frac{N(t)}{t} = \frac{1}{E(a_n)}. \qquad [1.1]$$

Note that in the case of an infinite buffer no customer will be lost and hence $\lambda_a = \lambda$, where λ is the average arrival rate. In the case of a finite buffer, it is possible for an arriving customer to be lost due to the buffer being full and hence $\lambda_a = \lambda(1 - P(loss))$.

A few words about the presentation of the materials in this book. The approach to the study of any queueing model can be classified into three categories: (1) embedded at departure epochs; (2) embedded at arrival epochs; and (3) at arbitrary times. The specific approach to be taken depends, most of the times, on the underlying queueing model. However, there are situations where one can have a choice in the approach.

One of the important aspects of any book is to provide different approaches for the same model so as to give a clear understanding of the underlying concepts to the reader. We take this into consideration and hence one will see the analysis of the same model from different perspectives. For example, the queueing model dealing with continuous time phase type (CPH) arrivals and CPH service times will be studied by looking at (a) the arrival epochs; (b) the service epochs; and (c) arbitrary time points. Similarly, the queueing model with CPH arrivals and exponential services will be discussed from both embedded at arrivals as well as at arbitrary time points of view.

In this chapter, our focus is on the study of queues by looking at departure epochs. That is, we focus on $M/G/1$-type queues. The chapter is organized as follows. In section 1.1, we present the most general single-server queueing model in which the arrivals occur according to a batch Markovian arrival process (BMAP) and the services are generally distributed. That is, we look at a $BMAP/G/1$ queue and its analysis. Illustrative examples covering a variety of scenarios follow the analysis. Special cases of the $BMAP/G/1$ queue are discussed very briefly in subsequent sections. Specifically, in sections 1.2, 1.3, 1.4, 1.5 and 1.6, respectively, we present the special cases involving $MAP/G/1$, $PH/G/1$, $M/G/1$, $PH/M/1$ and $M/M/1$ queues. However, more explanations will be provided when the special cases yield additional insights and/or explicit expressions.

1.1. *BMAP/G/*1 queue

Consider a single-server system in which the customers arrive according to a BMAP with representation matrices $\{D_k\}$ of order m. Recall (see Chapter 6 in Volume 1) that a BMAP is characterized by the parameter matrices $\{D_k\}$ such that D_0 governs transitions corresponding to no arrivals and D_k governs transitions related to an arrival of a batch of k customers. The service times are generally distributed with the (cumulative) distribution function (CDF) given by $F(.)$ and assumed to have a finite mean, μ', and a finite variance, σ^2. The service rate is denoted by μ.

Let $\boldsymbol{\pi}$ denote the steady-state probability vector of the underlying (irreducible) generator, Q, of the BMAP. That is:

$$Q = \sum_{k=0}^{\infty} D_k = D_0 + D, \qquad [1.2]$$

and:

$$\boldsymbol{\pi} Q = \mathbf{0}, \;\; \boldsymbol{\pi}\boldsymbol{e} = 1, \qquad [1.3]$$

Recall (see equation [6.4] in Volume 1) that the fundamental rate, λ, is defined as $\lambda = \boldsymbol{\pi} \sum_{k=1}^{\infty} kD_k\boldsymbol{e}$. The rate, λ_g, of group arrivals is given by $\lambda_g = \boldsymbol{\pi} D\boldsymbol{e}$ (see equation [6.5] in Volume 1).

We now look at the queueing system at departure epochs so as to study the system using Markov renewal process (MRP). Toward this end, let ϑ_n denote the epoch at which the nth departure (i.e. a service completion) occurs. Without loss of generality we take $\vartheta_0 = 0$. Suppose we define ζ_n and J_n, respectively, to be the number of customers in the system and the phase of the BMAP immediately after the nth departure. That is, we look at the system's state at epochs, ϑ_n^+. Note that the difference $\vartheta_{n+1} - \vartheta_n$ corresponds to the duration of the $(n+1)$st service.

We now define a number of auxiliary (matrix) probabilities for use in the following. Define $\hat{A}_n(t) = (\hat{a}_{i,j}(n,t))$ and $\hat{B}_n(t) = (\hat{b}_{i,j}(n,t))$ such that:

1) $\hat{a}_{i,j}(n,t)$ gives the conditional probability that the last departure left the server to be busy again with the BMAP in phase i and the next departure, which occurs no

later than time t, will see the BMAP in phase j with exactly n customers waiting in the system.

2) $\hat{b}_{i,j}(n,t)$ gives the conditional probability that the last departure left the server to be idle with the BMAP in phase i and the next departure, which occurs no later than time t, will see the BMAP in phase j with exactly n customers waiting in the system. Note that here a batch of one or more customers will have to arrive before time t and that a service completion must occur before time t. A pictorial description of this probability is given in Figure 1.1.

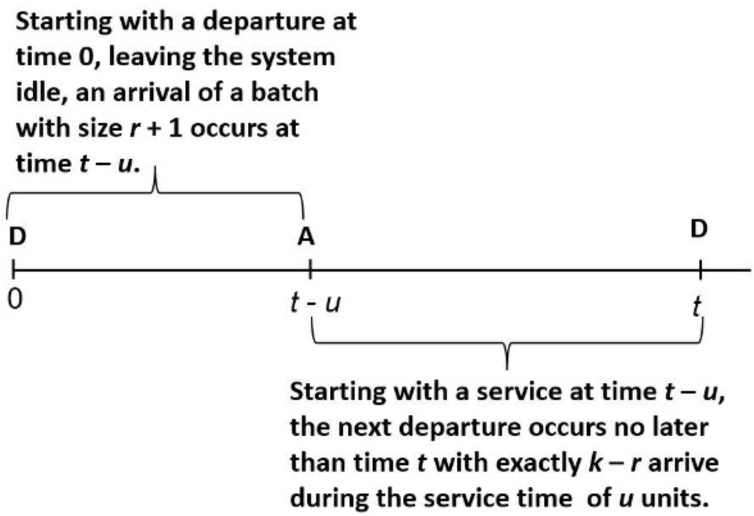

Figure 1.1. *Pictorial description for* $\hat{b}_{i,j}(n,t)$

Noting that $P(n,t)$ gives the (matrix) probability of the counting process associated with the BMAP (see equation [6.19] in Volume 1), we see that:

$$\hat{A}_n(t) = \int_0^t P(n,u)dF(u), \;\; n \geq 0, \; t \geq 0,$$

$$\hat{B}_n(t) = \int_0^t e^{D_0(t-u)} \sum_{r=0}^{n} D_{r+1}\hat{A}_{n-r}(u)du, \;\; n \geq 0, \; t \geq 0. \qquad [1.4]$$

While it is very clear how $\hat{A}_n(t)$ is obtained, a quick look at Figure 1.1 shows how one needs to split into two parts to get the expression for $\hat{B}_n(t)$.

The process, $\{(\zeta_n, J_n, \vartheta_{n+1} - \vartheta_n) : n \geq 0\}$, is an MRP on the state space $\{(i,j,t) : i \geq 0,\ 1 \leq j \leq m,\ t \geq 0\}$ and its TPM is given by:

$$\hat{P}(t) = \begin{bmatrix} \hat{B}_0(t) & \hat{B}_1(t) & \hat{B}_2(t) & \hat{B}_3(t) & \cdots \\ \hat{A}_0(t) & \hat{A}_1(t) & \hat{A}_2(t) & \hat{A}_3(t) & \cdots \\ & \hat{A}_0(t) & \hat{A}_1(t) & \hat{A}_2(t) & \cdots \\ & & \hat{A}_0(t) & \hat{A}_1(t) & \cdots \\ & & & \hat{A}_0(t) & \cdots \\ & & & & \ddots \end{bmatrix}, \qquad [1.5]$$

where the (matrix) entries are as given in equation [1.4]. For later use, we register a number of auxiliary matrices. The joint transform matrices, $A^*(z,s)$ and $B^*(z,s)$, respectively, of $\{\hat{A}_n(t)\}$ and $\{\hat{B}_n(t)\}$ are defined as:

$$A^*(z,s) = \sum_{n=0}^{\infty} z^n \int_0^{\infty} e^{-st} d\hat{A}_n(t),\ Re(s) \geq 0, |z| \leq 1, \qquad [1.6]$$

and:

$$B^*(z,s) = \sum_{n=0}^{\infty} z^n \int_0^{\infty} e^{-st} d\hat{B}_n(t),\ Re(s) \geq 0, |z| \leq 1. \qquad [1.7]$$

Let $D^*(z)$ denote the (matrix) probability generating function (PGF) of the BMAP. That is:

$$D^*(z) = \sum_{n=0}^{\infty} z^n D_n,\ |z| \leq 1. \qquad [1.8]$$

The following result (see Lucantoni (1993) and Neuts (1989)) establishes expressions for the transform matrices.

RESULT 1.1.– Suppose that $f^*(s)$ is the Laplace transform (LT) of $F(.)$. Then, the transform matrices $A^*(z,s)$ and $B^*(z,s)$ can be simplified as:

$$A^*(z,s) = \int_0^{\infty} e^{-st} e^{D(z)\,t} dF(t) = f^*(sI - D(z)),\ Re(s) \geq 0, |z| \leq 1, \qquad [1.9]$$

$$B^*(z,s) = z^{-1}(sI - D_0)^{-1}[D(z) - D_0]A^*(z,s),\ Re(s) \geq 0, |z| \leq 1. \qquad [1.10]$$

PROOF.– From the definition of $A^*(z,s)$ and using result 6.5 in Volume 1 (which says $P^*(z,t) = e^{D(z)t}$), we see:

$$A^*(z,s) = \int_0^{\infty} e^{-st} \sum_{n=0}^{\infty} z^n P(n,t) dF(t) = \int_0^{\infty} e^{-st} e^{D(z)\,t} dF(t), \qquad [1.11]$$

from which the stated result on $A^*(z,s)$ follows. From the definition of $B^*(z,s)$ and applying Leibnitz's rule, we get:

$$B^*(z,s) = \sum_{n=0}^{\infty} z^n \int_0^{\infty} e^{-st} d\hat{B}_n(t) = \sum_{n=0}^{\infty} z^n \int_0^{\infty} e^{-st} dt \sum_{r=0}^{n} D_{r+1} \hat{A}_{n-r}(t)$$
$$+ \sum_{n=0}^{\infty} z^n \int_0^{\infty} e^{-st} dt \int_0^t e^{D_0(t-u)} D_0 \sum_{r=0}^{n} D_{r+1} \hat{A}_{n-r}(u) du,$$

which upon simplification results in:

$$B^*(z,s) = \frac{1}{sz}\Big[(D(z)-D_0)A^*(z,s)+(sI-D_0)^{-1}D_0(D(z)-D_0)A^*(z,s)\Big],$$

from which the stated result follows immediately. Note that we used the fact that $\int_0^{\infty} e^{-st} \sum_{n=0}^{\infty} z^n \hat{A}_n(t) dt = \frac{1}{s} A^*(z,s)$. □

REMARK 1.1.– From equations [1.4], [1.9] and [1.10], it is easy to see:

$$A_n = \hat{A}_n(\infty) = \int_0^{\infty} P(n,t) dF(t), \; n \geq 0,$$
$$B_n = \hat{B}_n(\infty) = (-D_0)^{-1} \sum_{r=0}^{n} D_{r+1} A_{n-r}, \; n \geq 0,$$
$$A = A^*(1,0) = \sum_{n=0}^{\infty} A_n = \int_0^{\infty} e^{Qt} dF(t),$$
$$B = B^*(1,0) = (-D_0)^{-1}(Q - D_0)A = (I - D_0^{-1}Q)A. \quad [1.12]$$

REMARK 1.2.– On noting that the (i,j)th entry of $(-D_0^{-1} D_k)$ gives the conditional probability that a BP starts with an arrival of a batch of size k with the BMAP in state j given that the idle period started with the BMAP in state i, the entries of B_n as given in equation [1.12] are intuitively obvious.

REMARK 1.3.– It is easy to verify that the vector $\boldsymbol{\pi}$ is also the steady-state vector of A. This follows from the fact that $\boldsymbol{\pi} e^{Q\,t} = \boldsymbol{\pi}$, for all $t \geq 0$.

The following result (see Neuts (1989)) plays a key role and gives an expression for the vector of the conditional mean number of arrivals during a service given the phase of the BMAP.

RESULT 1.2.– The vector, $\boldsymbol{\beta}^* = \sum_{k=1}^{\infty} kA_k\boldsymbol{e}$, is given by:

$$\boldsymbol{\beta}^* = \frac{\lambda}{\mu}\boldsymbol{e} + (I - A)(\boldsymbol{e\pi} - Q)^{-1}\sum_{k=1}^{\infty} kD_k\boldsymbol{e}. \qquad [1.13]$$

PROOF.– Suppose that $M_k(t) = \sum_{n=k}^{\infty} \frac{n!}{(n-k)!}P(n,t)$. Then, (see Neuts (1989), Narayana and Neuts (1992) and Nielsen et al. (2007)), it is easy to verify:

$$\int_0^t dM_1(u) = \int_0^t M_1(x)Qdx + \int_0^t M_0(x)dx\sum_{r=0}^{\infty} rD_r, \qquad [1.14]$$

which on post-multiplying by $\boldsymbol{e}$ along with the fact that $M_0(t) = e^{Q\,t}$, we get:

$$M_1(t)\boldsymbol{e} = \int_0^t e^{Qx}dx\sum_{r=0}^{\infty} rD_r\boldsymbol{e}. \qquad [1.15]$$

Now applying result 4.25 in Volume 1, which gives an expression for the integral in equation [1.15], we get:

$$M_1(t)\boldsymbol{e} = t\lambda\boldsymbol{e} + [I - e^{Qt}](\boldsymbol{e\pi} - Q)^{-1}\sum_{r=0}^{\infty} rD_r\boldsymbol{e}. \qquad [1.16]$$

From definition, we have:

$$\boldsymbol{\beta}^* = \sum_{k=1}^{\infty} kA_k\boldsymbol{e} = \int_0^t M_1(t)\boldsymbol{e}dF(t) = \frac{\lambda}{\mu}\boldsymbol{e} + (I - A)(\boldsymbol{e\pi} - Q)^{-1}\sum_{k=1}^{\infty} kD_k\boldsymbol{e}. \qquad [1.17]$$

RESULT 1.3.– Neuts (1989) The $BMAP/G/1$ queue is stable if and only if $\lambda < \mu$.

PROOF.– Since $\boldsymbol{\pi\beta}^*$ gives the mean number of arrivals during a service time, the queue is stable if and only if this mean is less than one. Thus, using the expression for $\boldsymbol{\beta}^*$ as given in equation [1.17], we have:

$$\boldsymbol{\pi\beta}^* < 1 \;\; \Leftrightarrow \;\; \boldsymbol{\pi\beta}^* = \frac{\lambda}{\mu} < 1 \;\; \Leftrightarrow \;\; \lambda < \mu. \qquad [1.18]$$

Thus, we get the stated result. □

Steady-state probabilities at departures: Suppose we look at the embedded Markov chain (EMC) corresponding to the MRP with TPM, $\hat{P}(t)$ given in equation [1.5]. That is, we look at the TPM, $P = \hat{P}(\infty)$:

$$P = \begin{bmatrix} B_0 & B_1 & B_2 & B_3 & \cdots \\ A_0 & A_1 & A_2 & A_3 & \cdots \\ & A_0 & A_1 & A_2 & \cdots \\ & & A_0 & A_1 & \cdots \\ & & & A_0 & \cdots \\ & & & & \ddots \end{bmatrix}, \qquad [1.19]$$

where the (matrix) entries are as given in equation [1.12]. We assume that $P = \hat{P}(\infty)$ is irreducible and that the condition $\lambda < \mu$ holds good so that P is Ergodic. Suppose we denote by $\boldsymbol{x} = (\boldsymbol{x}_0, \boldsymbol{x}_1, \boldsymbol{x}_2, \cdots)$ the steady-state probability vector of P. That is:

$$\boldsymbol{x}P = \boldsymbol{x}, \;\; \boldsymbol{x}\boldsymbol{e} = 1. \qquad [1.20]$$

Note that the vectors, $\boldsymbol{x}_i, \; i \geq 0$, are all of dimension m.

The steady-state probability vector $\boldsymbol{x}$ of P satisfying equation [1.20] can be rewritten as:

$$\begin{aligned} \boldsymbol{x}_k &= \boldsymbol{x}_0 B_k + \sum_{j=1}^{k+1} \boldsymbol{x}_j \, A_{k-j+1}, \; k \geq 0, \\ \sum_{j=0}^{\infty} \boldsymbol{x_j} \, \boldsymbol{e} &= 1. \end{aligned} \qquad [1.21]$$

The following result gives an expression for the (vector) PGF, $\boldsymbol{X}(z) = \sum_{k=0}^{\infty} z^k \boldsymbol{x}_k$.

RESULT 1.4.– The (vector) PGF, $\boldsymbol{X}(z)$, of $\{\boldsymbol{x}_k\}$ is given by:

$$\boldsymbol{X}(z) = \boldsymbol{x}_0(-D_0)^{-1} D(z) A(z)[zI - A(z)]^{-1}, \;\; 0 < z < 1. \qquad [1.22]$$

PROOF.–In section 7.2 of Volume 1, we derived an expression for $\boldsymbol{X}(z)$ (see equation [7.36]), which is also valid here. Thus, we have:

$$\boldsymbol{X}(z) = \boldsymbol{x}_0[z\, B(z) - A(z)][z\, I - A(z)]^{-1}, \;\; 0 < z < 1. \qquad [1.23]$$

From equation [1.10], by taking $s = 0$, we get:

$$B(z) = B^*(z, 0) = \sum_{k=0}^{\infty} z^k B_k = z^{-1}(-D_0)^{-1}[D(z) - D_0]A(z), \; |z| \leq 1, \qquad [1.24]$$

which can be rewritten as:

$$zB(z) - A(z) = (-D_0)^{-1} D(z) A(z), \ |z| \leq 1. \tag{1.25}$$

In the above equations, $A(z) = A^*(z, 0) = \sum_{k=0}^{\infty} z^k A_k$ denotes the (matrix) PGF for $\{A_k\}$. The stated result follows by substituting the expression for $zB(z) - A(z)$ given in equation [1.25] into equation [1.23]. □

REMARK 1.4.– The PGF, $\boldsymbol{X}(z)$, is determined once $\boldsymbol{x}_0$ is computed. In section 7.2 of Volume 1, it was computed using the eigenvalue approach. Here, we take a different approach that has relevance to queueing. To compute $\boldsymbol{x}_0$, we first register the following key observations (see Neuts (1989)):

1) Note that the quantity x_{0j} gives the steady-state probability that a departure leaves the system empty with BMAP in phase j. Also, $\frac{1}{x_{0j}}$ gives the mean number of transitions in the EMC with TPM P between successive visits to $(0, j)$. Thus, to study the return time distributions to level $\mathbf{0}$, we need to study the first passage time distributions from level $\boldsymbol{i} + \mathbf{1}$ to level $\boldsymbol{i}$.

2) The EMC is skip-free to the left. That is, to reach level $\boldsymbol{j}$ from level $(\boldsymbol{i} + \mathbf{1})$, for $0 \leq j \leq i$, all the levels $\boldsymbol{j} + \mathbf{1}, \cdots, \boldsymbol{i}$ must be visited at least once.

3) Due to the homogeneity of the EMC, the first passage time distribution from $(\boldsymbol{i} + \mathbf{1})$ to level $\boldsymbol{i}$ is independent of i (away from the boundary state(s)).

The computation of $\boldsymbol{x}_0$ is done by first looking at the first passage time from level $(\boldsymbol{i} + \mathbf{1})$ to level $\boldsymbol{i}$, away from the boundary state $\mathbf{0}$, and then the recurrence time from level $\mathbf{1}$ to level $\mathbf{0}$. It should be pointed out that the first passage time from level $\boldsymbol{i}$ to level $(\boldsymbol{i} - \mathbf{1})$ is referred to as the fundamental period (see Neuts (1989)) and for the queueing model under study it is also the BP. In general, the BP may be different from the fundamental period. In general, studying the BPs is very involved. More on this will be seen later on.

First passage times from level $(\boldsymbol{i} + \mathbf{1})$ *to level* $\boldsymbol{i}$: we assume that the EMC is irreducible, which is the case in most applications. Let $T(i + r, j; i, k)$ denote the first passage time from state $(i + r, j)$ to the state (i, k), for $i, r \geq 1, 1 \leq j, k \leq m$. That is, $T(i+r, j; i, k)$ is the duration that the MRP, $\hat{P}(.)$, starting in state $(i+r, j)$ spends before visiting level $\boldsymbol{i}$ for the first time by entering the state (i, k). Let $V(i + r, j; i, k)$ denote the number of transitions involved during the first passage time $T(i+r, j; i, k)$.

The joint probability function of $T(i + r, j; i, k)$ and $V(i + r, j; i, k)$ plays a key role in $M/G/1$-type queues. We now define the matrix $G^{(r)}(n, t)$ whose (j, k)th entry gives the probability defined as (note that these matrices do not depend on i due to the structure), for $r \geq 0, 1 \leq j, k \leq m, \ n \geq 0$,

$$g_{j,k}^{(r)}(n, t) = P\{T(i + r, j; i, k) \leq t, V(i + r, j; i, k) = n\}, \tag{1.26}$$

where we take:

$$g_{j,k}^{(0)}(n,t) = \begin{cases} 1, \ j = k, \ n = 0, \ t = 0, \\ \quad\quad 0, \ \text{elsewhere.} \end{cases} \qquad [1.27]$$

Define the joint transform matrix, $G(z,s), |z| \leq 1, Re(s) \geq 0$, of $G(n,t)$ as:

$$G(z,s) = \sum_{n=1}^{\infty} z^n \int_0^{\infty} e^{-st} dG(n,t). \qquad [1.28]$$

RESULT 1.5 (Neuts 1989).– The joint transform matrix $G(z,s)$ satisfies:

$$G(z,s) = z \sum_{r=0}^{\infty} A_r(s) G^r(z,s), \ |z| \leq 1, \ Re(s) \geq 0, \qquad [1.29]$$

where $A_r(s) = \int_0^{\infty} e^{-st} d\hat{A}_r(t), \ r \geq 0.$

PROOF.– First note that the Markov property of the MRP $\hat{P}(.)$ (see equation [1.5]) implies that the pairs $\{(T(i+r, \zeta_{i+r}; i+r-1, \zeta_{i+r-1}), V(i+r, \zeta_{i+r}; i+r-1, \zeta_{i+r-1}))\}$ are mutually conditionally independent given the random variables, $\zeta_i, \cdots, \zeta_{i+r}$, that keep track of the phase of the BMAP at those epochs when the corresponding levels are visited. This leads to:

$$G^{(r)}(n,t) = \sum_S G^{(1)}(r_1,.) * G^{(1)}(r_2,.) * \cdots * G^{(1)}(r_p,t), \qquad [1.30]$$

where the set S is such that $S = \{(r_1, \cdots, r_p) : r_q \geq 1, 1 \leq q \leq p, \ r_1 + \cdots + r_p = r\}$. The equation [1.30] implies that $G^{(r)}(n,t) = G^r(n,t)$. That is, the r-fold convolution of $G(.,.)$ is nothing but the rth power of $G(.,.)$. By the definition of $G(.,.)$ it is clear that $\hat{A}_0(.)$ governs the probabilities of the process going from level $(\boldsymbol{i}+\mathbf{1})$ to level $\boldsymbol{i}$ in one transition. To have exactly $r, \ r \geq 2$, transitions to go from level $(\boldsymbol{i}+\mathbf{1})$ to level $\boldsymbol{i}$, the process has to stay or go to some level to the right of level $(\boldsymbol{i}+\mathbf{1})$, say level $(\boldsymbol{i}+\boldsymbol{k}), \ k \geq 1$, and then return to level $\boldsymbol{i}$ but one level down at a time due to the skip-free to the left property. Thus, we have the following relationship between $\hat{A}_r(.)$ and $G(.,.)$:

$$\begin{aligned} G(1,t) &= \hat{A}_0(t), \\ G(r,t) &= \sum_{k=1}^{\infty} \hat{A}_k(.) * G^{(k)}(r-1,t), \ r \geq 2. \end{aligned} \qquad [1.31]$$

Multiplying the first equation in [1.31] by z, the second equation in [1.31] by z^r, adding them over r, and taking the LST, we get:

$$\int_0^{\infty} e^{-st} \sum_{r=1}^{\infty} z^r dG(r,t) = zA_0(s) + \sum_{r=2}^{\infty} z^r \sum_{k=1}^{\infty} A_k(s) G^k(r-1,s), \qquad [1.32]$$

from which the stated result follows. □

The following result (see Lucantoni (1991) and Lucantoni and Neuts (1994)) gives a different way to look at the joint transform matrix $G(.,.)$. Before seeing the result, we need to define:

$$D[G(z,s)] = \sum_{k=0}^{\infty} D_k G^k(z,s),\ Re(s) \geq 0,\ |z| \leq 1. \qquad [1.33]$$

RESULT 1.6.– The joint transform matrix $G(z,s)$ satisfies:

$$G(z,s) = z \int_0^{\infty} e^{-st} e^{D[G(z,s)]t} dF(t), \qquad [1.34]$$

from which we see that the matrix G satisfies:

$$G = \int_0^{\infty} e^{D[G]t} dF(t), \qquad [1.35]$$

where:

$$D[G] = \sum_{k=0}^{\infty} D_k G^k. \qquad [1.36]$$

REMARK 1.5.– The matrix $G(n,t) = G^{(1)}(n,t), n \geq 1, t \geq 0$, is such that its (j,k)th entry, $g_{j,k}(n,t)$, is the conditional probability that the first passage time from $(i+1,j)$ to (i,k), for $i \geq 1, 1 \leq j,k \leq m$, occurs in exactly n transitions and no later than t.

REMARK 1.6.– The substochastic matrix, $\tilde{G}(t) = \sum_{n=1}^{\infty} G(n,t)$, gives the matrix distribution for the fundamental period. That is, $\tilde{g}_{j,k}(t)$ is the conditional probability that the first passage time from $(i+1,j)$ to (i,k), for $1 \leq j,k \leq m$, occurs no later than t.

REMARK 1.7.– The sequence, $\{\hat{G}(n) = G(n,\infty)\},\ n \geq 1$, of matrices, gives the (matrix) probability mass functions for the number of transitions during a fundamental period. That is, $\hat{g}_{j,k}(n), 1 \leq j,k \leq m, n \geq 1$, is the conditional probability that exactly n transitions occur during a first passage time from $(i+1,j)$ to (i,k).

REMARK 1.8.– The matrix $G = \sum_{n=1}^{\infty} \hat{G}(n)$ is such that its (j,k)th entry gives the conditional probability that the MRP eventually visits the state (i,k) for the first time starting in state $(i+1,j)$. This matrix plays an important role in the $M/G/1$-type queues (Neuts (1989)) like the matrix R in $GI/M/1$-type queues (Neuts (1981)).

REMARK 1.9.– If the MRP is irreducible, then G does not have zero rows. Furthermore, G has an eigenvalue of maximum modulus, which is positive.

REMARK 1.10.– The matrix G satisfies the following nonlinear matrix equation:

$$G = \sum_{i=0}^{\infty} A_i G^i, \qquad [1.37]$$

and is the minimal non-negative solution.

REMARK 1.11.– If the MRP is recurrent, then G is stochastic.

REMARK 1.12 (Structural properties of G (Neuts 1989)).– Some interesting structural properties are worth registering here. They are as follows:

1) To every zero column in A_0, the corresponding column in G is also zero.

2) If $(I - A_1)^{-1} A_0$ is irreducible, then G is also irreducible.

3) Suppose that A is reducible and is partitioned as:

$$A = \begin{bmatrix} A(1) & 0 & \cdots & 0 & 0 \\ 0 & A_1 & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & A(c) & 0 \\ \tilde{A}_1 & \tilde{A}_2 & \cdots & \tilde{A}_c & \tilde{A}_0 \end{bmatrix}, \qquad [1.38]$$

where $A(r), 1 \leq r \leq c$, are irreducible stochastic matrices. The last (block) row, which may be absent in some applications, consists of substochastic matrices (which, of course, will add to get a stochastic matrix). In this case, G is also reducible and has the same structure as A.

4) Earlier results (when A is assumed to be irreducible) hold good by looking at c such conditions like $\boldsymbol{\pi}(r)\boldsymbol{\beta}(r) \leq 1,\ 1 \leq r \leq c$, and c such G matrices.

5) $D[G]$ has a nice probabilistic interpretation (see Lucantoni et al. (1990)). It is as follows. When the queue is stable, $D[G]$ is the generator of a continuous-time Markov chain (CTMC) in which all BPs are excised. That is, the CTMC looks at the phase of the arrival process during the times when the system is idle.

6) The matrix G and $D(G)$ commute. That is, $GD(G) = D(G)G$. This can be seen immediately by looking at equation [1.36] and the fact that $D[G]$ and $e^{D[G]}$ commute (due to the nature of the exponential matrix).

Recurrence time of level **0**

Here, we study the first passage (or return or recurrence) time from level **0** to level **0**. Toward this end, let $\vartheta_{i,j}$ denote the time duration the underlying MRP takes to return to state $(0, j)$ given that it started in state $(0, i)$. Let $U_{i,j}$ denote the number of transitions required to return to state $(0, j)$ from state $(0, i)$.

Let $K(n,t) = \{k_{i,j}(n,t)\}, n \geq 1, t \geq 0$, be such that $k_{i,j}(n,t)$ is the conditional probability that the first passage (return or recurrence) time from $(0,i)$ to $(0,j)$, for $1 \leq i,j \leq m$, occurs in exactly n transitions and no later than t.

$$k_{i,j}(n,t) = P\{\vartheta_{i,j} \leq t, U_{i,j} = n\},\ n \geq 0, 1 \leq i,j \leq m. \qquad [1.39]$$

The joint transform matrix, $K(z,s), |z| \leq 1, Re(s) \geq 0$, is defined as:

$$K(z,s) = \sum_{n=1}^{\infty} z^n \int_0^{\infty} e^{-sx} dK(n,x). \qquad [1.40]$$

The following result is similar to result 1.5 and the proof is left as an exercise.

RESULT 1.7 (Neuts 1989).– The joint transform matrix $K(z,s)$ satisfies:

$$K(z,s) = z \sum_{r=0}^{\infty} B_r(s) G^r(z,s),\ |z| \leq 1,\ Re(s) \geq 0, \qquad [1.41]$$

where $B_r(s) = \int_0^{\infty} e^{-st} d\hat{B}_r(t),\ r \geq 0$.

Suppose we define $K = K(1,0) = \{k_{i,j}\}$ such that $k_{i,j}$ gives the conditional probability that the MRP with TPM $P(.)$ of the BMAP at departure epochs will eventually hit level **0** by visiting $(0,j)$ given that it started in state $(0,i)$.

RESULT 1.8.– The matrix K is given by:

$$K = K(1,0) = I - D_0^{-1} D[G], \qquad [1.42]$$

where $D[G]$ is as given in equation [1.36].

PROOF.– Using the form of B_n as given in equation [1.12] in the definition of K, we get:

$$K = K(1,0) = \sum_{n=0}^{\infty} B_n G^n = \sum_{n=0}^{\infty} -(D_0)^{-1} \sum_{r=0}^{n} D_{r+1} A_{n-r} G^n$$

$$= -(D_0)^{-1} \sum_{r=0}^{\infty} D_{r+1} \sum_{n=r}^{\infty} A_{n-r} G^n = -(D_0)^{-1} \sum_{r=0}^{\infty} D_{r+1} \sum_{n=0}^{\infty} A_n G^{n+r},$$

$$= -(D_0)^{-1} \sum_{r=0}^{\infty} D_{r+1} G^{r+1} = -(D_0)^{-1}(D[G] - D_0) = I - (D_0)^{-1}[D[G]. \qquad \square$$

Suppose that $\tilde{\boldsymbol{\mu}}_G$ is the vector (of dimension m) of means such that the jth component gives the expected number of transitions required to reach level $\boldsymbol{i}$ from state $(i+1,j)$. Let $\hat{\boldsymbol{\mu}}_G$ denote the vector (of dimension m) of means such that the

jth component gives the expected duration to reach level $\boldsymbol{i}$ starting from state $(i+1, j)$. The following result (due to Neuts (1989)) gives the expressions for these mean vectors. For use in the following, we define $\boldsymbol{g}$ to be the steady-state probability vector of G. That is:

$$\boldsymbol{g}G = \boldsymbol{g}, \;\; \boldsymbol{g}\boldsymbol{e} = 1. \qquad [1.43]$$

RESULT 1.9.– The vectors $\tilde{\boldsymbol{\mu}}_G$ and $\hat{\boldsymbol{\mu}}_G$ are given by:

$$\begin{aligned} \tilde{\boldsymbol{\mu}}_G &= (I - G + \boldsymbol{e}\,\boldsymbol{g})[I - A + (\boldsymbol{e} - \boldsymbol{\beta}^*)\boldsymbol{g}]^{-1}\boldsymbol{e}, \\ \hat{\boldsymbol{\mu}}_G &= \frac{1}{\mu}(I - G + \boldsymbol{e}\,\boldsymbol{g})[I - A + (\boldsymbol{e} - \boldsymbol{\beta}^*)\boldsymbol{g}]^{-1}\boldsymbol{e}. \end{aligned} \qquad [1.44]$$

where $\boldsymbol{\beta}^*$ is as given in equation [1.17].

PROOF.– We prove the result $\tilde{\boldsymbol{\mu}}_G$ as it is similar for $\hat{\boldsymbol{\mu}}_G$. By definition:

$$\begin{aligned} \tilde{\boldsymbol{\mu}}_G &= \frac{\partial}{\partial z} G(z,s)\Big]_{s=0,z=1} \boldsymbol{e} = \sum_{r=0}^{\infty} A_r G^r \boldsymbol{e} \\ &+ \sum_{r=1}^{\infty} A_r \sum_{k=0}^{r-1} G^k \frac{\partial}{\partial z} G(z,s)\Big]_{s=0,z=1} G^{r-1-k}\boldsymbol{e}, \end{aligned} \qquad [1.45]$$

which reduces to (using the fact that G is stochastic under the assumption that the stability condition $\rho < 1$ holds good):

$$\Big[I - \sum_{r=1}^{\infty} A_r \sum_{k=0}^{r-1} G^k\Big]\tilde{\boldsymbol{\mu}}_G = \boldsymbol{e}. \qquad [1.46]$$

Note that:

$$\begin{aligned} &\Big[I - \sum_{r=1}^{\infty} A_r \sum_{k=0}^{r-1} G^k\Big][I - G + \boldsymbol{e}\,\boldsymbol{g}] \\ &= [I - G + \boldsymbol{e}\,\boldsymbol{g} - \sum_{r=1}^{\infty} A_r \sum_{k=0}^{r-1} G^k (I - G) - \sum_{r=1}^{\infty} r A_r \boldsymbol{e}\,\boldsymbol{g}] \\ &= [I - A + (\boldsymbol{e} - \boldsymbol{\beta}^*)\boldsymbol{g}]. \end{aligned} \qquad [1.47]$$

First, note that $[I - A + (\boldsymbol{e} - \boldsymbol{\beta}^*)\boldsymbol{g}]$ is non-singular. To see this, suppose that $\boldsymbol{u}$ is a non-zero vector such that:

$$[I - A + (\boldsymbol{e} - \boldsymbol{\beta}^*)\boldsymbol{g}]\boldsymbol{u} = \boldsymbol{0} \;\Rightarrow\; (I - A)\boldsymbol{u} + (\boldsymbol{e} - \boldsymbol{\beta}^*)\boldsymbol{g}\,\boldsymbol{u} = \boldsymbol{0}. \qquad [1.48]$$

Now pre-multiplying the equation [1.48] by $\boldsymbol{\pi}$, we get $\boldsymbol{g}\,\boldsymbol{u} = 0$, from which we see that $(I - A)\boldsymbol{u} = \boldsymbol{0}$. This implies that $\boldsymbol{u} = c\boldsymbol{e}$ with $c > 0$ due to A being a stochastic matrix. This is not possible since $\boldsymbol{g}\,\boldsymbol{u} = 0 \Rightarrow c = 0$. This indicates that $[I - A + (\boldsymbol{e} - \boldsymbol{\beta}^*)\boldsymbol{g}]$ has to be non-singular.

Since both $[I - A + (\boldsymbol{e} - \boldsymbol{\beta}^*\boldsymbol{g})]$ and $(I - G + \boldsymbol{e}\,\boldsymbol{g})$ are non-singular, we see that $\left[I - \sum_{r=1}^{\infty} A_r \sum_{k=0}^{r-1} G^k\right]$ is also non-singular.

Now combining equations [1.46] and [1.47] we get the stated result. □

REMARK 1.13.– The connection between $\tilde{\boldsymbol{\mu}}_G$ and $\hat{\boldsymbol{\mu}}_G$ is very clear.

REMARK 1.14.– It is easy to verify that the following equalities hold good and will serve as internal accuracy checks for computing G and other items related to the BP to be seen later.

$$\boldsymbol{g}\,\tilde{\boldsymbol{\mu}}_{\boldsymbol{G}} = \frac{1}{1-\rho} \text{ and } \boldsymbol{g}\,\hat{\boldsymbol{\mu}}_{\boldsymbol{G}} = \frac{1}{\mu(1-\rho)}. \qquad [1.49]$$

RESULT 1.10.– The vector $\tilde{\boldsymbol{\mu}}_K$ is given by:

$$\tilde{\boldsymbol{\mu}}_K = (-D_0^{-1})[Q - D[G] + \boldsymbol{d}\,\boldsymbol{g}][I - A + (\boldsymbol{e} - \boldsymbol{\beta}^*)\boldsymbol{g}]^{-1}\boldsymbol{e}, \qquad [1.50]$$

where:

$$\boldsymbol{d} = \sum_{k=0}^{\infty} k D_k \boldsymbol{e}. \qquad [1.51]$$

PROOF.– By definition:

$$\begin{aligned}\tilde{\boldsymbol{\mu}}_K &= \frac{\partial}{\partial z} K(z,s)\Big]_{s=0,z=1} \boldsymbol{e} = \sum_{r=0}^{\infty} B_r G^r \boldsymbol{e} \\ &+ \sum_{r=1}^{\infty} B_r \sum_{k=0}^{r-1} G^k \frac{\partial}{\partial z} G(z,s)\Big]_{s=0,z=1} G^{r-1-k}\boldsymbol{e},\end{aligned} \qquad [1.52]$$

which reduces to (using the fact that G is stochastic under the assumption that the stability condition $\rho < 1$ holds good):

$$\begin{aligned}\tilde{\boldsymbol{\mu}}_K &= \boldsymbol{e} + \sum_{r=1}^{\infty} B_r \sum_{k=0}^{r-1} G^k \tilde{\boldsymbol{\mu}}_G \\ &= \boldsymbol{e} + \sum_{r=0}^{\infty} B_r (I - G^r + r\boldsymbol{e}\,\boldsymbol{g})[I - A + (\boldsymbol{e} - \boldsymbol{\beta}^*)\boldsymbol{g}]^{-1}\boldsymbol{e}.\end{aligned} \qquad [1.53]$$

The stated result follows on noting (recall $D = Q - D_0$) the following:

$$\sum_{r=0}^{\infty} B_r = (-D_0^{-1})DA,$$

$$\sum_{r=0}^{\infty} B_r G^r = (-D_0^{-1})(D[G] - D_0), \qquad [1.54]$$

$$\sum_{r=0}^{\infty} r B_r \boldsymbol{e}\,\boldsymbol{g} = (-D_0^{-1})\boldsymbol{d}\,\boldsymbol{g} - (-D_0^{-1})D\boldsymbol{e}\,\boldsymbol{g}.$$

□

Once we get an expression for $\boldsymbol{x}_0$, the PGF, $\boldsymbol{X}(z)$, as given in equation [1.22] can be used to solve for the rest of the steady-state probability vector $\boldsymbol{x}$ at departures. We demonstrate this by first establishing the following result that gives an expression for the vector $\boldsymbol{x}_0$.

RESULT 1.11.– The vector $\boldsymbol{x}_0$ is given by:

$$\boldsymbol{x}_0 = \frac{(1-\rho)}{\lambda}\boldsymbol{g}(-D_0), \qquad [1.55]$$

where $\boldsymbol{g}$ is the steady-state vector of G (see equation [1.43]).

PROOF.– Let κ denote the steady-state vector of the stochastic matrix K as given in equation [1.42]. That is:

$$\boldsymbol{\kappa}K = \boldsymbol{\kappa}, \;\; \boldsymbol{\kappa}\boldsymbol{e} = 1. \qquad [1.56]$$

Note that due to the form of K, we see $\boldsymbol{\kappa}$ can be obtained in terms of $\boldsymbol{g}$ as follows:

$$\boldsymbol{\kappa} = \frac{1}{\boldsymbol{g}D\boldsymbol{e}}\boldsymbol{g}(-D_0). \qquad [1.57]$$

To see this, note that $\boldsymbol{g}(-D_0)K = \boldsymbol{g}(-D_0) - \boldsymbol{g}D[G] = \boldsymbol{g}(-D_0)$, since $\boldsymbol{g}G = \boldsymbol{g} \Rightarrow \boldsymbol{g}D[G] = \mathbf{0}$. The uniqueness of $\boldsymbol{\kappa}$ yields the result. □

Applying result 2.60 from Volume 1, a classical result in MRP that relates the steady-state probability to the mean recurrence time, we note that:

$$\boldsymbol{x}_0 = \frac{1}{\boldsymbol{\kappa}\tilde{\boldsymbol{\mu}}_K}\boldsymbol{\kappa}. \qquad [1.58]$$

From equations [1.50] and [1.57], we note:

$$\boldsymbol{\kappa}\tilde{\boldsymbol{\mu}}_K = [\boldsymbol{g}D\boldsymbol{e}]^{-1}\boldsymbol{g}\Big[(Q - D[G] + \boldsymbol{d}\,\boldsymbol{g})(I - A + (\boldsymbol{e} - \boldsymbol{\beta}^*)\boldsymbol{g})^{-1}\boldsymbol{e}\Big]. \qquad [1.59]$$

Suppose we define $\boldsymbol{u} = [I - A + (\boldsymbol{e} - \boldsymbol{\beta}^*)\boldsymbol{g}]^{-1}\boldsymbol{e}$. Then, we see that:

$$[I - A + (\boldsymbol{e} - \boldsymbol{\beta}^*)\boldsymbol{g}]\boldsymbol{u} = \boldsymbol{e} \;\Rightarrow\; (I - A)\boldsymbol{u} + (\boldsymbol{e} - \boldsymbol{\beta}^*)\boldsymbol{g}\boldsymbol{u} = \boldsymbol{e}$$
$$\Rightarrow\; \boldsymbol{\pi}(\boldsymbol{e} - \boldsymbol{\beta}^*)\boldsymbol{g}\boldsymbol{u} = 1 \;\Rightarrow\; \boldsymbol{g}\boldsymbol{u} = \frac{1}{1-\rho}, \qquad [1.60]$$

which gives us:

$$(I - A)\boldsymbol{u} + \frac{1}{1-\rho}(\boldsymbol{e} - \boldsymbol{\beta}^*) = \boldsymbol{e}. \qquad [1.61]$$

Hence, noting that $\boldsymbol{g}D[G] = \mathbf{0}$, equation [1.59] can be simplified to:

$$\boldsymbol{\kappa}\tilde{\boldsymbol{\mu}}_K = [\boldsymbol{g}D\boldsymbol{e}]^{-1}\Big[\boldsymbol{g}Q[I - A + (\boldsymbol{e} - \boldsymbol{\beta}^*)\boldsymbol{g}]^{-1}\boldsymbol{e} + (1-\rho)^{-1}\boldsymbol{g}\,\boldsymbol{d}\,\boldsymbol{g}\Big]. \qquad [1.62]$$

Using the expression for $\boldsymbol{\beta}^*$ (see equation [1.17]) in equation [1.61] and routine simplifications yield:

$$(I - A)\Big[\boldsymbol{u} - (1-\rho)^{-1}(\boldsymbol{e}\,\boldsymbol{\pi} - Q)^{-1}\boldsymbol{d}\Big] = \mathbf{0}, \qquad [1.63]$$

where $\boldsymbol{d}$ is as given in equation [1.51]. Now, using the uniqueness of the right eigenvector of $(I - A)$ corresponding to the eigenvalue 1, we get:

$$\boldsymbol{u} = c\,\boldsymbol{e} + (1-\rho)^{-1}(\boldsymbol{e}\,\boldsymbol{\pi} - Q)^{-1}\boldsymbol{d}. \qquad [1.64]$$

Using equation [1.64] in the first expression within parentheses of equation [1.62], we get:

$$\boldsymbol{\kappa}\tilde{\boldsymbol{\mu}}_K = [(1-\rho)\boldsymbol{g}D\boldsymbol{e}]^{-1}\Big[\boldsymbol{g}Q(\boldsymbol{e}\,\boldsymbol{\pi} - Q)^{-1}\boldsymbol{d} + \boldsymbol{g}\,\boldsymbol{d}\,\boldsymbol{g}\Big]. \qquad [1.65]$$

Once we show that $\boldsymbol{g}Q(\boldsymbol{e}\,\boldsymbol{\pi} - Q)^{-1}\boldsymbol{d} = \lambda - \boldsymbol{g}\,\boldsymbol{d}$, the proof will be completed. Toward this end, suppose that $\boldsymbol{v} = (\boldsymbol{e}\,\boldsymbol{\pi} - Q)^{-1}\boldsymbol{d}$. Then, we have:

$$\boldsymbol{d} = \boldsymbol{e}\boldsymbol{\pi}\,\boldsymbol{v} - Q\boldsymbol{v} \;\Rightarrow\; \boldsymbol{\pi}\,\boldsymbol{d} = \boldsymbol{\pi}\,\boldsymbol{v} \;\Rightarrow\; \boldsymbol{\pi}\,\boldsymbol{v} = \lambda,\ (\because\ \boldsymbol{\pi}\,\boldsymbol{d} = \lambda)$$
$$\boldsymbol{g}\,\boldsymbol{d} = \boldsymbol{\pi}\,\boldsymbol{v} - \boldsymbol{g}Q\boldsymbol{v} \;\Rightarrow\; \boldsymbol{g}\,\boldsymbol{d} = \lambda - \boldsymbol{g}Q\boldsymbol{v} \;\Rightarrow\; \boldsymbol{g}Q\boldsymbol{v} = \lambda - \boldsymbol{g}\,\boldsymbol{d}. \qquad [1.66]$$

□

For the sake of completeness, we now summarize the results seen so far with regard to the steady-state vector and also recall the statements of theorem 7.2 in Volume 1 that pertain to our model here.

THEOREM 1.1.– When $\rho < 1$, the steady-state vector $\boldsymbol{x}$ is given by:

$$\boldsymbol{x}_0 = \frac{(1-\rho)}{\lambda}\boldsymbol{g}(-D_0),$$
$$\boldsymbol{x}_k = \Big[\boldsymbol{x}_0\tilde{B}_k + \sum_{j=1}^{k-1}\boldsymbol{x}_j\,\tilde{A}_{k-j+1}\Big](I - \tilde{A}_1)^{-1},\ k \geq 1, \qquad [1.67]$$

where:

$$\tilde{A}_k = \sum_{j=k}^{\infty} A_j\, G^{j-k}, \quad \tilde{B}_k = \sum_{j=k}^{\infty} B_j\, G^{j-k}, \;\; k \geq 1. \qquad [1.68]$$

REMARK 1.15.– When only the measures like the mean need to be computed, then we can use equation [1.35] to obtain the matrix G. The algorithmic steps for computing G are given below. This option does not require the knowledge of the matrices, $\{A_k\}$. However, if one needs to compute the steady-state vector $\boldsymbol{x}$, then we need to use equation [1.37]. The algorithmic steps in the computation of G as well as K matrices are spelled out in section 7.2 of Volume 1. Note that for this option we need to compute the matrices $\{P(n,t)\}$, which can be obtained using equation [6.36] in Volume 1 as follows (below we take $\theta = max_i\{-D_0\}_{ii}$):

$$A_n = \int_0^{\infty} P(n,t)dF(t) = \int_0^{\infty} \Big[\sum_{r=\lfloor \frac{n}{k^*} \rfloor k^*}^{\infty} e^{-\theta\, t}\, \frac{(\theta\, t)^r}{r!}\, \tilde{F}_{n,r} \Big] dF(t), \; n \geq 0. \quad [1.69]$$

The computation of A_n depends on the nature of the service time distribution function $F(.)$. For example, if the services are deterministic or have masses only at a finite number of points, then we need to evaluate $P(n,t)$ at a finite number of time points. If the services are assumed to be CPH, then A_n is computed without any numerical integration. Suppose we consider four cases: (1) constant services with a mean of, say, d; (2) service times with masses at $\{d_1, \cdots, d_L\}$ with probabilities $\{p_1, \cdots, p_L\}$; (3) services are of CPH with representation $(\boldsymbol{\beta}, S)$ of order n; and (4) services follow a (continuous) uniform distribution on (u_1, u_2), for $0 < u_1 < u_2 < \infty$. Then, we have, for $n \geq 0$,

$$A_n = \begin{cases} P(n,d), \quad \text{constant services}, \\ \sum_{i=1}^{L} p_i P(n,d_i), \quad \text{discrete services}, \\ \sum_{r=\lfloor \frac{n}{k^*} \rfloor k^*}^{\infty} \gamma_r\, \tilde{F}_{n,r}, \quad \text{CPH services}, \\ \frac{1}{u_2 - u_1} \sum_{i=1}^{N} P(n,a_i), \quad \text{uniform services}, \end{cases} \qquad [1.70]$$

where $\{\gamma_k\}$ follows a discrete PH distribution with representation $(\theta\boldsymbol{\beta}(\theta I - S)^{-1}, \theta(\theta I - S)^{-1})$ of order n, and $\{a_i\}$ is a finite partition of the interval (u_1, u_2) such that $u_1 = a_1 < a_2 < a_3 < \cdots < a_{N-1} < a_N = u_2$. The choice of the partition is important so as to not miss out capturing the number of BMAP arrivals in that small interval. We refer the reader to Volume 1 for illustrative numerical examples dealing with the computation of $\{P(n,t)\}$ under a variety of scenarios in the context of MAM, and Neuts and Li (1996) for full details on the algorithm itself. Here, we present a number of examples in the context of queueing theory.

Algorithm 1.1. Algorithmic steps in computing the matrix G based on equation [1.35]

Here, we use the uniformization technique. Toward that end, let $\theta = max_i\{-D_0\}_{ii},\ \gamma_r = \int_0^\infty e^{-\theta t}\frac{(\theta t)^r}{r!}dF(t), r \geq 0$.

Step 0: Set $k = 0, n = 0, G_k = 0, H_{0,k} = I$, for all k.

Step 1: Use Horner's method to get $D[G_k]$ (truncate $D[G]$ at some point, say, n_1, and the infinite sum of G at a point, say, n_2):

$$Y_0 = D_{n_1},\ Y_j = D_{n_1-j} + Y_{j-1}G_k,\ 1 \leq j \leq n_1,$$

and evaluate:

$$H_{r+1,k} = \left[I + \frac{1}{\theta}D[G_k]\right] H_{r,k},\ r = 0, 1, 2, \cdots, n_2.$$

Step 2: $k \leftarrow k+1,\ G_{k+1} = \sum_{r=0}^{n_2} \gamma_r H_{r,k}$.

Step 3: If $||G_{k+1} - G_k||_\infty < \epsilon$, where ϵ is a pre-determined small value, stop; otherwise go to step 1.

Steady-state probabilities at arbitrary times: Here, we look at the (joint) steady-state probability vector of the number in the system as well as the phase of the arrival process at arbitrary times. Let $\boldsymbol{y} = (\boldsymbol{y}_0, \boldsymbol{y}_1, \cdots)$ be the steady-state probability vector such that the jth component of $\boldsymbol{y}_i,\ i \geq 0$, gives the steady-state probability that there are i customers in the system and the phase of the arrival process is in state $j,\ 1 \leq j \leq m$, at an arbitrary time.

Suppose that $\hat{M}(t) = \sum_{n=0}^{\infty} \hat{P}^{(n)}(t)$ is the Markov renewal function associated with the MRP whose TPM is given in equation [1.5]. Recall that $d\hat{M}_{i_0,j_0;i,j}(u)$ is the conditional probability that the MRP enters into state (i, j) during the interval $(u, u + du)$ given that at time 0 it was in state (i_0, j_0). In the context of $BMAP/G/1$ queue, this corresponds to the conditional probability that at a departure epoch the system has i customers and the phase of the BMAP is j given that at time 0 the system has i_0 customers and the phase of the BMAP is j_0.

The following result gives an expression for $\boldsymbol{y}$.

THEOREM 1.2.– When $\rho < 1$, the steady-state vector $\boldsymbol{y}$ is given by:

$$\boldsymbol{y}_0 = (1-\rho)\boldsymbol{g},$$
$$\boldsymbol{y}_k = \Big[\sum_{i=0}^{k-1} \boldsymbol{y}_i D_{k-i} - \lambda(\boldsymbol{x}_{k-1} - \boldsymbol{x}_k)\Big](-D_0)^{-1},\ k \geq 1. \quad [1.71]$$

PROOF.– Let $y_{i,j}(t),\ i \geq 0, 1 \leq j \leq m,\ t \geq 0$, denote the probability that the queueing system under study has i customers in the system and the phase of the BMAP is in j at time t. Using the interpretation of $d\hat{M}_{i_0,j_0;i,j}(u)$ and the law of total probability by looking at all scenarios, it is easy to verify that:

$$y_{0,j}(t) = \sum_{r=1}^{m} \int_0^t d\hat{M}_{i_0,j_0;0,r}(u)\Big(e^{D_0\,(t-u)}\Big)_{r,j},$$

$$y_{k,j}(t) = \sum_{r=1}^{m} \int_0^t d\hat{M}_{i_0,j_0;0,r}(u) \int_0^{t-u} \Big(e^{D_0\,v} \sum_{i=1}^{k} D_i P(k-i, t-u-v)\Big)_{r,j} \times$$
$$[1 - F(t-u-v)]dv$$
$$+ \sum_{i=1}^{k} \sum_{r=1}^{m} \int_0^t d\hat{M}_{i_0,j_0;i,r}(u)\Big(P(k-i, t-u)\Big)_{r,j}[1 - F(t-u)],\ k \geq 1.$$

From the classical key renewal theorem for MRP (see result 2.58 in Volume 1), we get (in matrix form):

$$\boldsymbol{y}_0 = \lambda\boldsymbol{x}_0(-D_0)^{-1},$$
$$\boldsymbol{y}_k = \lambda\boldsymbol{x}_0(-D_0)^{-1} \sum_{i=1}^{k} D_i \int_0^{\infty} P(k-i,t)[1-F(t)]dt \quad [1.72]$$
$$+\lambda \sum_{i=1}^{k} \boldsymbol{x}_i \int_0^{\infty} P(k-i,t)[1-F(t)]dt,\ k \geq 1.$$

Let $\boldsymbol{Y}(z)$ denote the (vector) PGF of $\boldsymbol{y}$. Then, from equation [1.72] through multiplying by z^k and adding over k, we get:

$$\boldsymbol{Y}(z) = \boldsymbol{y}_0 + \lambda\boldsymbol{x}_0(-D_0)^{-1} \sum_{k=1}^{\infty} z^k \sum_{i=1}^{k} D_i \int_0^{\infty} P(k-i,t)[1-F(t)]dt$$

$$+\lambda \sum_{k=1}^{\infty} z^k \sum_{i=1}^{k} \boldsymbol{x}_i \int_0^{\infty} P(k-i,t)[1-F(t)]dt,$$

which on simplification yields:

$$\begin{aligned} \boldsymbol{Y}(z) &= \boldsymbol{y}_0 + \lambda \boldsymbol{x}_0(-D_0)^{-1}[D(z) - D_0] \int_0^\infty e^{D(z)\,t}[1 - F(t)]dt \\ &+ \lambda[\boldsymbol{X}(z) - \boldsymbol{x}_0] \int_0^\infty e^{D(z)\,t}[1 - F(t)]dt, \end{aligned} \qquad [1.73]$$

where $\boldsymbol{X}(z)$ is the (vector) PGF of the steady-state vector $\boldsymbol{x}$ and is given in equation [1.22]. Recalling $A(z) = \sum_{n=0}^{\infty} z^n A_n = \int_0^\infty e^{D(z)\,t} d\,F(t)$, and using the fact that:

$$\begin{aligned} &\int_0^\infty e^{D(z)\,t}[1 - F(t)]dt = \int_0^\infty e^{D(z)\,t} \int_t^\infty dF(u)dt = \int_0^\infty dF(u) \int_0^u e^{D(z)\,t}dt \\ &= [A(z) - I]D^{-1}(z), \ |z| \le 1, \end{aligned} \qquad [1.74]$$

equation [1.73] can now be rewritten as:

$$\begin{aligned} &\boldsymbol{Y}(z)D(z) = \lambda \boldsymbol{x}_0(-D_0)^{-1}D(z) + \lambda \boldsymbol{x}_0(-D_0)^{-1}[D(z) - D_0][A(z) - I] \\ &+\lambda[\boldsymbol{X}(z) - \boldsymbol{x}_0][A(z) - I] = \lambda(z - 1)\boldsymbol{X}(z), \ |z| \le 1. \end{aligned} \qquad [1.75]$$

Using the expression for $\boldsymbol{x}_0$ and the identification of like coefficients of $z^k, \ k \ge 0$, in equation [1.75], we get the stated result. □

The following result will serve as an accuracy check in the computation of the probability vectors, $\boldsymbol{x}$ and $\boldsymbol{y}$.

RESULT 1.12.– We have:

$$\begin{aligned} \boldsymbol{X}(1) &= \sum_{k=0}^{\infty} \boldsymbol{x}_k = \boldsymbol{\pi} + \boldsymbol{x}_0(-D_0)^{-1}Q\,A(I - A + \boldsymbol{e}\,\boldsymbol{\pi})^{-1}, \\ \boldsymbol{Y}(1) &= \sum_{k=0}^{\infty} \boldsymbol{y}_k = \boldsymbol{\pi}, \end{aligned} \qquad [1.76]$$

where $\boldsymbol{\pi}$ is the steady-state probability vector of the generator $Q = \sum_{k=0}^{\infty} D_k$ governing the BMAP.

PROOF.– The proof follows immediately by noting from equations [1.22] and [1.75] that $\boldsymbol{X}(1)[I - A] = \boldsymbol{x}_0(-D_0)^{-1}Q\,A$, and hence $\boldsymbol{Y}(1)Q = \mathbf{0}$. □

REMARK 1.16.– It is intuitively clear that $\boldsymbol{Y}(1) = \boldsymbol{\pi}$ due to the fact that the phase of BMAP in steady-state should be given by $\boldsymbol{\pi}$.

Stationary virtual waiting time: We assume that the FCFS rule is assumed. The virtual waiting time is the amount of time a virtual customer arriving at time t has to wait before receiving a service. That is, if an imaginary customer was to enter the system at time t, the virtual waiting time will be the time the customer has to wait before starting the service. The derivation of the Laplace-Stieltjes transform (LST) is quite involved and is due to Neuts (1989). This is the generalization of the well-known and classical Pollaczek-Khinchin formula for the $M/G/1$ queue to $BMAP/G/1$. We only state the result here and refer the reader to Neuts (1989) for full details.

RESULT 1.13.– Suppose that $\boldsymbol{w}(t)$ denote the vector whose jth component gives the joint probability that at an arbitrary time the arrival process is in phase j and that the virtual customer who arrives at that time waits at most t units of time before receiving a service. The LST, $\boldsymbol{w}^*(s)$, of $\boldsymbol{w}(t)$ is given by:

$$\boldsymbol{w}^*(s) = s(1-\rho)\boldsymbol{g}\Big(sI + D[f^*(s)]\Big)^{-1}, \; Re(s) \geq 0, \qquad [1.77]$$

where $f^*(s)$ is the LST of the service time distribution and $D[.]$ is as defined in equation [1.36].

REMARK 1.17.– Once various queue length densities and the waiting time distributions are obtained, we can compute the system performance measures like the mean, the median, the mode, the standard deviation and various percentiles, mostly through numerical computation. In the following examples, we present such measures.

Now we present a number of examples under various scenarios. These examples were generated using Fortran code (LF95 compiler), and a number of internal accuracy checks outlined earlier including the ones in result 1.12 were used to check the accuracy of the codes. Also, we used the exponential matrices, $e^{D_0\,t}$ and $e^{Q\,t}$, and compared them with the computed matrices $P(n,t)$ (see remark 6.10 in Volume 1 for details on this).

In the following examples, we let μ_a denote the mean of the batch size.

EXAMPLE 1.1.– In this example, we look at BMAP arrivals in which the inter-arrival times are Erlang order 3, and the service times are assumed to be constant with a mean of 0.9. Specifically, we have:

$$D_0 = \begin{pmatrix} -1.2 & 1.2 & 0 \\ 0 & -1.2 & 1.2 \\ 0 & 0 & -1.2 \end{pmatrix}, \;\; D_k = a_k \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1.2 & 0 & 0 \end{pmatrix}, \; k \geq 1.$$

Here, $\{a_k\}$ is assumed to be uniform on $\{1, 2, 3, 4\}$ so that $D_k = 0$, for $k \geq 5$.

Verify that we have $\mu_a = 2.5$, $\lambda_g = \frac{1}{2.5}$, $\lambda = 1$, $\sigma_g = 1.44338$, and the 1-lag correlation coefficient, ρ_g, is zero due to the independence of the successive inter-arrival times. The matrices G and K along with their invariant vectors, $\boldsymbol{g}$ and $\boldsymbol{\kappa}$, are:

$$G = \begin{pmatrix} 0.3552 & 0.4006 & 0.2441 \\ 0.0461 & 0.443 & 0.5109 \\ 0.0965 & 0.2297 & 0.6738 \end{pmatrix}, \quad K = \begin{pmatrix} 0.1888 & 0.3595 & 0.4517 \\ 0.1888 & 0.3595 & 0.4517 \\ 0.1888 & 0.3595 & 0.4517 \end{pmatrix},$$

$$\boldsymbol{g} = (0.1087, 0.3156, 0.5757), \quad \boldsymbol{\kappa} = (0.1888, 0.3595, 0.4517).$$

Selected queue length statistics are listed in Table 1.1 and the steady-state probabilities of the number of customers in the system at departure, arbitrary and arrival epochs are plotted in Figure 1.2.

Epoch type	*Mean*	*Median*	*Mode*	*Modal value*	*SD*
Departure	6.974	5	3	0.1016	6.451
Arbitrary	6.794	5	0	0.1000	6.461
Arrival	8.479	7	4	0.1133	6.488

Table 1.1. *Selected queue length statistics for example 1.1*

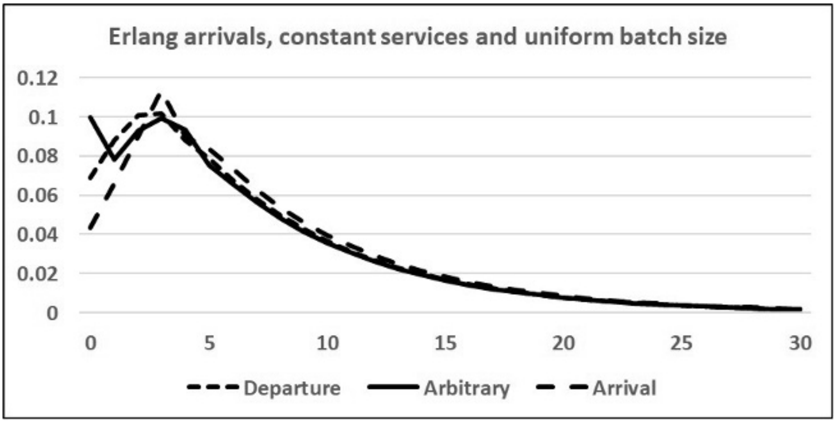

Figure 1.2. *Plots of steady-state probabilities for example 1.1*

EXAMPLE 1.2.– In this example, we look at BMAP arrivals in which the inter-arrival times are exponential, the service times are assumed to be constant with a mean of 0.9 and the batch sizes are uniformly distributed. That is, we have:

$$D_0 = (-0.4), \;\; D_k = (0.1), \; 1 \leq k \leq 4, \;\; D_k = (0), \; k \geq 5.$$

Verify that we have $\mu_a = 2.5,\ \lambda_g = \frac{1}{2.5},\ \lambda = 1,\ \sigma_g = 2.5$, and the 1-lag correlation coefficient, ρ_g, is zero due to the independence of successive inter-arrival times. Obviously, G, K, $\boldsymbol{g}$ and $\boldsymbol{\kappa}$ are all scalars and given by 1. Selected queue length statistics are displayed in Table 1.2 and the steady-state probabilities of the number of customers in the system at departure, arbitrary and arrival epochs are plotted in Figure 1.3.

Epoch type	*Mean*	*Median*	*Mode*	*Modal value*	*SD*
Departure	14.835	11	3	0.0545	14.305
Arbitrary	13.837	9	0	0.1000	14.270
Arrival	16.334	12	4	0.0613	14.316

Table 1.2. *Selected queue length statistics for example 1.2*

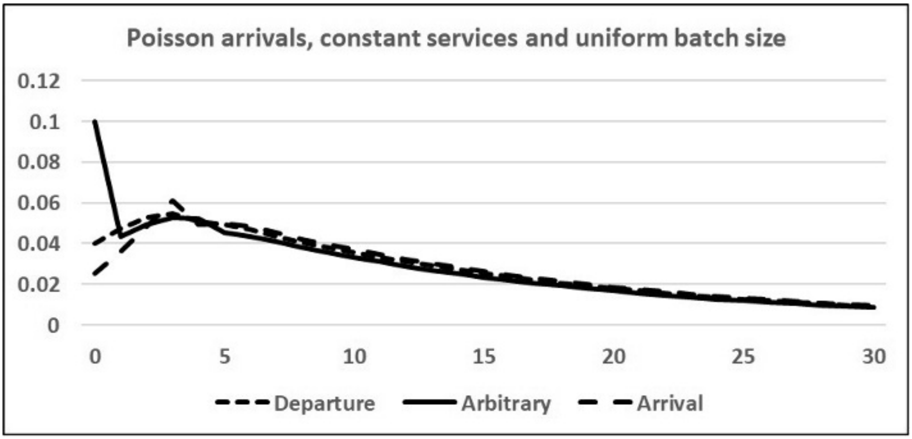

Figure 1.3. *Plots of steady-state probabilities for example 1.2*

EXAMPLE 1.3.– This is similar to the previous two examples, except that the inter-arrivals times are taken to be hyperexponential. Specifically, we have:

$$D_0 = \begin{pmatrix} -0.64 & 0 & 0 \\ 0 & -0.32 & 0 \\ 0 & 0 & -0.064 \end{pmatrix},\quad D_k = \frac{1}{4}\begin{pmatrix} 0.5120 & 0.0960 & 0.0320 \\ 0.2560 & 0.0480 & 0.0160 \\ 0.0512 & 0.0096 & 0.0032 \end{pmatrix},\ k \geq 1,$$

and $D_k = 0,\ k \geq 5$.

Verify that we have $\mu_a = 2.5$, $\lambda_g = \frac{1}{2.5}$, $\lambda = 1$, $\sigma_g = 5.0$, and the 1-lag correlation coefficient, ρ_g, is zero due to the independence of successive inter-arrival times. The matrices, G and K, and their invariant vectors, $\boldsymbol{g}$ and $\boldsymbol{\kappa}$, are displayed below:

$$G = \begin{pmatrix} 0.7251 & 0.0935 & 0.1815 \\ 0.0936 & 0.8032 & 0.1032 \\ 0.0210 & 0.0119 & 0.9670 \end{pmatrix}, \quad K = \begin{pmatrix} 0.4235 & 0.2113 & 0.3652 \\ 0.4235 & 0.2113 & 0.3652 \\ 0.4235 & 0.2113 & 0.3652 \end{pmatrix},$$

$$\boldsymbol{g} = (0.0941, 0.094, 0.8119), \quad \boldsymbol{\kappa} = (0.4235, 0.2113, 0.3652).$$

Selected queue length statistics are displayed in Table 1.3 and the steady-state probabilities of the number of customers in the system at departure, arbitrary and arrival epochs are plotted in Figure 1.4.

Epoch type	*Mean*	*Median*	*Mode*	*Modal value*	*SD*
Departure	44.499	31	3	0.0201	43.482
Arbitrary	40.480	27	0	0.1000	43.298
Arrival	45.969	33	4	0.0222	43.417

Table 1.3. *Selected queue length statistics for example 1.3*

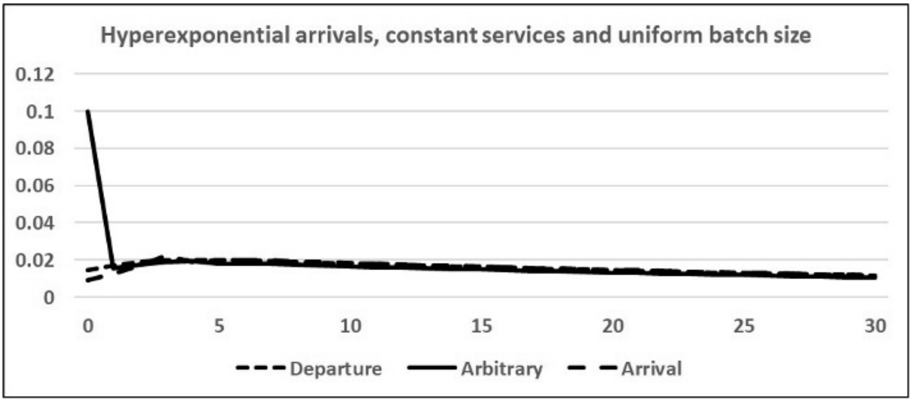

Figure 1.4. *Plots of steady-state probabilities for example 1.3*

EXAMPLE 1.4.– In this example, we look at BMAP arrivals and the service times are assumed to be constant with a mean of 0.9. We model the inter-arrival times with two distributions, one having a 1-lag negative correlation and the other having a 1-lag positive correlation. The BMAP representation for negatively correlated arrivals is given by:

$$D_0 = \begin{pmatrix} -0.5 & 0.5 & 0 \\ 0 & -0.5 & 0.5 \\ 0 & 0 & -1.0 \end{pmatrix}, \ D_k = \frac{1}{4}\begin{pmatrix} 0 & 0 & 0 \\ 0.005 & 0 & 0.495 \\ 0.990 & 0 & 0.010 \end{pmatrix}, \ 1 \le k \le 4,$$

and the representation for the positively correlated arrivals is given by:

$$D_0 = \begin{pmatrix} -0.5 & 0.5 & 0 \\ 0 & -0.5 & 0.5 \\ 0 & 0 & -1.0 \end{pmatrix}, \ D_k = \frac{1}{4}\begin{pmatrix} 0 & 0 & 0 \\ 0.495 & 0 & 0.005 \\ 0.010 & 0 & 0.099 \end{pmatrix}, \ 1 \le k \le 4,$$

and $D_k = 0, \ k \ge 5$, for both the cases. Note that both the BMAPs have the same mean and the same standard deviation for the inter-arrival times of successive groups. Like in those examples, we assume the batch sizes to follow a uniform distribution.

Verify that we have $\mu_a = 2.5, \ \lambda_g = \frac{1}{2.5}, \ \lambda = 1, \ \sigma_g = 2.3383$, and the 1-lag correlation coefficient, ρ_g, are, respectively, given by -0.32667 and 0.32667. We identify the nature of the correlated arrivals with superscripts NC and PC, respectively, for negative and positive 1-lag correlation. The matrices G and K along with their invariant vectors, $\boldsymbol{g}$ and $\boldsymbol{\kappa}$, are:

$$G^{(NC)} = \begin{pmatrix} 0.6588 & 0.3279 & 0.0132 \\ 0.1023 & 0.8396 & 0.0581 \\ 0.2336 & 0.3312 & 0.4352 \end{pmatrix}, \ K^{(NC)} = \begin{pmatrix} 0.2755 & 0.5102 & 0.2143 \\ 0.2755 & 0.5102 & 0.2143 \\ 0.4513 & 0.5078 & 0.0409 \end{pmatrix},$$

$$\boldsymbol{g}^{(NC)} = (0.2529, 0.6720, 0.0751), \ \ \boldsymbol{\kappa} = (0.3076, 0.5098, 0.1826).$$

$$G^{(PC)} = \begin{pmatrix} 0.6721 & 0.3277 & 0.0002 \\ 0.1607 & 0.8384 & 0.0009 \\ 0.1651 & 0.3327 & 0.5022 \end{pmatrix}, \ K^{(PC)} = \begin{pmatrix} 0.4901 & 0.5069 & 0.0030 \\ 0.4901 & 0.5069 & 0.0030 \\ 0.2548 & 0.5109 & 0.2343 \end{pmatrix},$$

$$\boldsymbol{g}^{(PC)} = (0.3289, 0.6698, 0.0013), \ \ \boldsymbol{\kappa}^{(PC)} = (0.4892, 0.5069, 0.0039).$$

Selected performance measures are listed in Table 1.4 and the steady-state probabilities of the number of customers in the system at departure, arbitrary and arrival epochs are plotted in Figure 1.5.

$Type\ of\ BMAP$	$Epoch\ Type$	$Mean$	$Median$	$Mode$	$Modal\ value$	SD
NC	$Departure$	11.798	9	3	0.0658	11.022
	$Arbitrary$	11.113	8	0	0.1000	11.044
	$Arrival$	13.304	10	4	0.0708	11.053
PC	$Departure$	420.718	268	0	0.0336	474.929
	$Arbitrary$	378.806	217	0	0.1000	467.027
	$Arrival$	421.867	270	4	0.0329	474.021

Table 1.4. *Selected queue length statistics for example 1.4*

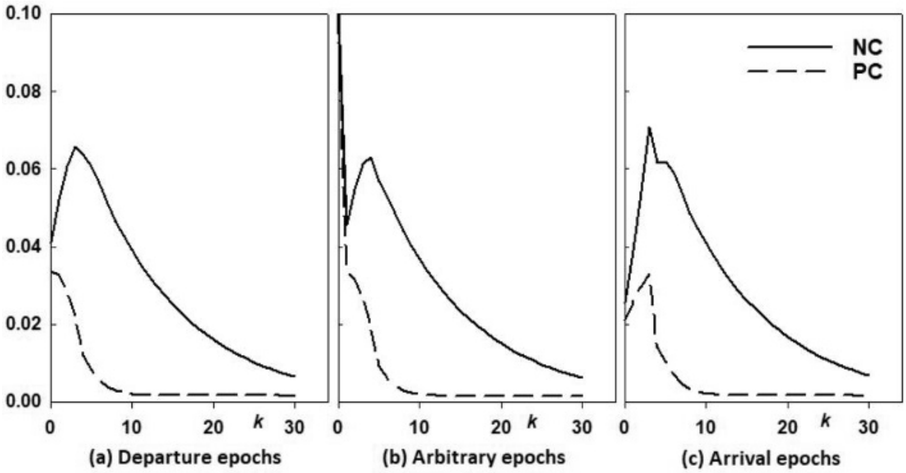

Figure 1.5. *Plots of steady-state probabilities for example 1.4*

EXAMPLE 1.5.– Referring to examples 1.1 through 1.4, suppose that the group arrivals are now modeled using a Poisson distribution with parameter 1.25 such that:

$$a_k = e^{-1.25} \frac{1.25^{k-1}}{(k-1)!}, \ k \geq 1.$$

In order to fix $\lambda = 1$ like in the previous examples for proper comparisons, we need to normalize $\{D_k\}$ appropriately. The representations for the BMAPs for the five cases to be discussed in this example are as follows:

$$D_0^{(ER)} = \frac{1}{3} \begin{pmatrix} -4 & 4 & 0 \\ 0 & -4 & 4 \\ 0 & 0 & -4 \end{pmatrix}, D_k^{(ER)} = \frac{a_k}{3} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 4 & 0 & 0 \end{pmatrix}, k \geq 1,$$

$$D_0^{(EX)} = \left(-\frac{1}{2.25} \right), \ D_k^{(EX)} = \left(\frac{a_k}{2.25} \right), \ k \geq 1,$$

$$D_0^{(HE)} = \frac{1}{45}\begin{pmatrix} -32 & 0 & 0 \\ 0 & -16 & 0 \\ 0 & 0 & -3.2 \end{pmatrix}, \; D_k^{(HE)} = \frac{a_k}{45}\begin{pmatrix} 25.60 & 4.80 & 1.60 \\ 12.80 & 2.40 & 0.80 \\ 2.56 & 0.48 & 0.16 \end{pmatrix}, \; k \geq 1,$$

$$D_0^{(NC)} = \frac{1}{9}\begin{pmatrix} -5 & 5 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & -10 \end{pmatrix}, \; D_k^{(NC)} = \frac{a_k}{9}\begin{pmatrix} 0 & 0 & 0 \\ 0.05 & 0 & 4.95 \\ 9.90 & 0 & 0.10 \end{pmatrix}, \; k \geq 1,$$

$$D_0^{(PC)} = \frac{1}{9}\begin{pmatrix} -5 & 5 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & -10 \end{pmatrix}, \; D_k^{(PC)} = \frac{a_k}{9}\begin{pmatrix} 0 & 0 & 0 \\ 4.95 & 0 & 0.95 \\ 0.10 & 0 & 9.90 \end{pmatrix}, \; k \geq 1,$$

Note that all these five BMAPs have the same mean batch size of 2.25 and the same arrival rate, $\lambda = 1.0$. However, the other parameters vary as follows. The standard deviations of these five arrival processes are, respectively, given by 1.2990, 2.25, 4.5, 2.3383 and 2.3383. Also, the 1-lag correlation, ρ_g, is as in example 1.4. For this example, G and K matrices along with their invariant vectors, $\boldsymbol{g}$ and $\boldsymbol{\kappa}$, are displayed along with their identifiers:

$$G^{(ER)} = \begin{pmatrix} 0.3203 & 0.4043 & 0.2754 \\ 0.0510 & 0.4204 & 0.5286 \\ 0.0979 & 0.2431 & 0.6589 \end{pmatrix}, \quad K^{(ER)} = \begin{pmatrix} 0.1852 & 0.3635 & 0.4513 \\ 0.1852 & 0.3635 & 0.4513 \\ 0.1852 & 0.3635 & 0.4513 \end{pmatrix},$$

$$\boldsymbol{g}^{(ER)} = (0.1068, 0.3164, 0.5767), \quad \boldsymbol{\kappa}^{(ER)} = (0.1852, 0.3635, 0.4513),$$

$$G^{(HE)} = \begin{pmatrix} 0.7033 & 0.1012 & 0.1955 \\ 0.1026 & 0.7848 & 0.1126 \\ 0.0234 & 0.0133 & 0.9634 \end{pmatrix}, \quad K^{(HE)} = \begin{pmatrix} 0.4292 & 0.2117 & 0.3591 \\ 0.4292 & 0.2117 & 0.3591 \\ 0.4292 & 0.2117 & 0.3591 \end{pmatrix},$$

$$\boldsymbol{g}^{(HE)} = (0.0966, 0.0953, 0.8081), \quad \boldsymbol{\kappa}^{(HE)} = (0.4292, 0.2117, 0.3591),$$

$$G^{(NC)} = \begin{pmatrix} 0.6322 & 0.3523 & 0.0155 \\ 0.1123 & 0.8265 & 0.0611 \\ 0.2468 & 0.3547 & 0.3986 \end{pmatrix}, \quad K^{(NC)} = \begin{pmatrix} 0.2789 & 0.5072 & 0.2139 \\ 0.2789 & 0.5072 & 0.2139 \\ 0.4543 & 0.5055 & 0.0402 \end{pmatrix},$$

$$\boldsymbol{g}^{(NC)} = (0.2549, 0.6704, 0.0747), \quad \boldsymbol{\kappa}^{(NC)} = (0.3108, 0.5069, 0.1822),$$

$$G^{(PC)} = \begin{pmatrix} 0.6476 & 0.3521 & 0.0002 \\ 0.1734 & 0.8257 & 0.0009 \\ 0.1773 & 0.3558 & 0.4669 \end{pmatrix}, \quad K^{(PC)} = \begin{pmatrix} 0.4921 & 0.5049 & 0.003 \\ 0.4921 & 0.5049 & 0.003 \\ 0.2544 & 0.5078 & 0.2378 \end{pmatrix},$$

$$\boldsymbol{g}^{(PC)} = (0.3298, 0.6689, 0.0013), \quad \boldsymbol{\kappa}^{(PC)} = (0.4912, 0.5049, 0.0039),$$

The plots of the steady-state probabilities at various epochs of the five arrival processes are displayed in Figure 1.6, and selected performance measures are listed in Table 1.5.

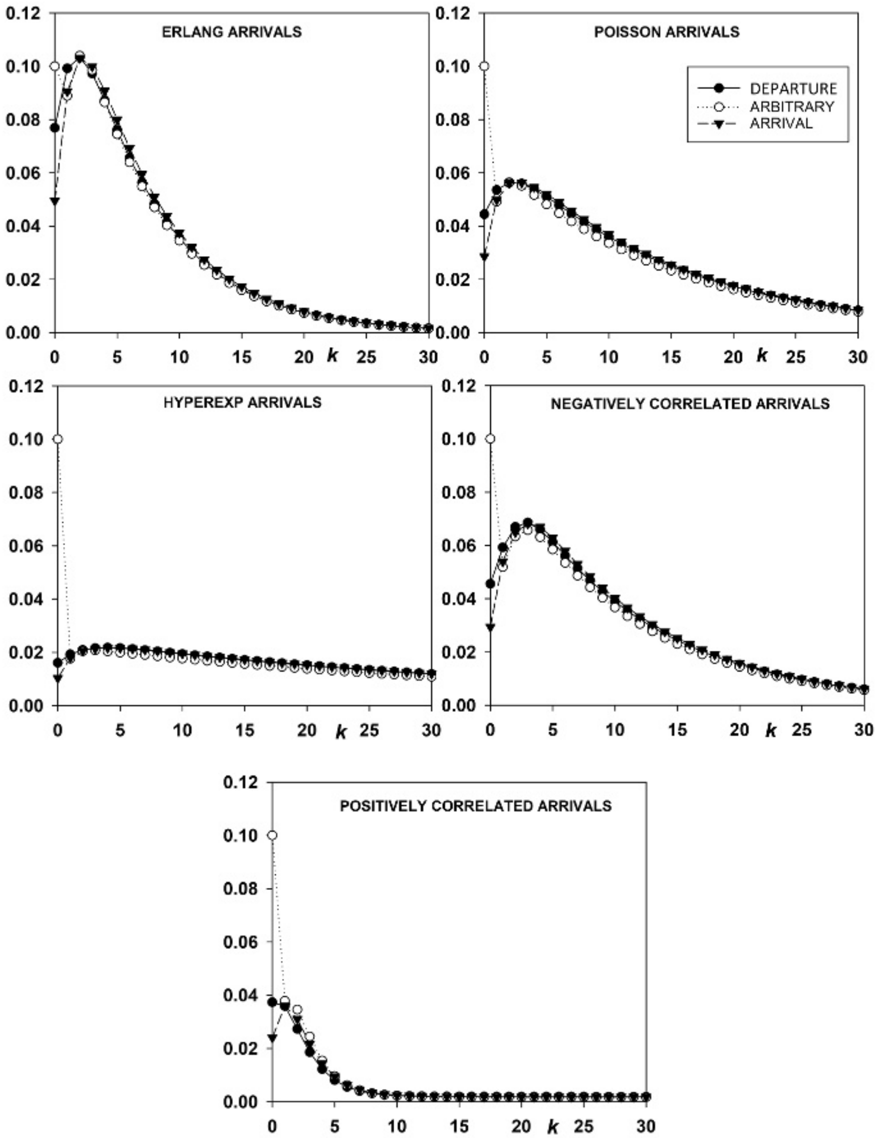

Figure 1.6. *Plots of steady-state probabilities at various epochs for example 1.5*

Type of BMAP	*Epoch Type*	*Mean*	*Median*	*Mode*	*Modal value*	*SD*
ER	*Departure*	6.797	5	2	0.1034	6.441
	Arbitrary	6.628	5	2	0.1040	6.419
	Arrival	8.143	6	3	0.1031	6.457
EX	*Departure*	13.877	10	2	0.0564	13.473
	Arbitrary	12.975	9	0	0.1000	13.431
	Arrival	15.223	11	4	0.0563	13.480
HE	*Departure*	40.618	28	4	0.0217	39.746
	Arbitrary	36.994	25	0	0.1000	39.584
	Arrival	41.943	30	5	0.0217	39.696
NC	*Departure*	11.136	8	3	0.0686	10.526
	Arbitrary	10.512	7	0	0.1000	10.518
	Arrival	12.482	9	4	0.0682	10.535
PC	*Departure*	379.179	242	0	0.0373	427.967
	Arbitrary	341.459	196	0	0.1	420.870
	Arrival	380.218	243	2	0.0360	427.168

Table 1.5. *Selected queue length statistics for example 1.5*

EXAMPLE 1.6.– This example is similar to example 1.5 except that the batch sizes are now modeled using a discrete PH distribution. The representation for this PH distribution is given by:

$$\boldsymbol{\alpha} = (0.7, 0.3), \quad T = \begin{pmatrix} 0.1 & 0.7 \\ 0.2 & 0.3 \end{pmatrix},$$

First, recall from section 3.1 (see result 3.1) from Volume 1 that the PMF, $\{a_k\}$, is obtained as $a_0 = 0$ and $a_k = \boldsymbol{\alpha} T^{k-1} \boldsymbol{T^0}, \ k \geq 1$. Verify first that the mean batch size, $\mu_a = 2.6735$, and hence the matrices $D_k, \ k \geq 0$, need to multiplied by an appropriate constant to get $\lambda = 1$. This constant, say, d, is obtained as follows. In example 1.5, the mean batch size is 2.25 and hence by taking $d = \frac{2.25}{2.6735} \simeq 0.8416$, we get the proper BMAP representations to guarantee $\lambda = 1$ for this example. For example, the representation for the Erlang arrivals for this example is given by:

$$D_0^{(ER)} = \frac{0.8416}{3} \begin{pmatrix} -4 & 4 & 0 \\ 0 & -4 & 4 \\ 0 & 0 & -4 \end{pmatrix}, D_k^{(ER)} = \frac{0.8416 \, a_k}{3} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 4 & 0 & 0 \end{pmatrix}, k \geq 1.$$

Thus, for this example, all these five BMAPs have the same mean batch size of 2.6735, the same arrival rate, $\lambda = 1.0$. However, the other parameters vary as follows. The standard deviations of these five arrival processes are, respectively, given by 1.5432, 2.6735, 5.3457, 2.7777 and 2.7777. Also, the 1-lag correlation, ρ_g, is the same as in example 1.4.

The matrices G and K matrices along with their invariant vectors, $\boldsymbol{g}$ and $\boldsymbol{\kappa}$, are:

$$G^{(ER)} = \begin{pmatrix} 0.3798 & 0.3981 & 0.2221 \\ 0.0488 & 0.4622 & 0.489 \\ 0.1075 & 0.2302 & 0.6623 \end{pmatrix}, \quad K^{(ER)} = \begin{pmatrix} 0.2198 & 0.371 & 0.4092 \\ 0.2198 & 0.371 & 0.4092 \\ 0.2198 & 0.371 & 0.4092 \end{pmatrix},$$

$$\boldsymbol{g}^{(ER)} = (0.1214, 0.3263, 0.5523), \quad \boldsymbol{\kappa}^{(ER)} = (0.2198, 0.371, 0.4092),$$

$$G^{(HE)} = \begin{pmatrix} 0.758 & 0.0885 & 0.1535 \\ 0.0994 & 0.814 & 0.0866 \\ 0.0222 & 0.0111 & 0.9667 \end{pmatrix}, \quad K^{(HE)} = \begin{pmatrix} 0.4677 & 0.2083 & 0.324 \\ 0.4677 & 0.2083 & 0.324 \\ 0.4677 & 0.2083 & 0.324 \end{pmatrix},$$

$$\boldsymbol{g}^{(HE)} = (0.1134, 0.101, 0.7856), \quad \boldsymbol{\kappa}^{(HE)} = (0.4677, 0.2083, 0.3240),$$

$$G^{(NC)} = \begin{pmatrix} 0.6762 & 0.3099 & 0.0139 \\ 0.1017 & 0.8334 & 0.0649 \\ 0.2407 & 0.299 & 0.4603 \end{pmatrix}, \quad K^{(NC)} = \begin{pmatrix} 0.2847 & 0.4649 & 0.2504 \\ 0.2847 & 0.4649 & 0.2504 \\ 0.4835 & 0.4732 & 0.0433 \end{pmatrix},$$

$$\boldsymbol{g}^{(NC)} = (0.2667, 0.6485, 0.0849), \quad \boldsymbol{\kappa}^{(NC)} = (0.3259, 0.4666, 0.2075),$$

$$G^{(PC)} = \begin{pmatrix} 0.6891 & 0.3107 & 0.0002 \\ 0.1617 & 0.8372 & 0.0011 \\ 0.1548 & 0.2929 & 0.5522 \end{pmatrix}, \quad K^{(PC)} = \begin{pmatrix} 0.5203 & 0.476 & 0.0037 \\ 0.5203 & 0.476 & 0.0037 \\ 0.2449 & 0.4609 & 0.2942 \end{pmatrix},$$

$$\boldsymbol{g}^{(PC)} = (0.3422, 0.6561, 0.0017), \quad \boldsymbol{\kappa}^{(PC)} = (0.5188, 0.4760, 0.0052).$$

The plots of the steady-state probabilities at various epochs of the five arrival processes are displayed in Figure 1.7, and selected performance measures are listed in Table 1.6.

EXAMPLE 1.7.– Here, we compare the impact of the batch size distribution when services are constant by looking at examples 1.1 through 1.6. A few selected measures are displayed in Table 1.7 and the steady-state probability vectors at departure epochs under various scenarios are plotted in Figure 1.8.

$Type\ of\ BMAP$	$Epoch\ Type$	$Mean$	$Median$	$Mode$	$Modal\ value$	SD
ER	$Departure$	11.354	8	1	0.0697	11.201
	$Arbitrary$	10.702	7	0	0.1000	11.100
	$Arrival$	12.501	9	2	0.0723	11.183
EX	$Departure$	19.795	14	1	0.0415	19.467
	$Arbitrary$	18.212	12	0	0.1000	19.260
	$Arrival$	20.878	15	2	0.0432	19.358
HE	$Departure$	51.933	36	4	0.0171	51.056
	$Arbitrary$	46.545	31	0	0.1000	49.892
	$Arrival$	52.446	38	5	0.0171	50.127
NC	$Departure$	16.507	12	3	0.0479	15.993
	$Arbitrary$	15.287	10	0	0.1000	15.869
	$Arrival$	17.623	13	4	0.0482	15.932
PC	$Departure$	453.714	290	0	0.0308	511.203
	$Arbitrary$	346.241	235	0	0.1000	385.220
	$Arrival$	392.159	292	2	0.0307	401.8610

Table 1.6. *Selected measures under DPH batch size for example 1.6*

$Batch$	$MEASURE$	ERL	EXP	HEX	NCA	PCA
UB	$MEAN$	6.974	14.836	44.499	11.798	420.718
	$MEDIAN$	5	11	31	9	268
	$MODE$	3	3	3	3	0
	$MVALUE$	0.1016	0.0545	0.0201	0.0658	0.0336
	SD	6.451	14.305	43.482	11.022	474.929
PB	$MEAN$	6.797	13.877	40.618	11.136	379.179
	$MEDIAN$	5	10	28	8	242
	$MODE$	2	2	4	3	0
	$MVALUE$	0.1034	0.0564	0.0217	0.0686	0.0373
	SD	6.441	13.473	39.746	10.526	427.967
DP	$MEAN$	11.354	19.796	51.933	16.507	453.714
	$MEDIAN$	8	14	36	12	290
	$MODE$	1	1	4	3	0
	$MVALUE$	0.0697	0.0415	0.0171	0.0479	0.0308
	SD	11.201	19.467	51.056	15.993	511.2030

Table 1.7. *Selected measures for example 1.7 under various batch sizes*

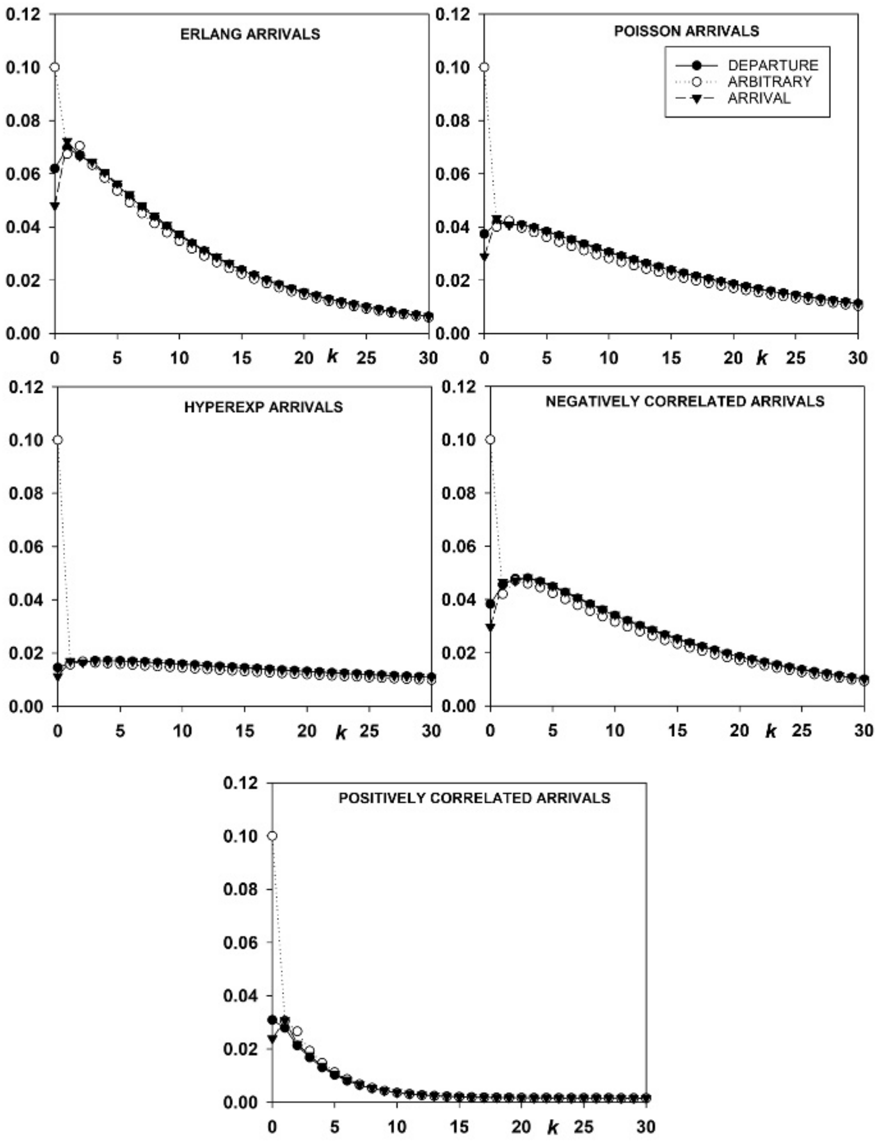

Figure 1.7. *Plots of steady-state probabilities at various epochs for example 1.6*

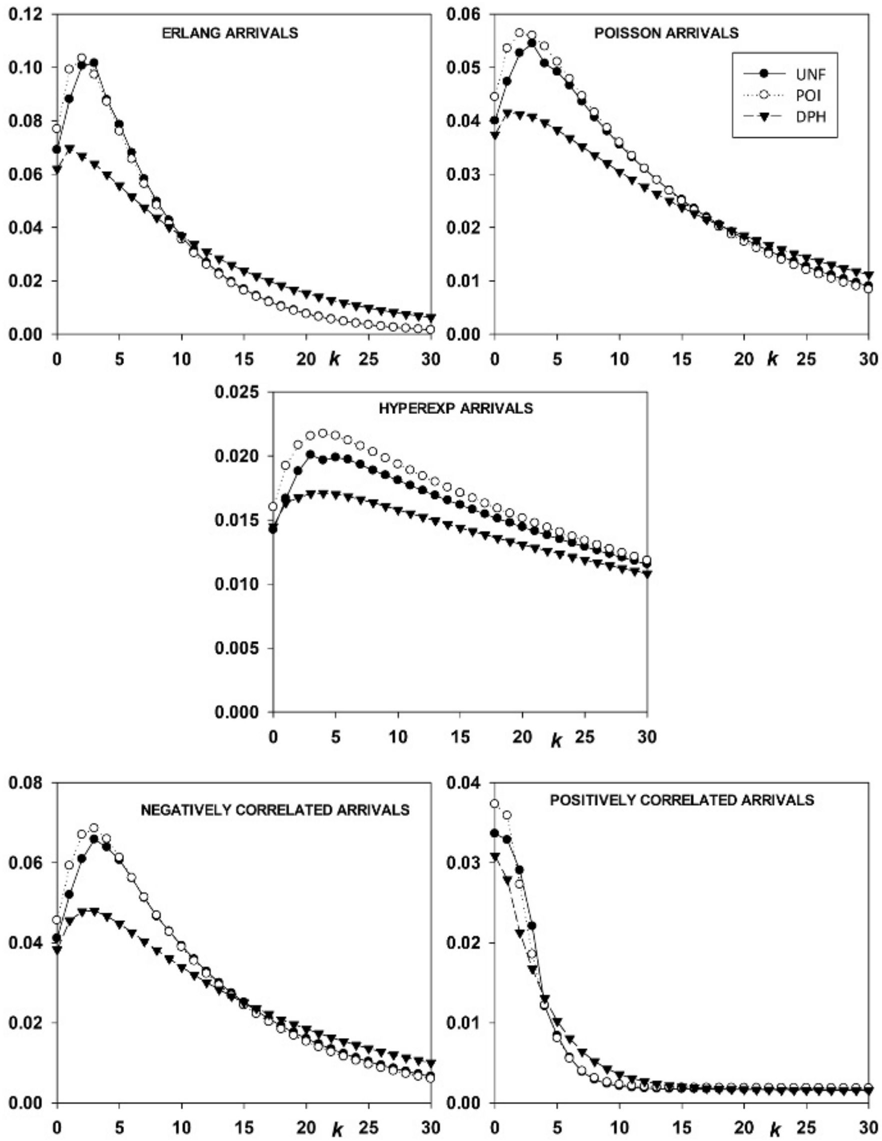

Figure 1.8. *Plots of steady-state probabilities at departure epochs for example 1.7*

EXAMPLE 1.8.– In this example, we look at BMAP arrivals as given in example 1.6 and consider two service time distributions. In the first one, we take the services to follow an arbitrary discrete probability function denoted as ARS for ease of reference in the table and figure below. Denoting by Y, the service time, we take $P(Y = 0.7) = 0.8, P(Y = 1.3) = 0.15$ and $P(Y = 2.9) = 0.05$. In the second one, we look at uniform services (UFS) on the interval $(0.4, 1.4)$.

Note that these two services have a mean of 0.9. However, the standard deviations are, respectively, given by 0.50596 and 0.28868. For batch sizes we look at uniform (UB) on {1, 2, 3, 4} as well as Poisson (PB) with parameter 1.25.

In Table 1.8, selected queue length statistics are displayed and in Figure 1.9 the plot of the steady-state vector under various scenarios is given.

$Service$	$Batch$	$MEASURE$	ERL	EXP	HEX	NCA	PCA
ARS	UB	$MEAN$	7.426	15.159	44.916	12.129	421.815
		$MEDIAN$	6	11	35	9	274
		$MODE$	2	3	3	3	0
		$MVALUE$	0.0906	0.0516	0.02	0.0615	0.0332
		SD	7.828	15.885	45.791	12.557	476.394
	PB	$MEAN$	5.998	11.787	37.36881	9.421	379.494
		$MEDIAN$	6	11	29	9	264
		$MODE$	2	2	3	3	0
		$MVALUE$	0.0916	0.0537	0.0213	0.0632	0.0368
		SD	6.618	14.376	43.781	11.115	430.142
UFS	UB	$MEAN$	7.428	15.362	45.298	12.292	424.522
		$MEDIAN$	5	11	32	8	269
		$MODE$	3	3	3	3	0
		$MVALUE$	0.0971	0.0534	0.02	0.0641	0.0334
		SD	7.031	15.076	44.930	11.719	485.652
	PB	$MEAN$	7.253	14.391	41.391	11.624	382.641
		$MEDIAN$	5	10	29	8	242
		$MODE$	2	2	4	3	0
		$MVALUE$	0.0988	0.0553	0.0215	0.0664	0.0371
		SD	7.0320	14.2090	41.1090	11.2050	437.6460

Table 1.8. *Selected queue length statistics at departure epochs for $BMAP/G/1$ queue of example 1.8*

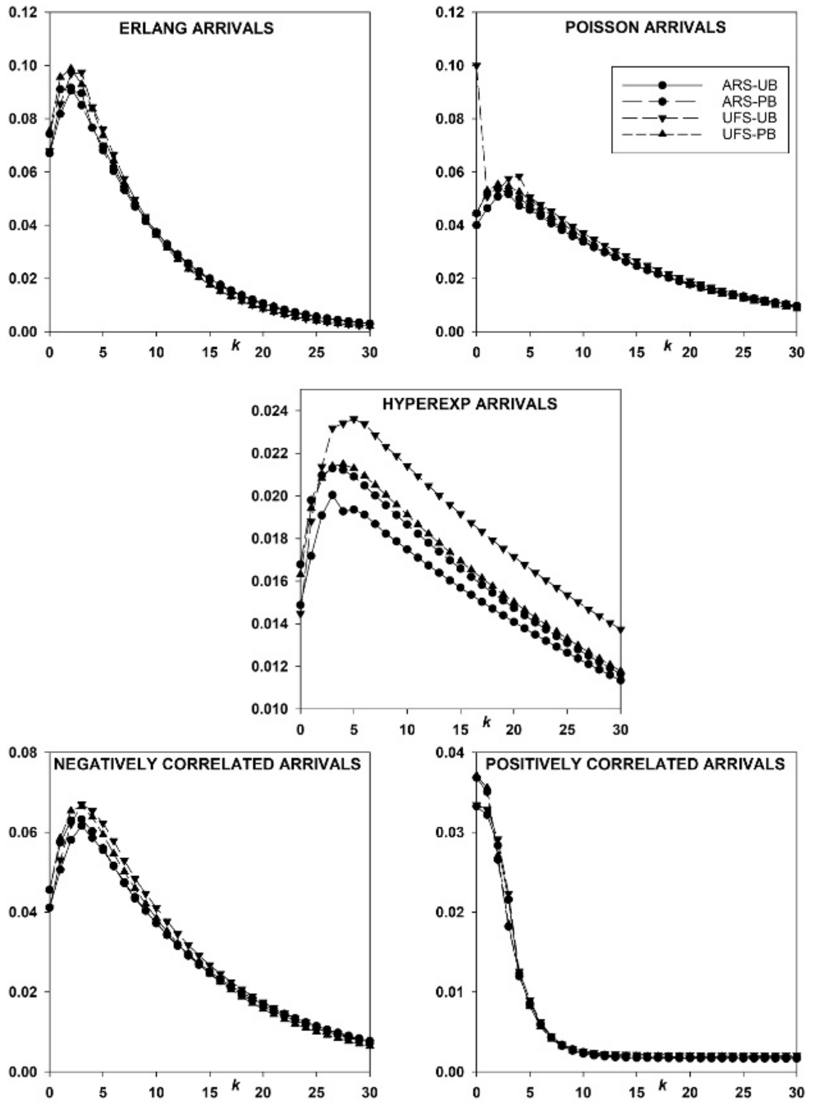

Figure 1.9. *Plots of steady-state probabilities at departure epochs for example 1.8*

EXAMPLE 1.9.– In this example, we look at $BMAP/PH/1$-type queues by referring to example 1.5, but here the service times are modeled using Erlang of order 3 with parameter $\frac{4}{3}$. The arrival rate, λ_g, of the groups will be normalized such that $\lambda = 1$. The service rate will be normalized so that the mean service time will be 0.9. Thus, in this example, arrivals occur in batches, which are modeled using Poisson distribution with parameter 1.25, and due to the shift in the masses the mean batch size is 2.25. We refer the reader to the representations given in example 1.5.

For this example, G and K matrices along with their invariant vectors, $\boldsymbol{g}$ and $\boldsymbol{\kappa}$, are:

$$G^{(ER)} = \begin{pmatrix} 0.3907 & 0.3636 & 0.2457 \\ 0.0574 & 0.4787 & 0.4639 \\ 0.1083 & 0.2235 & 0.6682 \end{pmatrix}, \quad K^{(ER)} = \begin{pmatrix} 0.2335 & 0.3580 & 0.4085 \\ 0.2335 & 0.3580 & 0.4085 \\ 0.2335 & 0.3580 & 0.4085 \end{pmatrix},$$

$$\boldsymbol{g}^{(ER)} = (0.1279, 0.3241, 0.5479), \quad \boldsymbol{\kappa}^{(ER)} = (0.2335, 0.358, 0.4085),$$

$$G^{(HE)} = \begin{pmatrix} 0.7321 & 0.0927 & 0.1752 \\ 0.1031 & 0.7930 & 0.1039 \\ 0.0242 & 0.0129 & 0.9629 \end{pmatrix}, \quad K^{(HE)} = \begin{pmatrix} 0.4601 & 0.2070 & 0.3329 \\ 0.4601 & 0.2070 & 0.3329 \\ 0.4601 & 0.2070 & 0.3329 \end{pmatrix},$$

$$\boldsymbol{g}^{(HE)} = (0.1095, 0.0985, 0.792), \quad \boldsymbol{\kappa}^{(HE)} = (0.4601, 0.207, 0.3329),$$

$$G^{(NC)} = \begin{pmatrix} 0.6588 & 0.3224 & 0.0188 \\ 0.1076 & 0.8268 & 0.0656 \\ 0.2324 & 0.3127 & 0.4549 \end{pmatrix}, \quad K^{(NC)} = \begin{pmatrix} 0.2777 & 0.4658 & 0.2565 \\ 0.2777 & 0.4658 & 0.2565 \\ 0.4808 & 0.4735 & 0.0457 \end{pmatrix},$$

$$\boldsymbol{g}^{(NC)} = (0.264, 0.6488, 0.0872), \quad \boldsymbol{\kappa}^{(NC)} = (0.3207, 0.4674, 0.2119),$$

$$G^{(PC)} = \begin{pmatrix} 0.6763 & 0.3234 & 0.0003 \\ 0.1683 & 0.8306 & 0.0011 \\ 0.1623 & 0.3079 & 0.5298 \end{pmatrix}, \quad K^{(PC)} = \begin{pmatrix} 0.5199 & 0.4764 & 0.0037 \\ 0.5199 & 0.4764 & 0.0037 \\ 0.2455 & 0.4626 & 0.2919 \end{pmatrix},$$

$$\boldsymbol{g}^{(PC)} = (0.3419, 0.6562, 0.0017), \quad \boldsymbol{\kappa}^{(PC)} = (0.5185, 0.4763, 0.0052),$$

The plots of the steady-state probabilities at various epochs of the five arrival processes are displayed in Figure 1.10, and selected performance measures are listed in Table 1.9.

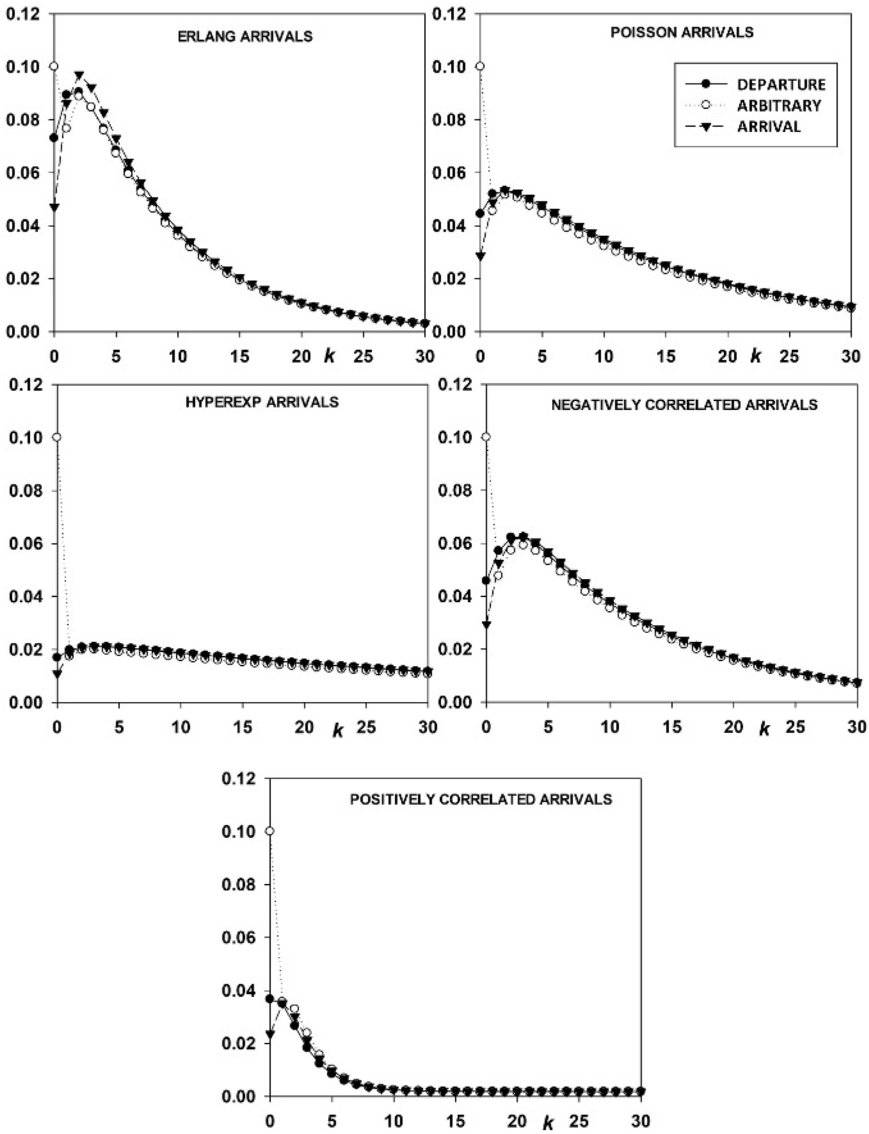

Figure 1.10. *Plots of steady-state probabilities at various epochs for example 1.9*

Type of BMAP	*Epoch Type*	*Mean*	*Median*	*Mode*	*Modal value*	*SD*
ER	*Departure*	8.104	6	2	0.09	7.885
	Arbitrary	7.934	6	0	0.1	7.889
	Arrival	9.456	7	3	0.091	7.913
EX	*Departure*	15.212	11	2	0.053	14.947
	Arbitrary	14.31	10	0	0.1	14.909
	Arrival	16.558	12	3	0.053	14.953
HE	*Departure*	42.164	29	3	0.021	41.454
	Arbitrary	38.536	26	0	0.1	41.303
	Arrival	43.495	31	4	0.021	41.421
NC	*Departure*	12.449	9	3	0.062	11.995
	Arbitrary	11.832	8	0	0.1	12.006
	Arrival	13.801	10	4	0.063	12.019
PC	*Departure*	380.543	243	0	0.037	429.336
	Arbitrary	342.823	197	0	0.1	422.251
	Arrival	381.582	244	2	0.035	428.5350

Table 1.9. *Selected quality statistics for $BMAP/ER/1$ queue of example 1.9*

EXAMPLE 1.10.– In this example, we look at $BMAP/PH/1$-type queues by referring to example 1.9, but here the service times are modeled using hyperexponential with parameters $\frac{19}{9}$ and $\frac{19}{90}$ with probabilities, respectively, given by 0.9 and 0.1. Note that the mean and the standard deviation of the services are 0.9 and 2.0203, respectively. The arrival rate, λ_g, of the groups will be normalized such that $\lambda = 1$. The service rate will be normalized so that the mean service time will be 0.9.

Note first that in this example the arrivals occur in batches, which are modeled using a Poisson distribution with parameter 1.25. We refer the reader to the representations given in example 1.5. Here, G and K matrices along with their invariant vectors, $\boldsymbol{g}$ and $\boldsymbol{\kappa}$, are:

$$G^{(ER)} = \begin{pmatrix} 0.5942 & 0.2669 & 0.1391 \\ 0.0556 & 0.6404 & 0.3040 \\ 0.1216 & 0.1567 & 0.7217 \end{pmatrix}, \quad K^{(ER)} = \begin{pmatrix} 0.3999 & 0.3327 & 0.2674 \\ 0.3999 & 0.3327 & 0.2674 \\ 0.3999 & 0.3327 & 0.2674 \end{pmatrix},$$

$$\boldsymbol{g}^{(ER)} = (0.1875, 0.3435, 0.4689), \quad \boldsymbol{\kappa}^{(ER)} = (0.3999, 0.3327, 0.2674),$$

$$G^{(HE)} = \begin{pmatrix} 0.8517 & 0.0614 & 0.0869 \\ 0.0961 & 0.8464 & 0.0575 \\ 0.0259 & 0.0110 & 0.9631 \end{pmatrix}, \quad K^{(HE)} = \begin{pmatrix} 0.6044 & 0.1930 & 0.2026 \\ 0.6044 & 0.1930 & 0.2026 \\ 0.6044 & 0.1930 & 0.2026 \end{pmatrix},$$

$$\boldsymbol{g}^{(HE)} = (0.2004, 0.1280, 0.6716), \quad \boldsymbol{\kappa}^{(HE)} = (0.6044, 0.1930, 0.2026),$$

$$G^{(NC)} = \begin{pmatrix} 0.7695 & 0.2098 & 0.0207 \\ 0.0832 & 0.8471 & 0.0697 \\ 0.1923 & 0.1731 & 0.6346 \end{pmatrix}, \quad K^{(NC)} = \begin{pmatrix} 0.2773 & 0.3024 & 0.4203 \\ 0.2773 & 0.3024 & 0.4203 \\ 0.6065 & 0.3425 & 0.0510 \end{pmatrix},$$

$$\boldsymbol{g}^{(NC)} = (0.3089, 0.5658, 0.1253), \quad \boldsymbol{\kappa}^{(NC)} = (0.3784, 0.3147, 0.3069),$$

$$G^{(PC)} = \begin{pmatrix} 0.7850 & 0.2145 & 0.0005 \\ 0.1373 & 0.8611 & 0.0015 \\ 0.0947 & 0.1443 & 0.7611 \end{pmatrix}, \quad K^{(PC)} = \begin{pmatrix} 0.6394 & 0.3537 & 0.0069 \\ 0.6394 & 0.3537 & 0.0069 \\ 0.1764 & 0.2642 & 0.5594 \end{pmatrix},$$

$$\boldsymbol{g}^{(PC)} = (0.3892, 0.6061, 0.0047), \quad \boldsymbol{\kappa}^{(PC)} = (0.6323, 0.3523, 0.0154).$$

Selected performance measures are listed in Table 1.10, and the plots of the steady-state probabilities at various epochs of the five arrival processes are displayed in Figure 1.11.

Type of BMAP	*Epoch Type*	*Mean*	*Median*	*Mode*	*Modal value*	*SD*
ER	*Departure*	26.909	17	0	0.063	29.763
	Arbitrary	26.719	17	0	0.100	29.794
	Arrival	28.26	19	2	0.060	29.782
EX	*Departure*	34.106	22	1	0.045	36.734
	Arbitrary	33.204	21	0	0.100	36.717
	Arrival	35.452	24	2	0.044	36.736
HE	*Departure*	62.614	42	1	0.024	64.93
	Arbitrary	58.909	38	0	0.100	64.687
	Arrival	63.94	44	2	0.024	64.882
NC	*Departure*	31.251	20	1	0.047	33.835
	Arbitrary	30.626	20	0	0.100	33.847
	Arrival	32.597	22	2	0.046	33.838
PC	*Departure*	399.891	256	0	0.034	449.547
	Arbitrary	362.204	210	0	0.100	442.603
	Arrival	400.95	257	2	0.031	448.791

Table 1.10. *Selected measures for $BMAP/HE/1$ queue*

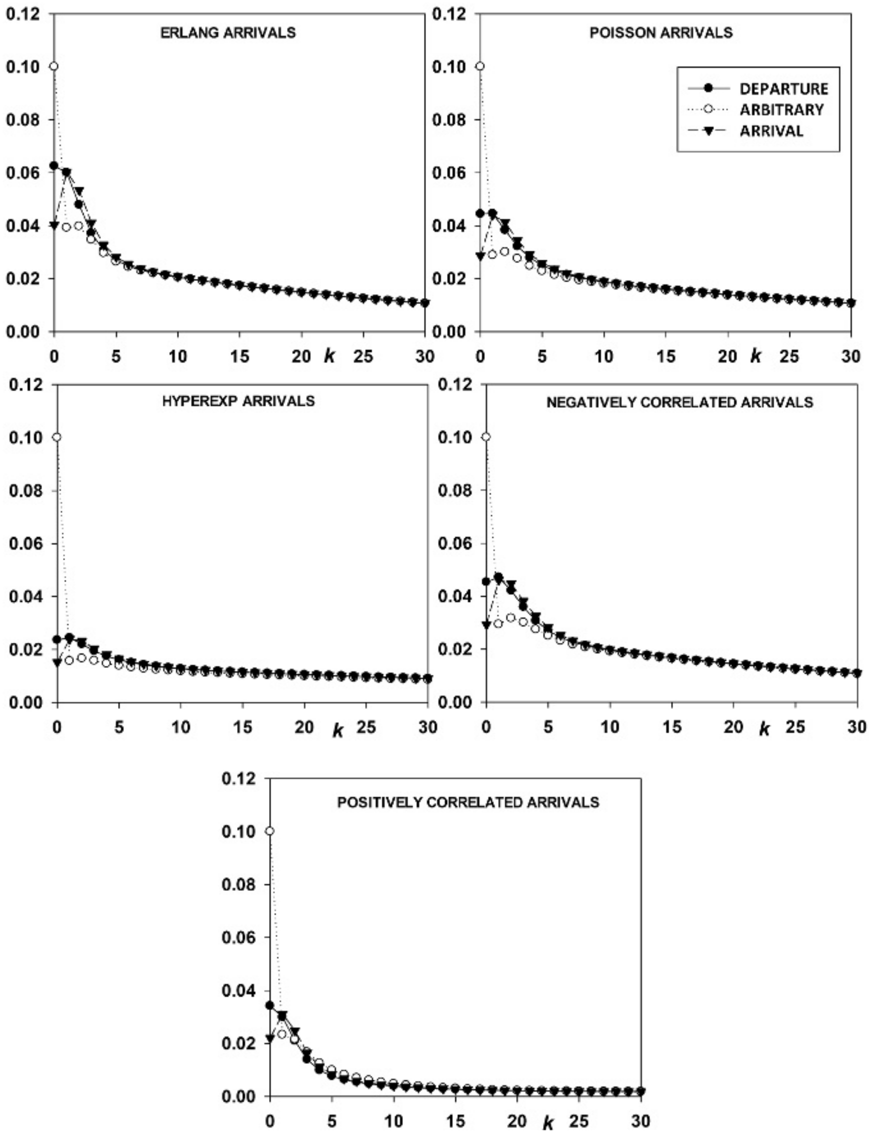

Figure 1.11. *Plots of steady-state probabilities at various epochs for example 1.10*

EXAMPLE 1.11.– In this example, we look at $BMAP/G/1$-type queues by looking at a batch MMPP and a batch MSPP for the arrivals and for the services we look at three cases: (1) CTS: constant services with a mean of 0.8; (2) ERS Erlang of order 3 with rate 3.75 in each state; (c) HES: hyperexponential with rates 2.625, 1.3125 and 0.2625 with mixing probabilities, respectively, given by 0.7, 0.2 and 0.1. Note that these three have the same mean of 0.8 and the standard deviations are 0, 0.46188 and 1.64251, respectively. The representations of the batch MMPP and the batch MSPP are as follows:

$$D_0 = \begin{pmatrix} -0.26345 & 0.03764 & 0.01882 & 0.01882 \\ 0.05645 & -0.56454 & 0.05645 & 0.07527 \\ 0 & 0.03764 & -0.65863 & 0.05645 \\ 0.01882 & 0.05645 & 0.07527 & -0.90326 \end{pmatrix},$$

$$D_k = a_k \begin{pmatrix} 0.18817 & 0 & 0 & 0 \\ 0 & 0.37637 & 0 & 0 \\ 0 & 0 & 0.56454 & 0 \\ 0 & 0 & 0 & 0.75272 \end{pmatrix},$$

$$D_0 = \begin{pmatrix} -1.20112 & 0 & 0 & 0 \\ 0 & -0.90085 & 0 & 0 \\ 0 & 0 & -0.60057 & 0 \\ 0 & 0 & 0 & -0.30029 \end{pmatrix},$$

$$D_k = a_k \begin{pmatrix} 0.30028 & 0.30028 & 0.30028 & 0.30028 \\ 0 & 0.30028 & 0 & 0.60057 \\ 0.06006 & 0.12011 & 0.06006 & 0.36034 \\ 0.09009 & 0.06006 & 0.12011 & 0.03003 \end{pmatrix}.$$

These two arrival processes have the same (batch) arrival rate of 0.5. However, MMPP process has a standard deviation of 2.37064 and its 1-lag correlation is 0.10657, whereas the standard deviation and the 1-lag correlation of MSPP are 2.49355 and -0.05798, respectively. For the batch size distribution, we look at four types: (1) Poisson (POI) with parameter 1, (2) Geometric (GEO) with parameter 0.5, (3) Uniform (UNF) on $\{1, 2, 3\}$ and (4) DPH with representation given by:

$$\boldsymbol{\alpha} = (0.1, 0.6, 0.3), \quad T = \begin{pmatrix} 0.2 & 0.2 & 0.1 \\ 0.1 & 0.1 & 0.2 \\ 0.1 & 0.1 & 0.4 \end{pmatrix}.$$

Note that the mean batch size for all four distributions is 2.0. In Figures 1.12 and 1.13, respectively, we display the steady-state probability vector at departure epochs for the batch MMPP and for the batch MSPP arrivals. Obviously, one can see the impact of the type of arrivals, services and the batch sizes on this steady-state vector.

In Figure 1.14, we display the steady-state vectors at various epochs for both arrival processes and for CTS so as to compare them.

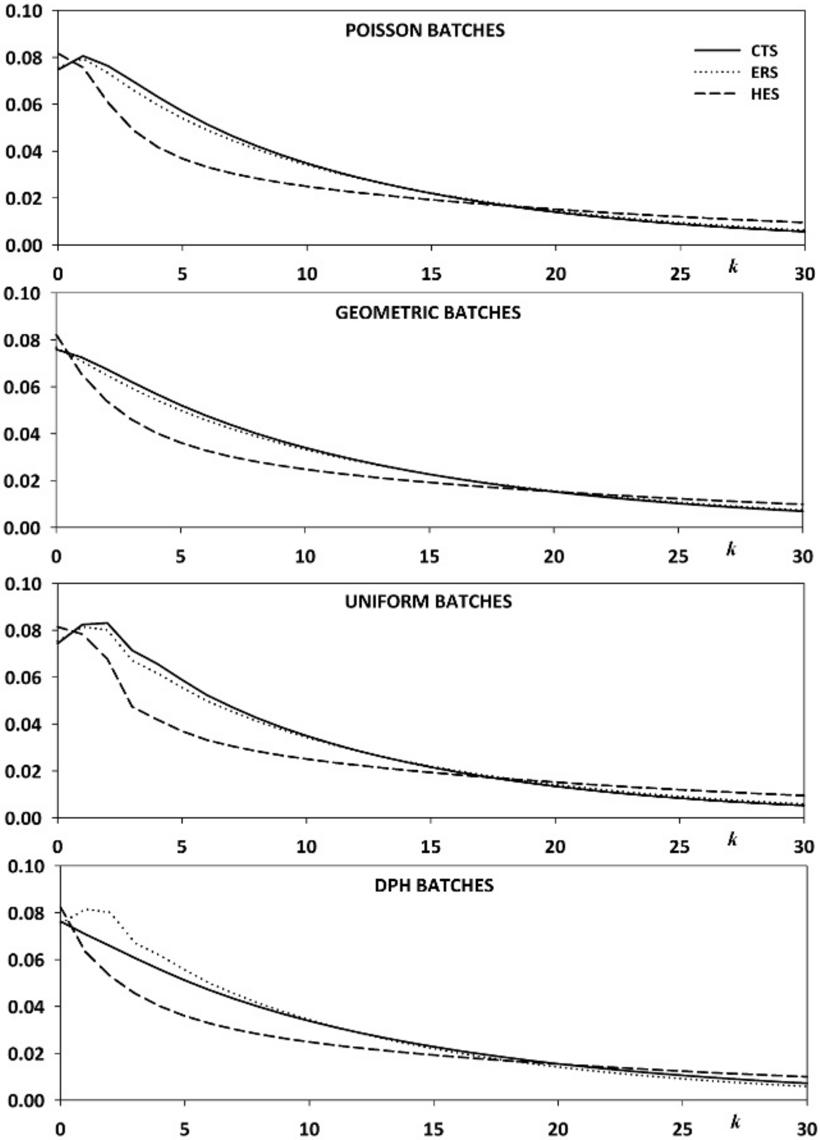

Figure 1.12. *Steady-state probabilities at departure epochs for batch MMPP for example 1.11*

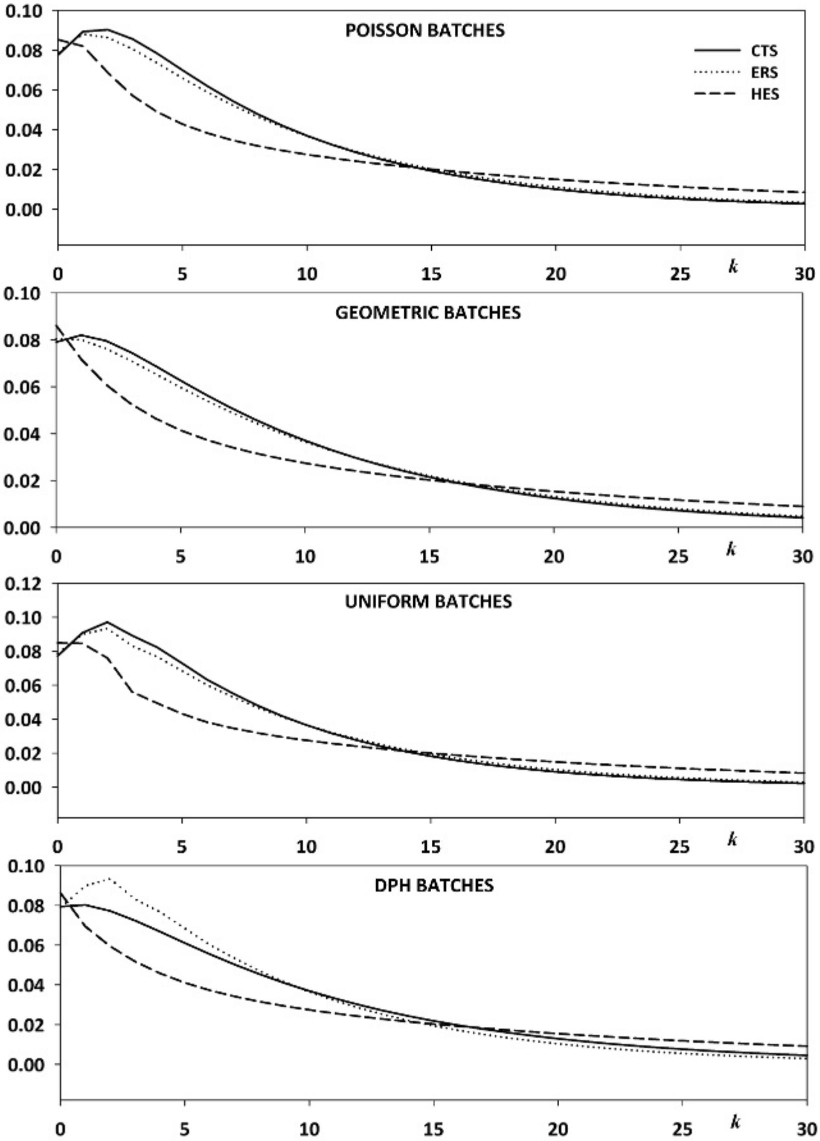

Figure 1.13. *Steady-state probabilities at departure epochs for batch MSPP for example 1.11*

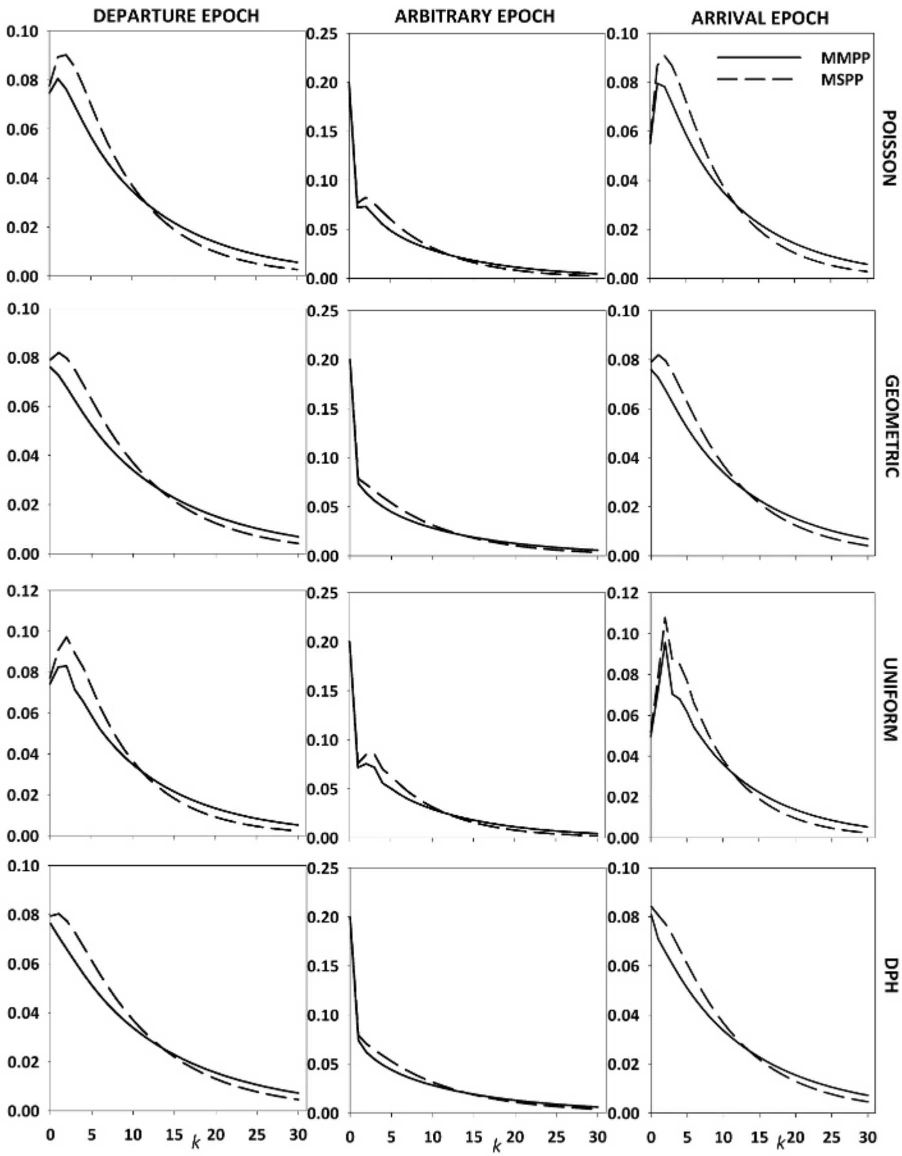

Figure 1.14. *Steady-state probabilities at various epochs for batch MMPP and MSPP and for CTS for example 1.11*

Using the notation $G^{(a,b)}$, where a stands for the arrival process abbreviated as MM for MMPP and MS for MSPP, and b for the type of batch size distribution abbreviated as PB for POI, GB for GEO, UB for UNF and DB for DPH, we display G and K matrices for CTS. Others are left as exercises.

$$G^{(\text{MM, PB})} = \begin{pmatrix} 0.9256 & 0.0353 & 0.0214 & 0.0177 \\ 0.0785 & 0.7826 & 0.0714 & 0.0675 \\ 0.0370 & 0.0713 & 0.8217 & 0.0700 \\ 0.0899 & 0.1071 & 0.1370 & 0.6660 \end{pmatrix}, \quad G^{(\text{MS, PB})} = \begin{pmatrix} 0.4140 & 0.0639 & 0.1126 & 0.4095 \\ 0.0088 & 0.5439 & 0.0452 & 0.4021 \\ 0.0120 & 0.0329 & 0.6667 & 0.2884 \\ 0.0128 & 0.0208 & 0.0504 & 0.9160 \end{pmatrix},$$

$$G^{(\text{MM, GB})} = \begin{pmatrix} 0.9260 & 0.0350 & 0.0213 & 0.0177 \\ 0.0760 & 0.7864 & 0.0701 & 0.0675 \\ 0.0339 & 0.0680 & 0.8286 & 0.0695 \\ 0.0833 & 0.1026 & 0.1334 & 0.6807 \end{pmatrix}, \quad G^{(\text{MS, GB})} = \begin{pmatrix} 0.4182 & 0.0698 & 0.1155 & 0.3965 \\ 0.0098 & 0.5491 & 0.0450 & 0.3961 \\ 0.0135 & 0.0358 & 0.6674 & 0.2833 \\ 0.0145 & 0.0226 & 0.0520 & 0.9109 \end{pmatrix},$$

$$G^{(\text{MM, UB})} = \begin{pmatrix} 0.9255 & 0.0354 & 0.0215 & 0.0176 \\ 0.0794 & 0.7812 & 0.0719 & 0.0675 \\ 0.0382 & 0.0726 & 0.8190 & 0.0703 \\ 0.0925 & 0.1088 & 0.1384 & 0.6603 \end{pmatrix}, \quad G^{(\text{MS, UB})} = \begin{pmatrix} 0.4123 & 0.0616 & 0.1114 & 0.4147 \\ 0.0085 & 0.5418 & 0.0452 & 0.4045 \\ 0.0114 & 0.0318 & 0.6664 & 0.2905 \\ 0.0121 & 0.0200 & 0.0497 & 0.9182 \end{pmatrix},$$

$$G^{(\text{MM, DB})} = \begin{pmatrix} 0.9261 & 0.0349 & 0.0213 & 0.0178 \\ 0.0755 & 0.7873 & 0.0698 & 0.0675 \\ 0.0332 & 0.0673 & 0.8302 & 0.0693 \\ 0.0818 & 0.1016 & 0.1325 & 0.6840 \end{pmatrix}, \quad G^{(\text{MS, DB})} = \begin{pmatrix} 0.4192 & 0.0711 & 0.1161 & 0.3936 \\ 0.0100 & 0.5503 & 0.0449 & 0.3948 \\ 0.0138 & 0.0365 & 0.6675 & 0.2822 \\ 0.0149 & 0.0230 & 0.0524 & 0.9097 \end{pmatrix},$$

$$K^{(\text{MM, PB})} = \begin{pmatrix} 0.6632 & 0.1353 & 0.1183 & 0.0832 \\ 0.1878 & 0.4827 & 0.1855 & 0.1440 \\ 0.0896 & 0.1422 & 0.6355 & 0.1327 \\ 0.1575 & 0.1749 & 0.2319 & 0.4357 \end{pmatrix}, \quad K^{(\text{MS, PB})} = \begin{pmatrix} 0.0705 & 0.1241 & 0.1948 & 0.6106 \\ 0.0154 & 0.1387 & 0.0750 & 0.7709 \\ 0.0378 & 0.1004 & 0.1235 & 0.7383 \\ 0.0816 & 0.1116 & 0.2609 & 0.5459 \end{pmatrix},$$

$$K^{(\text{MM, GB})} = \begin{pmatrix} 0.6633 & 0.1337 & 0.1177 & 0.0853 \\ 0.1796 & 0.4927 & 0.1817 & 0.1460 \\ 0.0823 & 0.1340 & 0.6519 & 0.1318 \\ 0.1445 & 0.1648 & 0.2230 & 0.4676 \end{pmatrix}, \quad K^{(\text{MS, GB})} = \begin{pmatrix} 0.0800 & 0.1346 & 0.1994 & 0.5860 \\ 0.0167 & 0.1510 & 0.0735 & 0.7589 \\ 0.0424 & 0.1087 & 0.1246 & 0.7243 \\ 0.0927 & 0.1207 & 0.2693 & 0.5173 \end{pmatrix},$$

$$K^{(\text{MM, UB})} = \begin{pmatrix} 0.6633 & 0.1359 & 0.1185 & 0.0823 \\ 0.1910 & 0.4789 & 0.1869 & 0.1432 \\ 0.0924 & 0.1456 & 0.6290 & 0.1330 \\ 0.1628 & 0.1790 & 0.2352 & 0.4230 \end{pmatrix}, \quad K^{(\text{MS, UB})} = \begin{pmatrix} 0.0669 & 0.1199 & 0.1929 & 0.6203 \\ 0.0148 & 0.1338 & 0.0755 & 0.7759 \\ 0.0360 & 0.0971 & 0.1229 & 0.7440 \\ 0.0774 & 0.1080 & 0.2576 & 0.5570 \end{pmatrix},$$

$$K^{(\text{MM, DB})} = \begin{pmatrix} 0.6633 & 0.1333 & 0.1176 & 0.0858 \\ 0.1778 & 0.4950 & 0.1809 & 0.1463 \\ 0.0807 & 0.1322 & 0.6556 & 0.1315 \\ 0.1416 & 0.1626 & 0.2210 & 0.4748 \end{pmatrix}, \quad K^{(\text{MS, DB})} = \begin{pmatrix} 0.0821 & 0.1369 & 0.2004 & 0.5806 \\ 0.0170 & 0.1537 & 0.0730 & 0.7563 \\ 0.0434 & 0.1106 & 0.1247 & 0.7213 \\ 0.0951 & 0.1227 & 0.2712 & 0.5110 \end{pmatrix}.$$

While we see the differences in the two arrival processes under various scenarios with regard to the steady-state probabilities at various epochs, the picture is even more clear when looking at the G and K matrices. For example, the entries of G matrices indicate a vast difference when going from MMPP to MSPP; however, they appear to be similar when looking at different batch size distributions when the arrival process is fixed. This is an interesting case and worth further investigation.

A number of queueing models can be seen as special cases of $BMAP/G/1$ and in some cases simplifications occur. We explore them in the next few sections and point out where additional simplifications are possible analytically. Where such simplifications are not possible, we add more illustrative examples that are not provided in the previous sections.

1.2. *MAP/G/*1 queue

Here, the arrivals occur singly resulting in many simplifications mainly for numerical implementation. That is, here we have $D_k = 0,\ k \geq 2$, and hence $B_n = (-D_0)^{-1} D_1 A_n,\ n \geq 0$. The single arrivals make it easy to obtain the measures numerically as compared to the group arrivals. Also, the vector $\boldsymbol{y}$ is simplified as:

$$\boldsymbol{y}_0 = (1-\rho)\boldsymbol{g},\ \ \boldsymbol{y}_k = [\boldsymbol{y}_{k-1} D_1 - \lambda(\boldsymbol{x}_{k-1} - \boldsymbol{x}_k)](-D_0)^{-1},\ k \geq 1. \quad [1.78]$$

REMARK 1.18.– It is easy to verify that the steady-state probabilities at arrival epochs are identical to the ones at departures. To see this, suppose that $\boldsymbol{z} = (z_1, z_2, \cdots)$ denotes the steady-state probability vector at arrivals such that z_i denotes the probability that an arriving customer will see $i,\ i \geq 1$, customers (including the arrived one). Then, we have $z_i = \frac{1}{\lambda}\boldsymbol{y}_{i-1} D_1 \boldsymbol{e},\ i \geq 1$. Now using equations [1.55] and [1.78] (in the case of single arrivals), we get $z_1 = \boldsymbol{x}_0\boldsymbol{e}$. Using the recursive scheme given in equation [1.78], we get the stated result.

EXAMPLE 1.12.– In this example, we look at different negatively as well as positively correlated processes as input to a $MAP/G/1$ queue with service times assumed to be constant (CTS) with a value of 0.9. These MAPs are selected by first setting $\lambda_1 = 1.25 + 0.5(k-2)$ and $\lambda_2 = 2\lambda_1$ and vary k from 2 to 9 (see section 6.3 in Volume 1). Specifically, the matrices D_0 and D_1 are such that

$$D_0 = \begin{pmatrix} E_k & 0 \\ 0 & -\lambda_2 \end{pmatrix},\ \ D_1 = \begin{pmatrix} \lambda_1\, p_1 \boldsymbol{e_1(k)e_1^T(k)} & \lambda_1\, q_1 \boldsymbol{e_1^T(k)} \\ \\ q_2\lambda_2 \boldsymbol{e_1^T(k)} & p_2\lambda_2 \end{pmatrix},$$

where p_i and q_i are such that $0 < p_i < 1, q_i = 1 - p_i,\ i = 1, 2$, and E_k denotes Erlang of order k with parameter λ_1 in each stage. First, verify Table 1.11 containing the values of the standard deviation and the 1-lag correlation coefficient. Also, note that these MAPs are qualitatively different even though they all have the same mean of 1.

k	TaP	$p_1 = p_2$	ρ	σ	TaP	$p_1 = p_2$	ρ	σ
2	NC_1	0.01	-0.3267	1.0392	PC_1	0.99	0.3267	1.0392
3	NC_2	0.01	-0.4804	1.0202	PC_2	0.99	0.4804	1.0202
4	NC_3	0.01	-0.5786	1.0123	PC_3	0.99	0.5786	1.0123
5	NC_4	0.01	-0.6454	1.0082	PC_4	0.99	0.6454	1.0082
6	NC_5	0.01	-0.6935	1.0059	PC_5	0.99	0.6935	1.0059
7	NC_6	0.01	-0.7296	1.0044	PC_6	0.99	0.7296	1.0044
8	NC_7	0.01	-0.7577	1.0035	PC_7	0.99	0.7577	1.0035
9	NC_8	0.01	-0.7802	1.0028	PC_8	0.99	0.7802	1.0028

Table 1.11. *Selected MAPs with* $\lambda_1 = 1.25 + 0.5(k-2)$ *and* $\lambda_2 = 2\lambda_1$

The graphs of the steady-state probability vector at departure epochs are plotted in Figure 1.15.

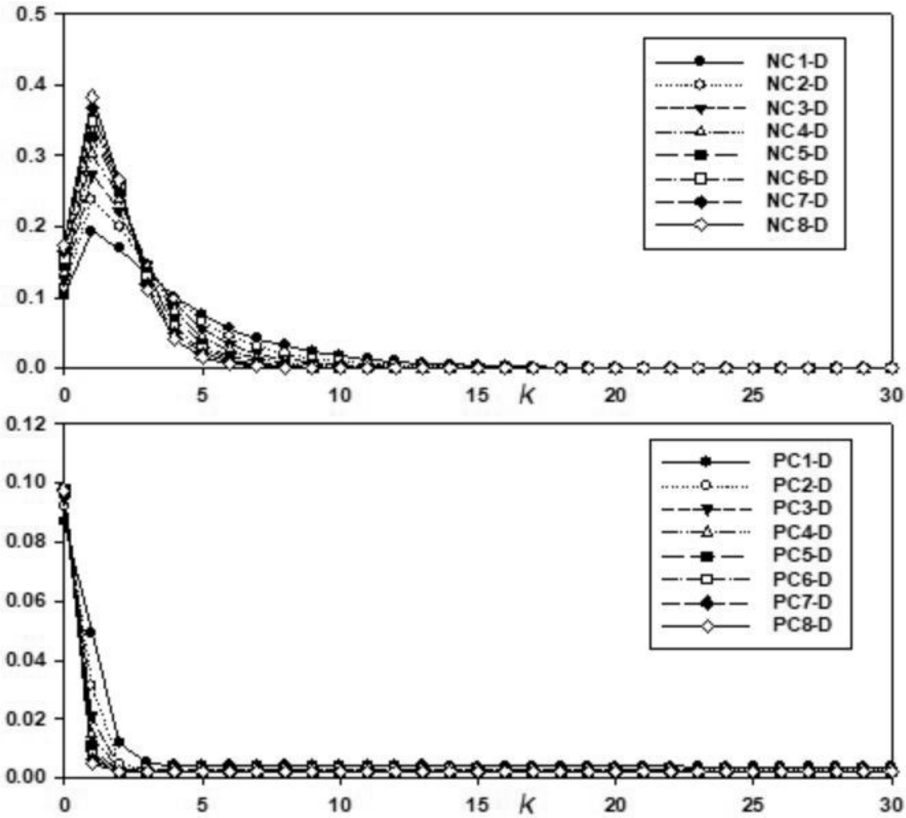

Figure 1.15. *Plots of steady-state probabilities at departure epochs for example 1.12*

These graphs clearly illustrate the significant role played by the (1-lag) correlation. For example, the tail probabilities for the negatively correlated arrivals tend to decay faster to zero, whereas that is not the case for the positively correlated ones.

EXAMPLE 1.13.– This example is similar to example 1.12 except now the services are modeled using an arbitrary distribution (ABS). Denoting by Y, the service time, we take $P(Y = 0.7) = 0.8, P(Y = 1.3) = 0.15$ and $P(Y = 2.9) = 0.05$. Verify that the mean service time is 0.9 and the standard deviation is 0.50596. The graphs of the steady-state probability vector at departure epochs are plotted in Figure 1.16.

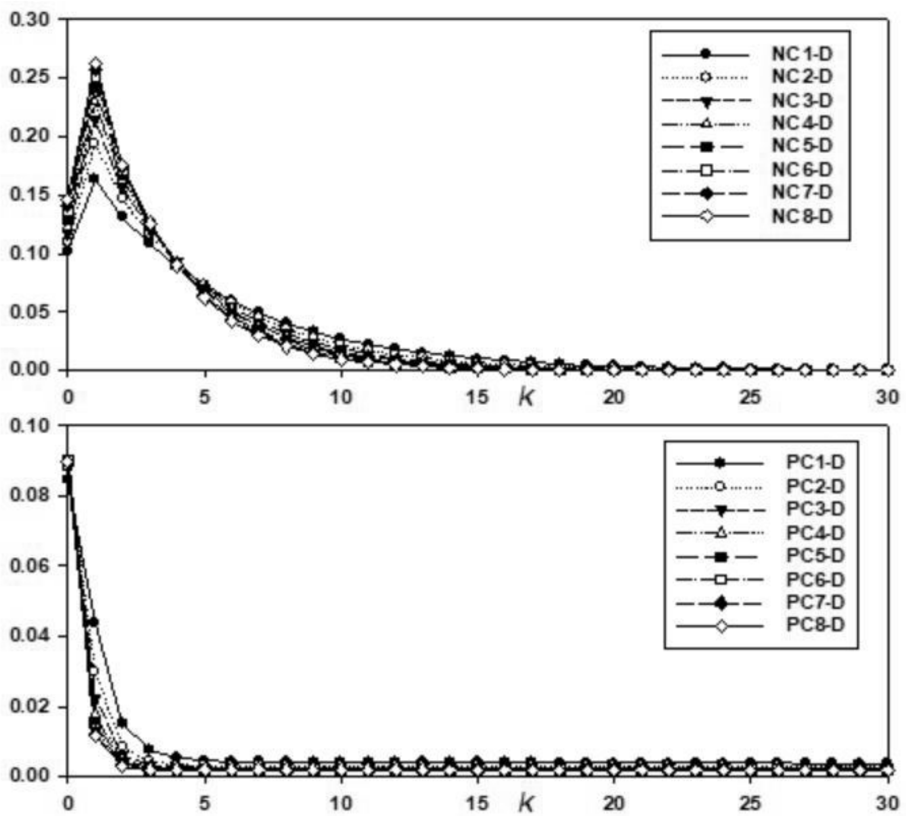

Figure 1.16. *Plots of steady-state probabilities at departure epochs for example 1.13*

Like in example 1.12, we see a similar trend indicating the significant role played by the (1-lag) correlation.

EXAMPLE 1.14.– Here again the example is similar to example 1.13 except that the service times are now uniform (UFS) on the interval $(0.4, 1.4)$ services. Verify that the mean and the standard deviation of the services are, respectively, given by 0.9 and 0.28868. The graphs of the steady-state probability vector at departure epochs are plotted in Figure 1.17.

It is obvious to infer from these figures that the positively correlated arrivals have a larger mean number and a higher median values as compared to those of the negatively correlated arrivals. Furthermore, in the case of positively correlated arrivals, the median values appear to be insensitive to the type of services; however, the median values depend on the magnitude of the correlation.

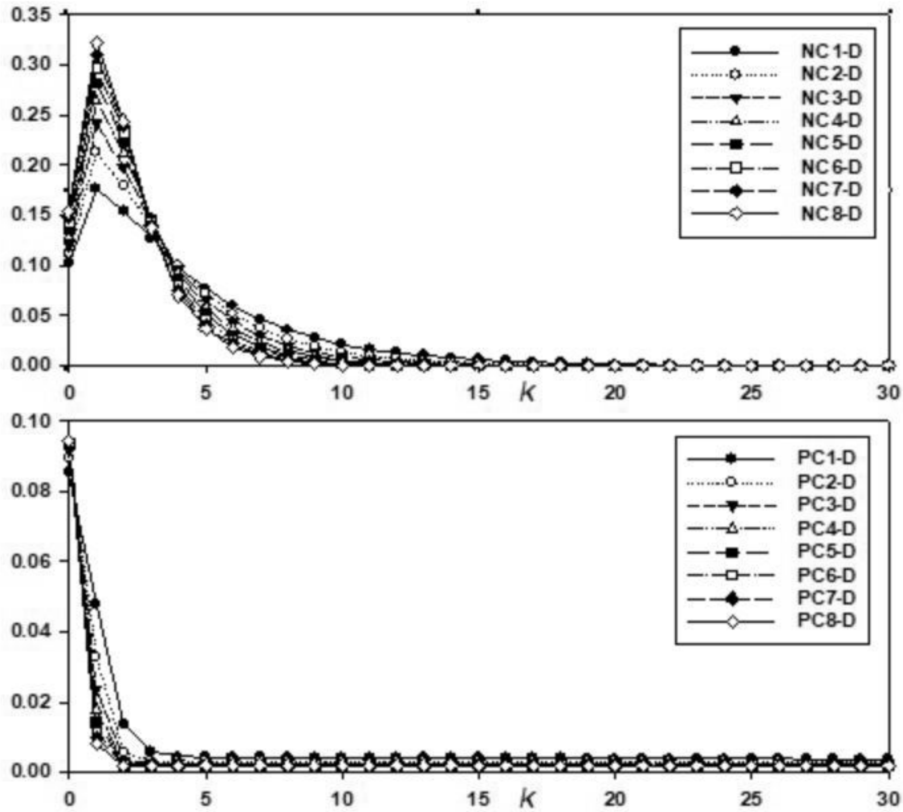

Figure 1.17. *Plots of steady-state probabilities at departure epochs for example 1.14*

EXAMPLE 1.15.– In this example, we compare the impact of the service time distributions, namely, CTS, ABS and UFS, considered in examples 1.12 through 1.14. Note that these services all have a mean of 0.9 but with different coefficient of variation. The graphs of the natural log of the mean number in system and the median of the steady-state vector at departure epochs are plotted in Figures 1.18 and 1.19, respectively.

EXAMPLE 1.16.– In this example, we look at the case of single arrivals with inter-arrivals times that are either negatively or positively correlated. The services are modeled using the minimum of three distributions: exponential with parameter 1, Erlang of order 2 with parameter 2 and a hyperexponential distribution with parameters 10 and 1 with mixing probabilities 0.9 and 0.1. Recall from section 4.1 (see result 4.8) in Volume 1 that this minimum is a CPH with:

$$\boldsymbol{\beta} = (0.9, 0.1, 0, 0), \quad S = \begin{pmatrix} -1.68584 & 0 & 0.68810 & 0 \\ 0 & -1.09752 & 0 & 0.68810 \\ 0 & 0 & -1.68584 & 0 \\ 0 & & 0 & -1.09752 \end{pmatrix}.$$

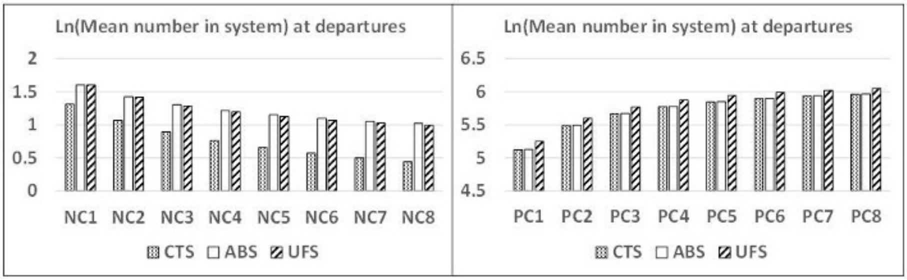

Figure 1.18. *Ln(mean number in system) at departure epochs for example 1.15*

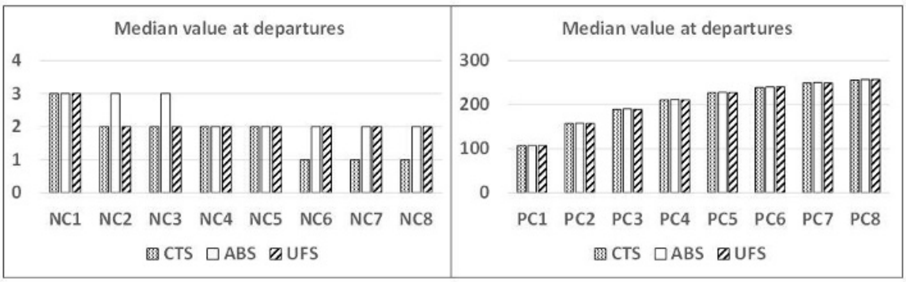

Figure 1.19. *Median of the number in system at departure epochs for example 1.15*

Verify that the above CPH has a mean of 0.9 and a standard deviation of 0.84534. The plots of the steady-state probabilities at departure epochs for the eight pairs of negatively and positively correlated MAPs (see example 1.12 for the details on these MAPs) are given in Figure 1.20. Also, in Table 1.12 we list the various measures under various scenarios.

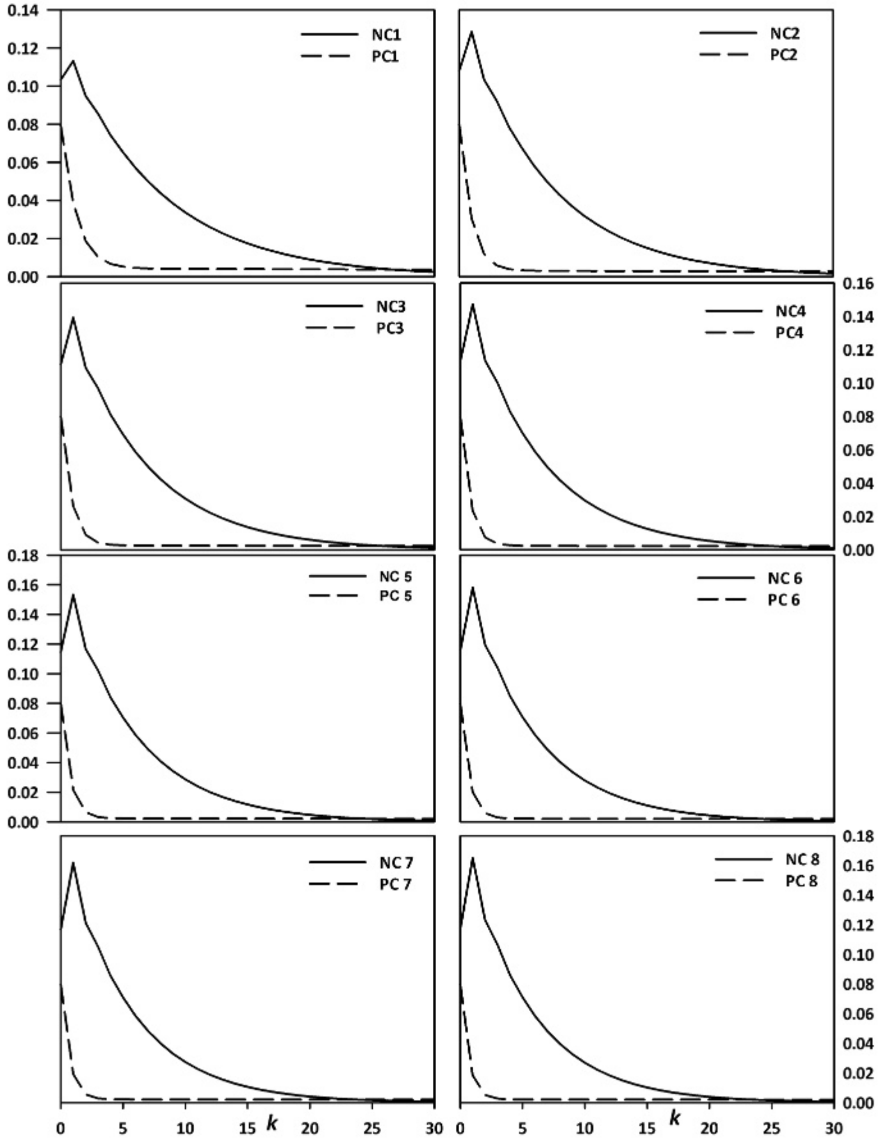

Figure 1.20. *Plots of the steady-state probabilities under various scenarios for example 1.16*

$Type\ of\ BMAP$	CC	$Mean$	$Median$	$Mode$	$Modal\ value$	SD
NCA	-0.7802	4.923	3	1	0.165	5.035
	-0.7577	5.027	3	1	0.162	5.148
	-0.7296	5.156	4	1	0.158	5.285
	-0.6935	5.321	4	1	0.153	5.458
	-0.6454	5.549	4	1	0.147	5.701
	-0.5786	5.873	4	1	0.139	6.048
	-0.4804	6.36	4	1	0.129	6.552
	-0.3267	7.204	5	1	0.113	7.426
PCA	0.3267	170.874	109	0	0.079	192.741
	0.4804	245.593	159	0	0.08	272.422
	0.5786	293.251	191	0	0.08	322.734
	0.6454	325.864	213	0	0.08	356.996
	0.6935	349.487	230	0	0.08	381.754
	0.7296	367.345	242	0	0.08	400.424
	0.7577	381.315	251	0	0.08	415.016
	0.7802	392.524	259	0	0.08	426.695

Table 1.12. *Selected measures for* $MAP/PH/1$ *queue for example 1.16*

1.3. *PH/G/*1 queue

Here, the arrivals occur singly and according to a CPH distribution with representation given by $(\boldsymbol{\alpha}, T)$ of order m. We assume that there is no mass at the origin so that the inter-arrival times are positive with probability 1. Verify that $D_0 = T$ and $D_1 = \boldsymbol{T^0}\boldsymbol{\alpha}$. The matrix G is obtained as solution to:

$$G = \int_0^\infty exp\{(T + \boldsymbol{T^0}\boldsymbol{\alpha}G)t\}dF(t).$$

The steady-state vector $\boldsymbol{g}$ of G is of the form: $\boldsymbol{g} = [\boldsymbol{\alpha}G(-T)^{-1}\boldsymbol{e}]^{-1}\ \boldsymbol{\alpha}G(-T)^{-1}$. The matrix K is given by $K = \boldsymbol{e}\,\boldsymbol{\alpha}G$ and hence $\boldsymbol{\kappa} = \boldsymbol{\alpha}G$.

Unlike for the $MAP/G/1$ queue, which required an iterative process involving a matrix, here the iterative process is on the vector $\boldsymbol{u}$, which is defined as $\boldsymbol{u} = \boldsymbol{\alpha}G$. With the knowledge of $\boldsymbol{u}$, should there be a need for obtaining G, we can compute it directly. The details are given below.

Algorithm 1.2. Algorithmic steps in computing the matrix G

From the definition of $\boldsymbol{u}$ and the form of G, we get:

$$\boldsymbol{u} = \sum_{n=0}^{\infty} \gamma_n \boldsymbol{\alpha} \left[I + \frac{1}{\theta}(T + \boldsymbol{T^0 u}) \right]^n, \text{ where } \theta = -\min\{T_{i,i}\}. \qquad [1.79]$$

The vector $\boldsymbol{u}$ is first computed using the uniformization method as follows: For a given $\epsilon > 0$, compute γ_n, where $\gamma_n = \int_0^\infty e^{-\theta t} \frac{(\theta t)^n}{k!} dF(t)$, for $0 \leq n \leq n^*$. The truncation point, n^*, is identified as $\sum_{n=0}^{n^*} \gamma_n > 1 - \epsilon$. Define $\boldsymbol{h}_n = \boldsymbol{\alpha} \left[I + \frac{1}{\theta}(T + \boldsymbol{T^0 u}) \right]^n, \ n \geq 0$.

Step 0: Set $k = 0, \boldsymbol{u}^{(0)} = \mathbf{0}$.

Step 1: Set $k \leftarrow k + 1$.

Step 1A: Set $n = 0, \boldsymbol{h}_0 = \boldsymbol{\alpha}$. $\boldsymbol{u}^{(k)} = \gamma_0 \boldsymbol{h}_0$.

Step 1B: $n \leftarrow n + 1$; $\boldsymbol{h}_n = \boldsymbol{h}_{n-1}[I + \frac{1}{\theta}(T + \boldsymbol{T^0 u})]$. $\boldsymbol{u}^{(k)} \rightarrow \boldsymbol{u}^{(k)} + \gamma_n \boldsymbol{h}_n$.

Step 1C: If $n < n^*$ go to step 1B. Otherwise go to step 2.

Step 2: If $||\boldsymbol{u}^{(k)} - \boldsymbol{u}^{(k-1)}||_\infty < \epsilon$ (note one can use the same ϵ used for finding n^* or a different one), go to step 3. If not, go to step 1.

Step 3: If G is required, then it is computed as $G = \sum_{n=0}^{n^*} \gamma_n \left[I + \frac{1}{\theta}(T + \boldsymbol{T^0 u}) \right]^n$. One can employ Horner's rule to compute this efficiently by first computing and storing the matrix $I + \frac{1}{\theta}(T + \boldsymbol{T^0 u})$.

EXAMPLE 1.17.– In this example, we look at a $PH/G/1$-type queue such that we consider Erlangs of order 1 through 10 for arrivals, all of which are normalized to have the arrival rate be 1. We look at three different service times (see examples 1.12–1.14), namely, constant (CTS), arbitrary (ABS) and uniform (UFS). These service time distributions will be parameterized to have a mean of 0.9. However, the standard deviations of these are 0, 0.50596 and 0.28868, respectively. The steady-state probability vectors at departure epochs are plotted in Figure 1.21. The mean and the median of the number in the system at departure epochs under various scenarios are plotted in Figure 1.22.

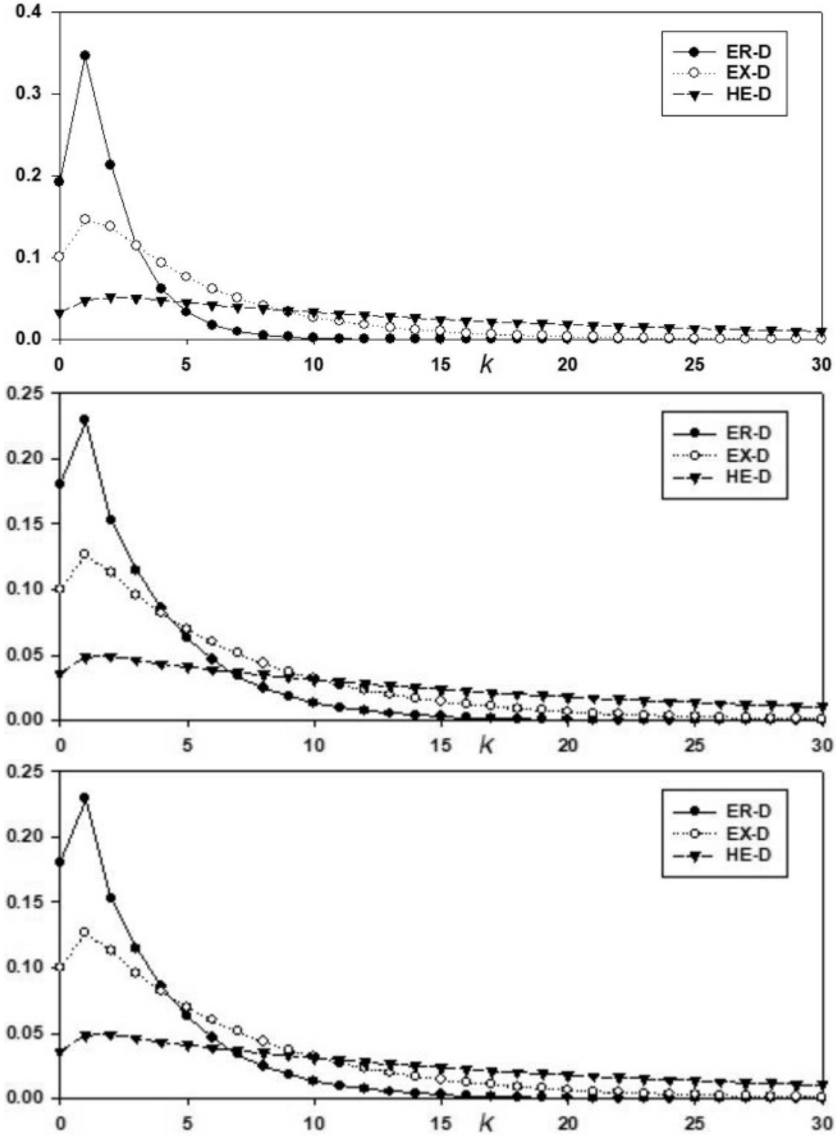

Figure 1.21. *Steady-state vectors at departure epochs for CTS (top), ABS (middle) and UFS (bottom) – example 1.17*

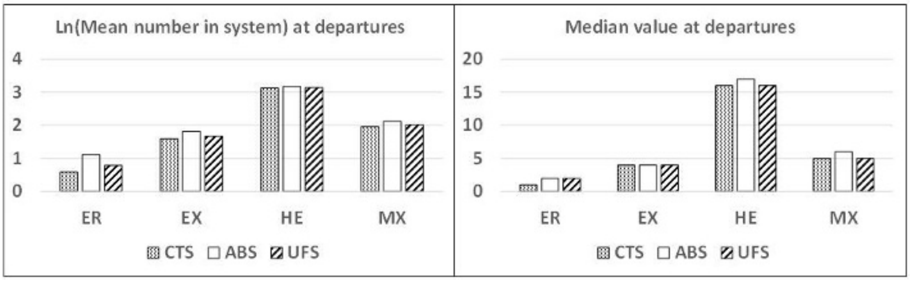

Figure 1.22. *Mean and median of the number in system at departure epochs for example 1.17*

EXAMPLE 1.18.– In this example, we look at four different arrival processes in the context of $PH/G/1$ and with three different service times: constant (CTS), arbitrary (ABS) and uniform (UFS). The four PH distributions considered are Erlang (ER) of order 3; exponential (EX); HE: hyperexponential (HE) with rates 10 and 1 and mixing probabilities 0.9 and 0.1; and a mixture of (EH) of Erlang of order 2 and HE with mixing equal mixing probabilities. These will be normalized to have a mean of 1. For services we look at constant (CTS) with a value of 0.95; arbitrary (ABS) distribution, which has masses at 0.3, 1.02, 2.22 and 3.22 with probabilities, respectively, given by 0.5, 0.3, 0.15 and 0.05; and uniform (UFS) on the interval $(0.45, 1.45)$. Note that all these three service times have a mean of 0.95 and the standard deviations are, respectively, given by 0, 0.28868 and 0.84398. Specifically, the CPH representations for the arrivals are as follows:

$$\mathbf{ER}: \ \boldsymbol{\alpha} = (1,0,0), \ T = \begin{pmatrix} -3 & 3 & 0 \\ 0 & -3 & 3 \\ 0 & 0 & -3 \end{pmatrix}, \mathbf{HE}: \ \boldsymbol{\alpha} = (0.9, 0.1), \ T = \begin{pmatrix} -10 & 0 \\ 0 & -1 \end{pmatrix},$$

$$\mathbf{EX}: \ \boldsymbol{\alpha} = 1, \ T = (-1), \ \mathbf{EH}: \ \boldsymbol{\alpha} = (0.5, 0, 0.45, 0.05), \ T = \begin{pmatrix} -2 & 2 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -10 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

The steady-state probability vectors at departure and at arbitrary epochs are plotted in Figures 1.23 and 1.24, respectively.

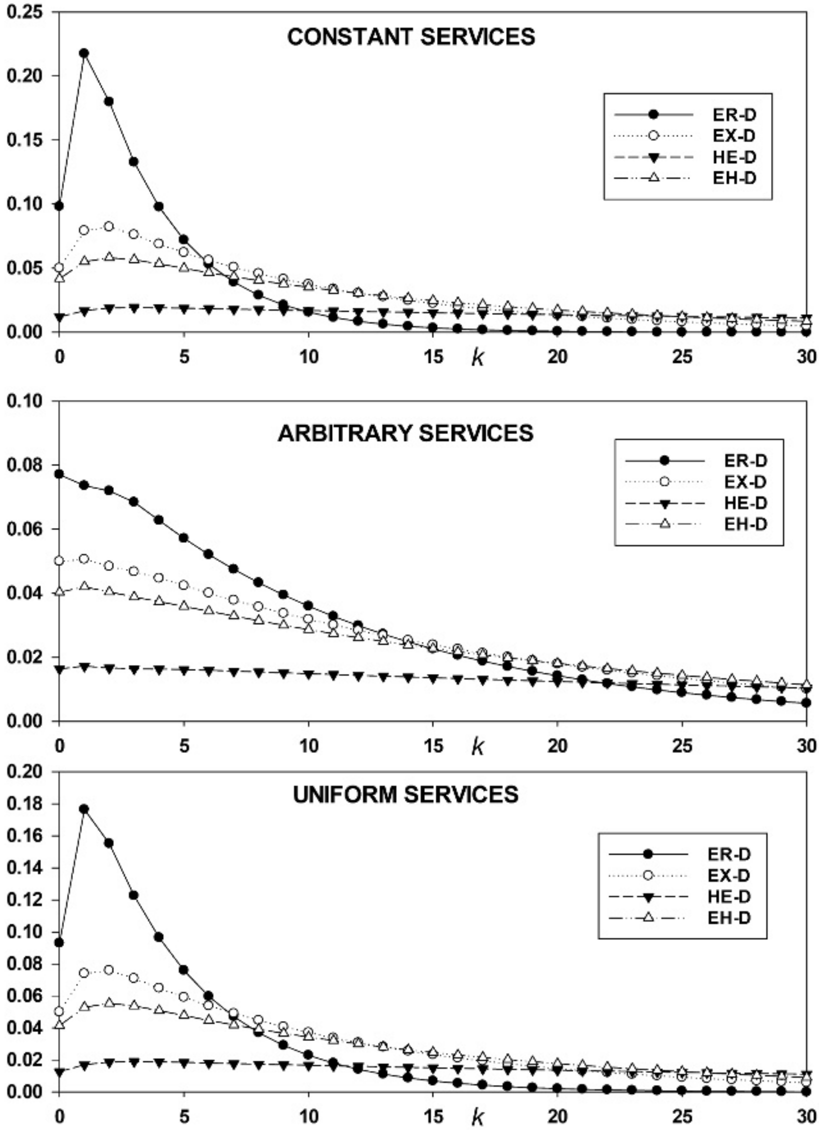

Figure 1.23. *The steady-state vector at departure epochs under various scenarios for example 1.18*

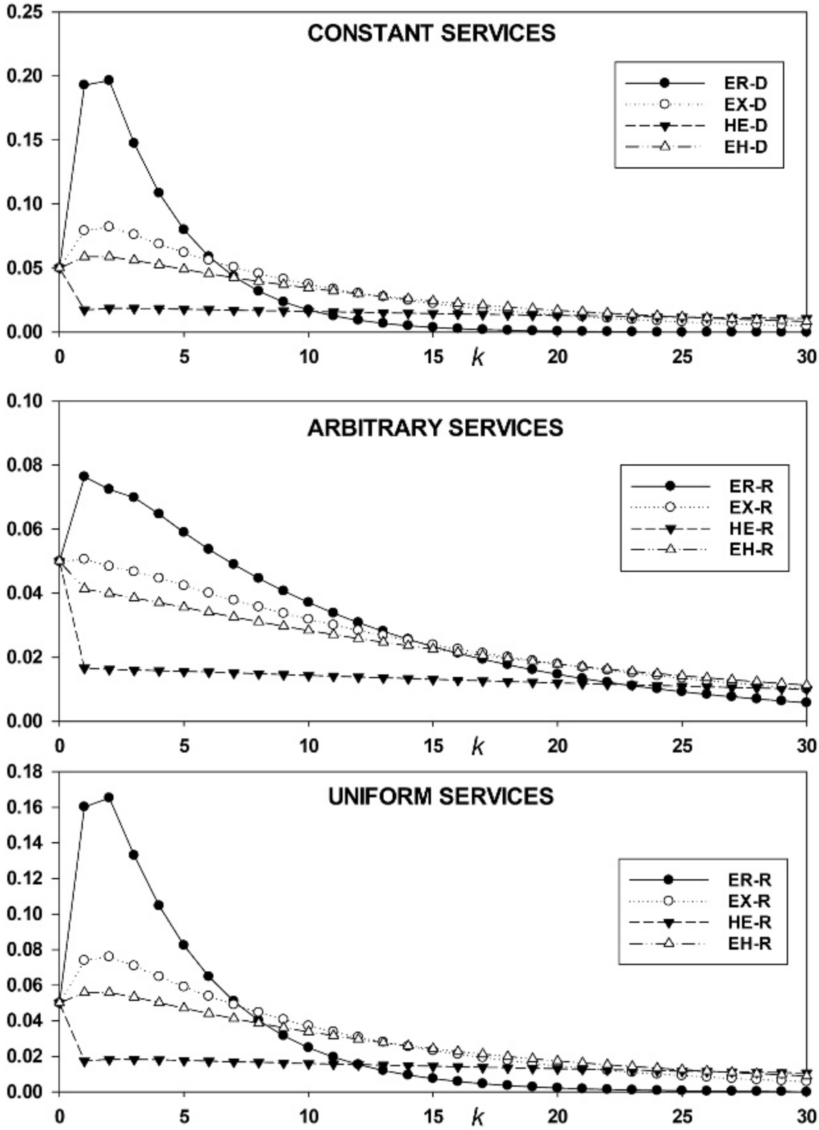

Figure 1.24. *The steady-state vector at arbitrary epochs under various scenarios for example 1.18*

1.4. *M/G/1* queue

The assumptions here are that (1) the arrivals occur singly according to a Poisson process with rate λ and (2) the service times are generally distributed with a finite mean and a finite variance. Let $F(.)$ denote the CDF of the service times and let $\rho = \frac{\lambda}{\mu}$, where μ is the service rate.

It is easy to verify that, for this model, we have (1) $D_0 = -\lambda$ and $D_1 = \lambda$; (2) G is a scalar and obviously equal to 1; (3) $A^*(z, s)$, which becomes a scalar, is given by $A^*(z, s) = f^*(s + \lambda(1 - z))$; and (d) $A(z)$, which is the (scalar) PGF of A_n, is $f^*(\lambda(1 - z))$.

The above observations result in the scalar probabilities:

$$A_n = \int_0^\infty e^{-\lambda t} \frac{(\lambda t)^n}{n!} dF(t), \; n \geq 0, \qquad [1.80]$$

and the PGFs of $\{x_k\}$ and $\{y_k\}$ are identical and are given by:

$$X(z) = Y(z) = \frac{(1 - \rho)(1 - z)A(z)}{A(z) - z}. \qquad [1.81]$$

The LT of the stationary waiting time distribution in the queue of an arriving (same as at an arbitrary time due to Poisson arrivals) customer is given by:

$$w_q^*(s) = s(1 - \rho)[s - \lambda(1 - f^*(s))]^{-1}. \qquad [1.82]$$

Note that $w_q^*(s)$ can be rewritten as:

$$w_q^*(s) = (1 - \rho)\Big[1 - \frac{\lambda(1 - f^*(s))}{s}\Big]^{-1} = (1 - \rho) \sum_{k=0}^{\infty} \rho^k \Big[\frac{\mu(1 - f^*(s))}{s}\Big]^k. \qquad [1.83]$$

Thus, the stationary waiting time distribution in the queue of an arrival in the $M/G/1$ queue is a geometric mixture of successive convolutions of the CDF given by:

$$\mu \int_0^t [1 - F(x)]dx.$$

The mean number, say, μ_{NS}, of customers at departures (as well as at an arbitrary time) in the system is given by (see Bhat (2015)):

$$\mu_{NS} = \rho + \frac{\rho^2(\mu^2\sigma^2 + 1)}{2(1-\rho)}, \qquad [1.84]$$

where σ^2 is the variance of the service times.

EXAMPLE 1.19.– Suppose that the arrivals occur according to a Poisson process with rate $\lambda = 1$. We consider three service times: (1) constant services (CTS) with a value of 0.95; (2) arbitrary services (ABS) with masses at 0.3, 1.02, 2.22 and 3.22 with probabilities, respectively, given by 0.5, 0.3, 0.15 and 0.05; and (3) uniform services (UFS) on the interval (0.45, 1.45). In all these three cases, the mean is 0.95. The standard deviations of CTS, ABS and UFS are 0, 0.843978 and 0.288675, respectively.

The steady-state probability vector at departure epochs (note that here this vector is also the vector at arbitrary epochs) for the three services is plotted in Figure 1.25.

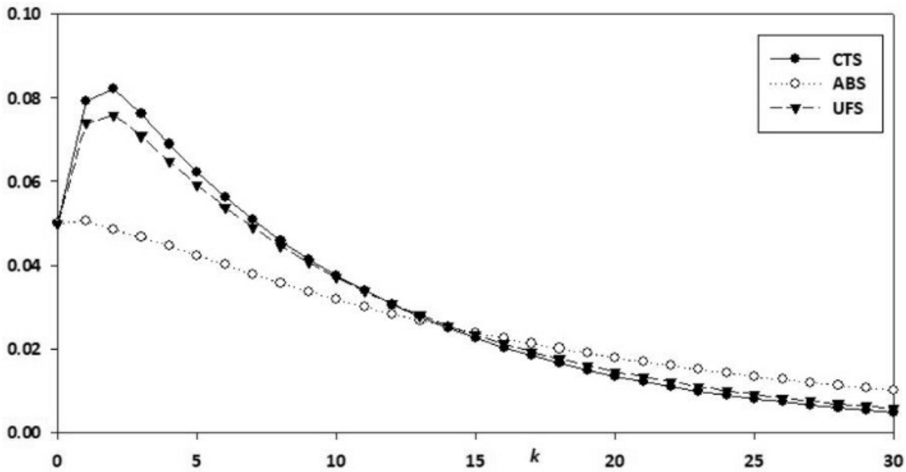

Figure 1.25. *The steady-state vector at arbitrary epochs under various three services for example 1.19*

The selected queue length statistics for these three cases are displayed in Table 1.13.

$Type\ of\ services$	$Mean$	$Median$	$Mode$	$Modal\ value$	SD
CTS	9.900	7	2	0.0822	9.626
ABS	16.964	12	1	0.0506	16.993
UFS	10.741	8	2	0.0759	10.504

Table 1.13. *Selected queue length statistics for the $M/G/1$ queue of example 1.19 under three scenarios*

Using the explicit expression for the mean as given in equation [1.84], verify the means for CTS, ABS, and UFS are, respectively, given by 9.975, 17.098 and 10.808.

EXAMPLE 1.20.– The purpose of this example is to see the behavior of the steady-state probabilities at departure epochs and other measures in the context of $M/D/1$ queue as ρ is varied. Toward this end, we first plot the graph of the steady-state probability as a function of k and ρ in Figure 1.26, and in Figure 1.27 we plot various measures at departure epochs.

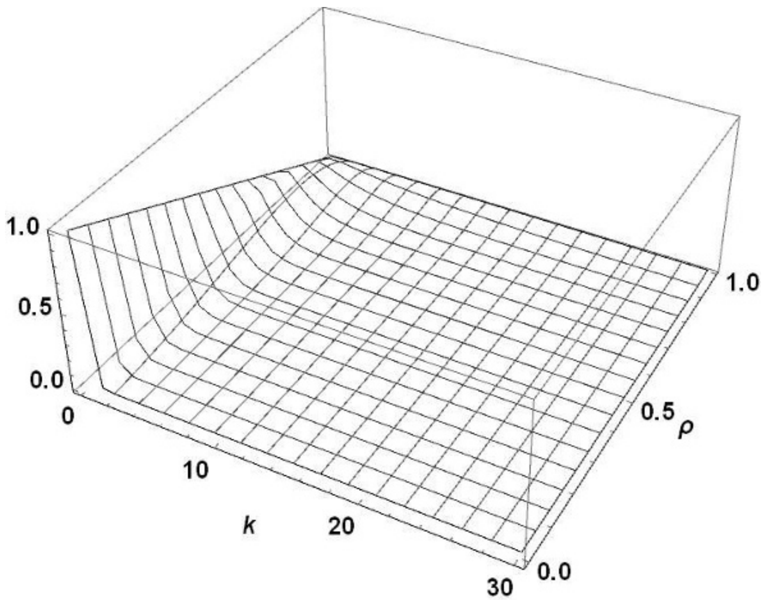

Figure 1.26. *The steady-state vector at arbitrary epochs for example 1.20*

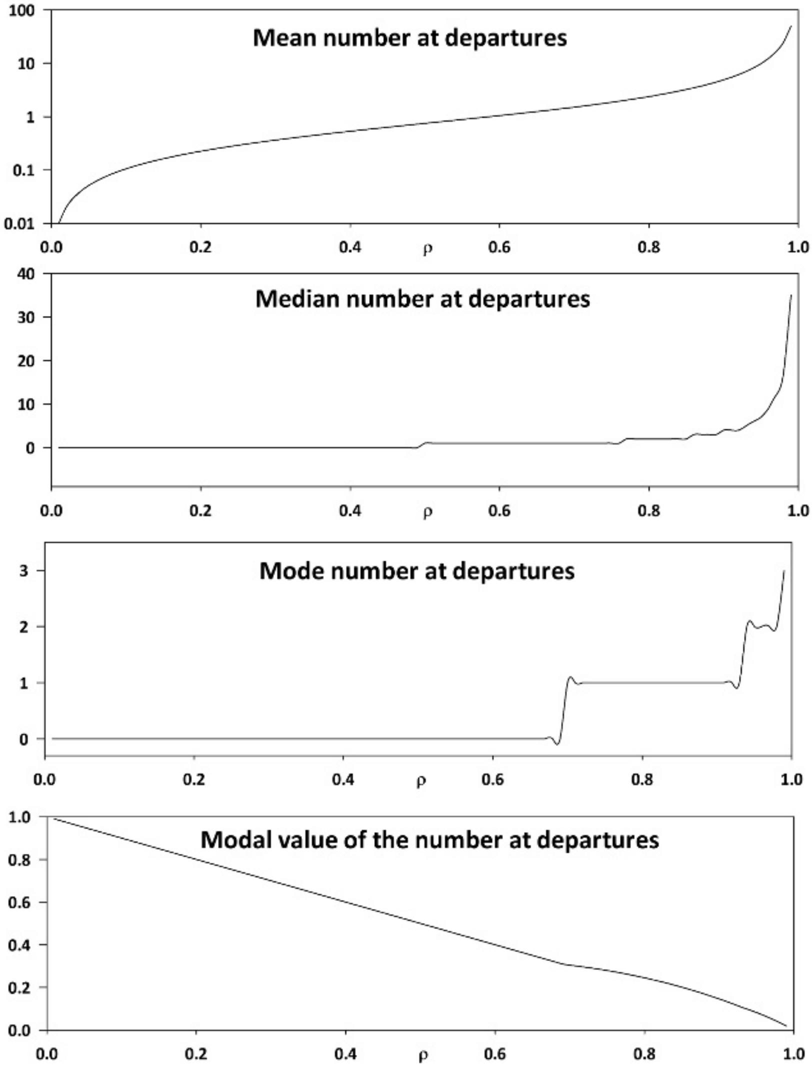

Figure 1.27. *Selected measures for $M/D/1$ queue for example 1.20*

1.5. *PH/M/1* queue

Here, the arrivals occur singly and according to a CPH distribution with representation given by $(\boldsymbol{\alpha}, T)$ of order m. Note that the arrival rate, λ, is $\lambda = [\boldsymbol{\alpha}(-T)^{-1}\boldsymbol{e}]^{-1}$. The service times are exponential with rate μ. It is easy to verify that:

$$A_n = \mu \int_0^\infty P(n,t)e^{-\mu t}dt, \; n \geq 0.$$

Suppose that $A(z) = \sum_{n=0}^{\infty} z^n A_n$ is the matrix PGF. Then, verify that (see equation [4.36] in Volume 1) $P^*(z,t) = e^{[T + z\boldsymbol{T^0}\boldsymbol{\alpha}]\,t}, \;\; t \geq 0, \; |z| \leq 1$:

$$A(z) = (\mu I - T - z\boldsymbol{T^0}\boldsymbol{\alpha})^{-1}, \; |z| \leq 1.$$

The matrix G is now obtained as a solution to:

$$G = \mu[\mu I - T - \boldsymbol{T^0}\boldsymbol{\alpha}G]^{-1}, \qquad [1.85]$$

from which we get the following matrix-quadratic equation:

$$\boldsymbol{T^0}\boldsymbol{\alpha}G^2 + (T - \mu I)G + \mu I = 0.$$

EXAMPLE 1.21.– Suppose that the inter-arrival times of customers entering into a single-server system are of phase type with representation given by:

$$\boldsymbol{\alpha} = (0.9, 0.1, 0, 0), \;\; T = \begin{pmatrix} -1.51726 & 0. & 0.61929 & 0. \\ 0. & -0.98776 & 0. & 0.61929 \\ 0. & 0. & -1.51726 & 0. \\ 0. & 0. & 0. & -0.98776 \end{pmatrix}.$$

The service times are exponential with mean 0.95. Compute the G matrix and its invariant vector, $\boldsymbol{g}$. Also, compute various other measures.

Using the matrix-quadratic equation for G, verify that:

$$G = \begin{pmatrix} 0.6512 & 0.0510 & 0.2521 & 0.0457 \\ 0.1610 & 0.5498 & 0.1022 & 0.1870 \\ 0.2902 & 0.0612 & 0.5937 & 0.0549 \\ 0.2379 & 0.0502 & 0.1510 & 0.5609 \end{pmatrix}, \; \boldsymbol{g} = (0.4214, 0.1084, 0.3379, 0.1323).$$

The mean, the median, the mode, the modal values and the standard deviation of the queue length in steady-state at departure epochs are 17.824, 12, 0, 0.053 and 18.277, respectively.

The mean, the median, the mode, the modal values and the standard deviation of the queue length in steady-state at an arbitrary epoch are 17.882, 12, 1, 0.050 and 18.279, respectively.

1.6. *M/M/1* queue

The simplest single-server queueing model in continuous-time is the one with Poisson arrivals and exponential services. Here, all the results are explicitly known. The approach taken here uses the embedded points at departure epochs. We briefly summarize them here as they are obtained by taking the simplest form for D_0 and D_1 matrices. In the current setup, $M/M/1$ queue is obtained by taking $D_0 = -\lambda$, $D_1 = -\lambda$ and $F(t) = 1 - e^{-\mu t}$. This results in:

$$A_n = B_n = \mu \int_0^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} e^{-\mu t} dt = \frac{\mu}{\lambda + \mu} \Big(\frac{\mu}{\lambda + \mu} \Big)^n, \; n \geq 0.$$

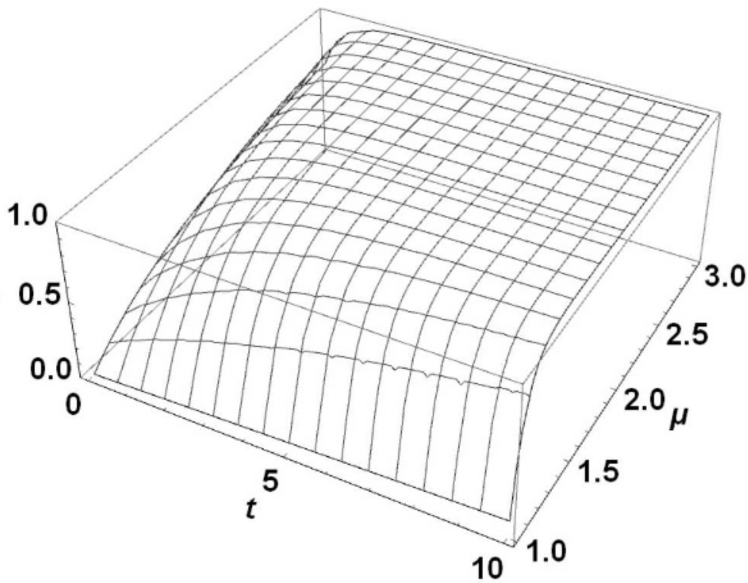

Figure 1.28. *The CDF of the waiting time in queue for $M/M/1$ queue*

Obviously, $G = K = 1$, and the steady-state probabilities are: $x_k = y_k = (1 - \rho)\rho^k, \; k \geq 0$. Using equation [1.82] by taking $f^*(s) = \dfrac{\mu}{s + \mu}$ for the current case

$\Big($note that here $\dfrac{\mu(1-f^*(s))}{s} = f^*(s)\Big)$, we get $w_q^*(s) = (1-\rho)\sum_{k=0}^{\infty}\rho^k\Big(\dfrac{\mu}{s+\mu}\Big)^k$, which can be inverted to obtain $F_q(t) = P(T_q \le t) = 1 - \rho e^{-\mu(1-\rho)\,t},\ t \ge 0$. In Figure 1.28, we display the graph of $F_q(t)$ by fixing $\lambda = 1$ and varying μ and t.

1.7. Exercises

Exercise 1.1. Referring to examples 1.1 through 1.4, develop and implement a code for computing various quantities and point out some key observations based on the measures displayed in the tables and the plots seen in the figures in those examples.

Exercise 1.2. Referring to example 1.5, develop and implement a code for computing various quantities and point out some key observations based on the measures displayed in the table and the plots seen in the figure.

Exercise 1.3. Referring to example 1.6, develop and implement a code for computing various quantities and point out some key observations based on the measures displayed in the table and the plots seen in the figure.

Exercise 1.4. Referring to example 1.7, develop and implement a code for computing various quantities and point out some key observations based on the measures displayed in the table and the plots seen in the figure.

Exercise 1.5. Referring to example 1.8, develop and implement a code for computing various quantities and point out some key observations based on the measures displayed in the table and the plots seen in the figure.

Exercise 1.6. Referring to example 1.9, develop and implement a code for computing various quantities and point out some key observations based on the measures displayed in the table and the plots seen in the figure.

Exercise 1.7. Referring to example 1.10, develop and implement a code for computing various quantities and point out some key observations based on the measures displayed in the table and the plots seen in the figure.

Exercise 1.8. Referring to example 1.11, develop and implement a code for computing various quantities and point out some key observations based on the measures displayed in the figures. Verify the G and K matrices given for constant services. Obtain these matrices for Erlang and hyperexponential services. Investigate further the role of these two BMAP under various scenarios.

Exercise 1.9. Referring to examples 1.12 through 1.16, develop and implement a code for computing various quantities and point out some key observations based on the measures displayed in the tables and the plots seen in the figures for these examples.

Exercise 1.10. Referring to examples 1.17 and 1.18, develop and implement a code for computing various quantities and point out some key observations based on the measures displayed in the tables and the plots seen in the figures for these examples.

Exercise 1.11. Referring to examples 1.19 and 1.20, develop and implement a code for computing various quantities and point out some key observations based on the measures displayed in the tables and the plots seen in the figures for these examples.

Exercise 1.12. Referring to example 1.21, verify the results for G and its invariant vector.

Note: For exercises dealing with computational ones, your answers may differ based on the error tolerance levels chosen.

Exercise 1.13. Suppose that the customers arrive in batches that are modeled using a Poisson process with parameter 0.5 and the inter-arrival times of the batches are Erlang of order 4 with parameter $7/3$. That is:

$$D_0 = \frac{7}{3}\begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad D_k = \frac{7}{3}e^{-0.5}\frac{0.5^{k-1}}{(k-1)!}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \ k \geq 1.$$

There is a single server to serve the customers on an FCFS basis and the service times are constant with a value of 0.8 unit. Answer the following questions:

a) Find G, K and their respective invariant vectors, $\boldsymbol{g}$ and $\boldsymbol{\kappa}$.

b) Compute the selected queue length statistics: the mean, the median, the mode, the modal value and the standard deviation at various epochs.

c) Compute the first few steady-state probability vectors.

Exercise 1.14. Suppose that the customers arrive in batches that are modeled using a geometric distribution with parameter $2/3$ and the inter-arrival times of the batches are Erlang of order 4 with parameter $7/3$. That is:

$$D_0 = \frac{7}{3}\begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad D_k = \frac{14}{9}\left(\frac{1}{3}\right)^{k-1}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \ k \geq 1.$$

There is a single server to serve the customers on an FCFS basis and the service times are constant with a value of 0.8 unit. Answer the following questions:

a) Find G, K and their respective invariant vectors, $\boldsymbol{g}$ and $\boldsymbol{\kappa}$.

b) Compute the selected queue length statistics: the mean, the median, the mode, the modal value and the standard deviation at various epochs.

c) Compute the first few steady-state probability vectors.

Exercise 1.15. Suppose that the customers arrive in batches that are modeled using a (discrete) uniform on $\{1, 2\}$ and the inter-arrival times of the batches are Erlang of order 4 with parameter $7/3$. That is:

$$D_0 = \frac{7}{3}\begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad D_k = 0.5\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$
$$k = 1, 2, \; D_k = 0, \; k \geq 3.$$

There is a single server to serve the customers on an FCFS basis and the service times are constant with a value of 0.8 unit. Answer the following questions:

a) Find G, K and their respective invariant vectors, $\boldsymbol{g}$ and $\boldsymbol{\kappa}$.

b) Compute the selected queue length statistics: the mean, the median, the mode, the modal value and the standard deviation at various epochs.

c) Compute the first few steady-state probability vectors.

Exercise 1.16. Suppose that the customers arrive according to a BMAP with representation:

$$D_0 = \frac{7}{3}\begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad D_k = (\boldsymbol{\alpha} T^{k-1} \boldsymbol{T^0})\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \; k \geq 1,$$

where:

$$\boldsymbol{\alpha} = (0.7397, 0.2603), \;\; T = \begin{pmatrix} 0.1000 & 0.2000 \\ 0.1512 & 0.2341 \end{pmatrix}, \; \boldsymbol{T^0} = \boldsymbol{e} - T\boldsymbol{e}.$$

There is a single server to serve the customers on an FCFS basis and the service times are constant with a value of 0.8 unit. Answer the following questions:

a) Find G, K and their respective invariant vectors, $\boldsymbol{g}$ and $\boldsymbol{\kappa}$.

b) Compute the selected queue length statistics: the mean, the median, the mode, the modal value and the standard deviation at various epochs.

c) Compute the first few steady-state probability vectors.

Exercise 1.17. Suppose that the customers arrive in batches that are modeled using a Poisson process with parameter 0.5 and the inter-arrival times of the batches are assumed to be hyperexponential. That is, set $a_k = \frac{7}{3}e^{-0.5}\frac{0.5^{k-1}}{(k-1)!},\ k \geq 1,$

$$D_0 = \begin{pmatrix} -1.4 & 0 & 0 & 0 \\ 0 & -0.7 & 0 & 0 \\ 0 & 0 & -0.35 & 0 \\ 0 & 0 & 0 & -0.175 \end{pmatrix},$$

$$D_k = a_k \begin{pmatrix} 0.7 & 0.42 & 0.21 & 0.07 \\ 0.35 & 0.21 & 0.105 & 0.035 \\ 0.175 & 0.105 & 0.0525 & 0.0175 \\ 0.0875 & 0.0525 & 0.02625 & 0.00875 \end{pmatrix}.$$

There is a single server to serve the customers on an FCFS basis and the service times are constant with a value of 0.8 unit. Answer the following questions:

a) Find G, K and their respective invariant vectors, $\boldsymbol{g}$ and $\boldsymbol{\kappa}$.

b) Compute the selected queue length statistics: the mean, the median, the mode, the modal value and the standard deviation at various epochs.

c) Compute the first few steady-state probability vectors.

Exercise 1.18. Suppose that the customers arrive in batches that are modeled using a geometric distribution with parameter $2/3$ and the inter-arrival times of the batches are assumed to be hyperexponential. That is, set $a_k = (2/3)(1/3)^{k-1},\ k \geq 1,$

$$D_0 = \begin{pmatrix} -1.4 & 0 & 0 & 0 \\ 0 & -0.7 & 0 & 0 \\ 0 & 0 & -0.35 & 0 \\ 0 & 0 & 0 & -0.175 \end{pmatrix},$$

$$D_k = a_k \begin{pmatrix} 0.7 & 0.42 & 0.21 & 0.07 \\ 0.35 & 0.21 & 0.105 & 0.035 \\ 0.175 & 0.105 & 0.0525 & 0.0175 \\ 0.0875 & 0.0525 & 0.02625 & 0.00875 \end{pmatrix}.$$

There is a single server to serve the customers on an FCFS basis and the service times are constant with a value of 0.8 unit. Answer the following questions:

a) Find G, K and their respective invariant vectors, $\boldsymbol{g}$ and $\boldsymbol{\kappa}$.

b) Compute the selected queue length statistics: the mean, the median, the mode, the modal value and the standard deviation at various epochs.

c) Compute the first few steady-state probability vectors.

Exercise 1.19. Suppose that the customers arrive in batches that are modeled using a (discrete) uniform on $\{1, 2\}$ and the inter-arrival times of the batches are assumed to be hyperexponential. That is, set $a_k = 0.5,\ k = 1, 2,\ a_k = 0,\ k \geq 3$,

$$D_0 = \begin{pmatrix} -1.4 & 0 & 0 & 0 \\ 0 & -0.7 & 0 & 0 \\ 0 & 0 & -0.35 & 0 \\ 0 & 0 & 0 & -0.175 \end{pmatrix},$$

$$D_k = a_k \begin{pmatrix} 0.7 & 0.42 & 0.21 & 0.07 \\ 0.35 & 0.21 & 0.105 & 0.035 \\ 0.175 & 0.105 & 0.0525 & 0.0175 \\ 0.0875 & 0.0525 & 0.02625 & 0.00875 \end{pmatrix}.$$

There is a single server to serve the customers on an FCFS basis and the service times are constant with a value of 0.8 unit. Answer the following questions:

a) Find G, K and their respective invariant vectors, $\boldsymbol{g}$ and $\boldsymbol{\kappa}$.

b) Compute the selected queue length statistics: the mean, the median, the mode, the modal value and the standard deviation at various epochs.

c) Compute the first few steady-state probability vectors.

Exercise 1.20. Suppose that the customers arrive according to a BMAP with representation:

$$D_0 = \begin{pmatrix} -1.4 & 0 & 0 & 0 \\ 0 & -0.7 & 0 & 0 \\ 0 & 0 & -0.35 & 0 \\ 0 & 0 & 0 & -0.175 \end{pmatrix},$$

$$D_k = a_k \begin{pmatrix} 0.7 & 0.42 & 0.21 & 0.07 \\ 0.35 & 0.21 & 0.105 & 0.035 \\ 0.175 & 0.105 & 0.0525 & 0.0175 \\ 0.0875 & 0.0525 & 0.02625 & 0.00875 \end{pmatrix},$$

where:

$$\boldsymbol{\alpha} = (0.7397, 0.2603), \;\; T = \begin{pmatrix} 0.1000 & 0.2000 \\ 0.1512 & 0.2341 \end{pmatrix}, \; \boldsymbol{T^0} = \boldsymbol{e} - T\boldsymbol{e}.$$

There is a single server to serve the customers on an FCFS basis and the service times are constant with a value of 0.8 unit. Answer the following questions:

a) Find G, K and their respective invariant vectors, $\boldsymbol{g}$ and $\boldsymbol{\kappa}$.

b) Compute the selected queue length statistics: the mean, the median, the mode, the modal value and the standard deviation at various epochs.

c) Compute the first few steady-state probability vectors.

Exercise 1.21. Suppose that the customers arrive in batches according to a BMAP with representation given by:

$$D_0 = \frac{1}{3}\begin{pmatrix} -5 & 5 & 0 & 0 \\ 0 & -5 & 5 & 0 \\ 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & -7 \end{pmatrix}, D_k = a_k \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0.01167 & 0 & 0 & 1.155 \\ 2.31 & 0 & 0 & 0.02333 \end{pmatrix}.$$

Consider the following four distributions for the batch size:

$$a_k^{(r)} = \begin{cases} e^{-0.5}\dfrac{0.5^{k-1}}{(k-1)!}, \; k \geq 1, \; r = 1, \\[2ex] (2/3)(1/3)^{k-1}, \; k \geq 1, \; r = 2, \\[2ex] 0.5, \; k = 1, 2, \; r = 3, \\[2ex] \boldsymbol{\alpha} T^{k-1} \boldsymbol{T^0}, \; k \geq 1, \; r = 4, \end{cases}$$

where:

$$\boldsymbol{\alpha} = (0.7397, 0.2603), \;\; T = \begin{pmatrix} 0.1000 & 0.2000 \\ 0.1512 & 0.2341 \end{pmatrix},$$

with $T\boldsymbol{e} + \boldsymbol{T^0} = \boldsymbol{e}$. There is a single server to serve the customers on an FCFS basis and the service times are constant with a value of 0.8 unit. Answer the following questions:

a) Find the 1-lag correlation of two successive inter-arrival times.

b) Find G, K and their respective invariant vectors, $\boldsymbol{g}$ and $\boldsymbol{\kappa}$, for these four types of group sizes.

c) Compute the selected queue length statistics: the mean, the median, the model, the modal value and the standard deviation at various epochs, for these four types of group sizes.

d) Compute the first few steady-state probability vectors, for these four types of group sizes.

Exercise 1.22. Suppose that the customers arrive in batches according to a BMAP with representation given by:

$$D_0 = \frac{1}{3}\begin{pmatrix} -5 & 5 & 0 & 0 \\ 0 & -5 & 5 & 0 \\ 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & -7 \end{pmatrix}, D_k = a_k \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1.155 & 0 & 0 & 0.01167 \\ 0.02333 & 0 & 0 & 2.31 \end{pmatrix}.$$

Consider the following four distributions for the batch size:

$$a_k^{(r)} = \begin{cases} e^{-0.5}\dfrac{0.5^{k-1}}{(k-1)!}, \; k \geq 1, \; r = 1, \\ (2/3)(1/3)^{k-1}, \; k \geq 1, \; r = 2, \\ 0.5, \; k = 1, 2, \; r = 3, \\ \boldsymbol{\alpha} T^{k-1} \boldsymbol{T^0}, \; k \geq 1, \; r = 4, \end{cases}$$

where:

$$\boldsymbol{\alpha} = (0.7397, 0.2603), \;\; T = \begin{pmatrix} 0.1000 & 0.2000 \\ 0.1512 & 0.2341 \end{pmatrix},$$

with $T\boldsymbol{e} + \boldsymbol{T^0} = \boldsymbol{e}$. There is a single server to serve the customers on an FCFS basis and the service times are constant with a value of 0.8 unit. Answer the following questions:

a) Find the 1-lag correlation of two successive inter-arrival times.

b) Find G, K and their respective invariant vectors, $\boldsymbol{g}$ and $\boldsymbol{\kappa}$, for these four types of group sizes.

c) Compute the selected queue length statistics: the mean, the median, the model, the modal value and the standard deviation at various epochs, for these four types of group sizes.

d) Compute the first few steady-state probability vectors, for these four types of group sizes.

Exercise 1.23. Suppose that the customers arrive in batches according to a BMAP with representation given by:

$$D_0 = \frac{7}{3}\begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad D_k = \frac{7}{3}a_k \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad k \geq 1.$$

Consider the following four distributions for the batch size:

$$a_k^{(r)} = \begin{cases} e^{-0.5}\dfrac{0.5^{k-1}}{(k-1)!}, \ k \geq 1, \ r = 1, \\ (2/3)(1/3)^{k-1}, \ k \geq 1, \ r = 2, \\ 0.5, \ k = 1, 2, \ r = 3, \\ \boldsymbol{\alpha} T^{k-1}\boldsymbol{T^0}, \ k \geq 1, \ r = 4, \end{cases}$$

where:

$$\boldsymbol{\alpha} = (0.7397, 0.2603), \quad T = \begin{pmatrix} 0.1000 & 0.2000 \\ 0.1512 & 0.2341 \end{pmatrix},$$

with $T\boldsymbol{e} + \boldsymbol{T^0} = \boldsymbol{e}$. There is a single server to serve the customers on an FCFS basis and the service times are uniform on the interval $(0.3, 1.3)$. Answer the following questions:

a) Find the 1-lag correlation of two successive inter-arrival times.

b) Find G, K and their respective invariant vectors, $\boldsymbol{g}$ and $\boldsymbol{\kappa}$, for these four types of group sizes.

c) Compute the selected queue length statistics: the mean, the median, the model, the modal value and the standard deviation at various epochs, for these four types of group sizes.

d) Compute the first few steady-state probability vectors, for these four types of group sizes.

Exercise 1.24. Referring to exercise 1.23, suppose that the customers arrive in batches according to a BMAP with representation given by:

$$D_0 = \begin{pmatrix} -1.4 & 0 & 0 & 0 \\ 0 & -0.7 & 0 & 0 \\ 0 & 0 & -0.35 & 0 \\ 0 & 0 & 0 & -0.175 \end{pmatrix},$$

$$D_k = a_k \begin{pmatrix} 0.7 & 0.42 & 0.21 & 0.07 \\ 0.35 & 0.21 & 0.105 & 0.035 \\ 0.175 & 0.105 & 0.0525 & 0.0175 \\ 0.0875 & 0.0525 & 0.02625 & 0.00875 \end{pmatrix}.$$

All other parameters remain the same as in exercise 1.23. Answer the following questions:

a) Find the 1-lag correlation of two successive inter-arrival times.

b) Find G, K and their respective invariant vectors, $\boldsymbol{g}$ and $\boldsymbol{\kappa}$, for these four types of group sizes.

c) Compute the selected queue length statistics: the mean, the median, the model, the modal value and the standard deviation at various epochs, for these four types of group sizes.

d) Compute the first few steady-state probability vectors, for these four types of group sizes.

Exercise 1.25. Referring to exercise 1.23, suppose that the customers arrive in batches according to a BMAP with representation given by:

$$D_0 = \frac{1}{3}\begin{pmatrix} -5 & 5 & 0 & 0 \\ 0 & -5 & 5 & 0 \\ 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & -7 \end{pmatrix}, D_k = a_k \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0.01167 & 0 & 0 & 1.155 \\ 2.31 & 0 & 0 & 0.02333 \end{pmatrix}.$$

All other parameters remain the same as in exercise 1.23. Answer the following questions:

a) Find G, K and their respective invariant vectors, $\boldsymbol{g}$ and $\boldsymbol{\kappa}$, for these four types of group sizes.

b) Compute the selected queue length statistics: the mean, the median, the model, the modal value and the standard deviation at various epochs, for these four types of group sizes.

c) Compute the first few steady-state probability vectors, for these four types of group sizes.

Exercise 1.26. Referring to exercise 1.23, suppose that the customers arrive in batches according to a BMAP with representation given by:

$$D_0 = \frac{1}{3}\begin{pmatrix} -5 & 5 & 0 & 0 \\ 0 & -5 & 5 & 0 \\ 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & -7 \end{pmatrix}, D_k = a_k \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1.155 & 0 & 0 & 0.01167 \\ 0.02333 & 0 & 0 & 2.31 \end{pmatrix}.$$

All other parameters remain the same as in exercise 1.23. Answer the following questions:

a) Find G, K and their respective invariant vectors, $\boldsymbol{g}$ and $\boldsymbol{\kappa}$, for these four types of group sizes.

b) Compute the selected queue length statistics: the mean, the median, the model, the modal value and the standard deviation at various epochs, for these four types of group sizes.

c) Compute the first few steady-state probability vectors, for these four types of group sizes.

Exercise 1.27. Suppose that the customers arrive in batches according to a BMAP with representation given by:

$$D_0 = \frac{7}{3}\begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad D_k = a_k \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \ k \geq 1,$$

where $\{a_k\}$ has one of the four distributions for the batch size as described in exercise 1.23. Suppose that the service times are modeled using an Erlang of order 2 with parameter 2.5 in each stage so that the mean service time is 0.8. Answer the following questions:

a) Find G, K and their respective invariant vectors, $\boldsymbol{g}$ and $\boldsymbol{\kappa}$, for these four types of group sizes.

b) Compute the selected queue length statistics: the mean, the median, the model, the modal value and the standard deviation at various epochs, for these four types of group sizes.

c) Compute the first few steady-state probability vectors, for these four types of group sizes.

Exercise 1.28. Referring to exercise 1.27, suppose that the customers arrive in batches according to a BMAP with representation given by:

$$D_0 = \begin{pmatrix} -1.4 & 0 & 0 & 0 \\ 0 & -0.7 & 0 & 0 \\ 0 & 0 & -0.35 & 0 \\ 0 & 0 & 0 & -0.175 \end{pmatrix},$$

$$D_k = a_k \begin{pmatrix} 0.7 & 0.42 & 0.21 & 0.07 \\ 0.35 & 0.21 & 0.105 & 0.035 \\ 0.175 & 0.105 & 0.0525 & 0.0175 \\ 0.0875 & 0.0525 & 0.02625 & 0.00875 \end{pmatrix},$$

where $\{a_k\}$ has one of the four distributions for the batch size as described in exercise 1.23. All other parameters remain the same as in exercise 1.27. Answer the following questions:

a) Find G, K and their respective invariant vectors, $\boldsymbol{g}$ and $\boldsymbol{\kappa}$, for these four types of group sizes.

b) Compute the selected queue length statistics: the mean, the median, the model, the modal value and the standard deviation at various epochs, for these four types of group sizes.

c) Compute the first few steady-state probability vectors, for these four types of group sizes.

Exercise 1.29. Referring to exercise 1.27, suppose that the customers arrive in batches according to a BMAP with representation given by:

$$D_0 = \frac{1}{3}\begin{pmatrix} -5 & 5 & 0 & 0 \\ 0 & -5 & 5 & 0 \\ 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & -7 \end{pmatrix}, D_k = a_k \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0.01167 & 0 & 0 & 1.155 \\ 2.31 & 0 & 0 & 0.02333 \end{pmatrix},$$

where $\{a_k\}$ has one of the four distributions for the batch size as described in exercise 1.23. All other parameters remain the same as in exercise 1.27. Answer the following questions:

a) Find G, K and their respective invariant vectors, $\boldsymbol{g}$ and $\boldsymbol{\kappa}$, for these four types of group sizes.

b) Compute the selected queue length statistics: the mean, the median, the model, the modal value and the standard deviation at various epochs, for these four types of group sizes.

c) Compute the first few steady-state probability vectors, for these four types of group sizes.

Exercise 1.30. Referring to exercise 1.27, suppose that the customers arrive in batches according to a BMAP with representation given by:

$$D_0 = \frac{1}{3}\begin{pmatrix} -5 & 5 & 0 & 0 \\ 0 & -5 & 5 & 0 \\ 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & -7 \end{pmatrix}, D_k = a_k \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1.155 & 0 & 0 & 0.01167 \\ 0.02333 & 0 & 0 & 2.31 \end{pmatrix},$$

where $\{a_k\}$ has one of the four distributions for the batch size as described in exercise 1.23. All other parameters remain the same as in exercise 1.27. Answer the following questions:

a) Find G, K and their respective invariant vectors, $\boldsymbol{g}$ and $\boldsymbol{\kappa}$, for these four types of group sizes.

b) Compute the selected queue length statistics: the mean, the median, the model, the modal value and the standard deviation at various epochs, for these four types of group sizes.

c) Compute the first few steady-state probability vectors, for these four types of group sizes.

Exercise 1.31. Suppose that arrivals in batches and the batch arrivals follow a Poisson process with a mean of $2/3$ per unit of time. The batch size PMF, $\{a_k\}$, is modeled using one of four types: (1) PoB: $a_k = e^{-0.5}\dfrac{0.5^{k-1}}{(k-1)!}$, $k \geq 1$; (2) GeB: $a_k = (2/3)(1/3)^{k-1}$, $k \geq 1$; (3) $UfB: \ a_k = 0.5, \ k = 1, 2; = 0$, elsewhere; and (4) $DpB: \ a_k = \boldsymbol{\alpha} T^{k-1}\boldsymbol{T^0}, \ k \geq 1$, where:

$$\boldsymbol{\alpha} = (0.7397, 0.2603), \quad T = \begin{pmatrix} 0.1000 & 0.2000 \\ 0.1512 & 0.2341 \end{pmatrix}.$$

There is a single server and the service times are modeled using one of four types: (1) CTS: constant with a value of 0.8 unit; (2) UFS: uniform on the interval $(0.3, 1.3)$; (3) ERS: Erlang of order 2 with parameter 2.5 in each state; (4) HES: hyperexponential with parameters 2.375 and 0.2375, with mixing probabilities, respectively, taken as 0.9 and 0.1. Compute the mean and the standard deviation of various steady-state probabilities under different scenarios.

Exercise 1.32. Suppose that the customers arrive singly according to a MAP (corresponding to Erlang inter-arrival times) with representation given by:

$$D_0 = \begin{pmatrix} -4 & 4 & 0 & 0 \\ 0 & -4 & 4 & 0 \\ 0 & 0 & -4 & 4 \\ 0 & 0 & 0 & -4 \end{pmatrix}, D_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 \end{pmatrix}.$$

There is a single server in the system and for services we look at four scenarios as displayed below:

A: CTS: Constant services with a mean of 0.8 unit and a standard deviation of 0.

B: UFS: Uniform services on the interval $(0.3, 1.3)$ with a mean of 0.8 unit and a standard deviation of 0.28868 unit.

C: ERS: Erlang services with two states with rate 2.5 in each state. The mean and the standard deviation of 0.8 unit and a standard deviation of 0.56569 unit.

D: HES: Hyperexponential services with parameters 2.375 and 0.2375 with mixing probabilities, respectively, given by 0.8 unit and a standard deviation of 1.79578 units.

Answer the following questions, where the superscripts contain the type of services used:

a) Find G, K and their respective invariant vectors, $\boldsymbol{g}$ and $\boldsymbol{\kappa}$, for these four types of services.

c) Compute the selected queue length statistics: the mean, the median, the model, the modal value and the standard deviation at various epochs, for these four types of services.

d) Compute the first few steady-state probability vectors, for these four types of services.

Exercise 1.33. Referring to exercise 1.32, we keep all parameters the same except the arrival process, which is taken to be a hyperexponential one with representation given by:

$$D_0 = \begin{pmatrix} -2.1 & 0 & 0 & 0 \\ 0 & -1.05 & 0 & 0 \\ 0 & 0 & -0.525 & 0 \\ 0 & 0 & 0 & -0.2625 \end{pmatrix},$$

$$D_1 = \begin{pmatrix} 1.05 & 0.63 & 0.315 & 0.105 \\ 0.525 & 0.315 & 0.1575 & 0.0525 \\ 0.2625 & 0.1575 & 0.07875 & 0.02625 \\ 0.13125 & 0.07875 & 0.03938 & 0.01312 \end{pmatrix}.$$

Answer the following questions, where the superscripts contain the type of services used:

a) Find G, K and their respective invariant vectors, $\boldsymbol{g}$ and $\boldsymbol{\kappa}$, for these four types of services.

c) Compute the selected queue length statistics: the mean, the median, the model, the modal value and the standard deviation at various epochs, for these four types of services.

d) Compute the first few steady-state probability vectors, for these four types of services.

Exercise 1.34. Referring to exercise 1.32, we keep all parameters the same except the arrival process, which is taken to be a MAP with a negative correlation, with representation given by:

$$D_0 = \begin{pmatrix} -1.75 & 1.75 & 0 & 0 \\ 0 & -1.75 & 1.75 & 0 \\ 0 & 0 & -1.75 & 0 \\ 0 & 0 & 0 & -3.5 \end{pmatrix}, D_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0.0175 & 0 & 0 & 1.7325 \\ 3.465 & 0 & 0 & 0.035 \end{pmatrix}.$$

Answer the following questions, where the superscripts contain the type of services used:

a) Find G, K and their respective invariant vectors, $\boldsymbol{g}$ and $\boldsymbol{\kappa}$, for these four types of services.

c) Compute the selected queue length statistics: the mean, the median, the model, the modal value and the standard deviation at various epochs, for these four types of services.

d) Compute the first few steady-state probability vectors, for these four types of services.

Exercise 1.35. Referring to exercise 1.32, we keep all parameters the same except the arrival process, which is taken to be a MAP with a negative correlation, with representation given by:

$$D_0 = \begin{pmatrix} -1.75 & 1.75 & 0 & 0 \\ 0 & -1.75 & 1.75 & 0 \\ 0 & 0 & -1.75 & 0 \\ 0 & 0 & 0 & -3.5 \end{pmatrix}, D_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1.7325 & 0 & 0 & 0.0175 \\ 0.035 & 0 & 0 & 3.465 \end{pmatrix}.$$

Answer the following questions, where the superscripts contain the type of services used:

a) Find G, K and their respective invariant vectors, $\boldsymbol{g}$ and $\boldsymbol{\kappa}$, for these four types of services.

c) Compute the selected queue length statistics: the mean, the median, the model, the modal value and the standard deviation at various epochs, for these four types of services.

d) Compute the first few steady-state probability vectors, for these four types of services.

Exercise 1.36. Show that G and $D[G]$ (see equation [1.36]) commute.

Exercise 1.37. Show that for $MAP/G/1$ queueing model (assuming the system is stable) that the steady-state probability vectors at arrival epochs and departure epochs are identical.

Exercise 1.38. Show that $\boldsymbol{\beta}^*$ is as given in equation [1.17].

Exercise 1.39. Prove the result for $\hat{\boldsymbol{\mu}}_G$ as given in equation [1.44].

Exercise 1.40. Prove the following result (see equation [1.53]):

$$\sum_{r=1}^{\infty} B_r \sum_{k=0}^{r-1} G^k \tilde{\boldsymbol{\mu}}_G = \sum_{r=0}^{\infty} B_r (I - G^r + r\boldsymbol{e}\,\boldsymbol{g})[I - A + (\boldsymbol{e} - \boldsymbol{\beta}^*)\boldsymbol{g}]^{-1}\boldsymbol{e}.$$

Exercise 1.41. Prove result 1.7.

Exercise 1.42. Prove result 1.14.

Exercise 1.43. Prove equation [1.59] from equations [1.50] and [1.57].

Exercise 1.44. Prove the simplification mentioned in equation [1.62].

Exercise 1.45. Prove result given in equation [1.65].

Exercise 1.46. Prove the expression given in equation [1.69].

Exercise 1.47. Prove result 1.13.

Exercise 1.48. Using equation [1.75] and with the help of the identification of the like coefficients, prove the expressions given in equation [1.71].

Exercise 1.49. Prove the expressions given in equation [1.78].

Exercise 1.50. Use the expressions given in equation [1.78] to show that the steady-state probabilities at arrivals and at departures are identical.

Exercise 1.51. For the $PH/G/1$ queue, show that the steady-state vector $\boldsymbol{g}$ of G is of the form: $\boldsymbol{g} = [\boldsymbol{\alpha}G(-T)^{-1}\boldsymbol{e}]^{-1}\ \boldsymbol{\alpha}G(-T)^{-1}$.

Exercise 1.52. For the $PH/G/1$ queue, show that the matrix K is given by $K = \boldsymbol{e}\,\boldsymbol{\alpha}G$ and hence $\boldsymbol{\kappa} = \boldsymbol{\alpha}G$.