1

Jordan Superalgebras

Consuelo MARTINEZ[1] **and Efim ZELMANOV**[2]
[1] *Department of Mathematics, University of Oviedo, Spain*
[2] *Department of Mathematics, University of California San Diego, USA*

1.1. Introduction

Superalgebras appeared in a physical context in order to study, in a unified way, supersymmetry of elementary particles. Jordan algebras grew out of quantum mechanics and gained prominence due to their connections to Lie theory. In this chapter, we survey Jordan superalgebras focusing on their connections to other subjects. In this section we introduce some basic definitions and in section 1.2 we give the Tits–Kantor–Koecher construction that shows the way in which Lie and Jordan structures are connected. In section 1.3, we show examples of some basic superalgebras (the so-called classical superalgebras). Section 1.4 is about the notion of brackets and explains how to construct superalgebras using different types of brackets. Section 1.5 explains Cheng–Kac superalgebras, an important class of superalgebras that appeared for the first time in the context of superconformal algebras. The classification of Jordan superalgebras is explained in section 1.6, and it includes the cases of an algebraically closed field of zero characteristics, the case of prime characteristic, both for Jordan superalgebras with semisimple even part and with non-semisimple even part, and the case of non-unital Jordan superalgebras. Finally, in section 1.7, we give some general ideas about Jordan superconformal algebras. Throughout the chapter, all algebras are considered over a field F, $\mathrm{char} F \neq 2$.

Algebra and Applications 1,
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DEFINITION 1.1.– *A (linear) Jordan algebra is a vector space J with a linear binary operation $(x, y) \mapsto xy$ satisfying the following identities:*

(J1) $xy = yx$ *(commutativity);*

(J2) $(x^2y)x = x^2(yx) \quad \forall x,\ y \in J$ *(Jordan identity).*

Instead of (J2) we can consider the corresponding linearized identity:

(J'2) $(xy)(zu) + (xz)(yu) + (xu)(yz) = ((xy)z)u + ((xu)z)y + ((yu)z)x \quad \forall x,\ y,\ z,\ u \in J.$

REMARK 1.1.– A Lie algebra L is a vector space with a linear binary operation $(x, y) \mapsto [x, y]$ satisfying the following identities:

(L1) $[x, y] = -[y, x]$ (anticommutativity);

(L2) $[[x, y], z] + [[y, z], x] + [[z, x], y] = 0$ for arbitrary elements $x,\ y,\ z \in J$ (Jacobi identity).

EXAMPLE 1.1.– If A is an associative algebra, then $(A^{(+)}, \cdot)$, where $a \cdot b = ab + ba$ is a Jordan algebra, and $(A^{(-)}, [\ ,\ ])$, where $[a, b] = ab - ba$ is a Lie algebra. Both $A^{(+)}$ and $A^{(-)}$ have the same underlying vector space as A.

DEFINITION 1.2.– *A superalgebra A is an algebra with a $\mathbb{Z}/2\mathbb{Z}$-grading. So $A = A_{\bar{0}} + A_{\bar{1}}$ is a direct sum of two vector spaces and*

$$A_{\bar{0}}A_{\bar{0}} \subseteq A_{\bar{0}}, \quad A_{\bar{1}}A_{\bar{1}} \subseteq A_{\bar{0}}, \quad \textit{and} \quad A_{\bar{0}}A_{\bar{1}}, A_{\bar{1}}A_{\bar{0}} \subseteq A_{\bar{1}}.$$

Elements of $A_{\bar{0}} \cup A_{\bar{1}}$ are called homogeneous elements. The parity of a homogeneous element a, denoted $|a|$, is defined by $|a| = 0$ if $a \in A_{\bar{0}}$ and $|a| = 1$ if $a \in A_{\bar{1}}$.

Elements in $A_{\bar{0}}$ are called even and elements in $A_{\bar{1}}$ are called odd.

Note that $A_{\bar{0}}$ is a subalgebra of A, but $A_{\bar{1}}$ is not, instead it can be seen as a bimodule over $A_{\bar{0}}$.

EXAMPLE 1.2.– If V is a vector space of countable dimension, then $G = G(V)$ denotes the Grassmann (or exterior) algebra over V, that is, the quotient of the tensor algebra over the ideal generated by the symmetric tensors $v \otimes w + w \otimes v$, $v,\ w \in V$. This algebra $G(V)$ is $\mathbb{Z}/2\mathbb{Z}$-graded. Indeed, $G(V) = G(V)_{\bar{0}} + G(V)_{\bar{1}}$, where the "even part" is the linear span of all tensors of even length and the "odd part" $G(V)_{\bar{1}}$ is the linear span of all tensors of odd length.

$G(V)$ is an example of a superalgebra.

DEFINITION 1.3.– *Consider a variety of algebras $\mathcal{V}$ defined by homogeneous identities (see Jacobson (1968) or Zhevlakov* et al. *(1982)). We say that a superalgebra $A = A_{\bar{0}} + A_{\bar{1}}$ is a $\mathcal{V}$-superalgebra if the even part of $A \otimes_F G(V)$ lies in the variety, that is*

$$A_{\bar{0}} \otimes G(V)_{\bar{0}} + A_{\bar{1}} \otimes G(V)_{\bar{1}} \in \mathcal{V}.$$

DEFINITION 1.4.– *The algebra $A_{\bar{0}} \otimes G(V)_{\bar{0}} + A_{\bar{1}} \otimes G(V)_{\bar{1}}$ is called the Grassmann envelope of the superalgebra A and will be denoted as $G(A)$.*

Let us consider $\mathcal{V}$ the variety of associative, commutative, anticommutative, Jordan or Lie algebras, respectively. Then we get:

EXAMPLE 1.3.– A superalgebra $A = A_{\bar{0}} + A_{\bar{1}}$ is an associative superalgebra if and only if it is a $\mathbb{Z}/2\mathbb{Z}$-graded associative algebra.

EXAMPLE 1.4.– A superalgebra $A = A_{\bar{0}} + A_{\bar{1}}$ is a commutative superalgebra if it satisfies:

$$xy = (-1)^{|x||y|}yx$$

for any x, y homogeneous elements of A.

EXAMPLE 1.5.– A superalgebra A is an anticommutative superalgebra if

$$xy = -(-1)^{|x||y|}yx$$

for every x, y homogeneous elements of A.

EXAMPLE 1.6.– A Jordan superalgebra is a superalgebra that is commutative and satisfies the graded identity:

$$\begin{aligned}&(xy)(zu) + (-1)^{|y||z|}(xz)(yu) + (-1)^{|u||y|+|u||z|}(xu)(yz)\\ &= ((xy)z)u + (-1)^{|u||y|+|z||y|+|z||u|}((xu)z)y + (-1)^{|x|(|y|+|z|+|u|)+|z||u|}((yu)z)x\end{aligned}$$

for every homogeneous elements x, y, z, $u \in A_{\bar{0}} \cup A_{\bar{1}}$.

EXAMPLE 1.7.– An anticommutative superalgebra A is a Lie superalgebra if it satisfies:

$$[[x, y], z] + (-1)^{|x||y|+|x||z|}[[y, z], x] + (-1)^{|x||z|+|y||z|}[[z, x], y] = 0$$

for every x, y, $z \in A_{\bar{0}} + A_{\bar{1}}$.

DEFINITION 1.5.– *If* $J = J_{\bar{0}} + J_{\bar{1}}$ *is a Jordan superalgebra and* x, y, $z \in J_{\bar{0}} \cup J_{\bar{1}}$, *then their triple product is defined by:*

$$\{x, y, z\} = (xy)z + x(yz) - (-1)^{|x||y|}y(xz).$$

Note that every algebra is a superalgebra with the trivial grading, that is, $A = A_{\bar{0}}$.

1.2. Tits–Kantor–Koecher construction

Tits (1962, 1966) made an important observation that relates Lie and Jordan structures. Let L be a Lie superalgebra whose even part $L_{\bar{0}}$ contains an $\mathfrak{sl}_2$-triple $\{e, f, h\}$, that is,

$$[e, f] = h, \quad [h, e] = 2e, \quad [h, f] = -2f.$$

DEFINITION 1.6.– *An* $\mathfrak{sl}_2$*-triple* e, f, h *is said to be "good" if* $\mathrm{ad}(h) : L \to L$ *is diagonalizable and the eigenvalues are only* -2, 0, 2.

In such a case, $L = L_{-2} + L_0 + L_2$ decomposes as a direct sum of eigenspaces. We can define a new product in L_2 by:

$$a \circ b = [[a, f], b] \qquad \forall a, b \in L_2.$$

With this new product, $J = (L_2, \circ)$ becomes a Jordan superalgebra.

Moreover, (Tits 1962, 1966; Kantor 1972) and (Koecher 1967) showed that every Jordan superalgebra can be obtained in this way. The corresponding Lie superalgebra is not unique, but any two such Lie superalgebras are centrally isogenous, that is, they have the same central cover. Let us recall the construction of $L = \mathrm{TKK}(J)$, the universal Lie superalgebra in this class (see Martin and Piard (1992)).

CONSTRUCTION.– Consider J a unital Jordan superalgebra, and $\{e_i\}_{i \in I}$ a basis of J.

Let

$$\{e_i, e_j, e_k\} = \sum_t \gamma_{ijk}^t e_t, \qquad \text{where } \gamma_{ijk}^t \in F.$$

Define a Lie superalgebra K by generators $\{x_j^+, x_j^-\}_{j\in I}$ and relations

$$[x_i^+, x_j^+] = 0 = [x_i^-, x_j^-],$$
$$[[x_i^+, x_j^-], x_k^+] = \sum_t \gamma_{ijk}^t x_t^+,$$
$$[[x_i^-, x_j^+], x_k^-] = \sum_t \gamma_{ijk}^t x_t^-, \qquad \text{for every } i,j,k \in I.$$

This Lie superalgebra has a short grading $K = K_{-1} + K_0 + K_1$ where

$$K_{-1} = F(x_i^-)_{i\in I}, \quad K_1 = F(x_i^+)_{i\in I}, \quad K_0 = F([x_i^-, x_j^+])_{i,j\in I}.$$

K is the universal Tits–Kantor–Koecher Lie superalgebra of the unital Jordan superalgebra J:

$$K = \mathrm{TKK}(J).$$

1.3. Basic examples (classical superalgebras)

Let $A = A_{\bar 0} + A_{\bar 1}$ be an associative superalgebra. The new operation in the underlying vector space A given by:

$$a \circ b = \frac{1}{2}(ab + (-1)^{|a||b|}ba) \quad \forall a,b \in A_{\bar 0} \cup A_{\bar 1},$$

defines a structure of a Jordan superalgebra on A that is denoted $A^{(+)}$.

DEFINITION 1.7.– *Those Jordan superalgebras that can be obtained as subalgebras of a superalgebra $A^{(+)}$, with A an associative superalgebra, are called special. Superalgebras that are not special are called exceptional.*

REMARK 1.2.– If we consider in the original associative superalgebra the new product given by:

$$[a, b] = ab - (-1)^{|a||b|}ba \quad \forall a,b \in A_{\bar 0} \cup A_{\bar 1},$$

we get a Lie superalgebra that is denoted as $A^{(-)}$.

DEFINITION 1.8.– *A superalgebra A is simple if it does not have non-trivial graded ideals. A graded ideal is an ideal $I \trianglelefteq A$ such that for every $a = a_0 + a_1 \in I$, it follows that a_0, $a_1 \in I$. So every graded ideal I satisfies $I = (A_{\bar 0} \cap I) + (A_{\bar 1} \cap I)$.*

Wall (1963, 1964) proved that an arbitrary simple finite dimensional superalgebra over an algebraically closed field is isomorphic to one of the following two types:

I) $A = M_{m+n}(F) = A_{\bar{0}} + A_{\bar{1}}, \quad A_{\bar{0}} = \left\{ \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \begin{matrix} m \\ n \end{matrix} \right\}, \quad A_{\bar{1}} = \left\{ \begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix} \right\}.$

II) $A = Q(n) = \left\{ \begin{pmatrix} a & b \\ b & a \end{pmatrix} \;\middle|\; a, b \in M_n(F) \right\} \leq M_{n+n}(F).$

Consequently, we can easily get the first examples of simple finite dimensional Jordan superalgebras as explained above.

EXAMPLE 1.8.– $J = M^{(+)}_{m+n}(F)$, $m \geq 1, n \geq 1$.

EXAMPLE 1.9.– $J = Q(n)^{(+)}$, $n \geq 2$.

DEFINITION 1.9.– *Let A be an associative superalgebra. A map $* : A \to A$ is a superinvolution if it satisfies:*

i) $(a^*)^* = a, \quad \forall a \in A$;

ii) $(ab)^* = (-1)^{|a||b|} b^* a^*, \quad \forall a, b \in A_{\bar{0}} \cup A_{\bar{1}}$.

If $* : A \to A$ is a superinvolution of the associative superalgebra A, then the set of symmetric elements $H(A, *)$ is a Jordan superalgebra of $A^{(+)}$. Similarly, the subspace of skew-symmetric elements $K(A, *) = \{a \in A \mid a^* = -a\}$ is a Lie subsuperalgebra of $A^{(-)}$.

The following two subsuperalgebras of $M^{(+)}_{m+n}$ are of this type.

EXAMPLE 1.10.– Let I_n, I_m be the identity matrices and let t be the transposition. Let us denote $U = \begin{pmatrix} 0 & -I_m \\ I_m & 0 \end{pmatrix}$.

Then $U^t = U^{-1} = -U$, and $* : M_{n+2m}(F) \to M_{n+2m}(F)$ given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^* = \begin{pmatrix} I_n & 0 \\ 0 & U \end{pmatrix} \begin{pmatrix} a^t & -c^t \\ b^t & d^t \end{pmatrix} \begin{pmatrix} I_n & 0 \\ 0 & U^{-1} \end{pmatrix}$$

is a superinvolution.

The superalgebras

$$\mathrm{osp}_{n,2m} = K(M_{n+2m}(F), *) \quad \text{and} \quad \mathrm{Josp}_{n,2m}(F) = H(M_{n+2m}(F), *)$$

are the Lie and Jordan orthosymplectic superalgebras, respectively.

EXAMPLE 1.11.– The associative superalgebra $M_{n+n}(F)$ has another superinvolution given by:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{\sigma} = \begin{pmatrix} d^t & -b^t \\ c^t & a^t \end{pmatrix}.$$

The Lie and Jordan superalgebras (respectively) that consist of skew-symmetric and symmetric elements, respectively, are denoted as $\mathrm{P}_n(F)$ and $\mathrm{JP}_n(F)$ (and are also called "strange series").

EXAMPLE 1.12.– The three-dimensional Kaplansky superalgebra $\mathrm{K}_3 = Fe + (Fx + Fy)$ with multiplication table:

$$e^2 = e, \quad ex = \frac{1}{2}x, \quad ey = \frac{1}{2}y, \quad x^2 = y^2 = 0, \quad x \cdot y = -y \cdot x = e$$

is a simple Jordan superalgebra. Note that K_3 is not unital.

EXAMPLE 1.13.– The one-parametric family of four-dimensional superalgebras $D(t)$ defined as $D(t) = (Fe_1 + Fe_2) + (Fx + Fy)$ with the product

$$e_i^2 = e_i, \;\; e_1e_2 = 0, \;\; e_ix = \frac{1}{2}x, \;\; e_iy = \frac{1}{2}y, \;\; xy = e_1 + te_2, \; t \in F, \;\; i = 1, 2.$$

The superalgebra $D(t)$ is simple if $t \neq 0$. In the case $t = -1$, the superalgebra $D(-1)$ is isomorphic to $M_{1+1}(F)^{(+)}$.

EXAMPLE 1.14.– Let V be a vector space that is $\mathbb{Z}/2\mathbb{Z}$-graded, $V = V_{\bar{0}} + V_{\bar{1}}$, and has a superform $(\,|\,) : V \times V \to F$, which is symmetric on $V_{\bar{0}}$, skew-symmetric in $V_{\bar{1}}$ and $(V_{\bar{0}}|V_{\bar{1}}) = 0 = (V_{\bar{1}}|V_{\bar{0}})$. Then $J = F1 + V = (F1 + V_{\bar{0}}) + V_{\bar{1}}$ is a Jordan superalgebra, where the product of two arbitrary elements $\alpha 1 + v$ and $\beta 1 + w$ in J is given by

$$(\alpha 1 + v) \cdot (\beta 1 + w) = \alpha\beta 1 + (v|w)1 + (\alpha w + \beta v)$$

for arbitrary $v, w \in V$.

We will refer to J as the superalgebra of a superform.

J is simple if and only if the form $(\,|\,)$ is non-degenerate.

EXAMPLE 1.15.– Kac (1977a) introduced a 10-dimensional Jordan superalgebra J whose even part has dimension 6 and splits as the direct sum of a superalgebra of a superform and a one-dimensional algebra. Thus, $J_{\bar{0}} = (Fe + \sum_{1 \le i \le 4} Fv_i) \oplus F_f$ has the multiplication given by:

$$e^2 = e, \quad ev_i = v_i, \quad v_1 v_2 = 2e = v_3 v_4, \quad f^2 = f$$

and any other product of two basic elements is 0.

The odd part $J_{\bar{1}}$ has a basis $\{x_1, x_2, y_1, y_2\}$ and the following multiplication table:

$$x_i y_i = e - 3f, \quad x_i^2 = 0 = y_i^2, \quad x_1 y_2 = v_3 = -y_2 x_1, \quad x_2 y_1 = v_4 = -y_1 x_2,$$
$$x_1 x_2 = 0 = x_2 x_1, \quad y_1 y_2 = v_2 = -y_2 y_1, \quad i = 1, 2.$$

Finally, the action of $J_{\bar{0}}$ over $J_{\bar{1}}$ is given by:

$$ex_j = \frac{1}{2} x_j, \quad ey_j = \frac{1}{2} y_j, \quad fx_j = \frac{1}{2} x_j, \quad fy_j = \frac{1}{2} y_j, \quad j = 1, 2$$
$$v_1 x_j = 0, \; v_1 y_1 = x_2, \; v_1 y_2 = -x_1, \; v_2 y_j = 0, \; v_2 x_1 = -y_2, \; v_2 x_2 = y_1$$
$$v_3 x_1 = 0 = v_3 y_2, \quad v_3 x_2 = x_1, \quad v_3 y_1 = y_2,$$
$$v_4 x_2 = 0 = v_4 y_1, \quad v_4 x_1 = x_2, \quad v_4 y_2 = y_1.$$

This superalgebra is simple if $\operatorname{char} F \neq 3$. In case of $\operatorname{char} F = 3$, it has an ideal of dimension 9 that is spanned by e, v_i, $1 \le i \le 4$, x_j, y_j, $1 \le j \le 2$. It is called degenerated Kac superalgebra and is denoted by K_9.

In Medvedev and Zelmanov (1992), it is proved that the Kac superalgebra K_{10} is not a homomorphic image of a special Jordan superalgebra.

Benkart and Elduque (2002) realized K_{10} as the space of $\mathrm{K}_3 \otimes \mathrm{K}_3 + F1$ (with a new product) where K_3 is the Kaplansky superalgebra.

Another (octonionic) construction of K_{10} was suggested in Racine and Zelmanov (2015).

1.4. Brackets

DEFINITION 1.10.– *Let A be an associative commutative superalgebra. A binary map $\{\ ,\ \} : A \times A \to A$ is called a Poisson bracket if*

1) $(A, \{\ ,\ \})$ is a Lie superalgebra;

2) $\{ab, c\} = a\{b, c\} + (-1)^{|b||c|}\{a, c\}b$ *for arbitrary* $a, b, c \in A_{\bar{0}} \cup A_{\bar{1}}$.

EXAMPLE 1.16.– Let $F[p_1, \ldots, p_n, q_1, \ldots, q_n]$ be a polynomial algebra in $2n$ variables. The classical Hamiltonian bracket:

$$\{f, g\} = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} - \frac{\partial g}{\partial p_i} \frac{\partial f}{\partial q_i} \right), \qquad f, g \in F[p_1, \ldots, p_n, q_1, \ldots, q_n]$$

is a Poisson bracket.

EXAMPLE 1.17.– Let

$$G(n) = \langle \xi_1, \ldots, \xi_n \rangle$$

be the Grassmann algebra over an n-dimensional vector space V. Then the bracket

$$\{f, g\} = \sum_{i=1}^{n} (-1)^{|f|} \frac{\partial f}{\partial \xi_i} \frac{\partial g}{\partial \xi_i}$$

for arbitrary $f, g \in G(V)_{\bar{0}} \cup G(V)_{\bar{1}}$ is a Poisson bracket.

DEFINITION 1.11.– *Given* $A = A_{\bar{0}} + A_{\bar{1}}$ *an associative commutative superalgebra, a bilinear map* $\{\ ,\ \} : A \times A \to A$ *is called a contact bracket if:*

i) $(A, \{\ ,\ \})$ *is a Lie superalgebra;*

ii) $\{ab, c\} = a\{b, c\} + (-1)^{|b||c|}\{a, c\}b + abD(c)$ *for arbitrary homogeneous elements a, b, c in A.*

Note that a Poisson bracket is a contact bracket with $D = 0$.

EXAMPLE 1.18.– Let $F[t]$ be the polynomial algebra. Then the bracket $\{f(t), g(t)\} = f'(t)g(t) - f(t)g'(t)$ is a contact bracket.

EXAMPLE 1.19.– Let $\Lambda(1 : n)$ be the polynomial superalgebra in one (even) Laurent variable and n (odd) Grassmann variables $\xi_1, \ldots, \xi_n$, $\Lambda(1 : n) = F[t, t^{-1}, \xi_1, \ldots, \xi_n]$. Consider $D = \frac{\partial}{\partial t}$ or $D = t\frac{\partial}{\partial t}$ a derivation of $F[t, t^{-1}]$. The bilinear map defined on generators of $\Lambda(1 : n)$ by:

$$[t, \xi_i] = \frac{1}{2}\xi_i D(t), \quad [\xi_i, \xi_j] = \delta_{ij}, \quad 1 \leq i, j \leq n$$

can be extended to a contact bracket on $\Lambda(1 : n)$.

DEFINITION 1.12.– *[Kantor double] Let A be an associative commutative superalgebra with a bilinear map $\{\ ,\ \} : A \times A \to A$. Assume that $\{A_{\bar{i}}, A_{\bar{j}}\} \subseteq A_{\overline{i+j}}$. Consider a direct sum of vector spaces* $\mathrm{KJ}(A, \{\ ,\ \}) = A + Av$ *where $|v| = 1$. Define a new product in $J(A, \{\ ,\ \})$ that coincides with the original one in A and is given by:*

$$a(bv) = (ab)v, \quad (bv)a = (-1)^{|a|}(ba)v, \quad (av)(bv) = (-1)^{|b|}\{a, b\}.$$

The superalgebra $\mathrm{KJ}(A, \{\ ,\ \})$ *is called the Kantor double of* $(A, \{\ ,\ \})$.

Kantor (1990) proved that if the bracket $\{\ ,\ \}$ is a Poisson bracket, then $\mathrm{KJ}(A, \{\ ,\ \})$ is a Jordan superalgebra.

DEFINITION 1.13.– *The bilinear map $\{\ ,\ \}$ is called a Jordan bracket on the superalgebra A if* $\mathrm{KJ}(A, \{\ ,\ \})$ *is a Jordan superalgebra (see King and McCrimmon (1992)).*

Cantarini and Kac (2007) showed that there is a 1-1 correspondence between Jordan brackets and contact brackets. Indeed, if $[a, b]$ is a contact bracket with derivation D, $D(a) = [a, 1]$, then the new bracket

$$\{a, b\} = [a, b] + \frac{1}{2}(D(a)b - aD(b))$$

is a Jordan bracket.

Applying this to example 1.19, we get the following:

EXAMPLE 1.20.– The values $\{\xi_i, t\} = 0$, $\{\xi_i, \xi_j\} = \delta_{ij}$ for $1 \leq i, j \leq n$ extend to a Jordan bracket of $\Lambda(1 : n)$. Applying the Kantor double process to this bracket, we get Jordan superalgebras $J_n = \mathrm{KJ}(\Lambda(1 : n), \{\ ,\ \})$.

1.5. Cheng–Kac superalgebras

Given an arbitrary unital associative commutative (super)-algebra with an (even) derivation $d : Z \to Z$, Martínez and Zelmanov constructed (Martínez and Zelmanov 2010) a Jordan superalgebra $\mathrm{JCK}(Z, d)$ named the Cheng–Kac Jordan superalgebra.

The even part of $\mathrm{JCK}(Z, d)$ is a rank 4 free module over Z with basis $\{1, w_1, w_2, w_3\}$, $J_{\bar{0}} = Z + \sum\limits_{i=1}^{3} w_i Z$, and multiplication given by $w_i w_j = 0$ if $1 \leq i \neq j \leq 3$, $w_1^2 = 1 = w_2^2$, $w_3^2 = -1$. The odd part of this superalgebra is also a rank 4 free module over Z with basis $\{x, x_1, x_2, x_3\}$, $J_{\bar{1}} = xZ + \sum_{i=1}^{3} x_i Z$.

The action of the even part on the odd part and the products of two elements, respectively, are given by the following multiplication tables:

	g	$w_j g$
xf	$x(fg)$	$x_j(fg^d)$
$x_i f$	$x_i(fg)$	$x_{i\times j}(fg)$

	xg	$x_j g$
xf	$f^d g - fg^d$	$-w_j(fg)$
$x_i f$	$w_i(fg)$	0

where $x_{i\times i} = 0$, $x_{1\times 2} = -x_{2\times 1} = x_3$, $x_{1\times 3} = -x_{3\times 1} = x_2$, $-x_{2\times 3} = x_1 = x_{3\times 2}$.

The superalgebra $\mathrm{JCK}(Z, d)$ is simple if and only if Z is d-simple, that is, Z does not contain proper d-invariant ideals (see Martínez and Zelmanov (2010)).

Let us remark that for $Z = \mathbb{C}[t, t^{-1}]$ the above construction leads to the Cheng–Kac superconformal algebra, that is, $\mathrm{CK}(6) = \mathrm{TKK}(\mathrm{JCK}(6))$, where

$$\mathrm{JCK}(6) = \mathrm{JCK}(\mathbb{C}[t, t^{-1}], D)$$

with $D = \frac{\partial}{\partial t}$ (see section 1.8).

1.6. Finite dimensional simple Jordan superalgebras

1.6.1. *Case F is algebraically closed and* $\mathrm{char}\, F = 0$

Let us assume now that F is algebraically closed and $\mathrm{char}\, F = 0$. Kac derived the classification of finite dimensional simple Jordan F-superalgebras from his classification of simple finite dimensional Lie superalgebras via the Tits–Kantor–Koecher construction.

THEOREM 1.1 (see Kac (1977a) and Kantor (1990)).– Let $J = J_{\bar{0}} + J_{\bar{1}}$ be a simple Jordan superalgebra over an algebraically closed field F, $\mathrm{char}\, F = 0$. Then J is isomorphic to one of the superalgebras in examples 1.8, 1.9 and 1.10–1.15 or it is the Kantor double of the Poisson bracket in example 1.17.

REMARK 1.3.– We will assume always in this section that $J_{\bar{1}} \neq (0)$.

1.6.2. *Case* $\mathrm{char}\, F = p > 2$*, the even part $J_{\bar{0}}$ is semisimple*

Let us assume next that $\mathrm{char}\, F = p > 2$ and the even part $J_{\bar{0}}$ is a semisimple Jordan algebra.

Recall that a semisimple Jordan algebra is a direct sum of finitely many simple ideals.

This case was addressed in Racine and Zelmanov (2003) and the classification essentially coincides with the one of zero characteristic, expect of some differences if $\mathrm{char}\, F = 3$.

EXAMPLE 1.21.– Let $H_3(F)$, $K_3(F)$ denote the symmetric and skew-symmetric 3×3 matrices over F, $\mathrm{char}\, F = 3$. Consider $J_{\bar{0}} = H_3(F)$ and $J_{\bar{1}} = \overline{K_3(F)} \oplus \overline{\overline{K_3(F)}}$ the sum of two copies of $K_3(F)$. We have a Jordan superalgebra structure on $J = J_{\bar{0}} + J_{\bar{1}}$ via $a \bullet b = a \cdot b$ in $M_3(F)^+$ if $a, b \in H_3(F)$, that is,

$$a \bullet b = ab + ba,$$

$$\bar{c} \bullet \bar{\bar{d}} = cd + dc \in H_3(F) \quad \text{for } c, d \in K_3(F),$$

$$\overline{K_3(F)} \bullet \overline{K_3(F)} = (0) = \overline{\overline{K_3(F)}} \bullet \overline{\overline{K_3(F)}}, \quad \text{and}$$

$$a\bar{c} = \overline{\overline{ac + ca}}, \;\; a\bar{\bar{c}} = \overline{ac + ca} \quad \text{if } a \in H_3(F), \; c \in K_3(F).$$

This superalgebra is simple.

EXAMPLE 1.22.– Let $B = B_{\bar{0}} + B_{\bar{1}}$ with $B_{\bar{0}} = M_2(F)$, $B_{\bar{1}} = Fm_1 + Fm_2$, where F is a field, $\mathrm{char}\, F = 3$. The product in $B_{\bar{1}}$ is given by:

$$m_1^2 = -e_{21}, \quad m_2^2 = e_{12}, \quad m_1 m_2 = e_{11}, \quad m_2 m_1 = -e_{22}.$$

The action of $B_{\bar{0}}$ over $B_{\bar{1}}$ is defined as follows:

$$e_{11} m_1 = m_1, \quad e_{11} m_2 = 0 = m_1 e_{11}, \quad m_2 e_{11} = m_2,$$

$$m_1 e_{22} = m_1, \quad e_{22} m_1 = 0 = m_2 e_{22}, \quad e_{22} m_2 = m_2,$$

$$e_{12} m_1 = m_2, \quad e_{12} m_2 = 0 = m_2 e_{12}, \quad m_1 e_{12} = -m_2,$$

$$e_{21} m_2 = m_1, \quad e_{21} m_1 = 0 = m_1 e_{21}, \quad m_2 e_{21} = -m_1.$$

Shestakov (1997) proved that B is an alternative superalgebra and has a natural involution $*$ given by $(a + m)^* = \bar{a} - m$, $a \in B_{\bar{0}}$, where $a \mapsto \bar{a}$ is the symplectic involution, and $m \in B_{\bar{1}}$.

If $H_3(B, *)$ denotes the symmetric matrices with respect to the involution $*$, then $H_3(B, *)$ is a simple Jordan superalgebra. It is i-exceptional, that is, it is not a homomorphic image of a special Jordan superalgebra.

THEOREM 1.2 (Racine and Zelmanov (2003)).– Let $J = J_{\bar{0}} + J_{\bar{1}}$ be a finite dimensional central simple Jordan superalgebra over an algebraically closed field F

of $\mathrm{char}\, F = p > 2$. If $J_{\bar{1}} \neq (0)$ and $J_{\bar{0}}$ is semisimple, then J is isomorphic to one of the superalgebras in examples 1.8, 1.9, 1.10–1.14 or $\mathrm{char}\, F = 3$ and J is the nine-dimensional degenerate Kac superalgebra (see example 1.15) or J is isomorphic to one of the superalgebras in examples 1.21 and 1.22.

1.6.3. *Case* $\mathrm{char}\, F = p > 2$*, the even part* $J_{\bar{0}}$ *is not semisimple*

This case shows similarities with infinite dimensional superconformal Jordan algebras (see section 1.8) in characteristic 0.

Let us denote $B(m) = F[a_1, \dots, a_m \mid a_i^p = 0]$ the algebra of truncated polynomials in m variables. Let $B(m,n) = B(m) \otimes G(n)$ be the tensor product of $B(m)$ with the Grassmann algebra $G(n) = \langle 1, \xi_1, \dots, \xi_n \rangle$. Then $B(m,n)$ is an associative commutative superalgebra.

THEOREM 1.3 (Martínez and Zelmanov (2010)).– Let $J = J_{\bar{0}} + J_{\bar{1}}$ be a finite dimensional simple unital Jordan superalgebra over an algebraically closed field F of characteristic $p > 2$. If the even part $J_{\bar{0}}$ is not semisimple, then there exist integers m, n and a Jordan bracket $\{\ ,\ \}$ on $B(m,n)$ such that $J = B(m,n) + B(m,n)v = \mathrm{KJ}(B(m,n), \{\ ,\ \})$ is a Kantor double of $B(m,n)$ or J is isomorphic to a Cheng–Kac Jordan superalgebra $\mathrm{JCK}(B(m), d)$ for some derivation $d : B(m) \to B(m)$.

1.6.4. *Non-unital simple Jordan superalgebras*

Finally, let us consider non-unital simple Jordan superalgebras. As we have seen, K_3 the three-dimensional Kaplansky superalgebra and K_9 the nine-dimensional degenerate Kac superalgebra are examples of such superalgebras.

EXAMPLE 1.23.– Let Z be a unital associative commutative algebra, $D : Z \to Z$ a derivation. Assume that Z is D-simple and that the only constants are elements $\alpha 1$, $\alpha \in F$.

Let us consider in Z the bracket $\{\ ,\ \}$ given by:

$$\{a, b\} = (aD)b - a(bD) \quad \forall a, b \in Z.$$

The above bracket is a Jordan bracket, so the Kantor double $V(Z, D) = Z + Zv = \mathrm{KJ}(Z, \{\ ,\ \})$ is a simple unital Jordan superalgebra.

Now we will change the product in $V(Z, D)$, modifying only the action of the even part on the odd part and preserving the product of two even (respectively, two odd) elements. Denote with juxtaposition the product on $V(Z, D)$. Let a, $b \in Z$. We define the new product $\cdot$ by:

$$a \cdot b = ab, \quad a \cdot bv = \frac{1}{2}(ab)v, \quad av \cdot bv = \{a, b\} = (av)(bv).$$

In this way, we get another Jordan superalgebra $V_{1/2}(Z, D)$ that is simple but not unital.

It was proved in Zelmanov (2000) that:

THEOREM 1.4.– Let J be a finite dimensional simple central non-unital Jordan superalgebra over a field F. Then J is isomorphic to one of the superalgebras on the list:

i) the Kaplansky superalgebra K_3 (example 1.12);

ii) the field F has characteristic 3 and J is the degenerate Kac superalgebra (example 1.15);

iii) a superalgebra $V_{1/2}(Z, D)$ (example 1.23).

DEFINITION 1.14.– *Let A be a Jordan superalgebra and let N be its radical, that is, the largest solvable ideal of A. The superalgebra A is said to be semisimple if $N = (0)$.*

EXAMPLE 1.24.– Let B be a simple non-unital Jordan superalgebra and let $H(B) = B + F1$ be its unital hull. Then $H(B)$ is a semisimple Jordan superalgebra that is not simple.

THEOREM 1.5 (Zelmanov (2000)).– Let J be a finite dimensional Jordan superalgebra. Then J is semisimple if and only if

$$J \cong \bigoplus_{i=1}^{s} (J_{i1} \oplus \cdots \oplus J_{ir_i} + K_i 1) + J_{(1)} \oplus \cdots \oplus J_{(t)}$$

where $J_{(1)}, \ldots, J_{(t)}$ are simple Jordan superalgebras and for every $i = 1, \ldots, s$, the superalgebras J_{ij} are simple non-unital Jordan superalgebras over the field extension K_i of F.

1.7. Finite dimensional representations

Jacobson (1968) developed the theory of bimodules over semisimple finite dimensional Jordan algebras.

In this section, we discuss representations (bimodules) of finite dimensional Jordan superalgebras.

DEFINITION 1.15.– *The rank of a Jordan superalgebra J is the maximal number of pairwise orthogonal idempotents in the even part.*

Unless otherwise stated we will assume $\mathrm{char}\, F = 0$.

DEFINITION 1.16.– *Let V be a $\mathbb{Z}/2\mathbb{Z}$-graded vector space with bilinear mappings $V \times J \to V$, $J \times V \to V$. We call V a Jordan bimodule if the split null extension $V + J$ is a Jordan superalgebra.*

Recall that in the split null extension the multiplication extends the multiplication on J, products $V \cdot J$ and $J \cdot V$ are defined via the bilinear mappings above and $V \cdot V = (0)$.

Let $V = V_{\bar{0}} + V_{\bar{1}}$ be a Jordan bimodule over J. Consider the vector space $V^{\mathrm{op}} = V_{\bar{1}}^{\mathrm{op}} + V_{\bar{0}}^{\mathrm{op}}$, where $V_{\bar{i}}^{\mathrm{op}}$ is a copy of $V_{\bar{i}}$ with different parity. Define the action of J on V^{op} via

$$av^{\mathrm{op}} = (-1)^{|a|}(av)^{\mathrm{op}}, \quad v^{\mathrm{op}}a = (va)^{\mathrm{op}}.$$

Then V^{op} is also a Jordan bimodule over J. We call it the opposite module of V.

Let V be the free Jordan J-bimodule on one free generator.

DEFINITION 1.17.– *The associative subsuperalgebra $U(J)$ of $\mathrm{End}_F\, V$ generated by all linear transformations $R_V(a) : V \to V$, $v \mapsto va$, $a \in J$, is called the universal multiplicative enveloping superalgebra of J.*

Every Jordan bimodule over J is a right module over $U(J)$.

DEFINITION 1.18.– *A bimodule V over J is called a one-sided bimodule if $\{J, V, J\} = (0)$.*

Let $V(1/2)$ be the free one-sided Jordan J-bimodule on one free generator.

DEFINITION 1.19.– *The associative subsuperalgebra $S(J)$ of $\mathrm{End}_F\, V(1/2)$ generated by all linear transformations $R_{V(1/2)}(a) : V(1/2) \to V(1/2)$, $v \mapsto va$, $a \in J$, is called the universal associative enveloping algebra of J.*

Every one-sided Jordan J-bimodule is a right module over $S(J)$.

Finally, let J be a unital Jordan superalgebra with the identity element e. Let $V(1)$ denote the free unital J-bimodule on one free generator. The associative subsuperalgebra $U_1(J)$ of $\mathrm{End}_F\, V(1)$ generated by $\{R_{V(1/2)}(a)\}_{a\in J}$ is called the universal unital enveloping algebra of J.

For an arbitrary Jordan bimodule V, the Peirce decomposition

$$V = \{e, V, e\} \oplus \{e, V, 1-e\} \oplus \{1-e, V, 1-e\}$$

is a decomposition of V into a direct sum of unital and one-sided bimodules. Hence $U(J) \cong U_1(J) \oplus S(J)$.

1.7.1. *Superalgebras of rank* ≥ 3

In this section, we consider Jordan bimodules over finite dimensional simple Jordan superalgebras of rank ≥ 3, that is, superalgebras $M_{m+n}^{(+)}$, $m+n \geq 3$; $\mathrm{Josp}(n, 2m)$, $n+m \geq 3$; $Q(n)^{(+)}$, $n \geq 3$; $\mathrm{JP}(n)$, $n \geq 3$.

In this case, the universal multiplicative enveloping superalgebra $U(J)$ is finite dimensional and semisimple (Martin and Piard 1992). Hence every Jordan bimodule is completely reducible, as in the case of Jordan algebras.

The superalgebras $\mathrm{Josp}(n, 2m)$ and $\mathrm{JP}(n)$ are of the type

$$H(A, *) = \{a \in A \mid a^* = a\}$$

where A is a simple finite dimensional associative superalgebra and $*$ is an involution.

EXAMPLE 1.25.– An arbitrary right module over A is a one-sided module over $H(A, *)$.

EXAMPLE 1.26.– The subspace $K(A, *) = \{k \in A \mid k^* = -k\}$ with the action $k \cdot a = ka + ak$; $k \in K(A, *)$, $a \in H(A, *)$ is a unital $H(A, *)$-bimodule.

THEOREM 1.6 (Martin and Piard (1992)).– An arbitrary irreducible Jordan bimodule over $\mathrm{Josp}(n, 2m)$, $n+m \geq 3$, or $\mathrm{JP}(n)$, $n \geq 3$, is a bimodule of examples 1.25 and 1.26 or the regular bimodule.

The superalgebras $M_{m+n}^{(+)}$, $Q(n)^{(+)}$ are of the type $A^{(+)}$, where A is a simple finite dimensional associative superalgebra.

EXAMPLE 1.27.– Every right module over A gives rise to a one-sided Jordan bimodule over $A^{(+)}$.

Suppose now that the superalgebra A is equipped with an involution $*$.

EXAMPLE 1.28.– The subspaces $H(A, *)$ and $K(A, *)$ become Jordan $A^{(+)}$-bimodules with the actions:

1) $h \circ a = ha + a^* h$;

2) $h \circ a = ha^* + ah$;

3) $k \circ a = ka + a^* k$;

4) $k \circ a = ka^* + ak$;

where $h \in H(A, *)$, $k \in K(A, *)$, $a \in A$.

The associative superalgebras $M_{m+n}(F)$, where both m, n are odd, and $Q(n)$, are not equipped with an involution, but they are equipped with a pseudoinvolution.

DEFINITION 1.20.– *A graded linear map* $* : A \to A$ *is called a pseudoinvolution if* $(ab)^* = (-1)^{|a|\cdot|b|} b^* a^*$, $(a^*)^* = (-1)^{|a|} a$ *for arbitrary elements* a, $b \in A_{\bar{0}} \cup A_{\bar{1}}$.

EXAMPLE 1.29.– The mapping $\begin{pmatrix} a & b \\ b & a \end{pmatrix} \mapsto \begin{pmatrix} a^t & \sqrt{-1} b^t \\ \sqrt{-1} b^t & a^t \end{pmatrix}$ is a pseudoinvolution on $Q(n)$.

EXAMPLE 1.30.– The mapping $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a^t & -c^t \\ b^t & d^t \end{pmatrix}$ is a pseudoinvolution on $M_{m+n}(F)$.

Replacing the involution $*$ in example 1.28 with the pseudoinvolutions of examples 1.29 and 1.30, we get unital Jordan bimodules over $M_{m+n}(F)$, where m, n are odd, and over $Q(n)^{(+)}$.

THEOREM 1.7 (see Martin and Piard (1992), Martínez *et al*. (2010)).– An arbitrary irreducible Jordan bimodule over $M_{m+n}^{(+)}$, $m + n \geq 3$ or $Q(n)^{(+)}$, $n \geq 3$, is one of examples 1.27 and 1.28 with an involution or a pseudoinvolution, or a regular bimodule.

The exceptional Jordan superalgebra K_{10} has rank 3. Jordan bimodules over K_{10} have been classified by Shtern (1987).

THEOREM 1.8 (Shtern (1987)).– All Jordan bimodules over K_{10} are completely reducible. The only irreducible Jordan bimodules over K_{10} are the regular bimodule and its opposite.

1.7.2. *Superalgebras of rank* ≤ 2

If J is a Jordan superalgebra of rank ≤ 2, then, generally speaking, it is no longer true that its universal multiplicative algebra is finite dimensional and that any Jordan bimodule is completely reducible.

1.7.2(a) In the case $J = Q(2)^{(+)}$, however, it is true (see Martínez *et al.* (2010)). The universal multiplicative enveloping algebra $U(Q(2)^{(+)})$ is finite dimensional and semisimple and the description of irreducible Jordan bimodules is similar to that of $Q(n)^{(+)}$, $n \geq 3$.

1.7.2(b) Let us discuss bimodules over Kantor superalgebras. Recall that the Kantor superalgebras $\mathrm{Kan}(n)$ are Kantor doubles of the Grassmann superalgebras $G(n)$, $n \geq 1$, with respect to the Poisson bracket

$$[f, g] = \sum_{i=1}^{n} (-1)^{|f|} \frac{\partial f}{\partial \xi_i} \frac{\partial g}{\partial \xi_i}.$$

Let $\mathrm{Kan}(n) = G(n) + G(n)v$. The Grassmann superalgebra $G(n)$ is embeddable in the associative commutative superalgebra $A = F[t, \xi_1, \dots, \xi_n] = F[t] \otimes_F G(n)$.

For an arbitrary scalar $\alpha \in F$, the Poisson bracket $[\ ,\ ]$ extends to the Jordan bracket on A defined by $[t, \xi_i] = 0$, $[\xi_i, \xi_j] = -\delta_{ij}$, $[\xi_i, 1] = 0$, $[t, 1] = \alpha t$. The Kantor double $\mathrm{Kan}(n) = G(n) + G(n)v$ embeds in the Kantor double $\mathrm{Kan}(A, [\ ,\ ]) = A + Av$. The subspace $V(\alpha) = tG(n) + tG(n)v$ is an irreducible unital Jordan bimodule over $K(n)$. The square of the multiplication operator by the element v acts on $V(\alpha)$ as the scalar multiplication by α.

The simple superalgebras $\mathrm{Kan}(n)$, $n \geq 2$ are exceptional (see Martínez *et al.* (2001)). Therefore, they do not have non-zero one-sided Jordan bimodules.

THEOREM 1.9 (Stern (1995), Martínez and Zelmanov (2009), Solarte and Shestakov (2016)).– Every finite dimensional irreducible Jordan bimodule over $\mathrm{Kan}(n)$, $n \geq 2$ is isomorphic to $V(\alpha)$ or $V(\alpha)^{\mathrm{op}}$, $\alpha \in F$.

In Solarte and Shestakov (2016), the theorem above was proved for algebras over a field of characteristic $p > 2$.

1.7.2(c) Jordan superalgebras of a superform. Let $V = V_{\bar{0}} + V_{\bar{1}}$ be a $\mathbb{Z}/2\mathbb{Z}$-graded vector space with a non-degenerate supersymmetric bilinear form. Assume $\dim_F V_{\bar{0}} = m$, $\dim_F V_{\bar{1}} = 2n$ and choose a basis $e_1, \dots, e_m$ in $V_{\bar{0}}$ with $\langle e_i, e_j \rangle = \delta_{ij}$ and a basis $v_1, w_1, \dots, v_n, w_n$ in $V_{\bar{1}}$ such that

$$\langle v_i, w_j \rangle = \delta_{ij}, \quad \langle v_i, v_j \rangle = \langle w_i, w_j \rangle = 0, \quad 1 \leq i, j \leq n.$$

Let $\mathrm{Cl}(m)$ be the Clifford algebra of the restriction of the form $\langle\ ,\ \rangle$ to $V_{\bar{0}}$, and let

$$W_n = \langle 1, x_i, y_i, 1 \leq i \leq n \mid [x_i, y_j] = \delta_{ij},\ [x_i, x_j] = [y_i, y_j] = 0 \rangle$$

be the simple Weyl algebra.

Then the tensor product $S = \mathrm{Cl}(m) \otimes_F W_n$ is the universal associative enveloping superalgebra of the Jordan superalgebra $J = V + F \cdot 1$.

Since the algebra W_n, $n \geq 1$ is infinite dimensional, it follows that the superalgebra J does not have non-zero finite dimensional one-sided Jordan bimodules unless $n = 0$.

Consider in the algebra S the chain of subspaces

$$\ldots \subseteq S_{-1} \subseteq S_0 = F \cdot 1 \subseteq S_1 \subseteq \ldots,$$

where $S_r = (0)$ for $r < 0$, $S_r = \sum_{i \leq r} \underbrace{V \cdots V}_{i}$ for $r \geq 1$. Clearly, $S = \bigcup_{r \geq 0} S_r$.

THEOREM 1.10 (Martin and Piard (1992)).–

1) For every $r \geq 1$, S_r/S_{r-2} is a unital irreducible Jordan bimodule over J.

2) Let $V' = Fu \oplus V$, where $|u| = 0$. Extend the bilinear form $\langle \ , \ \rangle$ to V' via $\langle u, u \rangle = 1$, $\langle u, V \rangle = (0)$. Then for every even $r \geq 0$, the quotient uS_r/uS_{r-2} is a unital irreducible Jordan bimodule over J.

3) Every unital irreducible finite dimensional J-bimodule is isomorphic to S_r/S_{r-2} or to uS_r/uS_{r-2} for even r.

The classification of irreducible Jordan bimodules over $M_{1+1}(F)^{(+)}$, $D(t)$, K_3, $\mathrm{JP}(2)$ is too technical for an Encyclopedia survey. For a detailed description of finite dimensional irreducible Jordan bimodules, (see Martínez and Zelmanov (2003), Martin and Piard (1992), Martínez and Zelmanov (2006), Martínez and Shestakov (2020)). We will make only some general comments.

1.7.2(d) The universal associative enveloping superalgebra of $J = M_{1+1}(F)^{(+)}$ is infinite dimensional, and finite dimensional one-sided Jordan bimodules over J are not necessarily completely reducible.

There is a family of 4-dimensional unital Jordan J-bimodules $V(\alpha, \beta, \gamma)$, which are parameterized by scalars α, β, $\gamma \in F$. If $\gamma^2 - 1 - 4\alpha\beta \neq 0$, then the bimodule $V(\alpha, \beta, \gamma)$ is irreducible. If $\gamma^2 - 1 - 4\alpha\beta = 0$, then it has a composition series with 2-dimensional irreducible factors.

Every irreducible finite dimensional unital Jordan J-bimodule is isomorphic to $V(\alpha, \beta, \gamma)$, $\gamma^2 - 1 - 4\alpha\beta \neq 0$, or to a factor of a composition series of $V(\alpha, \beta, \gamma)$, $\gamma^2 - 1 - 4\alpha\beta = 0$ (see Martínez and Zelmanov (2009); Martínez and Shestakov (2020)).

1.7.2(e) Now let us discuss the superalgebras $D(t)$ and K_3. Recall that

$$D(t) = Fe_1 + Fe_2 + Fx + Fy;$$

$e_1^2 = e_1$, $e_2^2 = e_2$, $e_1e_2 = 0$, $e_ix = \frac{1}{2}x$, $e_iy = \frac{1}{2}y$, $xy = e_1 + te_2$.

Then $D(0) = F \cdot 1 + \mathrm{K}_3$, $D(-1) \cong M_{1+1}(F)^{(+)}$, $D(1)$ is a Jordan superalgebra of a superform.

We will assume therefore that $t \neq -1, 1$.

One-sided bimodules. The superalgebra K_3 does not have any non-zero one-sided Jordan bimodules (it has non-zero one-sided bimodules if $\operatorname{char} F > 0$). All finite dimensional one-sided Jordan bimodules over $D(t)$, $t \neq -1, 1$, are completely reducible. The superalgebra $D(t)$ does not have non-zero one-sided bimodules unless $t = -\dfrac{m}{m+1}$ or $t = -\dfrac{m+1}{m}$, where $m \in \mathbb{Z}$, $m \geq 1$. For $t = -\dfrac{m}{m+1}$, there exists one (up to opposites) irreducible one-sided J-bimodule of dimension $2m+3$; for $t = -\dfrac{m+1}{m}$, there exists one (up to opposites) irreducible one-sided J-bimodule of dimension $2m+1$.

Unital bimodules. If $J = D(t)$ and t cannot be represented as $-\dfrac{m}{m+2}$, $m \in \mathbb{Z}$, $m \geq 0$ or $-\dfrac{m+2}{m}$, $m \in \mathbb{Z}$, $m \geq 1$, then there is one (up to opposites) series of irreducible finite dimensional unital J-bimodules parameterized by positive integers. All finite dimensional bimodules in this case are completely reducible.

If $t = -\dfrac{m+2}{m}$, $m \in \mathbb{Z}$, $m \geq 1$, or $t = -\dfrac{m}{m+2}$, $m \in \mathbb{Z}$, $m \geq 0$, then there is one (up to opposites) additional irreducible bimodule.

Remark. A finite dimensional irreducible bimodule over K_3 is a unital finite dimensional irreducible bimodule over $D(0)$. Hence the above description of unital finite dimensional irreducible bimodules over $D(0)$ applies to K_3.

The detailed description of irreducible and indecomposable $D(t)$-bimodules is contained in Martínez and Zelmanov (2003), Martínez and Zelmanov (2006). Trushina (2008) extended the description above to superalgebras over fields of positive characteristics.

1.7.2(f) Jordan bimodules over $\mathrm{JP}(2)$ Representation theory of $\mathrm{JP}(2)$ is essentially different from that of $\mathrm{JP}(n)$, $n \geq 3$.

The universal associative enveloping superalgebra $S(\mathrm{JP}(2))$ is isomorphic to $M_{2+2}(F[t])$, where $F[t]$ is the polynomial algebra in one variable (see Martínez and

Zelmanov (2003)). Hence, irreducible one-sided bimodules are parameterized by scalars $\alpha \in F$ and have dimension 4, whereas indecomposable bimodules are parameterized by Jordan blocks.

Let V be an irreducible finite dimensional bimodule over $\mathrm{JP}(2)$. Let $L = K(\mathrm{JP}(2))$ be the Tits–Kantor–Koecher Lie superalgebra of $\mathrm{JP}(2)$, $L = L_{-1} + L_0 + L_1$. The superalgebra L has one-dimensional center. Fix $0 \neq z \in L_0$, then $L/Fz \cong P(3)$ (see Martinez and Zelmanov (2001)). The Lie superalgebra L_0 acts on the module V (see Jacobson (1968); Martin and Piard (1992)), and the element z acts as a scalar multiplication.

DEFINITION 1.21.– *We say that V is a module of level $\alpha \in F$ if z acts on V as the scalar multiplication by α.*

For an arbitrary scalar $\alpha \in F$, there are exactly two (up to opposites) non-isomorphic unital irreducible finite dimensional Jordan bimodules over $\mathrm{JP}(2)$ of level α. For their explicit realization, see (Martínez and Zelmanov (2014)).

Kashuba and Serganova (2020) described indecomposable finite dimensional Jordan bimodules over $\mathrm{Kan}(n)$, $n \geq 1$ and $\mathrm{JP}(2)$.

1.8. Jordan superconformal algebras

In this section, we will discuss connections between infinite dimensional Jordan superalgebras and the so-called superconformal algebras that originated in mathematical physics.

In view of importance of the Virasoro algebra and (especially) its central extensions in physics, (Neveu and Schwarz 1971; Ramond 1971) and others considered superextensions of the algebra Vir. These superextensions became known as superconformal algebras. Kac and van de Leur (1989) put the theory on a more formal footing and recognized that all known superconformal algebras are in fact infinite dimensional superalgebras of Cartan type considered in Kac (1977b). Following (Kac and van de Leur 1989), we call a $\mathbb{Z}$-graded Lie superalgebra $L = \sum\limits_{i\in\mathbb{Z}} L_i$ a *superconformal algebra* if

i) L is graded simple;

ii) $\mathrm{Vir} \subseteq L_{\bar{0}}$;

iii) the dimensions $\dim L_i$, $i \in \mathbb{Z}$ are uniformly bounded.

EXAMPLE 1.31.– The Lie superalgebra of superderivations

$$W(1:n) = \mathrm{Der}\,\mathbb{C}[t, t^{-1}, \xi_1, \ldots, \xi_n]$$

graded by degrees of t is a superconformal algebra.

EXAMPLE 1.32.– Let $\alpha \in \mathbb{C}$. Then $S(n,\alpha) = \{D \in W(1:n) \mid \mathrm{div}(t^{\alpha} D) = 0\} < W(1:n)$. Here if

$$D = f_0 \frac{\partial}{\partial t} + \sum_{i=1}^{n} f_i \frac{\partial}{\partial \xi_i}, \quad f_i \in \mathbb{C}[t, t^{-1}, \xi_1, \ldots, \xi_n]$$

then

$$\mathrm{div}\, D = \frac{\partial f_0}{\partial t} + \sum_{i=1}^{n} (-1)^{|f_i|} \frac{\partial f_i}{\partial \xi_i}.$$

If $\alpha \in \mathbb{Z}$, then $[S(n,\alpha), S(n,\alpha)]$ is a proper ideal in $S(n,\alpha)$ and the superalgebra $[S(n,\alpha), S(n,\alpha)]$ is simple. This family of superalgebras appeared in physics literature (Ademollo *et al*. 1976; Schwimmer and Seiberg 1987) under the name "SU_2-superconformal algebras".

EXAMPLE 1.33.– Consider the associative commutative superalgebra

$$\Lambda(1:n) = \mathbb{C}[t, t^{-1}, \xi_1, \ldots, \xi_n]$$

and the contact bracket $[\ ,\ ]$ of example 1.19. The superalgebra $K(n) = (\Lambda(1:n), [\ ,\ ])$ is simple unless $n = 4$. For $n = 4$, the commutator ideal $[K(4), K(4)]$ has codimension 1 and $[K(4), K(4)]$ is a simple superalgebra. This series is known in physics literature as "SO_n-superconformal algebras" (Ademollo *et al*. 1976; Schoutens 1987).

EXAMPLE 1.34.– In section 1.5, for an arbitrary associative commutative algebra Z with a derivation $d : Z \to Z$, we constructed the Jordan superalgebra $\mathrm{JCK}(Z,d)$. The Lie superalgebra $\mathrm{CK}(Z,d)$ is the Tits–Kantor–Koecher construction of $\mathrm{JCK}(Z,d)$. Taking $Z = \mathbb{C}[t, t^{-1}]$ and $d = \frac{\partial}{\partial t}$ or $d = t\frac{\partial}{\partial t}$, we get the Cheng–Kac superalgebras $\mathrm{CK}(6)$ discovered by Cheng and Kac (1997) and independently by Grozman *et al.* (2001).

Kac and van de Leur (1989) conjectured that examples 1.31–1.34 exhaust all superconformal algebras.

The superalgebras $K(n)$ and $\mathrm{CK}(6)$ appear as Tits–Kantor–Koecher constructions of Jordan superalgebras.

Let $[\ ,\ ]$ be the Jordan bracket on $\Lambda(1:n) = \mathbb{C}[t, t^{-1}, \xi_1, \ldots, \xi_n]$ of example 1.20, $J_n = K(\Lambda(1:n), [\ ,\ ])$ is the Kantor double of this bracket. Then

$$K(n+3) \cong \mathrm{TKK}(J_n), \quad n \geq 0.$$

In Kac *et al.* (2001), we classified "superconformal" Jordan superalgebras.

THEOREM 1.11 (Kac *et al.* (2001)).– Let $J = J_{\bar{0}} + J_{\bar{1}}$ be a $\mathbb{Z}$-graded Jordan superalgebra that is graded simple and the dimensions of all homogeneous components $\dim J_i$, $i \in \mathbb{Z}$ are uniformly bounded. Then either

1) J has finitely many negative (respectively, positive) non-zero homogeneous components;

2) or J is isomorphic to one of the superalgebras J_n, $\mathrm{JCK}(6)$, $n \geq 1$ or a twisted version of it.

This theorem agrees with the Kac–van de Leur conjecture on classification of superconformal algebras.

1.9. References

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