1

Graphs

This chapter is an introduction to our subject. After a few words about the history of graphs, we explain what we now call a "graph" in mathematics and introduce the main definitions useful to understand what follows. Then, we study some particular classes of graphs that we can meet in philosophy and give some examples of them.

1.1. Graph theory: a brief history

The origin of graph drawings is not well known, but we have already found some of them in ancient China.

As in the terrestrial order, where the ancient Chinese used cryptography to restrict the transmission of certain messages to the initiates, communication with the heavenly forces required in China a form adapted to the addressee, and conversely the message of celestial forces could not take over the form of a terrestrial message. Thus came the *fu*, these graphs, most often Daoist, which allowed followers to enter into communication with spirits and spirits to communicate with the earthly world (see Figure 1.1).

In this region of the world, some graphic constructions may also be found in decorative motives of architecture or jails (see Figure 1.2) which follow certain logics (He and Schnabel, 2018).

1 For a color version of all the figures in this chapter, see http://www.iste.co.uk/parrochia/graphs.zip.

Figure 1.1. *Taoist Fu surmounted by a constellation of 5 stars (right); official of fate (left) (BNF, Coins, Medals and Antiques - CF A-135) (source: BNF)*

Summer Palace (Beijing) - photo Z. Guo

Lingering Garden (Suzhou)

Forms of motives

Figure 1.2. *Chinese decorative graphs (photo: Z. Guo)*

It seems that one of the earliest forms of them in western countries are probably that of Morris and Mill games, as shown in Figure 1.3, where the nodes of the graph drawing are the positions that game counters can occupy, the edges indicating how game counters can move between nodes (see Kruskal 1960, pp. 272–286).

Figure 1.3. *Depiction of Morris gameboards*

The origin of Mill gameboards comes from stone carvings in Ancient Egypt but the date of the earliest examples of Mill gameboards to be drawn in a book only goes back to the end of the 13th century.

However, early graph drawings were not exclusively an invention of Asia or the Middle East.

Quipus, which are bundles of cords whose knots and combinations replaced the writing (see Figure 1.4), were used by the Incas from the 13th to 16th centuries for accounting purposes and especially to register important facts and events. Of the few hundred surviving examples, roughly 25% exhibit a hierarchical structure and therefore qualify as tree drawings (see Ascher and Ascher 1997).

Figure 1.4. *Incas' quipu (source: Wikipedia article: Quipu)*

Another example of ancient graph drawings is family trees which decorated the atria of the patrician roman villas. Though no example has survived, it has been described by Pliny the Elder and Seneca. It was not until the Middle Ages that we saw them reappear.

During the 14th century, there appeared trees depicting the categories of different kinds (vices, virtues), while logicians often use them to represent abstract information (Porphyrian tree) or to illustrate the system of logical arguments (square of opposition).

After that, nothing comes before the famous paper of 1736 written by Leonhard Euler on the seven bridges of Kœnigsberg[1] and regarded as the first paper in the history of graph theory (see Biggs et al. 1986; Parrochia 1993b). This

1 The city of Kœnigsberg (East Prussia), today called Kaliningrad (Russia), being built around two islands located on the Pregel and connected by a bridge, six other bridges connect the banks of the river to each other. A famous problem was whether you could go around the city by crossing each bridge only once. L. Euler resolved this problem in the negative. This resulted in the notion of the "Eulerian cycle".

paper, as well as the one written by Vandermonde on the knight problem in chess, carried on with the dream of the so-called “analysis situs” initiated by Leibniz and the ancestor of topology.

Euler’s formula relating the number of edges, vertices and faces of a convex polyhedron was studied and generalized by Cauchy (1813) and L’Huillier (1812-1813) and represents the beginning of this new branch of mathematics: algebraical topology.

More than one century after Euler’s paper and while Listing was introducing the concept of topology, Cayley was led by an interest in particular analytical forms arising from differential calculus to study a particular class of graphs, the trees (see Cayley 1857). This study was partly motivated by important problems in theoretical chemistry. The techniques Cayley used mainly concerned the enumeration of graphs with specific properties. Enumerative graph theory then arose from his results and also the fundamental results published between 1935 and 1937. These were then generalized by De Bruijn in 1959. The links established by Cayley between his results on trees and contemporary studies of chemical composition (see Cayley 1875) influenced the development of a standard terminology in graph theory.

For example, the term “graph” was introduced by Sylvester in a paper published in 1878 in *Nature*, where he draws an analogy between “quantic invariants” and “co-variants” of algebra and molecular diagrams (see Sylvester 1878):

> Every invariant and co-variant thus becomes expressible by a graph precisely identical with a Kekuléan diagram or chemicograph. I give a rule for the geometrical multiplication of graphs, i.e. for constructing a graph to the product of in- or co-variants whose separate graphs are given.

The first textbook on graph theory was written by Dénes König and published in 1936 (see König 1990). Another book by Frank Harary, published in 1969, was “considered the world over to be the definitive textbook on the subject” (see Gardner 1992, p. 203) and enabled mathematicians, chemists, electrical engineers and social scientists to talk to each other. In France, Claude Berge (see Berge 1970) has also published in the 1970s a very important book about graphs and hypergraphs.

Let us now enter some particular problems in graph theory.

One of the most famous and stimulating problems in this field is the four color problem: "Is it true that any map drawn in the plane may have its regions colored with four colors, in such a way that any two regions having a common border have different colors?" This problem was first posed by Francis Guthrie in 1852 and its first written record is in a letter of De Morgan addressed to Hamilton the same year. Many incorrect proofs have been proposed, including those by Cayley, Kempe and others. The study and the generalization of this problem by Tait, Heawood, Ramsey and Hadwiger led to the study of the colorings of the graphs embedded on surfaces with arbitrary genus. Tait's reformulation generated a new class of problems, the factorization problems, particularly studied by Petersen and König. The works of Ramsey on colorations and more specially the results obtained by Turán in 1941 were at the origin of another branch of graph theory, extremal graph theory.

The four-color problem remained unsolved for more than a century. In 1969, Heinrich Heesch (see He and Schnabel 2018) published a method for solving the problem using computers. A computer-aided proof produced in 1976 by Kenneth Appel and Wolfgang Haken makes the fundamental use of the notion of "discharging" developed by Heesch (see Appel and Haken 1977a, 1977b). The proof involved checking the properties of 1,936 configurations by computer, and was not fully accepted at the time due to its complexity. A simpler proof considering only 633 configurations was given 20 years later by Robertson et al. (1997).

The autonomous development of topology from 1860 to 1930 fertilized graph theory back through the works of Jordan, Kuratowski and Whitney. Another important factor of common development of graph theory and topology came from the use of the techniques of modern algebra. The first example of such a use comes from the work of the physicist Gustav Kirchhoff, who published in 1845 his Kirchhoff's circuit laws for calculating the voltage and current in electric circuits.

The introduction of probabilistic methods in graph theory, especially in the study of Erdös and Rényi of the asymptotic probability of graph connectivity, gave rise to yet another branch, known as *random graph theory*, which has been a fruitful source of graph-theoretic results.

The consideration of flows and networks and the development of the Ford–Fulkerson method provide a new approach in graph theory, combinatorics and then algorithmics.

We will first explore the already large universe of graphs.

1.2. Basic definitions

Let us begin with some definitions.

DEFINITION 1.1.– *In its modern mathematical sense, a graph is an ordered pair $G(V, E)$ comprising a set V of vertices or nodes or points together with a set E of edges or arcs or lines, which are 2-element subsets of V (i.e. an edge is associated with two vertices, and that association takes the form of the unordered pair comprising those two vertices). This is not exactly, indeed, the most general definition. To avoid ambiguity, we will see later that this type of graph may be described precisely as undirected and simple. This characteristics are sufficient for the moment.*

REMARK 1.1.– *The vertices belonging to an edge are called the ends or end vertices of the edge. A vertex may exist in a graph and not belong to an edge.*

REMARK 1.2.– *The sets V and E are usually taken to be finite, and many of the well-known results are not true (or are rather different) for infinite graphs because many of the arguments fail in the infinite case.*

DEFINITION 1.2.– *The order of a graph is $|V|$, its number of vertices. The size of a graph is $|E|$, its number of edges. The degree or valency of a vertex is the number of edges that connect to it, where an edge that connects a vertex to itself (a loop) is counted.*

For an edge $\{x, y\}$, graph theorists usually use the somewhat shorter notation xy.

1.3. Different types of graphs

As we have seen, many graphs have been discovered throughout the history of the discipline, so that today, even restricting us to interesting or well-known graphs, which are already very numerous, we are dealing with an extremely large set, in fact a very bushy forest, not to say a real jungle.

Exploring this collection supposes that we have identified some distinctive features that can guide us and allow us to define classification criteria.

There are, in fact, various types of graphs depending upon the number of vertices, the number of edges, their interconnectivity and their overall structure. We will discuss only a certain few important types of graphs in this chapter.

DEFINITION 1.3 (NULL GRAPH).– *A graph having no edges is called a null graph.*

Philosophical example: At the end of his book *The Unique and His Property*, the hymn to egoism, the notion that the individual is the measure of all things, Max Stirner states "that the individual is a world history for himself, and possesses his property in the rest of the world's history". It ensures "that every higher essence above be it God, be it the human being, weakens the feeling of my uniqueness, and only before the sun of this awareness". On the contrary, he assures: "If I base my affair on myself, the unique, then it stands on the transient, the mortal creator, who consumes himself, and I may say: I have based my affair on nothing" (see Stirner 2017, pp. 376–377). We cannot better illustrate both the idea of a graph with a single vertex and the denomination of this one as "null graph".

DEFINITION 1.4 (TRIVIAL GRAPH).– *A graph with only one vertex is called a trivial graph.*

Philosophical example: The circular $J - set$, with one vertex and one loop, in Finsler's theory of sets (see Booth and Ziegler 1996), is a beautiful example of trivial graph[2].

DEFINITION 1.5 (NON-DIRECTED GRAPH).– *A graph which contains edges but in which the edges are not directed ones is a non-directed graph.*

DEFINITION 1.6 (DIRECTED GRAPHS).– *A graph in which each edge has a direction is a directed graph (sometimes, it may be called a network).*

DEFINITION 1.7 (SIMPLE GRAPH).– *A graph with no loops and no parallel edges is called a simple graph.*

DEFINITION 1.8 (REGULAR GRAPH).– *A graph G is said to be regular, if all its vertices have the same degree. In a graph, if the degree of each vertex is k, then the graph is called a k-regular graph.*

DEFINITION 1.9 (COMPLETE GRAPH).– *A simple graph with n mutual vertices is called a complete graph and it is denoted by K_n.*

REMARK 1.3.– *In other words, if a vertex is connected to all other vertices in a graph, then it is called a complete graph.*

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2 On the metaphysical sense of the J-set, see Parrochia (1999).

DEFINITION 1.10 (CYCLE GRAPH).– *A simple graph with* n *vertices (*$n \geq 3$*) and* n *edges is called a cycle graph if all its edges form a cycle of length* n*.*

REMARK 1.4.– *If the degree of each vertex in the graph is two, then it is called a cycle graph.*

Philosophical example: As in the system of Hegel, the sequence of the main concepts of the philosophy of Eric Weil forms a cycle graph.

DEFINITION 1.11 (WHEEL GRAPH).– *A wheel graph is obtained from a cycle graph* C_{n-1} *by adding a new vertex. That new vertex is called a Hub which is connected to all the vertices of* C_n*.*

Philosophical example: In Leibnizian philosophy, God is a *Hub* because He is connected to all the monads. Conversely, each monad is connected to all aspects of the world. So, the Leibnizian system is, from a graphic point of view, a wheel of wheels.

DEFINITION 1.12 (CYCLIC GRAPH).– *A graph with at least one cycle is called a cyclic graph.*

Philosophical example: There are a lot of cycles in the classic philosophical systems of Leibniz or Hegel.

DEFINITION 1.13 (ACYCLIC GRAPH).– *A graph with no cycles is called an acyclic graph.*

Philosophical example: In Descartes' philosophy, the "long chains of reasons" are examples of acyclic graphs. But Descartes' system itself is not acyclic (see Beyssade 1979; Parrochia 1993b, p. 183, 2012, pp. 114–115).

DEFINITION 1.14 (BIPARTITE GRAPH).– *A simple graph* $G = (V, E)$ *with a vertex partition* $V = \{V_1, V_2\}$ *is called a bipartite graph if every edge of E joins a vertex in* V_1 *to a vertex in* V_2*.*

Philosophical example: According to Gilles Deleuze (see Deleuze 2003, pp. 42–43), in the whole graph of Spinoza's *Ethics*, we can make a partition into two big classes of vertices associated with statements: on the one hand, the propositions, corollaries and proofs, which form the deductive part of the book; on the other hand, the scholia, a discontinuous broken chain which do not belong to the deduction. However, scholia are connected to propositions and corollaries (or their proofs) by some edges. All that do not constitute really a bipartite graph.

DEFINITION 1.15 (COMPLETE BIPARTITE GRAPH).– *A bipartite graph $G = (V, E)$ with partition $V = \{V_1, V_2\}$ is said to be a complete bipartite graph if every vertex in V_1 is connected to every vertex of V_2.*

DEFINITION 1.16 (STAR GRAPH).– *A complete bipartite graph of the form $K_{1,n-1}$ is a star graph with n vertices. A star graph is a complete bipartite graph if a single vertex belongs to one set and all the remaining vertices belong to the other set.*

1.4. More on the list of graphs

As far as we know, there is no real classification of graphs. However, we can present a list of well-known graphs that we can put together according to their name:

1) First, a catchy "name" for a graph often comes from some drawing of the graph. A first part of the "graph menagerie" is formed by the most simple graphs. For example, among the graphs with at most five vertices, we get the following: triangle, claw, paw, kite, house, bull, bowtie, dart, etc. These simple graphs are sometimes used to decompose more complex graphs or to identify graphs which do not include them: for example, triangle-free graphs, paw-free graphs, etc.

2) The discovery of some graphs happened in the development of graph theory in connection with some particular problems (e.g. a counterexample to a conjecture or the solution of some extremal problem). In this context, we meet some famous graphs named after the mathematician who found them.

Here are, for example, the following: Petersen graph, Balaban 10-cage, Dürer graph, Herschel graph, Sousselier graph, Sylvester graph, Tutte graph, Wagner graph, etc. No need to define them for now: we will do so when we encounter them if this is the case.

3) We can also identify some symmetric or regular graphs: highly symmetric graphs, strongly regular graphs, symmetric graphs, semi-symmetric graphs, etc. A graph which is not symmetric is an asymmetric graph. We will be concerned with some of them.

4) There are also big graph families, with well-characterized properties: complete graphs, complete bipartite graphs, cycles, friendship graphs, fullerene graphs, platonic solids, truncated solids, snarks, stars, wheel graphs, etc.

Some websites like "The house of graphs" give a catalog of interesting graphs with particular properties: for example, a certain number of vertices, edges, cycles or some more sophisticated properties like the diameter, the girth, the chromatic number or the number of cliques, automorphisms, etc.

When you look for a particular graph, you can guess it has always existed and that its properties are already well known. If not, you have a chance to be the discoverer of a new class of graphs.

An important point is that a graph generally has different kinds of drawings, but all of these are isomorphic. So, a graph is in fact a class of graphs generally defined up to an isomorphism.

So far, philosophers, for structuring thought, have used only a very small part of the possibilities listed in mathematics, especially in graph theory.

Generally, a few concepts (often less than 20) are enough for a philosopher to allow them to represent the world. So far, philosophers have only explored one possibility each time – in other words, the choice was made blind.

But today, we can, by combinatorial enumeration, generate all possible graphs containing a given number of vertices and edges. The results defy the imagination. Thus, for a number of vertices less than 20, the number of graphs likely to be generated is given below (this sequence S corresponds to graphs with 0 vertices, graphs with 1 vertex, graphs with 2 vertices, etc.). The last number is the number of graphs with 19 vertices:

$$S = 1, 1, 2, 4, 11, 34, 156, 1044, 12346, 274668, 12005168, 1018997864,$$
$$165091172592, 50502031367952, 29054155657235488,$$
$$31426485969804308768, 64001015704527557894928,$$
$$245935864153532932683719776, 1787577725145611700547878190848,$$
$$24637809253125004524383007491432768.$$

To give a speaking example of what we are putting forward, here is a small sample of the graphs $G(X, U)(|X| = 6)$ that we can construct. Many different architectonics could thus be generated from the same basic corpus (see Figure 1.5).

A philosopher may say that this list has no more interest than the *One Hundred Billion Poems* by Queneau (1961) and could well be, like the late research of Oulipo, a free game where content gets lost. What obviously makes a good philosopher, like what makes a good poet, probably has little to do with combinatorics. But the counts remain interesting and put our constructions into perspective. They should, at least, make those who take themselves for masters more modest.

So, as we can see, there exist an immense zoo of graphs, where we can find, indeed, a lot of non-trivial forms of conceptual architectonic structures.

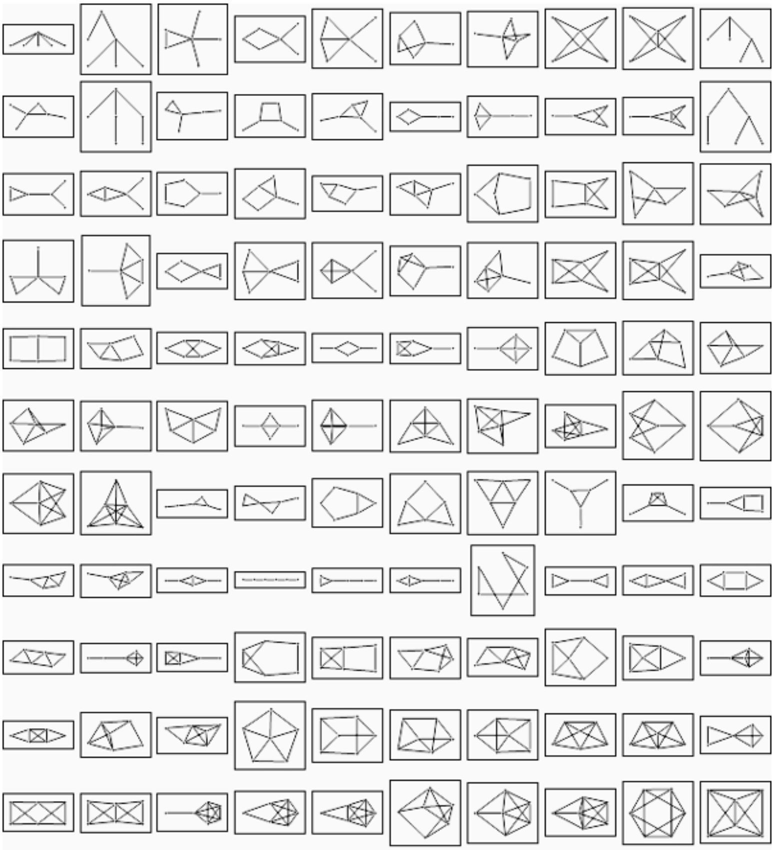

Figure 1.5. *Some graphs* $G(X, U)$ *with* $|X| = 6$

1.5. Graphs and vertices

Once we know whether the graph is labeled or not, the simplest criteria that can be retained are obviously the number of vertices. The number of the first graphs of unlabeled vertices is given in Table 1.1. Let v be the number of vertices of a graph, n be the number of connected graphs and m be the number of all possible graphs. As we can see, for $v = 6, n, m > 100$ and for $v > 7$, these numbers are growing extremely fast. For $v = 16$, these very numbers challenge the imagination. But it is not uncommon to deal with graphs whose number of vertices are greater. For a philosopher, for example, who has to organize the relations between 20 categories or more, this is quite possible. This tends to call into question the possibility of constructing a philosophy on a deductive mode: namely, from a finite set of speculative categories, attempting to try every possible arrangement to choose the best one. Of course, we cannot proceed like this.

Indeed, we think it could be useful to a philosopher, to know a few things about some simple graphs that could be bases of some cognitive systematization.

Vertices	Connected graphs	All graphs
1	1	1
2	1	2
3	2	4
4	6	11
5	21	34
6	112	156
7	853	1044
8	11117	12346
9	261080	274668
10	11716571	12005168
11	1006700565	1018997864
12	164059830476	165091172592
13	50335907869219	50502031367952
14	29003487462848061	29054155657235488
15	31397381142761241960	31426485969804308768
16	63969560113225176176277	64001015704527557894928

Table 1.1. *Number of graphs with a fixed number of vertices*

The main graphs we have to deal with in philosophy are connected graphs, though some philosophical doctrines, based on a collection of aphorisms, present a disconnected structure. Note also that, since Nietzsche and Kierkegaard, most of the philosophers have abandoned (rightly or wrongly – maybe wrongly –) the project to construct a *system* in the classical sense, i.e. a coherent and definitive organization of the whole of thought based on principles of economy and harmony (for details, see section 2.5 of Chapter 2, and also Chapter 5).

DEFINITION 1.17.– *A graph is said to be connected in the sense of a topological space if there is a path from any point to any other point in the graph. A graph that is not connected is said to be disconnected.*

REMARK 1.5.– *This definition means that the null graph and the singleton graph are considered connected, while empty graphs on $n \geq 2$ nodes are disconnected.*

All the connected graphs $G(V, E)$ with $|V| \leq 5$ are shown in Figure 1.6.

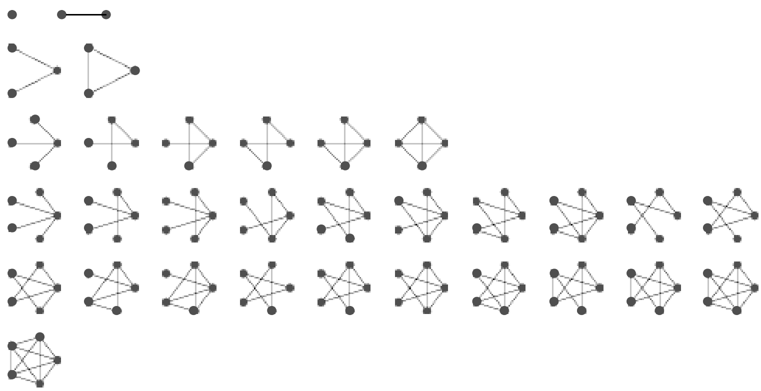

Figure 1.6. *Connected graphs $G(V, E)$ with $V \leq 5$*

1.6. Some operations on graphs

DEFINITION 1.18.– *The complement or inverse of a graph G is a graph H on the same vertices such that two distinct vertices of H are adjacent if and only if they are not adjacent in $G.H$ is sometimes written as $\bar{G}$.*

THEOREM 1.1.– *If G is disconnected, then its complement $\bar{G}$ is connected.*

REMARK 1.6.– *However, the converse is not true, as can be seen using the examples of the 4-vertex path graph and the 5-vertex cycle graph which are both connected and isomorphic to their complement.*

Here are some binary operations between two simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$:

1) The union is $G_1 \cup G_2 = (V_1 \cup V_2, E_1 \cup E_2)$ (simple graph).

2) The intersection $G_1 \cap G_2 = (V_1 \cap V_2, E_1 \cap E_2)$ (simple graph).

3) The ring sum $G_1 \oplus G_2$ is the subgraph of $G_1 \cup G_2$ induced by the edge set $E_1 \oplus E_2$ (simple graph).

The examples are presented in Figure 1.7.

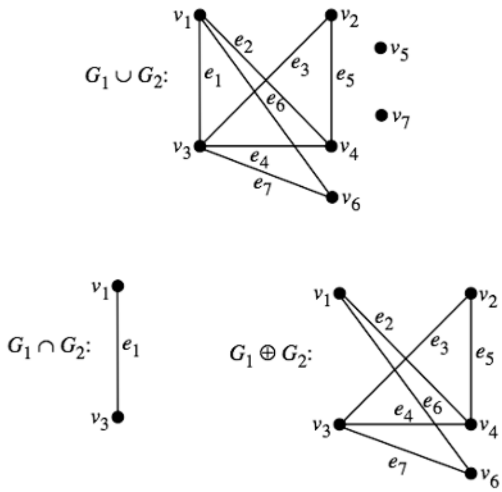

Figure 1.7. *Three binary operations between two simple graphs*

NOTE.– The set operation $\oplus$ is the symmetric difference, i.e.

$$E_1 \oplus E_2 = (E_1 - E_2) \cup (E_2 - E_1).$$

Since the ring sum is a subgraph induced by an edge set, there are no isolated vertices. All three operations are commutative and associative.

The operations $\cup$, $\cap$ and $\oplus$ can also be defined for more general graphs other than simple graphs. Naturally, we have to "keep track" of the multiplicity of the edges. In this case:

1) $\cup$: The multiplicity of an edge in $G_1 \cup G_2$ is the larger of its multiplicities in G_1 and G_2.

2) $\cap$: The multiplicity of an edge in $G_1 \cap G_2$ is the smaller of its multiplicities in G_1 and G_2.

3) $\oplus$: The multiplicity of an edge in $G_1 \oplus G_2$ is $|m_1 - m_2|$, where m_1 is its multiplicity in G_1 and m_2 is its multiplicity in G_2.

1.7. Graph isomorphisms

1.7.1. *Self-complementary graphs*

DEFINITION 1.19.– *A graph which is isomorphic to its graph complement is called a self-complementary graph (see Figure 1.8).*

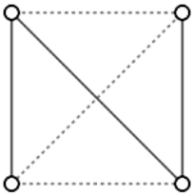

Figure 1.8. *An example of a self-complementary graph*

The numbers of simple self-complementary graphs on n = 1, 2, ... nodes are 1, 0, 0, 1, 2, 0, 0, 10, ... The first of them is shown in Figure 1.9.

Every self-complementary graph is not only connected, but also traceable. A trivial example of a self-complementary graph in philosophy is the world's image of the line in Plato's *Republic* (510e–511a), which can be associated with P_4, while the self-complementary graph C_5 is the figure circumscribing the Pythagorean pentagram.

Here, we allow ourselves to draw the attention of philosophers to an important methodological point.

The existence of self-complementary graphs proves that it is not always relevant to take the complement of a conceptual graph, contrary to the method advocated by Jacques Derrida and implemented in his work on the notion of "parergon" (see Derrida and Owens 1979)[3]. If the graph in question is a self-complementary graph, then, it is absolutely the same, from the structural point of view, to study the conceptual graph or its complementary.

3 The English version is a translation of section II of the four-part essay entitled "Parergon" published in Derrida 1978. Parts of this text originally appeared in the French review *Digraphe* 2 (see Derrida 1974 and our commentary in Parrochia 1993a). In Derrida's idea, what is at the edge of a philosopher's work and seems to fall outside their official theses sometimes allows them to be reinterpreted to the point of revealing what their discourse tends to veil. In the case of self-complementary graphs, it is not clear whether the so-called "parergonal border logic" could possibly reveal more than the graph itself.

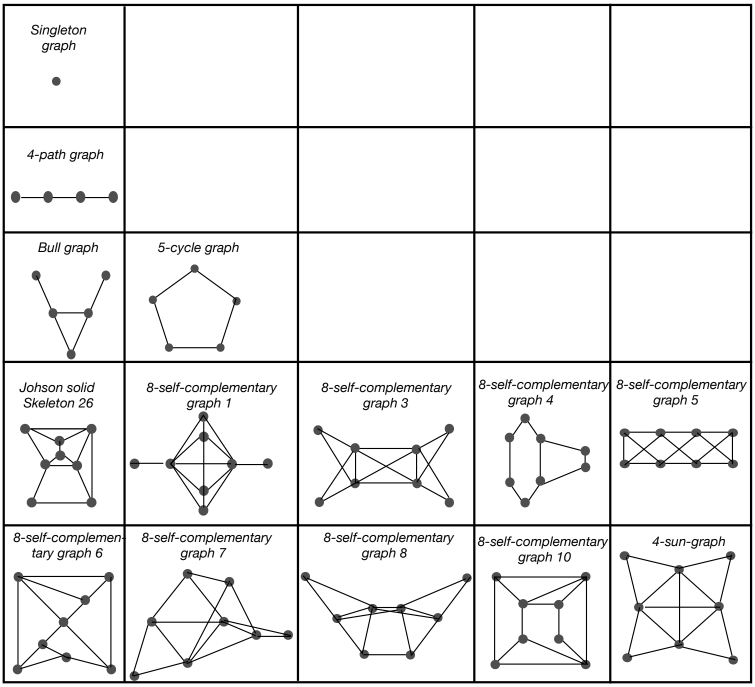

Figure 1.9. *The first self-complementary graphs*

1.7.2. *Properties of self-complementary graphs*

For a self-complementary graph on $n \geq 5$ vertices, there is a graph cycle of length ℓ for every integer $3 \leq \ell \leq n - 2$ (see Rao 1977; Farrugia 1999, p. 51). As a result, the graph circumference of a self-complementary graph on $n > 5$ vertices is either n (i.e. the graph is Hamiltonian), $n - 1$ or $n - 2$.

By definition, a self-complementary graph must have exactly half the total possible number of edges, i.e. $n(n-1)/4$ edges for a self-complementary graph on n vertices. Since $n(n-1)$ must be divisible by 4, it follows that $n = 0$ or $1(\text{mod.}4)$.

We may also remark that there is a polynomial-time condition to determine if a self-complementary graph is Hamiltonian.

And as the number of self-complementary graphs on n nodes can be derived from Pólya's enumeration theorem, using self-complementary graphs in philosophy should allow us to process a kind of deductive philosophy.

1.8. Symmetric and asymmetric graphs

Once we have seen whether a graph is finite or infinite, whether it is directed or not, simple or not, etc., we may search whether a graph is symmetric or not. The symmetry property is characteristic of mathematical graphs and leads to beautiful figures. But there is no reason for a philosophical graph, i.e. a graph which represents the whole world or some part of it, to be symmetric[4]. Moreover, symmetry is not a basic property. It is a consequence of more fundamental properties like the fact of being edge-transitive and vertex-transitive. Let us explain that with the help of some definitions.

DEFINITION 1.20.– *An edge-transitive graph is a graph G such that, given any two edges e_1 and e_2 of G, there is an automorphism of G that maps e_1 to e_2.*

DEFINITION 1.21.– *A vertex-transitive graph is a graph G in which, given any two vertices v_1 and v_2 of G, there is some automorphism:*

$$f : V(G) \to V(G)$$

such that:

$$f(v_1) = v_2$$

DEFINITION 1.22 (SYMMETRIC GRAPH).– *A symmetric graph is a graph that is both edge- and vertex-transitive (Holton and Sheehan 1993, p. 209). However, care must be taken with this definition since arc-transitive or 1-arc-transitive graphs are sometimes also known as symmetric graphs (Godsil and Royle 2001, p. 59).*

Of course, edge-transitive graphs are not necessarily vertex-transitive. Some graphs like the Folkman graph (20 vertices) (see Figure 1.10), the Gray graph (54 vertices), the Iofinova–Ivanov graph (110 vertices), the Ljubljana graph (112 vertices) and the Tutte-12 cage (126 vertices) are examples of such a case.

Conversely, there exist vertex-transitive graphs which are not edge-transitive. The graph corresponding to a triangular prism (see Figure 1.11), for example, is

4 We know that natural evolution seems to show a preference for symmetric structures, which can be explained for algorithmic reasons: they are easier to obtain at random than asymmetric structures, which are generally much more complex (see Louis et al. 2022). But in the evolution of thought, and so, in philosophy, no such algorithmic process exists and "simple" forms do not seem to be favored. However, for reasons of comprehension as much as of aesthetics, the system must surely be presented in a form that is economical and quick to use.

vertex-transitive but not edge-transitive. More generally, two copies G, G' of the complete graph K_n for $n \geq 3$ that are linked by n edges (one edge between v_i and v'_i for each i) is vertex-transitive but not edge-transitive.

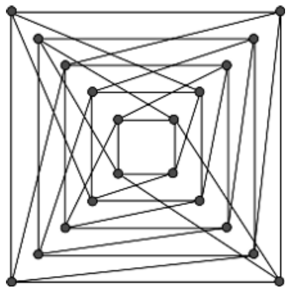

Figure 1.10. *The Folkman graph (source: Wikipedia article: Folkman graph (modified from Wolfram Mathworld))*

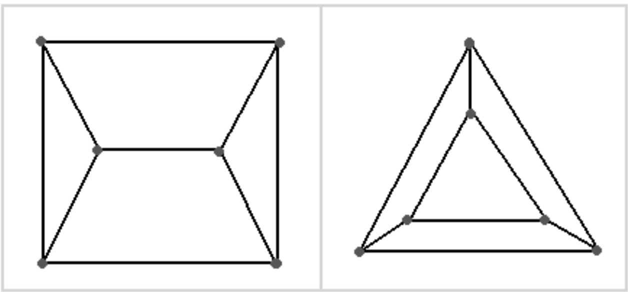

Figure 1.11. *The triangular prism*

DEFINITION 1.23 (ASYMMETRIC GRAPH).– *A graph which is not symmetric is said to be asymmetric. An asymmetric graph has no nontrivial symmetries, i.e. it is a graph for which there are no other automorphisms than the trivial automorphism (the identity mapping of the graph onto itself).*

An example is the Frucht graph (12 vertices, 18 edges), a 3-regular pancyclic Halin graph first described by Robert Frucht in 1939 (see Figure 1.12).

Can we measure the degree of asymmetry of a graph? The answer is positive. As Erdös and Réniy have shown (see Erdös and Rényi 1963), any asymmetric graph can be made symmetric by deleting some of its edges and by adding certain new edges connecting its vertices. This is called a *symmetrization* of G. Now, let us take the sum of the number of deleted edges – say r – and the number of new edges – say s. The degree of asymmetry $A[G]$ of a graph G is the minimum of $r + s$ where the minimum is taken over all possible symmetrizations of the graph G. If in order to make a graph symmetric, we delete r of its edges and add s new edges, we shall say that we changed $r + s$ edges. Clearly, the asymmetry of a symmetric graph is according to this definition equal to 0, while the asymmetry of any asymmetric graph is a positive integer. The question arises: how large can the degree of asymmetry of a graph of the order n (i.e. a graph which has n vertices) be? We shall denote by $A(n)$ the maximum of $A[G]$ for all graphs G of order n ($n = 2, 3, ...$). We put further $A(1) = +\infty$. The authors prove that for $n = 2$ to 5, $A(n) = 0$. Some graphs can be made symmetric by adding one edge (Figure 1.13(a)) or by omitting one edge (Figure 1.13(b)).

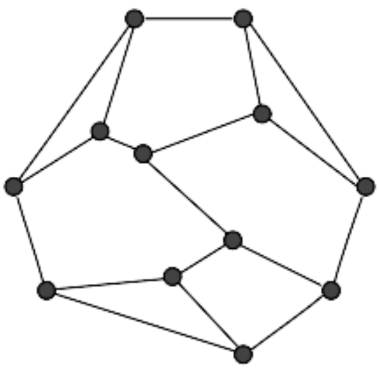

Figure 1.12. *The Frucht graph*

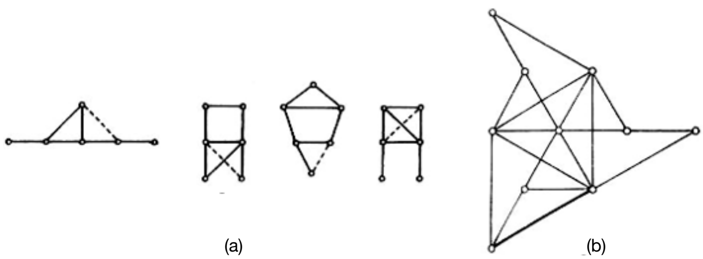

Figure 1.13. *Symmetrization of some graphs (from Erdös and Renyi 1963, p. 297).*

The authors also get the following results:

THEOREM 1.2.– *The asymmetry of a graph of order n cannot exceed $\frac{n-1}{2}$ if n is odd, while if n is even the asymmetry of a graph of order n cannot exceed $\frac{n}{2} - 1$.*

This is the best estimation, so we have, in other words:

THEOREM 1.3.–

$$\lim_{n\to\infty} \frac{A(n)}{n} = \frac{1}{2}.$$

The degree of asymmetry of a graph is therefore at most slightly lower than the number of its edges. In other words, if the world were an asymmetric graph of order n, it would be necessary at most to change a little less than half of the number of its edges to transform it into a symmetric graph.

THEOREM 1.4.– *The probability that a graph with n vertices and N edges chosen at random should be asymmetric tends to be 1 for $n \to \infty$.*

THEOREM 1.5.– *Assume now that Γ is an infinite graph. The probability P that Γ should be symmetric is such that $P = 1$.*

These last theorems mean that almost all finite graphs are asymmetric, while almost all infinite graphs are symmetric. So, if the world graph is finite, it has a big chance of being asymmetric, while if it is infinite, it has a big chance of being symmetric.

Perhaps there exist some exceptions. In Spinoza's philosophy, the infinite modes of thought and extension (and the unknown others) are infinite chains, so they are infinite edge-transitive graphs. But the whole graph collecting these chains is not connected. And so, it is not vertex-transitive. However, there is a correspondence between any pairs of chains (*Ethic*, book 2, prop. 7), which is just a consequence of the fact that such a graph of a bichain is necessarily bipartite (see Biggs 1993, p. 118).

1.9. Extremal graphs

In this part of mathematics called "analysis", we know that it can be very useful to study the extrema of a function. We draw from it information about its derivative and the direction of variation of the function. Extrema are critical points which allow us to characterize it.

In the same way, this part of graph theory named *extremal graph theory* studies extremal (maximal or minimal) graphs that satisfy a certain property, which allow us to get a better understanding of the graph. Extremality can be taken with respect to different graph invariants, such as order, size or girth. More abstractly, it also studies how global properties of a graph influence the local substructures of it.

For example, a question like "which acyclic graphs on n vertices have the maximum number of edges?" is typically an extremal graph theory question. The extremal graphs which answer this question are trees on n vertices with $n - 1$ edges. More generally, questions are of the following type: given a graph property P, an invariant u and a set of graphs S, we wish to find the minimum value of m such that every graph in S with $u > m$ possesses property P. In the example above, S was the set of n-vertex graphs, P was the property of being cyclic and u was the number of edges in the graph. Thus, every graph on n vertices with more than $n - 1$ edges must contain a cycle.

Several foundational results in extremal graph theory are questions of the above-mentioned form. For instance, the question of the largest number of edges that we can add to an n-vertex graph without creating clique of size k is answered by Turán's theorem. Instead of cliques, if the same question is asked for complete multi-partite graphs, the answer is given by the Erdös–Stone theorem.

Historically, extremal graph theory, in its strictest sense, has been developed by Hungarians. It started in 1941 when Turán proved his theorem determining those graphs of order n, not containing the complete graph K_k of order k, and extremal with respect to size (i.e. with as many edges as possible).

Turán's theorem answers the following question: what is the maximum possible number of edges in an undirected graph G with n vertices which does not contain K_3 (the graph with three vertices A, B, C and three edges AB, AC, BC, i.e. a triangle) as a subgraph? The complete bipartite graph where the partite sets differ in their size by at most 1 is the only extremal graph with this property. It contains:

$$\left\lfloor \frac{n^2}{4} \right\rfloor$$

edges.

Similar questions have been studied with various other subgraphs H instead of K_3. For instance, the Zarankiewicz problem concerns the largest graph that does not contain a fixed complete bipartite graph as a subgraph, and the even circuit theorem concerns the largest graph without a fixed-length even cycle. Turán also found the (unique) largest graph not containing K_k which is named after him, namely the Turán graph (see Figure 1.14). This graph is the complete join of $k-1$ independent sets (as equisized as possible) and has at most:

$$\left\lfloor \frac{(k-2)n^2}{2(k-1)} \right\rfloor = \left\lfloor \left(1 - \frac{1}{k-1}\right) \frac{n^2}{2} \right\rfloor$$

edges. For C_4 (the square), the largest graph on n vertices not containing C_4 has:

$$\left(\frac{1}{2} + o(1)\right) n^{3/2}$$

edges.

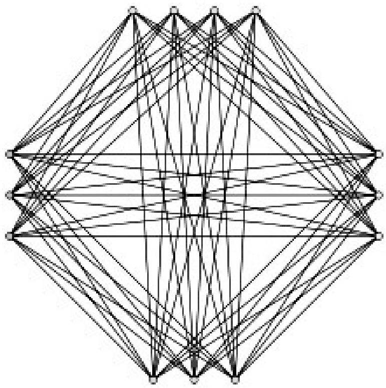

Figure 1.14. *Turán's graph (source: Wikipedia article: Turan graph)*

The preceding theorems give conditions for a small object to appear within (perhaps) a very large graph. At the opposite extreme, we might search for conditions which force the existence of a structure which covers every vertex. But it is possible for a graph with:

$$\binom{n-1}{2}$$

edges to have an isolated vertex – even though almost every possible edge is present in the graph – which means that even a graph with a very high density may have no interesting structure covering every vertex. Simple edge counting conditions, which give no indication as to how the edges in the graph are distributed, thus often tend to give uninteresting results for very large structures.

Instead, we introduce the concept of *minimum degree*. The minimum degree of a graph G is defined to be:

$$\delta(G) = \min_{v \in G} d(v).$$

Specifying a large minimum degree removes the objection that there may be a few "pathological" vertices. If the minimum degree of a graph G is 1, for example, then there can be no isolated vertices (even though G may have very few edges).

A classic result in this context is Dirac's theorem, which states that every graph G with n vertices and minimum degree at least $n/2$ contains a Hamilton cycle.

This last result contains very interesting information for philosophers who want to construct philosophical systems with (or without) the Hamilton cycle.

1.10. Independence, non-separability, reconstruction conjecture

Regarding recent philosophy, there is one observation that the mathematician must make: much of what philosophers believe is generally inaccurate or erroneous. Indeed, most of the statements that the philosophical current known as "French theory" have been able to express are either trivial or false.

For example, the notion of "difference" and the privilege of simulacra, in Deleuze, is a kind of poor Heraclitism which seems to ignore the scientific notion of the invariant. But an antireflexive, antisymmetric and antitransitive world is hardly plausible: any theory would be stillborn there[5]. We can also object that there are scientific laws, and that the question is not that everything changes all the time in the world (trivial observation). It is rather to detect what does not change in what is changing, which is precisely the task of reason.

Likewise, one of Jacques Derrida's major ideas, which consisted of introducing the concept of "deconstruction", has no real relevance. Initial translation of the German words "Destruktion" and "Abbau" used by Heidegger in "Being and Time", and essentially intended to reintegrate time into the philosophical problematic, this concept took on a whole new meaning with

5 It will be observed that: 1) the simulacrum constantly differs from itself (see Plato's "sophist", or even Diderot's "nephew of Rameau" (see Parrochia 2006, pp. 167–173)); 2) as it does not resemble itself, nor can it resemble another (it is therefore also defined by antisymmetry). Resemblance not being, in any case, transitive, the simulacrum cannot be integrated into any transitive relation whatsoever. It could not therefore exist – if it had the least possible existence – only in an antireflexive, antisymmetrical and antitransitive universe.

Derrida's *deconstruction*. The philosopher has in fact used this word to signify an operation aimed at revealing the structure or the traditional architecture of the founding concepts of the whole of ontology or of Western metaphysics. The critique encompasses not only the analyzed object, the analyst and their language, but is supposed to show an inextricable entanglement of the content and the container of the analytical production in which everything in itself, reduced to a pure system of signs, finally reveals its nothingness in the frameworks drawn by classical metaphysics. It is therefore the whole of philosophy that collapses to make room for a commentary which, indefinitely, will exhaust itself in saying this impossibility[6].

Of course, mathematics – and in particular graph theory – is capable of finely formalizing the subtleties likely to be encountered in certain real situations. Thus, the notions of dependence and independence, of separability and inseparability, of composition and decomposition then take on a precise meaning and can no longer be confused. The philosophical world of "the more or less" can only give way, in the long term, to the scientific universe of precision.

DEFINITION 1.24 (INDEPENDENT SET).– *An independent set (also named "stable set", "coclique" or "anticlique") is a set of vertices in a graph, no two of which are adjacent. That is, it is a set I of vertices such that for every two vertices in I, there is no edge connecting the two. The size of an independent set is the number of vertices it contains.*

A *maximal independent set* is an independent set that is not a proper subset of any other independent set. The problem of finding such a set is called "the maximum independent set problem" and is an NP-hard optimization problem. As such, it is unlikely that there exists an efficient algorithm for finding a maximum independent set of a graph. Figure 1.15 gives an example of a maximal independent set in the Petersen graph.

If we see the whole of ontology or of Western metaphysics as a very complicated graph, we can guess it contains some independent sets, and conjecture that there is, among them, a maximal one. This possible existence of stability seems to have escaped the supporters of the "French theory", too eager to put an end to metaphysics.

Let us take a glance now to separable and unseparable graphs.

6 Michel Foucault immediately saw this inevitable consequence of Derrida's theses (see Foucault 1972).

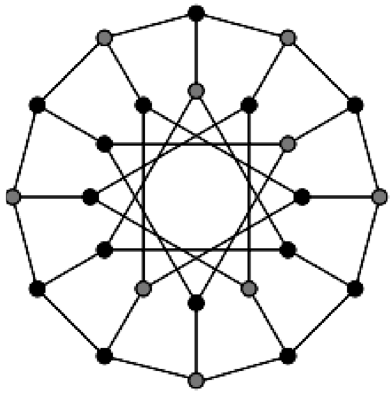

Figure 1.15. *Petersen graph GP(12, 4) (source: Wikipedia article: Independent set (graph theory): Generalized Petersen graph GP(12,4))*

DEFINITION 1.25 (CUT).– *A vertex v of a graph G is a cut vertex or an articulation vertex of G if the graph $G - v$ consists of a greater number of components than G (see Figure 1.16).*

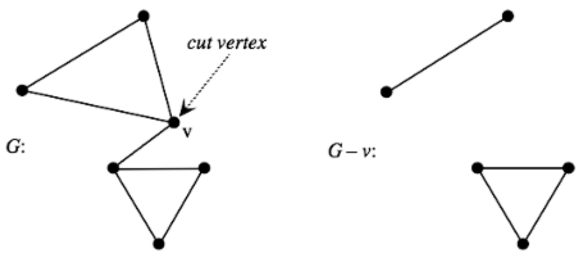

Figure 1.16. *A separable graph*

DEFINITION 1.26.– *More explicitly, we can say that a graph is separable if it is not connected or if there exists at least one cut vertex in the graph. Otherwise, the graph is non-separable (see Figure 1.17).*

Even when graphs are non-separable, they always admit subgraphs. Informally speaking, the reconstruction conjecture says that graphs are determined uniquely by their subgraphs. Let us introduce some new definitions to be able to express this more rigorously.

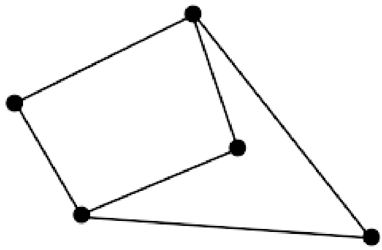

Figure 1.17. *A non-separable graph*

DEFINITION 1.27 (VERTEX-DELETED SUBGRAPH).– *Given a graph* $G = (V, E)$*, a vertex-deleted subgraph of G is a subgraph formed by deleting exactly one vertex from G. By definition, it is an induced subgraph of G.*

DEFINITION 1.28 (VERTEX-DECK).– *For a graph G, the vertex-deck (or simply "deck") of G, denoted* $D(G)$*, is the multiset of isomorphism classes of all vertex-deleted subgraphs of G. Each graph in* $D(G)$ *is called a card.*

DEFINITION 1.29 (HYPOMORPHIC GRAPHS).– *Two graphs that have the same deck are said to be hypomorphic.*

With these definitions, the conjecture can be stated like that.

Reconstruction conjecture: Any two hypomorphic graphs with at least three vertices are isomorphic.

REMARK 1.7.– *The requirement that the graphs have at least three vertices is necessary because both graphs with two vertices have the same decks.*

Harary (1964) suggested a stronger version of the conjecture.

Set reconstruction conjecture: Any two graphs with at least four vertices with the same sets of vertex-deleted subgraphs are isomorphic.

DEFINITION 1.30 (EDGE-DELETED SUBGRAPH).– *Given a graph* $G = (V, E)$*, an edge-deleted subgraph of G is a subgraph formed by deleting exactly one edge from G.*

DEFINITION 1.31 (EDGE-DECK).– *For a graph G, the edge-deck of G, denoted* $ED(G)$*, is the multiset of all isomorphism classes of edge-deleted subgraphs of G. Each graph in* $ED(G)$ *is called an edge-card.*

Edge reconstruction conjecture (see Harary 1964): any two graphs with at least four edges and having the same edge-decks are isomorphic.

As we can see, the "reconstruction conjecture" can be seen as a kind of methodological opposite to the famous Derrida's "deconstruction conjecture", a theory which, for lack of a rational method using effective formal tools, does not allow us to get any basic elements of the onto-theological discourse and to manage any new discovery within the frame of metaphysical knowledge. At the same time, it does not allow us to leave traditional metaphysics either and can only carry it out indefinitely, by limiting itself, here and there, to pointing out a gap between the texts and their contemporary criticism that nothing, ever, does not just fill.

On the contrary, the "reconstruction conjecture" (for a survey or the problem, see Harary 1974; Nash-Williams 1978) reveals in the graphs of metaphysics the subgraphs which generate them and, suddenly, allow, once combined differently, to escape the old determinations.

We hope that this method can be implemented in philosophy and will find full use of its immense possibilities.

This totally new approach to philosophy can be further explored by noting the following.

DEFINITION 1.32 (RECOGNIZABLE PROPERTIES).– *In the context of the reconstruction conjecture, a graph property is called recognizable if we can determine the property from the deck of a graph.*

The following properties of graphs are recognizable: order of a graph, number of edges, degree sequence of vertices, vertex-connectivity, Tutte polynomial, planarity, types of spanning trees and chromatic polynomial.

Both the reconstruction and set reconstruction conjectures have been verified for all graphs with at most 11 vertices by Brendan McKay (1997).

In a probabilistic sense, it has been shown by Béla Bollobás that almost all graphs are reconstructible (see Bollobás 1990). This means that the probability that a randomly chosen graph with n vertices is not reconstructible goes to 0 as n goes to infinity. In fact, it was shown that not only are almost all graphs reconstructible, but in fact that the entire deck is not necessary to reconstruct them: almost all graphs have the property that there exist three cards in their deck that uniquely determine the graph.

The conjecture has been verified for a number of infinite classes of graphs (and, trivially, their complements): regular graphs, trees, disconnected graphs, unit interval graphs, separable graphs without end vertices, maximal planar graphs, maximal outerplanar graphs, outerplanar graphs and critical blocks.

We get the following theorems:

THEOREM 1.6.– *The reconstruction conjecture is true if all 2-connected graphs are reconstructible (see Yang 1988).*

THEOREM 1.7 (DUALITY).– *If G can be reconstructed from its vertex deck $D(G)$, then its complement G' can be reconstructed from $D(G')$ as follows: Start with $D(G')$, take the complement of every card in it to get $D(G)$, use this to reconstruct G, then take the complement again to get G'.*

REMARK 1.8.– *Edge reconstruction does not obey any such duality: indeed, for some classes of edge-reconstructible graphs, it is not known if their complements are edge reconstructible.*

REMARK 1.9.– *Some structures are not generally reconstructible. Such are digraphs[7], hypergraphs (see Kocay 1987), infinite graphs[8], locally finite graphs[9].*

Despite these limitations, the concept of "reconstruction" and the approach that goes with it are interesting. By taking them as far as they can lead in philosophy, we would perhaps be able to think from new structures, not only non-logocentric, but also non-Derridian ones, thus leaving definitively the cesspool where Derrida wants to lock us[10].

7 Infinite families of non-reconstructible digraphs are known, including tournaments (see Stockmeyer 1977) and non-tournaments (see Stockmeyer 1981). A tournament is reconstructible if it is not strongly connected (see Harary and Palmer 1967). A weaker version of the reconstruction conjecture has been conjectured for digraphs.

8 Let T be a tree with an infinite number of vertices such that every vertex has infinite degree, and let nT be the disjoint union of n copies of T. These graphs are hypomorphic, and thus not reconstructible. Every vertex-deleted subgraph of any of these graphs is isomorphic: they all are the disjoint union of an infinite number of copies of T.

9 The question of reconstructibility for locally finite infinite trees (the Harary–Schwenk–Scott conjecture from 1972) was a longstanding open problem until 2017, when a non-reconstructible tree of maximum degree 3 was found by Bowler et al. (see Bowler et al. 2017).

10 It will not be objected that "Digraph" is precisely the name of the journal and of the collection where Derrida published some of his works, and that the choice of this title could make sense: in truth, the absence of possibility of reconstructing the mathematical digraphs does not mean that this exception would be able to confirm Derrida's idea of "deconstruction" in the least. Moreover, Jacques Derrida is not known as a theoretician of graphs.